# Riemann surfaces with boundaries and the theory of vertex operator algebras

Yi-Zhi Huang

**Abstract**

The connection between Riemann surfaces with boundaries and the theory of vertex operator algebras is discussed in the framework of conformal field theories defined by Kontsevich and Segal and in the framework of their generalizations in open string theory and boundary conformal field theory. We present some results, problems, conjectures, their conceptual implications and meanings in a program to construct these theories from representations of vertex operator algebras.

## 1 Introduction

Quantum field theory is one of the greatest gifts that physicists have brought to mathematicians. In physics, the quantum-field-theoretic models of electromagnetism, weak and strong interactions are among the most successful theories in physics. Besides its direct applications in building realistic physical models, quantum field theory is also an important ingredient and a powerful tool in string theory or M theory, which is being developed by physicists as a promising candidate of a unified theory of all interactions including gravity. To mathematicians, the most surprising thing is that ideas and intuition in quantum field theory have been used successfully by physicists to make deep mathematical conjectures and to discover the unity and beauty of different branches of mathematics in connection with the real world. The quantum invariants of knots and three-dimensional manifolds, Verlinde formula, mirror symmetry and Seiberg-Witten theory are among the most famous examples. The results predicted by these physical ideas and intuition also suggest that many seemingly-unrelated mathematical branches are in fact different aspects of a certain yet-to-be-constructed unified theory. The success of physical ideas and intuition in mathematics has shown that, no matter how abstract it is, our mathematical theory is deeply related to the physical world in which we live.

One of the most challenging problem for mathematicians is to understand precisely the deep ideas and conjectures obtained by physicists using quantum field theories. In particular, quantum field theories must be formulated precisely and constructed mathematically. The modern approach to quantum field theory used by physicists is the one based on path integrals. Though path integrals are in general not defined mathematically, their conjectured properties can be extracted to give axiomatic definitions of various notions of quantum field theory. When a definition of quantum field theory is given mathematically, a well-formulated problem is how to construct such a theory. The most successful such theories are topological quantum field theories, since the state spaces of these theories are typically finite-dimensional and consequently many existing mathematical theories can readily be applied to the construction. Outside the realm of topological theories, quantum field theories seem to be very difficult to construct and study because any nontrivial nontopological theory must have an infinite-dimensional state space.

Fortunately there is one simplest class of nontopological quantum field theories—two-dimensional conformal field theories—which has been studied extensively by physicists and mathematicians using various approaches. It is expected that a complete mathematical understanding of these conformal field theories will not only solve problems related to these theories, but will also give hints and insights to other classes of quantum field theories.

These theories arose in both condensed matter physics and string theory. The systematic development of (two-dimensional) conformal field theories in physics started with the seminal paper by Belavin, Polyakov and Zamolodchikov [BPZ] and the study of their geometry was initiated by Friedan and Shenker in [FS]. Mathematically, around the same time, using the theory of vertex operators developed in the early days of string theory and in the representation theory of affine Lie algebras, Frenkel, Lepowsky and Meurman [FLM1] gave a construction of what they called the "moonshine module," an infinite-dimensional representation of the Fischer-Griess Monster finite simple group and, based on his insight in representations of affine Lie algebras and the moonshine module, Borcherds [B] introduced the notion of vertex algebra. The theory of vertex operator algebras was further developed by Frenkel, Lepowsky and Meurman in [FLM2] and, in the same work, the statement by Borcherds in [B] that the moonshine module has a structure of a vertex operator algebra was proved. Motivated by the path integral formulation of string theory, I. Frenkel initiated a program to construct geometric conformal field theories in a suitable sense in 1986. Also around the same time, Y. Manin [Man] pointed out that the moduli space of algebraic curves of all genera should play the same role in the representation theory of the Virasoro algebra as the space $G/P$ ($P$ is a parabolic subgroup) in the representation theory of a semisimple Lie group $G$. Soon M. Kontsevich [Ko] and A. Beilinson and V. Schechtman [BS] found the relationship between the Virasoro algebra and the determinant line bundles over the moduli spaces of curves with punctures.

In 1987, Kontsevich and Segal independently gave a precise definition of conformal field theory using the properties of path integrals as axioms (see [S1]). To explain the rich structure of chiral parts of conformal field theories, Segal in [S2] and [S3] further introduced the notions of modular functor and weakly conformal field theory and sketched how to obtain a conformal field theory from a suitable weakly conformal field theory. In [Va], C. Vafa also gave, on a physical level of rigor, a formulation of conformal field theories using Riemann surfaces with punctures and local coordinates vanishing at these punctures.

Starting from Segal's axioms and some additional properties of rational conformal field theories, Moore and Seiberg [MS1] [MS2] constructed modular tensor categories and proved the Verlinde conjecture, which states that the fusion rules are diagonalized simultaneously by the modular transformation corresponding to $\tau \mapsto -1/\tau$. Around the same time, Tsuchiya, Ueno and Yamada [TUY] constructed the the algebro-geometric parts of conformal field theories corresponding to the Wess-Zumino-Novikov-Witten models. The algebro-geometric parts of conformal field theories corresponding to the minimal models were later constructed by Beilinson, Feigin and Mazur [BFM]. Though these algebro-geometric parts of conformal field theories do not give (weakly) conformal field theories in the sense of Segal, it seems that they are needed, explicitly or implicitly, in any future construction of full theories.

The definition of conformal field theory by Kontsevich and Segal is based on Riemann surfaces with boundaries. It was first observed by I. Frenkel that vertex operators, when modified slightly, actually correspond to the unit disk with two smaller disks removed. Based on a thorough study of the sewing operation for the infinite-dimensional moduli space of spheres with punctures and local coordinates and the determinant line bundle over this moduli space, the author in [H1], [H2] and [H3] gave a geometric definition of vertex operator algebra and proved that the geometric and the algebraic definitions are equivalent. In [Z], Y. Zhu proved that for a suitable vertex operator algebra, the $q$-traces of products of vertex operators associated to modules are modular invariant in a certain sense. The moduli space mentioned above contains the moduli space of genus-zero Riemann surfaces with boundaries and analytic boundary parametrizations, and therefore we expect that vertex operator algebras will play a crucial role in the construction of conformal field theories.

