# Lower-bounded and grading-restricted twisted modules for affine vertex (operator) algebras

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**Abstract**

We apply the construction of the universal lower-bounded generalized twisted modules by the author to construct universal lower-bounded and grading-restricted generalized twisted modules for affine vertex (operator) algebras. We prove that these universal twisted modules for affine vertex (operator) algebras are equivalent to suitable induced modules of the corresponding twisted affine Lie algebra or quotients of such induced modules by explicitly given submodules.

## 1 Introduction

In [Hua5], the author constructed universal lower-bounded generalized twisted modules for a grading-restricted vertex algebra. In the present paper, we apply this construction to construct and identify explicitly universal lower-bounded and grading-restricted generalized twisted modules for affine vertex (operator) algebras. In particular, general classes of lower-bounded and grading-restricted generalized twisted modules can be studied using these universal ones.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with a nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)$ and $g$ an automorphism of $\mathfrak{g}$. Then an induced module $M(\ell,0)$ of level $\ell\in\mathbb{C}$ for the affine Lie algebra $\hat{\mathfrak{g}}$ generated by the trivial module $\mathbb{C}$ for $\mathfrak{g}$ has a structure of grading-restricted vertex algebra. In the case that $\mathfrak{g}$ is simple and $\ell\neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, $M(\ell,0)$ has a conformal vector and is thus a vertex operator algebra. Let $L(\ell,0)$ be the irreducible quotient of $M(\ell,0)$. Then $L(\ell,0)$ is also a graidng-restricted vertex algebra and, when $\ell+h^\vee\neq 0$, is a vertex operator algebra. An automorphism $g$ of $\mathfrak{g}$ induces automorphisms, still denoted by $g$, of $\hat{\mathfrak{g}}$, $M(\ell,0)$ and $L(\ell,0)$. There is also a twisted affine Lie algebra $\hat{\mathfrak{g}}^{[g]}$ constructed using $\mathfrak{g}$, $(\cdot,\cdot)$ and $g$. Note that $\mathfrak{g}$ has a rich automorphism group containing the Lie group corresponding to $\mathfrak{g}$. Automorphisms of $\mathfrak{g}$, $M(\ell,0)$ and $L(\ell,0)$ are mostly of infinite orders and many of them do not act on $\mathfrak{g}$, $M(\ell,0)$ and $L(\ell,0)$ semisimply.

Twisted modules associated to automorphisms of finite orders of a vertex operator algebra were introduced and studied first by Frenkel, Lepowsky and Meurman in [FLM1], [FLM2] and [FLM3] and by Lepowsky in [Le1] and [Le2]. In [Hua1], the author introduced twisted modules associated to general automorphisms of a vertex operator algebra, including in particular, automorphisms which do not act on the vertex operator algebra semisimply. A particular class of examples associated to such general automorphisms were also given in [Hua1]. In [Hua5], the author gave a construction of universal lower-bounded generalized twisted modules associated to such general automorphisms of a grading-restricted vertex algebra.

Applying the construction in [Hua5] to $M(\ell, 0)$, we construct universal lower-bounded (grading-restricted) generalized $g$-twisted $M(\ell, 0)$-modules generated by a vector space (a finite-dimensional module for a suitable subalgebra $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$ of $\hat{\mathfrak{g}}^{[g]}$) with actions of $g$, its semisimple and unipotent parts, and some other operators and annihilated by the positive part of $\hat{\mathfrak{g}}^{[g]}$ when $M(\ell, 0)$ is viewed as a grading-restricted vertex algebra. When $\ell + h^{\vee} \neq 0$ and $M(\ell, 0)$ is viewed as a vertex operator algebra, we also construct universal lower-bounded (grading-restricted) generalized $g$-twisted $M(\ell, 0)$-modules generated by a vector space (a finite-dimensional module for $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$) with additional structures as above. These universal lower-bounded and grading-restricted generalized $g$-twisted $M(\ell, 0)$-modules as $\hat{\mathfrak{g}}^{[g]}$-modules are then proved to be equivalent to suitable induced modules for $\hat{\mathfrak{g}}^{[g]}$. The proofs of these equivalences use the results in Section 2 of [Hua6] on a linearly independent set of generators of the universal lower-bounded generalized twisted modules constructed in Section 5 of [Hua5]. In the case that $M(\ell, 0)$ is viewed as a vertex operator algebra, we also give explicit formulas for the Virasoro operators on the universal lower-bounded generalized twisted $M(\ell, 0)$-modules. These formulas are needed in the proof of their equivalences to suitable induced module for $\hat{\mathfrak{g}}^{[g]}$.

