Quantum Hall states and the representation theory of vertex operator algebras

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Outline

1. Quantum Hall systems
   - Quantum Hall effect
   - Abelian and nonabelian anyons
   - Topological quantum computation
2. Representation theory of vertex operator algebras
   - Vetrex operator algebras, modules and intertwining operators
   - The category of modules for a rational vertex operator algebra
   - Wave functions for quantum Hall states and vertex operator operator algebras
   - Study of intertwining operator algebras (nonabelian anyons)
3. Applications
   - From wavefunctions to modular tensor categories
   - An approach to a fundamental conjecture

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Hall effect

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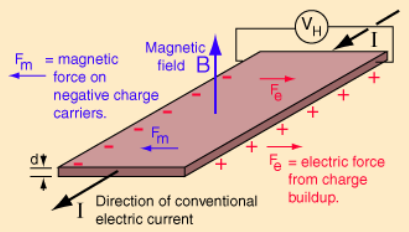
- Hall effect: When a magnetic field is applied perpendicular to the direction of a current flowing through a conductor, a measurable voltage is developed in the third perpendicular direction.



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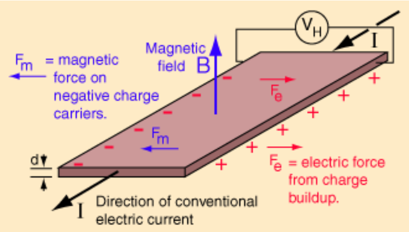

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- Hall resistance: The ratio

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Quantum Hall effect

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- When magnetic field is strong and the temperature is low, the Hall resistance R_H is quantized:

$$R_H = \frac{h}{\nu e^2},$$

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Quantum Hall states

- In a two-dimensional quantum system, a state is given by a wavefunction on the complex plane. When there are N electrons, a state is given by a wavefunction of the form $\Psi(z_1, \ldots, z_N)$ where $z_1, \ldots, z_N$ are complex variables.
- Laughlin states: In the case that $\nu = \frac{1}{3}$, Robert B. Laughlin found in 1983 the $k = 3$ case of the wavefunction

$$\prod_{i>j}(z_i - z_j)^k e^{-\sum_k \frac{|z_k|^2}{4\ell_0}}$$

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Quantum Hall states

- Moore-Read Pfaffian states: In the case that $\nu = \frac{5}{2}$, using conformal fiels theory, Gregory Moore and Nick Read found in 1991 the wavefunction

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^m e^{-\sum_k \frac{|z_k|^2}{4\ell_0}},$$

where for a skew-symmetric matrix A, the Pfaffian $\mathrm{Pf}(A)$ is the square root of the determinant of A.

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Abelian anyons

- The wavefunctions above are for electrons only. If there are some impurity in the material and so on, there might be excitations called quasi-particles. If these quasi-particles are indistinguishable, then we can exchange them using paths in the configuration space of n-tuples $(z_1, \dots, z_n)$ satisfying $z_i \neq z_j$. These paths form the braid group B_n.
- If the wavefunction after the exchange of quasi-particles is equal to a complex number of absolute value 1 multiplying the wavefunction before the exchange, these quasi-particles are called abelian anyons.
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Nonabelian anyons

- If the wavefunctions with quasi-particles form a Hilbert space and the exchange of quasi-particles gives a unitary operator on this Hilbert space, these quasi-particles are called nonabelian anyons.
- Nonabelian anyons are still to be found in experiments. An announcement by a group of physicists about finding nonabelian anyons has not been confirmed by other groups of physicists.

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History

- Abelian anyons were first suggested by Jon Leinaas and Jan Myrheim in 1977. They are derived rigorously by Gerald Goldin, Ralph Menikoff and David Sharp in 1980 and 1981 using representations of local nonrelativistic current algebra and the corresponding diffeomorphism group. Frank Wilczek in 1982 introduced the term anyons and proposed a model for abelian anyons in connection with fractional spin in two dimensions.

History

- Nonabelian anyons were considered theoretically by Sander (F.A.) Bais (1980) Gerald Goldin, Ralph Menikoff and David Sharp (1985), Gregory Moore and Nathan Seiberg (1988), Edward Witten (1989), Klaus Fredenhagen, Karl-Henning Rehren and Bert Schroer (1989) Jürg Fröhlich and Fabrizio Gabbiani (1990). Gregory Moore and Nick Read in 1991 suggested that the quasi-particles in a system whose ground states are given by the Moore-Read Pfaffian wavefunctions above are nonabelian anyons.