Recently, in addition to the continuing development of conformal field theories, several new directions have attracted attentions. For example, boundary conformal field theories first developed by Cardy in [C1] and [C2] play a fundamental role in many problems in condensed matter physics, and they have also become one of the main tools in the study of $D$-branes, which are nonperturbative objects in string theory. In the framework of topological field theories, boundary topological field theories (open-closed topological field theories) have been studied in detail by Lazaroiu [L] and by Moore and Segal [Mo] [S4]. Some part of the analogue in the conformal case of the work in [L], [Mo] and [S4] has been done mathematically by Felder, Fröhlich, Fuchs and Schweigert [FFFS] and by Fuchs, Runkel and Schweigert [FRS1] (see also [FRS2]). But in this conformal case, even boundary conformal field theories ("open-closed conformal field theories") on disks with parametrized boundary segments still need to be fully constructed and studied and the mathematics involved is obviously very hard but also very deep. On the other hand, many of the ideas, mathematical tools and structures used in the study of conformal field theories can be adapted to the study of open-closed conformal field theories and $D$-branes, which can be viewed as substructures in open-closed conformal field theories. Besides the obvious problem of constructing and classifying open-closed conformal field theories, the study of $D$-branes and their possible use in geometry have led to exciting and interesting mathematical problems.

Logarithmic conformal field theories initiated by Gurarie in [Gu] provide another new direction. These theories, which describe disordered systems in condensed matter physics, also occur naturally in the mathematical study of conformal field theories. In [Mil], Milas formulated and studied some of the basic ingredients of logarithmic conformal field theories in terms of representations of vertex operator algebras. One expects that there should also be a geometric formulation of logarithmic conformal field theory. The main mathematical problems would consist of the construction and classification of such theories. The representation theory of the Virasoro algebra with central charge 0 provides a crucial example of such a theory. We shall not discuss logarithmic conformal field theories in this paper. Instead, we refer the reader to the papers mentioned above and many other papers by physicists (for example, the expositions [Ga] by Gaberdiel, [F] by Flohr and the references there).

In the present paper, we shall discuss some results, problems, conjectures, their conceptual implications and meanings in a program to construct conformal field theories and their generalizations from representations of vertex operator algebras. In the next section, we recall a definition of (closed) conformal field theory by Kontsevich and Segal and other important notions introduced by Segal. We discuss the program of constructing conformal field theories in Section 3. In Section 4, we discuss open-closed conformal field theories which incorporate the axiomatic properties of conformal field theories and boundary conformal field theories.

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## 2 Conformal field theories

In this section, we recall the definition of conformal field theory in the sense of Kontsevich and Segal and other notions introduced by Segal.

First, we recall the definition of conformal field theory. For more details, see [S1], [S2] and [S3]. Consider the following symmetric monoidal category $\mathcal{B}$ constructed geometrically: Objects of $\mathcal{B}$ are finite ordered sets of copies of the unit circle $S^1$. Given two objects, morphisms from one object to another are conformal equivalence classes of Riemann surfaces (including degenerate ones, e.g., circles, and possibly disconnected) with oriented and ordered boundary components together with analytic parametrizations of these components such that the copies of $S^1$ in the domain and codomain parametrize the negatively oriented and positively oriented boundary components, respectively. For an object containing $n$ unit circles, the identity on it is the degenerate surface given by the $n$ unit circles with the trivial parametrizations of the boundary components. Given two composable morphisms, we can compose them using the boundary parametrizations in the obvious way. It is easy to see that the composition satisfies the associativity and any morphism composed with an identity is equal to itself. Thus we have a category. This category has a symmetric monoidal category structure defined by disjoint unions of objects and morphisms. We shall use $[\Sigma]$ to denote the conformal equivalence class of a Riemann surfaces $\Sigma$ with oriented, ordered and parametrized boundary components.

We also have a symmetric tensor category $\mathcal{T}$ of complete locally convex topological vector spaces over $\mathbb{C}$ with nondegenerate bilinear forms. A projective functor from $\mathcal{B}$ to $\mathcal{T}$ is a functor from $\mathcal{B}$ to the projective category of $\mathcal{T}$ which has the same objects as $\mathcal{T}$ but the morphisms are one-dimensional spaces of morphisms of $\mathcal{T}$. Note that for any functor or projective functor $\Phi$ of monoidal categories from $\mathcal{B}$ to $\mathcal{T}$, if $\Phi(S^1) = H$, then the image under $\Phi$ of the object containing $n$ ordered copies of $S^1$ must be $H^{\otimes n}$.

A *conformal field theory* (or *closed conformal field theory*) is a projective functor $\Phi$ from the symmetric monoidal category $\mathcal{B}$ to the symmetric tensor category $\mathcal{T}$ satisfying the following axioms: (i) Let $[\Sigma]$ be a morphism in $\mathcal{B}$ from $m$ ordered copies of $S^1$ to $n$ ordered copies of $S^1$. Let $[\Sigma_{\widehat{\widehat{i,j}}}]$ be the morphism from $m-1$ copies of $S^1$ to $n-1$ copies of $S^1$ obtained from $[\Sigma]$ by identifying the boundary component of $\Sigma$ parametrized by the $i$-th copy of $S^1$ in the domain of $[\Sigma]$ with the boundary component of $\Sigma$ parametrized by the $j$-th copy of $S^1$ in the codomain of $[\Sigma]$. (Note that the two copies of $S^1$ identified might or might not be on a same connected component of $\Sigma$.) Then the trace between the $i$-th tensor factor of the domain and the $j$-th tensor factor of the codomain of $\Phi([\Sigma])$ exists and is equal to $\Phi([\Sigma_{\widehat{\widehat{i,j}}}])$. (ii) Let $[\Sigma]$ be a morphism in $\mathcal{B}$ from $m$ ordered copies of $S^1$ to $n$ ordered copies of $S^1$. Let $[\Sigma_{i\to n+1}]$ be the morphism from the set of $m-1$ ordered copies of $S^1$ to the set of $n+1$ ordered copies of $S^1$ obtained by changing the $i$-th copy of $S^1$ of the domain of $[\Sigma]$ to the $n+1$-st copy of $S^1$ of the codomain of $[\Sigma_{i\to n+1}]$.