When $\mathfrak{g}$ is simple and $\ell \in \mathbb{Z}_{+}$ and $L(\ell, 0)$ is viewed as a vertex operator algebra, we construct universal lower-bounded (grading-restricted) generalized $g$-twisted $M(\ell, 0)$-modules generated by a vector space (a finite-dimensional $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$-module) with additional structures as above. We also prove that these universal lower-bounded and grading-restricted generalized $g$-twisted $L(\ell, 0)$-modules as $\hat{\mathfrak{g}}^{[g]}$-modules are equivalent to quotients by certain explicitly given submodules of the induced modules for $\hat{\mathfrak{g}}^{[g]}$ equivalent to the universal lower-bounded and grading-restricted generalized $g$-twisted $M(\ell, 0)$-modules. To prove these equivalences, we also generalize a result of Kac (see Proposition 8.1 in [K]) on automorphisms of finite orders of a finite-dimensional simple Lie algebra to semisimple automorphisms of arbitrary orders.

Immediate consequences of the universal properties satisfied by those universal twisted modules are that lower-bounded and grading-restricted generalized $g$-twisted $M(\ell, 0)$- and $L(\ell, 0)$-modules generated by subspaces and finite-dimensional $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$-submodules with additional structures as above are quotients of these universal ones. Thus we can study these types of twisted modules, including untwisted ones, using our results on the universal ones in the present paper.

In the case that $g$ is of finite order, Li gave the relationship between weak twisted modules for an affine vertex operator algebra and restricted modules for the corresponding twisted affine Lie algebra in [Li]. In [B], Bakalov introduced twisted affine Lie algebras in the case that $g$ is a general automorphsim of $\mathfrak{g}$ and gave the relationship between weak twisted modules for an affine vertex operator algebra and restricted modules for the corresponding twisted affine Lie algebra. In the present paper, we do not study these most general weak twisted modules and restricted modules. We study only lower-bounded and grading-restricted generalized twisted modules for affine vertex (operator) algebras and lower-bounded and grading-restricted modules for the twisted affine Lie algebras. We would like to emphasize that in the representation theory of vertex (operator) algebras, to obtain substantial results, we have to restrict ourselves to grading-restricted generalized (twisted) modules and we often have to further restrict ourselves to such modules of finite lengths. On the other hand, lower-bounded generalized (twisted) modules always appear in various constructions and proofs. One of the difficult problems is to prove that these lower-bounded generalized (twisted) modules appearing in our constructions and proofs are actually grading-restricted generalized (twisted) modules of finite lengths. So these two types of twisted modules are what we are mainly interested. Moreover, for such modules of finite lengths, we can reduce their study to those modules generated by subspaces annihilated by the positive part of the twisted affine Lie algebra. This is the reason why we choose to construct, identify and study these types of twisted modules in this paper. Though weak twisted modules are more general, usually we need them only in the formulations of certain notions in the representation theory of vertex (operator) algebras.

As is mentioned in the preceding paragraph, one of the difficult problems in the representation theory of vertex operator algebras is to prove that suitable lower-bounded generalized twisted modules are actually grading restricted. In fact, universal lower-bounded generalized twisted modules are in a certain sense analogous to Verma modules in the representation theory of finite-dimensional Lie algebras. In the case of finite-dimensional Lie algebras, we know that a Verma module generated from a highest weight vector has a finite-dimensional quotient module if and only if the highest weight is dominant integral. For vertex operator algebra, we can ask an analogous question: Under what conditions, a universal lower-bounded generalized twisted module has a grading-restricted quotient. In this paper, our construction and identification of grading-restricted generalized twisted modules for $M(\ell, 0)$ and $L(\ell, 0)$ gives an answer to this question for affine vertex (operator) algebras.