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Modular tensor categories

- Nonabelian anyons can be described or even defined by modular tensor categories.
- A modular tensor category is a semisimple rigid balanced braided tensor category with finitely many irreducible objects $W_1, \ldots, W_n$ such that the matrix $(\mathrm{Tr}\, R_{W_i W_j} \circ R_{W_j W_i})$ is nondegenerate.
- If the algebraic structure of modular tensor categories can be realized in a physical system such as a quantum Hall system, then one can use such a physical system to do quantum computation. Since modular tensor category is essentially topological, this is called topological quantum computation.

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Representation theory of vertex operator algebras and two-dimensional conformal field theory

- Two-dimensional conformal field theory was first developed by physicists; in particular, by Alexander Belavin, Alexander Polyakov, Alexander Zamolodchikov, John Cardy, Daniel Friedan, Stephen Shenker, Eric Verlinde, Gregory Moore, Nathan Seiberg and many others. Maxim Kontsevich and Graem Segal gave a mathematical definition of conformal field theory.
- Two-dimensional conformal field theory can be viewed as the representation theory of vertex operator algebras, a class of algebras introduced and studied first by Alexander Belavin, Alexander Polyakov, Alexander Zamolodchikov, Richard Borcherds, Igor Frenkel, James Lepowksy and Arne Meurman.

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Representation theory of vertex operator algebras and two-dimensional conformal field theory

- In this mathematical theory, we can
  - Introduce new mathematical concepts.
  - Formulate precise conjectures and theorems.
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  - Develop mathematical tools and intuitions.
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Vertex operator algebras

A vertex operator algebra consists the following data:

- A $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$.
- A vertex operator map

$$\begin{aligned} Y_V : V \otimes V &\to V[[z, z^{-1}]], \\ u \otimes v &\mapsto Y(u, z)v. \end{aligned}$$

- A vacuum $\mathbf{1} \in V_{(0)}$.
- A conformal vector $\omega \in V_{(2)}$.

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These data satisfy the following axioms:

- Grading-restriction property: $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $V_{(n)} = 0$ when n is sufficiently negative.
- Lower-truncation property: For $u, v \in V$, $Y(u, z)v$ contains only finitely many negative power terms.
- Axioms for the vacuum: For $u \in V$, $Y(\mathbf{1}, z)u = u$ and $\lim_{z \to 0} Y(u, z)\mathbf{1} = u$.
- Axioms for the conformal element: Let $L(n) : V \to V$ be defined by $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, then $[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12}(m^3 - m)\delta_{m+b,0}$, $\frac{d}{dz} Y(u, z) = Y(L(-1)u, z)$ ($L(-1)$-derivative property) for $u \in V$ and $L(0)u = nu$ for $u \in V_{(n)}$ ($L(0)$-grading property).

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Vertex operator algebras

- Duality property: For $u_1, u_2, v \in V$, $v' \in V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$, the series

$$\langle v', Y(u_1, z_1)Y(u_2, z_2)v\rangle$$
$$\langle v', Y(u_2, z_2)Y(u_1, z_1)v\rangle$$
$$\langle v', Y(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

Modules

- Let V be a vertex operator algebra. A V-module is an $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ equipped with a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all those axioms for V which still make sense.
- An $\mathbb{N}$-gradable weak V-module is an $\mathbb{N}$-graded vector space $W = \coprod_{n\in\mathbb{N}} W_{\langle n\rangle}$ equipped with a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all those axioms for V which still make sense, except the $L(0)$-grading property.

Modules

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- An $\mathbb{N}$-gradable weak V-module is an $\mathbb{N}$-graded vector space $W = \coprod_{n \in \mathbb{N}} W_{\langle n \rangle}$ equipped with a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all those axioms for V which still make sense, except the $L(0)$-grading property.