Then $\Phi([\Sigma])$ and $\Phi([\Sigma_{i\to n+1}])$ are related by the map from $\mathrm{Hom}(H^{\otimes m}, H^{\otimes n})$ to $\mathrm{Hom}(H^{\otimes m-1}, H^{\otimes (n+1)})$ obtained using the map $H \to H^*$ corresponding to the bilinear form $(\cdot, \cdot)$.

A *real conformal field theory* is a conformal field theory together with an anti-linear involution $\theta$ from $H$ to itself satisfying the following additional axiom: Let $[\Sigma]$ be a morphism in $\mathcal{B}$ from $m$ ordered copies of $S^1$ to $n$ ordered copies of $S^1$ and $[\overline{\Sigma}]$ the morphism in $\mathcal{B}$ from $n$ ordered copies of $S^1$ to $m$ ordered copies of $S^1$ obtained by taking the complex conjugate complex structure of the one on $[\Sigma]$ (note that the orientations of the boundary components are reversed). Then $\Phi([\overline{\Sigma}]) = \theta^{\otimes m} \circ \Phi^*([\Sigma]) \circ (\theta^{-1})^{\otimes n}$ where $\Phi^*([\Sigma])$ is the adjoint of $\Phi([\Sigma])$.

The definitions above do not reveal many important ingredients in concrete models. In particular, they do not give the detailed structure of chiral and anti-chiral parts of conformal field theories, that is, parts of conformal field theories depending on the moduli space parameters analytically and anti-analytically. It is known that meromorphic fields in a conformal field theory form a vertex operator algebra. The representations of this vertex operator algebra form the chiral parts of the theory. Therefore to construct conformal field theories from vertex operator algebras, it is necessary to study first chiral and anti-chiral parts of conformal field theories. Axiomatically, chiral and anti-chiral parts of conformal field theories are weakly conformal field theories defined by G. Segal in [S2] and [S3] and are generalizations of conformal field theories.

To describe weakly conformal field theories, we first need to describe modular functors. Instead of Riemann surfaces with parametrized boundary components, we need *Riemann surfaces with labeled, oriented and parametrized boundaries*, which are Riemann surfaces with oriented and parametrized boundaries and an assignment of an element of a fixed set $\mathcal{A}$ to each boundary component. The set $\mathcal{A}$ is typically the set of equivalence classes of irreducible modules for a vertex operator algebra. We consider a category whose objects are conformal equivalence classes of Riemann surfaces with labeled, oriented and parametrized boundaries and whose morphisms are given by the sewing operation, that is, if one such equivalent class can be obtained from another using the sewing operation, then the procedure of obtaining the second surface from the first one is a morphism. We use $[\Sigma]$ to denote the conformal equivalence class of the surface $\Sigma$.

A *modular functor* is a functor $E$ from the above category of Riemann surfaces with labeled, oriented and parametrized boundaries to the category of finite-dimensional vector spaces over $\mathbb{C}$ satisfying the following conditions: (i) $E([\Sigma_1 \sqcup \Sigma_2])$ is naturally isomorphic to $E([\Sigma_1]) \otimes E([\Sigma_2])$. (ii) If $\Sigma$ is obtained from another surface $\Sigma_a$ by sewing two boundary components with opposite orientations but with the same label $a \in \mathcal{A}$ of $\Sigma_a$, then $E([\Sigma])$ is naturally isomorphic to $\oplus_{b\in\mathcal{A}} E([\Sigma_b])$ where for $b \neq a$, $\Sigma_b$ is the surface obtained from $\Sigma_a$ by changing the label $a$ to $b$ on the boundary components to be sewn. (iii) $\dim E([S^2]) = 1$. (iv) $E([\Sigma])$ depends on $\Sigma$ holomorphically.

From a modular functor $E$, we can construct a symmetric monoidal category $\mathcal{B}_E$ extending the category $\mathcal{B}$ as follows: Objects of $\mathcal{B}_E$ are ordered sets of pairs of the form of a copy of the unit circle $S^1$ and an element of the set $\mathcal{A}$. Morphisms of $\mathcal{B}_E$ are pairs of the form of an equivalence class $[\Sigma]$ of Riemann surface with labeled, oriented and parametrized boundaries and the vector space $E([\Sigma])$, such that the labels of the boundary components of $\Sigma$ match with the labels of the copies of $S^1$ in the domain and codomain. Let $E$ be a modular functor labeled by $\mathcal{A}$. A *weakly conformal field theory over $E$* is a functor $\Phi$ from the symmetric monoidal category $\mathcal{B}_E$ to the symmetric tensor category $\mathcal{T}$ satisfying the obvious axioms similar to those for conformal field theories.

# 3 Vertex operator algebras and conformal field theories

The definitions of the notions in the preceding section of conformal field theory, modular functor and weakly conformal field theory are simple, beautiful and conceptually satisfactory. The obvious first problem is the following:

**Problem 3.1** *Construct conformal field theories, real conformal field theories, modular functors and weakly conformal field theories.*

Since the definition of (weakly) conformal field theory involves algebra, geometry and analysis, it is not surprising that the construction of examples is very hard. Up to now, for only the free fermion theories has a complete construction been sketched, by Segal in an unpublished manuscript; this construction has been further clarified recently by I. Kriz [Kr]. Also the method used for free fermion theories does not generalize to other cases. In [S2], Segal described a construction of conformal field theories from weakly conformal field theories satisfying a "unitarity" condition. Thus the main hard problem is actually the construction of modular functors and weakly conformal field theories. In this section, we discuss results and problems in a program to construct weakly conformal field theories in the sense of Segal from representations of suitable vertex operator algebras.