One main goal of studying these twisted modules is to use their properties and structures to study twisted intertwining operators among them (see [Hua3]). We expect that the constructions and results in the present paper will play an important role in the study of twisted intertwining operators for affine vertex operator algebras.

The present paper is organized as follows: In Section 2, we recall some basic material on the affine Lie algebra $\hat{\mathfrak{g}}$ of a finite-dimensional Lie algebra $\mathfrak{g}$, an automorphism $g$ of $\mathfrak{g}$ and the twisted affine Lie algebra $\hat{\mathfrak{g}}^{[g]}$. In Section 3, we recall vertex (operator) algebras $M(\ell, 0)$ and $L(\ell, 0)$ associated to affine Lie algebras and their automorphisms induced from those of $\mathfrak{g}$. The construction, identification and basic properties of lower-bounded and grading-restricted generalized twisted modules for $M(\ell, 0)$ are given in Section 4. In Subsection 4.1, we construct and identify explicitly lower-bounded and grading-restricted generalized twisted modules for $M(\ell, 0)$ viewed as a grading-restricted vertex algebra. In Subsection 4.2, we construct and identify explicitly such twisted modules for $M(\ell, 0)$ viewed as a vertex operator algebra. In Subsection 4.3, basic properties of these twisted modules for $M(\ell, 0)$, including their universal properties and their quotients, are given. The construction, identification and basic properties of lower-bounded and grading-restricted generalized twisted modules for $L(\ell, 0)$ are given in Section 5.

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# 2 Twisted affine Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $(\cdot, \cdot)$ a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Recall that the affine Lie algebra $\hat{\mathfrak{g}}$ is the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ equipped with the bracket operation

$$\begin{aligned}
[a \otimes t^m, b \otimes t^n] &= [a, b] \otimes t^{m+n} + (a, b)m\delta_{m+n,0}\mathbf{k}, \\
[a \otimes t^m, \mathbf{k}] &= 0,
\end{aligned}$$

for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Let $\hat{\mathfrak{g}}_{\pm} = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}]$. Then

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{-} \oplus \mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \hat{\mathfrak{g}}_{+}.$$

Let $g$ be an automorphism of $\mathfrak{g}$. Assume also that $(\cdot, \cdot)$ is invariant under $g$. This is true in the case that $\mathfrak{g}$ is semisimple and $(\cdot, \cdot)$ is proportional to the Killing form. Since $\mathfrak{g}$ is finite dimensional, there exist operators $\mathcal{L}_g$, $\mathcal{S}_g$ and $\mathcal{N}_g$ on $\mathfrak{g}$ such that $g = e^{2\pi i \mathcal{L}_g}$ and $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent parts of $\mathcal{L}_g$, respectively. Then $g$, $\mathcal{L}_g$, $\mathcal{S}_g$ and $\mathcal{N}_g$ induce operators, still denoted by $g$, $\mathcal{L}_g$, $\mathcal{S}_g$ and $\mathcal{N}_g$, on the affine Lie algebra $\hat{\mathfrak{g}}$ such that $g$ is also an automorphism of $\hat{\mathfrak{g}}$.

Let

$$P_{\mathfrak{g}} = \{\alpha \in \mathbb{C} \mid \Re(\alpha) \in [0, 1), e^{2\pi i \alpha} \text{ is an eigenvalue of } g\}.$$

Then

$$\mathfrak{g} = \coprod_{\alpha \in P_{\mathfrak{g}}} \mathfrak{g}^{[\alpha]},$$

where for $\alpha \in P_{\mathfrak{g}}$, $\mathfrak{g}^{[\alpha]}$ is the generalized eigenspace of $g$ (or the eigenspace of $e^{2\pi i \mathcal{S}_g}$) with the eigenvalue $e^{2\pi i \alpha}$.