Intertwining operators

Let W_1, W_2 and W_3 be V-modules. An intertwining operator of type $\binom{W_3}{W_1\, W_2}$ is a linear map $\mathcal{Y} : W_1 \otimes W_2 \to W_3\{z\}$, where $W_3\{z\}$ is the space of all series in complex powers of z with coefficients in W_3, satisfying all those axioms for V which still make sense, that is, a lower-truncation property, an $L(-1)$-derivative property and a duality property. Intertwining operators are the quantum fields for nonabelian anyons. The following theorem gives an algebraic structure to the quantum fields of nonabelian anyons associated to a vertex operator algebra.

Theorem (H. 1995)

For a vertex operator algebra satisfying certain conditions, intertwining operators for this vertex operator algebra have an algebraic structure called intertwining operator algebra.

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Examples

- Free bosons: Representations of infinite-dimensional Heisenberg algebras.
- Free bosons on tori: Vertex operator algebras, modules and classical vertex operators (intertwining operators) associated to lattices.
- Wess-Zumino-Novikov-Witten models: Representations of affine Lie algebras.
- Minimal models: Representations of the Virasoro algebra.
- Fermion theories: Representations of infinite-dimensional Clifford algebras, affine Lie superalgebras and superconformal algebras.
- Orbifolds, cosets and $\mathcal{W}$-algebras, including in particular the moonshine module.

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Outline

1. Quantum Hall systems
   - Quantum Hall effect
   - Abelian and nonabelian anyons
   - Topological quantum computation
2. Representation theory of vertex operator algebras
   - Vetrex operator algebras, modules and intertwining operators
   - The category of modules for a rational vertex operator algebra
   - Wave functions for quantum Hall states and vertex operator operator algebras
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   - From wavefunctions to modular tensor categories
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Modular tensor category structure

Theorem (H. 2005)

Let V be a simple vertex operator algebra satisfying the following conditions:

1. $V_{(n)} = 0$ *for* $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ *and* V' *is isomorphic to* V *as a* V*-module.*
2. *Every* $\mathbb{N}$*-gradable weak* V*-module is completely reducible.*
3. V *is* C_2*-cofinite, that is,* $\dim V/C_2(V) < \infty$ *where* $C_2(V)$ *is the subspace of* V *spanned by elements of the form* $\mathrm{Res}_z z^{-2} Y(u, z)v$ *for* $u, v \in V$.

Then the category of V*-modules has a natural structure of modular tensor category.*

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Examples and the classification problem

- Laughlin state wavefunctions can be obtained from correlation functions for the vertex operator algebra of a free boson on a circle.
- Moore-Read Pfaffian wavefunctions can be obtained from correlation functions for the tensor product of the vertex operator algebra of a free boson on a circle and the vertex operator algebra for the minimal model of central charge $1/2$.
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Analogy between codes, lattices and intertwining operator algebras (nonabelian anyons)

- Frenkel, Lepowsky and Meurman observed an analogy among codes, lattices and vertex operator algebras. But this analogy is not complete. For example, in the category of vertex operator algebras, there is no functor corresponding to the dual functor for codes and lattices.
- In 2004, I gave a complete analogy among codes, lattices and intertwining operator algebras. In particular, given an intertwining operator algebra, there is a "dual" intertwining operator algebra. Using this analogy, we can formulate many conjectures concerning intertwining operator algebras. Since intertwining operator algebras describe exactly nonabelian anyons, the results and conjectures on intertwining operator algebras are actually results and conjectures for nonabelian anyons.

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Analogy between codes, lattices and intertwining operator algebras (nonabelian anyons)

- Ongoing research project: Develop a representation theory of intertwining operator algebras. Introduced and constructed modules for an intertwining operator algebras. Established an equivalence between intertwining operator algebras containing a certain vertex operator algebra and braided tensor subcategories in the braided tensor category of modules for the vertex operator algebra. The goal is to prove some of the conjectures using this representation theory.

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Outline of the steps in the program

1. Obtain ground state wavefunctions (with only electrons) from experimental data.
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A fundamental conjecture and its proof in a special case

- Conjecture: The braid group representations given by the wavefunctions of quantum Hall states are the same as those given by the representations of the corresponding vertex operator algebras.
- Parsa Bonderson, Victor Gurarie, Chetan Nayak in 2010 proved the case of Moore-Read Pfaffian wavefunctions.

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General case

- The approach used in the proof by Bonderson, Gurarie and Nayak is not easy to be generalized to the general case.
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Thank you!