Since Riemann surfaces with labeled, oriented and parametrized boundaries can be decomposed into genus-zero Riemann surfaces with labeled, oriented and parametrized boundaries, the first step is to construct the genus-zero weakly conformal field theories, that is, to construct spaces associated to the unit circle with labels, to construct maps associated to genus-zero surfaces and to prove the axioms which make sense for genus-zero surfaces. This step has been worked out by the author using the theory of intertwining operator algebras developed by the author and the tensor product theory for modules for a vertex operator algebra of Lepowsky and the author. See [H1]–[H11] and [HL3]–[HL6] for the construction and the theories used.

For basic notions in the theory of vertex operator algebras, we refer the reader to [FLM2] and [FHL]. We need the following notions to state the main result: Let $V$ be a vertex operator algebra and $W$ a $V$-module. Let $C_1(W)$ be the subspace of $W$ spanned by elements of the form $u_{-1}w$ for $u \in V_+ = \coprod_{n>0} V_{(n)}$ and $w \in W$. If $\dim W/C_1(W) < \infty$, we say that $W$ is $C_1$-*cofinite* or $W$ satisfies the $C_1$-*cofiniteness condition*. A *generalized* $V$-*module* is a pair $(W, Y)$ satisfying all the conditions for a $V$-module except for the two grading-restriction conditions.

The main result in the genus-zero case is the following:

**Theorem 3.2** *Let $V$ be a vertex operator algebra satisfying the following conditions:*

1. *Every generalized $V$-module is a direct sum of irreducible $V$-modules.*

2. *There are only finitely many inequivalent irreducible $V$-modules and they are all $\mathbb{R}$-graded.*

3. *Every irreducible $V$-module satisfies the $C_1$-cofiniteness condition.*

*Then there is a natural structure of genus-zero weakly conformal field theory with the set $\mathcal{A}$ of equivalence classes of irreducible $V$-modules as the set of labels and suitable locally convex topological completions $H^a$ of representatives of equivalence classes $a \in \mathcal{A}$ as the spaces corresponding to the unit circle with label $a$.*

For vertex operator algebras associated to minimal models, WNZW models, $N=1$ superconformal minimal models and $N=2$ superconformal unitary models, the conditions of Theorem 3.2 are satisfied.

The second logical step is to construct genus-one theories, that is, to construct maps associated to genus-one surfaces and prove the axioms which make sense for genus-one surfaces. The first result in this step was obtained by Zhu [Z]. To state this result, we need some further notions: Let $V$ be a vertex operator algebra and $C_2(V)$ is the subspace of $V$ spanned by elements of the form $u_{-2}v$ for $u, v \in V$. Then we say that $V$ is *$C_2$-cofinite* or satisfies the *$C_2$-cofiniteness condition* if $V/C_2(V)$ is finite-dimensional. An *$\mathbb{N}$-gradable weak $V$-module* is a pair $(W = \coprod_{n\in\mathbb{N}} W_{(n)}, Y)$ satisfying all the conditions for a $V$-module except that the grading-restriction conditions are not required and the $L(0)$-grading condition is replaced by the condition that $u_nW_{(k)} \subset W_{(m-n-1+k)}$ for $u \in V_{(m)}$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. The following is Zhu's result (see [Z]):

**Theorem 3.3** *Let $V$ be a vertex operator algebra of central charge $c$ satisfying the following conditions:*

1. *$V$ has no nonzero elements of negative weights.*
2. *Every $\mathbb{N}$-gradable $V$-module is completely reducible.*
3. *There are only finitely many irreducible $V$-modules.*
4. *$V$ satisfies the $C_2$-cofiniteness condition.*
5. *$V$ as a module for the Virasoro algebra is generated by lowest weight vectors.*

*Let $\{M_1, \dots, M_m\}$ be a complete set of representatives of equivalence classes of (inequivalent) irreducible $V$-modules. Then for any $n \in \mathbb{Z}_+$, we have:*

1. *For $i = 1, \dots, m$, $u_1, \dots, u_n \in V$,*

$$\mathrm{Tr}_{M_i} Y(z_1^{L(0)}u_1, z_1) \cdots Y(z_n^{L(0)}u_n, z_n) q^{L(0)-\frac{c}{24}}$$

*is absolutely convergent when $1 > |z_1| > \cdots > |z_n| > |q| > 0$ and can be analytically extended to a meromorphic function in the region $1 > |z_1|, \dots, |z_n| > |q| > 0$, $z_i \neq z_j$ when $i \neq j$.*

*2. Let*

$$S_{M_i}((u_1, z_1), \dots, (u_n, z_n), \tau)$$

*for $i = 1, \dots, m$ be the analytically extensions above with $z_j$ replaced by $e^{2\pi i z_j}$ for $j = 1, \dots, n$, and $q$ replaced by $e^{2\pi\tau}$. For $i = 1, \dots, m$, $u_1, \dots, u_n \in V$ and*

$$\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \in \mathrm{SL}(2, \mathbb{Z}),$$

$$S_{M_i}\left(\left(\left(\frac{1}{\gamma\tau+\delta}\right)^{L(0)} u_1, \frac{z_1}{\gamma\tau+\delta}\right), \dots, \left(\left(\frac{1}{\gamma\tau+\delta}\right)^{L(0)} u_n, \frac{z_n}{\gamma\tau+\delta}\right), \tau\right)$$

*is a linear combination of*

$$S_{M_j}((u_1, z_1), \dots, (u_n, z_n), \tau)$$

*for $j = 1, \dots, m$.*

In [H11], it was proved that the conclusion of Theorem 3.2 is also true if for $n < 0$, $V_{(n)} = 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$ and Conditions 2-4 in Theorem 3.3 hold.