For $\alpha, \beta \in [0, 1) + i\mathbb{R}$, let

$$s(\alpha, \beta) = \begin{cases} \alpha + \beta & \Re(\alpha + \beta) < 1 \\ \alpha + \beta - 1 & \Re(\alpha + \beta) \geq 1. \end{cases}$$

Then $\Re(s(\alpha, \beta)) \in [0, 1)$ and for $\alpha, \beta \in P_{\mathfrak{g}}$, $s(\alpha, \beta) \in (P_g + P_g) \cup (P_g + P_g - 1)$.

**Lemma 2.1** *For $\alpha, \beta \in P_{\mathfrak{g}}$, $[\mathfrak{g}^{[\alpha]}, \mathfrak{g}^{[\beta]}] \subset \mathfrak{g}^{[s(\alpha,\beta)]}$. In particular, in the case that $[\mathfrak{g}^{[\alpha]}, \mathfrak{g}^{[\beta]}] \neq 0$, $s(\alpha, \beta) \in P_g \cap ((P_g + P_g) \cup (P_g + P_g - 1)) \subset P_g$ and $e^{2\pi i(\alpha+\beta)}$ is an eigenvalue of $g$.*

*Proof.* For $a \in \mathfrak{g}^{[\alpha]}$ and $b \in \mathfrak{g}^{[\beta]}$, we have

$$\begin{aligned}
&(g - e^{2\pi i(\alpha+\beta)})[a, b]\\
&\quad = [ga, gb] - e^{2\pi i(\alpha+\beta)}[a, b]\\
&\quad = e^{2\pi i(\alpha+\beta)}[e^{2\pi i\mathcal{N}_g}a, e^{2\pi i\mathcal{N}_g}b] - e^{2\pi i(\alpha+\beta)}[a, b]\\
&\quad = e^{2\pi i(\alpha+\beta)}\left([(1_{\mathfrak{g}} + (e^{2\pi i\mathcal{N}_g} - 1_{\mathfrak{g}}))a, (1_{\mathfrak{g}} + (e^{2\pi i\mathcal{N}_g} - 1_{\mathfrak{g}}))b] - [a, b]\right)\\
&\quad = e^{2\pi i(\alpha+\beta)}\left([(e^{2\pi i\mathcal{N}_g} - 1_{\mathfrak{g}})a, b] + [a, (e^{2\pi i\mathcal{N}_g} - 1_{\mathfrak{g}})b] + [(e^{2\pi i\mathcal{N}_g} - 1_{\mathfrak{g}})a, (e^{2\pi i\mathcal{N}_g} - 1_{\mathfrak{g}})b]\right).
\end{aligned}$$

Then there exists $\widetilde{K} \in \mathbb{Z}_+$ such that

$$(g - e^{2\pi i(\alpha+\beta)})^{\widetilde{K}}[a, b] = 0.$$

(Note that we can always take $\widetilde{K} = \dim \mathfrak{g}$.) If $[a, b] = 0$, we have $[a, b] \subset \mathfrak{g}^{[s(\alpha,\beta)]}$. If $[a, b] \neq 0$, it is a generalized eigenvector of $g$ with eigenvalue $e^{2\pi i(\alpha+\beta)}$ and thus is in $\mathfrak{g}^{[s(\alpha,\beta)]}$. In this case, $\mathfrak{g}^{[s(\alpha,\beta)]} \neq 0$. So $s(\alpha, \beta) \in P_g$. We also have either $s(\alpha, \beta) = \alpha + \beta \in P_g + P_g$ or $s(\alpha, \beta) = \alpha + \beta - 1 \in P_g + P_g - 1$. Thus $s(\alpha, \beta) \in P_g \cap ((P_g + P_g) \cup (P_g + P_g - 1))$. ∎