By modifying Zhu's method, Dong, Li, Mason [DLM] and Miyamoto [Miy1] [Miy2] generalized Zhu's result above in a number of directions. In particular, it was shown in [DLM] that Condition 5 is not needed in Theorem 3.3. But to construct genus-one theories completely, we need to generalize Zhu's result to intertwining operator algebras. In this general case, Zhu's method cannot be modified to study the $q$-traces of products of more than one intertwining operators. Recently, the author has solved this problem. Here we describe the results.

First we have:

**Theorem 3.4** *Let $V$ be a vertex operator algebra of central charge $c$ such that the associativity for intertwining operators for $V$ (see [H3], [H6] and [H9]) hold, $W_i$ $V$-modules satisfying the $C_2$-cofiniteness condition, $\tilde{W}_i$, $i = 1, \dots, n$, $V$-modules, and $\mathcal{Y}_i$, $i = 1, \dots, n$, intertwining operators of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$, respectively, where, for convenience, we use the convention $\tilde{W}_0 = \tilde{W}_n$. Then for any $w_i \in W_i$, $i = 1, \dots, n$, the series*

$$\mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) q^{L(0) - \frac{c}{24}}$$

*is absolutely convergent in the region $1 > |z_1| > \cdots > |z_n| > |q| > 0$ and can be analytically extended to a (multivalued) analytic function in the region $1 > |z_1|, \dots, |z_n| > |q| > 0$, $z_i \neq z_j$ when $i \neq j$.*

Let $A_j$, $j \in \mathbb{Z}_+$, be complex numbers defined by

$$\frac{1}{2\pi i}\log(1+2\pi i w) = \left(\exp\left(\sum_{j\in\mathbb{Z}_+} A_j w^{j+1}\frac{\partial}{\partial w}\right)\right) w$$

(see [H5]). Then we have

$$\begin{aligned}
&\frac{1}{2\pi i}\log(1+e^{-2\pi i z}(w-z))\\
&\quad = \left(\frac{e^{-2\pi i z}}{2\pi i}\right)^{(w-z)\frac{\partial}{\partial(w-z)}}\left(\exp\left(\sum_{j\in\mathbb{Z}_+} A_j (w-z)^{j+1}\frac{\partial}{\partial(w-z)}\right)\right)(w-z).
\end{aligned}$$

Note that the composition inverse of $\frac{1}{2\pi i}\log(1+2\pi i w)$ is $\frac{1}{2\pi i}(e^{2\pi i w}-1)$ and thus we have

$$\frac{1}{2\pi i}(e^{2\pi i w}-1) = \left(\exp\left(-\sum_{j\in\mathbb{Z}_+} A_j w^{j+1}\frac{\partial}{\partial w}\right)\right) w.$$

Let $V$ be a vertex operator algebra. For any $V$-module $W$, we shall denote the operator $\sum_{j\in\mathbb{Z}_+} A_j L(j)$ on $W$ by $L^+(A)$. Then

$$e^{-\sum_{j\in\mathbb{Z}_+} A_j L(j)} = e^{-L^+(A)}.$$

Let $W_i$, $\tilde{W}_i$ and $w_i \in W_i$ for $i = 1, \dots, n$ be as in Theorem 3.4. For any intertwining operators $\mathcal{Y}_i$, $i = 1, \dots, n$, of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$, respectively, let

$$F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_n; z_1,\dots,z_n; \tau)$$

be the analytic extension of

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1((2\pi i e^{2\pi i z_1})^{L(0)}e^{-L^+(A)}w_1, e^{2\pi i z_1})\cdots\\
&\qquad\qquad \cdots\mathcal{Y}_n((2\pi i e^{2\pi i z_n})^{L(0)}e^{-L^+(A)}w_n, e^{2\pi i z_n})q^{L(0)-\frac{c}{24}}
\end{aligned}$$

in the region $1 > |e^{2\pi i z_1}|, \dots, |e^{2\pi i z_n}| > |q| > 0$, $z_i \neq z_j$ when $i \neq j$, where $q = e^{2\pi i \tau}$. We now consider the vector space $\mathcal{F}_{w_1,\dots,w_n}$ spanned by all such functions. Then we have the following result obtained in [H12]:

**Theorem 3.5** *Let $V$ be a vertex operator algebra satisfying the following conditions:*

1. *For $n < 0$, $V_{(n)} = 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$.*
2. *Every $\mathbb{N}$-gradable $V$-module is completely reducible.*
3. *$V$ satisfies the $C_2$-cofiniteness condition.*

*Then for any $V$-modules $W_i$ and $\tilde{W}_i$ and any intertwining operators $\mathcal{Y}_i$ of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$ ($i = 1, \ldots, n$), respectively, and any*

$$\left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in SL(2, \mathbb{Z}),$$

*we have*

$$F_{\mathcal{Y}_1, \ldots, \mathcal{Y}_n}\left(\left(\frac{1}{\gamma\tau+\delta}\right)^{L(0)} w_1, \ldots, \left(\frac{1}{\gamma\tau+\delta}\right)^{L(0)} w_n; \frac{z_1}{\gamma\tau+\delta}, \ldots, \frac{z_n}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) \in \mathcal{F}_{w_1, \ldots, w_n}$$

We also need to show that the traces corresponding to more general self-sewing of spheres also give modular invariants. This is related to the following problem on the uniformization of annuli:

**Problem 3.6** *Given an annulus, we know that it is conformally equivalent to a standard one. Is it possible to obtain the conformal map from the standard one to the given one by sewing disks with analytically parametrized boundaries to pants successively?*

In the problem, by obtaining a conformal map by sewing disks with analytically parametrized boundaries to pants we mean the following: Cut a hole in an annulus with analytically parametrized boundary components, give the new boundary component an analytic parametrization and then sew a disk with analytically parametrized boundary to the resulting pants. The result is another annulus. If we sew such disks many times and choose the boundary parametrizations such that the resulting annulus is conformally equivalent to the original one. Then we in fact obtain this conformal equivalence by sewing disks to pants successfully.

We conjecture that this is possible, but probably one might need to sew infinitely many times. If we do need to sew infinitely many times, then we will need to take limits and will have to work with an analytic theory.

Using all the results above in genus-one, we can construct genus-one weakly conformal field theories.

The third step is to construct weakly conformal field theories in any genus. In particular, we need to construct locally convex completions of irreducible modules for the vertex operator algebra. In fact, if higher-genus correlation functions are constructed and their sewing property is proved, one can construct the locally convex completions from these correlation functions using the same method as the one used in the construction of genus-zero completions in [H8] and [H10] (see also [CS] for some properties of the topology constructed in [H8]): We first construct completions involving only genus-zero surfaces and then add elements associated to higher-genus surfaces. Here by elements associated to higher-genus surfaces, we mean the following: Consider a Riemann surface with one positively oriented boundary component. If there is a conformal field theory, this surface gives a map from $\mathbb{C}$ to $H$. The image of $1 \in \mathbb{C}$ is called the *element associated to the surface*. For example, in the genus-one case, for any $V$-module $W$, the one point function $\mathrm{Tr}_W Y(e^{2\pi i z L(0)}e^{-L^+(A)}\cdot, e^{2\pi i z})q^{L(0)-\frac{c}{24}}$ should be viewed as an element of $V^*$ and should belong to the final completion of the module $V^*$. In this step, the main nontrivial problem is actually the convergence of series corresponding to the sewing of higher-genus Riemann surfaces with parametrized boundaries. The higher-genus modular invariance can be obtained from the genus-zero duality (associativity) and the genus-one modular invariance. The details involve deep results in the analytic theory of Teichmüller spaces and moduli spaces and their generalizations by Radnell in [R] and have been intensively studied by Radnell and the author.

The final step is to put two copies of a weakly conformal field theories together suitably to get a conformal field theory. In [S2], Segal gave a description of how this step can be done for a weakly conformal field theories satisfying a certain unitarity condition.

Even though the problem of constructing conformal field theories has not been worked out completely, the existing results already give many useful and interesting results, for example, the Verlinde formula and the modular tensor category structure mentioned above. Here we discuss briefly the problem of whether a vertex operator algebra or its suitable completion is a double loop space. We need the notion of operad which was first formulated by May in

[May]. See for example, [May], [KM] and [H5] for the concepts needed in the theorem below, including algebras over operads and the little disk operad. The following theorem is obtained in [H8] and [H10] based on the geometry of vertex operator algebras in [H1], [H2], [H5] and its reformulation in [HL1] and [HL2] using the language of (partial) operads:

**Theorem 3.7** *Let $V$ be a vertex operator algebra. Then there exists a unique minimal locally convex completion $H^V$ of $V$ such that $H^V$ is a genus-zero holomorphic conformal field theory. Here by minimal we mean that any genus-zero holomorphic conformal field theory containing $V$ must contain $H^V$. In particular, $H^V$ is an algebra over the little disk operad.*

In [May], May proved the following recognition principle for double loop spaces:

**Theorem 3.8** *A space over the little disk operad has the weak homotopy type of a double loop space.*

Combining these two theorems, we conclude immediately that the locally convex completion of a vertex operator algebra and the subset of its nonzero elements have the weak homotopies of double loop spaces. But actually the operad underlying the completion of a vertex operator algebra is much richer than the little disk operad. So we can ask:

**Problem 3.9** *Can the operad underlying the completion of a vertex operator algebra recognize more "space-time" properties and structures of the completion of the algebra? Here by "space-time" properties and structures we mean, for example, whether the algebra has a structure of a double loop space and so on. Especially, can it recognize "space-time" properties and structures homeomorphically, not only (weak) homotopically? Can it recognize geometric properties and structures, not only topological ones?*

An answer to the questions in this problem will undoubtedly provide a deep understanding of the topological and geometric properties of conformal field theories.

# 4 Open-closed conformal field theories

Boundary conformal field theories [C1] [C2] are natural generalizations of conformal field theories. In physics, boundary conformal field theories describe realistic physical phenomena and also describe $D$-branes in string theory. In this section, we generalize Kontsevich's and Segal's notion of conformal field theory to a notion of open-closed conformal field theory which incorporates the axiomatic properties of conformal field theories and boundary conformal field theories. In the topological case, a notion of open-closed topological field theory has been introduced and studied by Lazaroiu [L] and by Moore and Segal [Mo] [S4]. Some part of the analogue in the conformal case of the work [L], [Mo] and [S4] has been done mathematically by Felder, Fröhlich, Fuchs and Schweigert [FFFS] and by Fuchs, Runkel and Schweigert [FRS1] (see also [FRS2]). But in this case, the full analogues of (noncommutative) associative algebras in the topological case discussed in [L], [Mo] and [S4] should be algebras over the so-called "Swiss-cheese" operad. These algebras still need to be constructed and studied mathematically. The goal of our project on open-closed (or boundary) conformal field theories is to establish the corresponding formulations and results in the conformal case. Here we shall discuss only the case that the surfaces involved are orientable. The unorientable theories can be formulated similarly using unorientable surfaces with conformal structures and the corresponding results can be obatined using double covering surfaces of the unoriented surfaces and certain operations in the state spaces. To avoid confusion, in the present section, we shall call a conformal field theory discussed in the preceding two sections explicitly a closed conformal field theory.

Closed conformal field theories describe closed string theory perturbatively. So geometrically they are defined in terms of Riemann surfaces with boundaries, where the boundary components correspond to closed strings. In string theory, there are also open strings. Also, $D$-branes are submanifolds of the space-time on which the end points of open strings move. To describe open strings perturbatively, one still uses Riemann surfaces with boundaries, but each boundary component is divided into two parts: the part corresponding to open strings and the part corresponding to the motions of the end points of open strings.