**Corollary 2.2** *The operators $e^{2\pi i\mathcal{S}_g}$ and $e^{2\pi i\mathcal{N}_g}$ are also automorphisms of $\mathfrak{g}$. The operator $\mathcal{N}_g$ is a derivation of the Lie algebra $\mathfrak{g}$.*

*Proof.* Let $a \in \mathfrak{g}^{[\alpha]}$ and $b \in \mathfrak{g}^{[\beta]}$. By Lemma 2.1,

$$e^{2\pi i\mathcal{S}_g}[a, b] = e^{2\pi i(\alpha+\beta)}[a, b] = [e^{2\pi i\alpha}a, e^{2\pi i\alpha}b] = [e^{2\pi i\mathcal{S}_g}a, e^{2\pi i\mathcal{S}_g}b].$$

So $e^{2\pi i\mathcal{S}_g}$ is an automorphism of $\mathfrak{g}$. Therefore $e^{-2\pi i\mathcal{S}_g}$ is also an automorphism of $\mathfrak{g}$. Thus $e^{2\pi i\mathcal{N}_g} = e^{-2\pi i\mathcal{S}_g}g$ is an automorphism of $\mathfrak{g}$.

For $a, b \in \mathfrak{g}$, we have

$$\begin{aligned}
(\mathrm{ad}\, e^{2\pi i\mathcal{N}_g}a)b &= [e^{2\pi i\mathcal{N}_g}a, b]\\
&= e^{2\pi i\mathcal{N}_g}[a, e^{-2\pi i\mathcal{N}_g}b]\\
&= e^{2\pi i\mathcal{N}_g}(\mathrm{ad}\, a)e^{-2\pi i\mathcal{N}_g}b\\
&= ((\mathrm{Ad}\, 2\pi i\mathcal{N}_g)(\mathrm{ad}\, a))b.
\end{aligned}$$

Thus

$$\begin{aligned}
[\mathcal{N}_g a, b] &= \frac{1}{2\pi i}(\mathrm{ad}\, \log e^{2\pi i\mathcal{N}_g}a)b\\
&= \frac{1}{2\pi i}(\log(\mathrm{Ad}\, 2\pi i\mathcal{N}_g)(\mathrm{ad}\, a))b\\
&= ((\mathrm{ad}\, \mathcal{N}_g)(\mathrm{ad}\, a))b\\
&= \mathcal{N}_g[a, b] - [a, \mathcal{N}_g b],
\end{aligned}$$

proving that $\mathcal{N}_g$ is a derivation of $\mathfrak{g}$. ∎

**Lemma 2.3** *If $\alpha+\beta \notin \{0,1\}$, then $\mathfrak{g}^{[\alpha]}$ and $\mathfrak{g}^{[\beta]}$ are orthogonal. If $\alpha+\beta \in \{0,1\}$, then $(\cdot,\cdot)$ restricted to $\mathfrak{g}^{[\alpha]} \times \mathfrak{g}^{[\beta]}$ is nondegenerate.*

*Proof.* For $a \in \mathfrak{g}^{[\alpha]}$, there exists $p \in \mathbb{Z}_+$ such that $(g - e^{2\pi i\alpha})^p a = 0$. On the other hand, since $\alpha+\beta \notin \{0,1\}$, the restriction of $(g^{-1} - e^{2\pi i\alpha})^p$ to $\mathfrak{g}^{[\beta]}$ is a linear isomorphism from $\mathfrak{g}^{[\beta]}$ to itself. If there exist $a \in \mathfrak{g}^{[\alpha]}$ and $b \in \mathfrak{g}^{[\beta]}$ such that $(a,b) \neq 0$. then there exists $c \in \mathfrak{g}^{[\beta]}$ such that $b = (g^{-1} - e^{2\pi i\alpha})^p c$. Then $(a, (g^{-1} - e^{2\pi i\alpha})^p c) = (a,b) \neq 0$. But since $(\cdot,\cdot)$ is invariant under $g$, we have $(a, (g^{-1} - e^{2\pi i\alpha})^p c) = ((g - e^{2\pi i\alpha})^p a, c) = 0$. Contradiction. So we must have $(a,b) = 0$ for $a \in \mathfrak{g}^{[\alpha]}$ and $b \in \mathfrak{g}^{[\beta]}$.