We first give a geometric symmetric monoidal category $\mathcal{B}^{OC}$ for open-closed conformal field theories. The object of $\mathcal{B}^{OC}$ are ordered sets of finitely many copies of $[0,1]$ and $S^1$. The morphisms are conformal equivalence classes of Riemann surfaces with oriented, ordered and analytically parametrized segments of boundary components (parametrized by the copies of $[0,1]$ in the objects) and oriented, ordered and analytically parametrized boundary components (parametrized by the copies of $S^1$ in the objects). Note that by the definition of boundary component, the boundary components containing segments parametrized by $[0,1]$ and the boundary compoents parametrized by $S^1$ are certainly disjoint. Also it is possible that there are boundary components which are not parametrized at all, that is, do not contain any segments parametrized by $[0,1]$ and are also not parametrized by $S^1$. The monoidal structure is also given by disjoint union as in the case of $\mathcal{B}$. Note that the category $\mathcal{B}$ discussed in Section 1 is a subcategory of $\mathcal{B}^{OC}$. Also note that for any functor or projective functor $\Phi$ of monoidal categories from $\mathcal{B}^{OC}$ to $\mathcal{T}$, if $\Phi(S^1) = H^C$ and $\Phi([0,1]) = H^O$, then the images under $\Phi$ of objects of must be in the tensor subcategory of $\mathcal{T}$ generated by $H^C$ and $H^O$.

An *open-closed conformal field theory* is a projective functor $\Phi$ from $\mathcal{B}^{OC}$ to the category $\mathcal{T}$ satisfying the following conditions: (i) Let $\Sigma$ be a morphism in $\mathcal{B}^{OC}$ from the ordered set of $m$ copies of $S^1$ and $p$ copies of $[0,1]$ to the ordered set of $n$ copies of $S^1$ and $q$ copies of $[0,1]$. Let $\Sigma_{\widehat{\widehat{i,j}}}$ (or $\Sigma_{\widehat{i,j}}$) be the morphism from the ordered set of $m-1$ copies of $S^1$ and $p$ copies of $[0,1]$ (or the ordered set of $m$ copies of $S^1$ and $p-1$ copies of $[0,1]$) to the ordered set of $n-1$ copies of $S^1$ and $q$ copies of $[0,1]$ (or the ordered set of $n$ copies of $S^1$ and $q-1$ copies of $[0,1]$) obtained from $\Sigma$ by identifying the $i$-th copy of $S^1$ (or the $i$-th copy of $[0,1]$) in the domain of $\Sigma$ with the $j$-th copy of $S^1$ (or the $j$-th copy of $[0,1]$) in the codomain of $\Sigma$. Then the trace between the $i$-th copy of $H^C$ (or the $i$-th copy of $H^O$) in the domain of $\Phi(\Sigma)$ and the $j$-th copy of $H^C$ (or the $j$-th copy of $H^O$) in the codomain of $\Phi(\Sigma)$ exists and is equal to $\Phi(\Sigma_{\widehat{\widehat{i,j}}})$ (or $\Phi(\Sigma_{\widehat{i,j}})$). (ii) Let $\Sigma$ be a morphism in $\mathcal{B}^{OC}$ from the ordered set of $m$ copies of $S^1$ and $p$ copies of $[0,1]$ to the ordered set of $n$ copies of $S^1$ and $q$ copies of $[0,1]$. Let $\Sigma_{i\to n+1}$ (or $\Sigma^{i\to n+1}$) be the morphism from the ordered set of $m-1$ copies of $S^1$ and $p$ copies of $[0,1]$ (or the ordered set of $m$ copies of $S^1$ and $p-1$ copies of $[0,1]$) to the ordered set of $n+1$ copies of $S^1$ and $q$ copies of $[0,1]$ (or the ordered set of $n$ copies of $S^1$ and $q+1$ copies of $[0,1]$) obtained from $\Sigma$ by changing the $i$-th copy of $S^1$ (or the $i$-th copy of $[0,1]$) in the domain of $\Sigma$ to the $n+1$-st copy of $S^1$ (or the $q+1$-st copy of $[0,1]$) in the codomain of $\Sigma_{i\to n+1}$ (or $\Sigma^{i\to n+1}$). Then $\Phi(\Sigma)$ and $\Phi(\Sigma_{i\to n+1})$ (or $\Phi(\Sigma)$ and $\Phi(\Sigma_{i\to n+1})$) are related by the map obtained using the map $H^C \to (H^C)^*$ ($H^O \to (H^O)^*$) corresponding to the bilinear form $(\cdot,\cdot)_C$ (or

$(\cdot,\cdot)_O$).

We consider an open-closed conformal field theory with the spaces $H^C$ and $H^O$. Consider a Riemann surface $\Sigma_C^O$ with two boundary components, one of its boundary component being negatively oriented and parametrized by $S^1$ and the other boundary component having one and only one positively oriented segment parametrized by $[0,1]$. Then by definition, the map corresponding to the equivalence class of $\Sigma_O^C$ is a map $\iota_{\Sigma_O^C}$ from $H^C$ to $H^O$. Similarly, associated to such a surface but with the opposite orientations on the boundary components, denoted by $\Sigma_O^C$, we have a map $\iota_{\Sigma_O^C}$ from $H^O$ to $H^C$. Note that in the case of open-closed topological field theories studied in [L] and [Mo] [S4], such maps are unique since all these surfaces are topologically equivalent. But in an open-closed conformal field theory, these maps depend on the equivalence classes of the surface $\Sigma_C^O$ or $\Sigma_O^C$.

It is also not difficult to see that the important Cardy condition [C2] is a consequence of the definition above. In fact, a cylinder whose boundary are not parametrized at all can be obtained from a rectangle, with two opposite sides parametrized by $[0,1]$ and the other two sides not parametrized, by identifying the two parametrized opposite sides. But it can also be obtained by sewing three cylinders: The first has one boundary component which contains no segments parametrized by $[0,1]$ (so this component is not parametrized at all) and has one positively oriented boundary component parametrized by $S^1$; the second one has one negatively oriented and one positively oriented boundary components which are both parametrized by $S^1$; the third has one boundary component which contains no segments parametrized by $[0,1]$ (so as in the case of the first cylinder, this componenet is not parametrized at all) and has one negatively oriented boundary component parametrized by $S^1$. According to the axioms, the map corresponding to the cylinder obtained from a rectangle by identifying two opposite sides is the trace of the map corresponding to the rectangle. According to the axioms again, the other way of obtaining this cylinder means that it is in fact the composition of three maps: The first is a map from $\mathbb{C}$ to $H^C$; the second is a map from $H^C$ to $H^C$; the third is a map from $H^C$ to $\mathbb{C}$. Since the first and the last map are actually equivalent to an element of $H^C$ and an element of $(H^C)^*$, respectively, we see that the composition of the three maps above is equivalent to the matrix element between these two elements of $H^C$ and $(H^C)^*$ of the map from $H^C$ to $H^C$. Now by the axioms again, we see that the trace of the map corresponding to the rectangle is equal to this matrix element. This is exactly the *Cardy condition*.

As in the case of closed conformal field theories, the most important problem is the following:

**Problem 4.1** *Construct open-closed conformal field theories satisfying the definition above.*

From the definition, we see that open-closed conformal field theories must have closed conformal field theories as subtheories. Therefore it is natural to try to construct open-closed conformal field theories from closed conformal field theories. Here is a strategy used by physicists: A Riemann surface for open-closed conformal field theories can be doubled to obtain a Riemann surface for closed conformal field theories. This procedure establishes a connection, at least geometrically, between closed conformal field theories and open-closed conformal field theories. This connection provides a concrete way to construct open-closed conformal field theories from closed conformal field theories.

For closed conformal field theories, one starts with the genus-zero case. For open-closed conformal field theories, the construction of genus-zero theories is again the first step. The simplest genus-zero surfaces in this case are Riemann surfaces with only one boundary component and with segments of the boundary component parametrized by $[0, 1]$. In particular, these surfaces do not have boundary components parametrized by $S^1$.

We consider certain such special Riemann surfaces. They are the closed upper half unit disk with some smaller disks inside and some smaller upper half disks centered on the real line removed and with the obvious parametrizations for all the full and half circles. These Riemann surfaces form an operad and is called the "Swiss cheese" operad by Voronov [Vo]. Just as in the case of closed conformal field theories, our first step is to construct algebras over this operad. In fact, in an open-closed topological field theory discussed in [L] and [Mo], there is an associative algebra which in general is not commutative. Algebras over the Swiss cheese operad should be viewed as analogues of such algebras in open-closed conformal field theories.

To construct algebras over the Swiss cheese operad, we form doubles of these surfaces. These doubles are disks with smaller disks removed and are special elements of the little disk operad. So any algebra over the little disk operad is automatically an algebra over the Swiss cheese operad. In particular, the locally convex completion of a vertex operator algebra is an algebra over the Swiss cheese operad. But there are more such algebras. First, the vertex operator algebras corresponding to the upper halves and to the lower halves of the doubles can differ by an isomorphism. Second, since the middle disks are always centered on the real line, we do not have to worry about the multivalued property of the corresponding operators and thus we can place elements of modules for the vertex operator algebras at these middle disks. In particular, we can construct algebras over the Swiss cheese operad from suitable subalgebras of the intertwining operator algebras constructed from modules and intertwining operators for vertex operator algebras. This method of constructing algebras over the Swiss cheese operad is studied by Kong and the author in [HK]. In particular, in [HK], a notion of open-string vertex algebra is introduced and studied and examples of such algebras are constructed. It is established in [HK] that the category of open-string vertex algebras is equivalent to the category of so-called differentiable-meromorphic algebras over a partial operad which is an extension of the swiss cheese operad.

Note that in this case the vertex operator algebras corresponding to the upper and lower half disks can differ by an isomorphism. Thus we have to make sure that they agree on the real line and this requirement is called a *boundary condition*. In general it is certainly not true that these boundary conditions for vertex operators hold when the vertex operators act on all elements of a module for the algebra. But it is possible that when the vertex operators act on certain special elements of modules for the algebra, a boundary condition is satisfied. These special elements are called *boundary states* for the given boundary condition. In all known examples in physics (certainly still to be constructed mathematically), the Hilbert space $H^O$ for the open-closed conformal field theory can be decomposed as a direct sum of spaces $H^O_{ab}$ which are suitable completions of the spaces of boundary states obtained by imposing boundary conditions labeled by $a$ and $b$ at the boundary segments corresponding to the movements of the two end points 0 and 1, respectively, of the open string described by $[0, 1]$. Note that for boundary states, it is important to have locally convex completion of modules for the vertex operator algebra since boundary states are in general infinite series of elements of the homogeneous subspaces of modules. A boundary condition at the boundary segment corresponding to the movement of an end point of an open string amounts to exactly a so-called *D-brane* in the string theory.

In analogy with the problem of whether a vertex operator algebra or its completion is a double loop space, for open-closed conformal field theories, we also have the analogous problem of whether an algebra or a space over the Swiss cheese operad has a certain loop space structure. Given a topological space with a base point, consider paths from the base point to arbitrary points. Now consider based loops in the space of all such paths. It is easy to see that the Swiss cheese operad acts on the space of such loops.

**Problem 4.2** *Can we recognize such a loop space structure from a structure of a space over the Swiss cheese operad?*

If possible, then we have a topological space such that open strings (paths) move in it. Such a picture might be helpful in the study of $D$-branes in a genegral conformal field theory background since they can then be viewed as subsets in the topological space.

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Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019

*E-mail address*: yzhuang@math.rutgers.edu