In the case that $\alpha+\beta \in \{0,1\}$, if $(\cdot,\cdot)$ restricted to $\mathfrak{g}^{[\alpha]} \times \mathfrak{g}^{[\beta]}$ is degenerate, then there exists $a \in \mathfrak{g}^{[\alpha]} \setminus \{0\}$ such that $(a,b) = 0$ for $b \in \mathfrak{g}^{[\beta]}$. But for $\beta \in P_g$ such that $\alpha+\beta \notin \{0,1\}$, we just proved that $(a,b) = 0$ for $b \in \mathfrak{g}^{[\beta]}$. Thus $(a,b) = 0$ for $b \in \mathfrak{g}$. Contradiction to the nondegeneracy of $(\cdot,\cdot)$. ∎

**Proposition 2.4** *The nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)$ is also invariant under $e^{2\pi i\mathcal{S}_g}$ and $e^{2\pi i\mathcal{N}_g}$.*

*Proof.* Let $a \in \mathfrak{g}^{[\alpha]}$ and $b \in \mathfrak{g}^{[\beta]}$. If $\alpha+\beta \in \{0,1\}$, then $(e^{2\pi i\mathcal{S}_g}a, e^{2\pi i\mathcal{S}_g}b) = e^{2\pi i(\alpha+\beta)}(a,b) = (a,b)$. If $\alpha+\beta \notin \{0,1\}$, then by Lemma 2.3, $(e^{2\pi i\mathcal{S}_g}a, e^{2\pi i\mathcal{S}_g}b) = 0 = (a,b)$. So $(\cdot,\cdot)$ is invariant under $e^{2\pi i\mathcal{S}_g}$.

Since $e^{2\pi i\mathcal{N}_g} = e^{-2\pi i\mathcal{S}_g}g$ and certainly $(\cdot,\cdot)$ is also invariant under $e^{-2\pi i\mathcal{S}_g}$, $(\cdot,\cdot)$ is invariant under $e^{2\pi i\mathcal{N}_g}$. ∎

**Corollary 2.5** *For $a,b \in \mathfrak{g}$, we have $(\mathcal{N}_g a, b) + (a, \mathcal{N}_g b) = 0$.*

*Proof.* For $a,b \in \mathfrak{g}$, we have

$$\begin{aligned}(\mathcal{N}_g a, b) &= \left(\frac{1}{2\pi i}(1_{\mathfrak{g}} + (\log e^{2\pi i\mathcal{N}_g} - 1_{\mathfrak{g}}))a, b\right)\\ &= \left(a, \frac{1}{2\pi i}(1_{\mathfrak{g}} + (\log e^{-2\pi i\mathcal{N}_g} - 1_{\mathfrak{g}})b\right)\\ &= -(a, \mathcal{N}_g b).\end{aligned}$$

∎

**Remark 2.6** Note that if $\Re\{\alpha\} = \Re\{\beta\} = 0$, then $\Re\{\alpha+\beta\} = \Re\{s(\alpha,\beta)\} = 0$. In particular,

$$\coprod_{\Re\{\alpha\}=0} \mathfrak{g}^{[\alpha]}$$

is a subalgebra of $\mathfrak{g}$. The fixed-point subalgebra $\mathfrak{g}^{[0]}$ is a subalgebra of this subalgebra.

The decomposition

$$\mathfrak{g} = \coprod_{\alpha \in P_{\mathfrak{g}}} \mathfrak{g}^{[\alpha]}$$

induces decompositions

$$\hat{\mathfrak{g}} = \coprod_{\alpha \in P_{\mathfrak{g}}} \hat{\mathfrak{g}}^{[\alpha]}$$

where $\hat{\mathfrak{g}}^{[\alpha]}$ for $\alpha \in P_{\mathfrak{g}}$ are the generalized eigenspaces of $g$ (or the eigenspaces of $e^{2\pi i \mathcal{S}_g}$) on $\hat{\mathfrak{g}}$ with the eigenvalue $e^{2\pi i\alpha}$.

We now define the twisted affine Lie algebra associated to $\mathfrak{g}$, $(\cdot, \cdot)$ and $g$ (see, for example, [K] and [B]). Let

$$\hat{\mathfrak{g}}^{[g]} = \coprod_{\alpha \in P_{\mathfrak{g}}} \mathfrak{g}^{[\alpha]} \otimes t^{\alpha}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}.$$

We define a bracket operation on $\hat{\mathfrak{g}}^{[g]}$ by

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m(a, b)\delta_{m+n,0}\mathbf{k} + (\mathcal{N}_g a, b)\delta_{m+n,0}\mathbf{k}, \tag{2.1}$$

$$[\mathbf{k}_1, a \otimes t^m] = 0, \tag{2.2}$$

for $a \in \mathfrak{g}^{[\alpha]}$, $b \in \mathfrak{g}^{[\beta]}$, $m \in \alpha + \mathbb{Z}$, $n \in \beta + \mathbb{Z}$, $\alpha, \beta \in P_{\mathfrak{g}}$. Then it is straightforward to verify that the vector space $\hat{\mathfrak{g}}^{[g]}$ equipped with the bracket operation defined above is a Lie algebra.

Let

$$\begin{aligned}
\hat{\mathfrak{g}}^{[g]}_{+} &= \left(\bigoplus_{\alpha \in P_{\mathfrak{g}}, \Re\{\alpha\}>0} \mathfrak{g}^{[\alpha]} \otimes t^{\alpha}\mathbb{C}[t]\right) \oplus \left(\bigoplus_{\alpha \in P_{\mathfrak{g}}, \Re\{\alpha\}=0} \mathfrak{g}^{[\alpha]} \otimes t^{\alpha+1}[t]\right),\\
\hat{\mathfrak{g}}^{[g]}_{-} &= \bigoplus_{\alpha \in P_{\mathfrak{g}}} \mathfrak{g}^{[\alpha]} \otimes t^{\alpha-1}\mathbb{C}[t^{-1}],\\
\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}} &= \left(\bigoplus_{\alpha \in P_{\mathfrak{g}}, \Re\{\alpha\}=0} \mathfrak{g}^{[\alpha]} \otimes \mathbb{C}t^{\alpha}\right),\\
\hat{\mathfrak{g}}^{[g]}_{0} &= \hat{\mathfrak{g}}^{[g]}_{\mathbb{I}} \oplus \mathbb{C}\mathbf{k}.
\end{aligned}$$

Then $\hat{\mathfrak{g}}^{[g]}_{+}$, $\hat{\mathfrak{g}}^{[g]}_{-}$, $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$ and $\hat{\mathfrak{g}}^{[g]}_{0}$ are subalgebras of $\hat{\mathfrak{g}}^{[g]}$ and $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$ is a subalgebra of $\hat{\mathfrak{g}}^{[g]}_{0}$. Moreover, we have a triangular decomposition

$$\hat{\mathfrak{g}}^{[g]} = \hat{\mathfrak{g}}^{[g]}_{-} \oplus \hat{\mathfrak{g}}^{[g]}_{0} \oplus \hat{\mathfrak{g}}^{[g]}_{+}.$$

In this paper, we are interested in only those $\hat{\mathfrak{g}}^{[g]}$-modules with lower-bounded $\mathbb{C}$-gradings compatible with the grading of $\hat{\mathfrak{g}}^{[g]}$ and with actions of $g$. To be precise, we give the following definition:

**Definition 2.7** A *graded* $\hat{\mathfrak{g}}^{[g]}$*-module* is a $\hat{\mathfrak{g}}^{[g]}$-module $W$ with a $\mathbb{C}$-grading $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ such that $\hat{\mathfrak{g}}^{[g]}_{[m]} W_{[n]} \subset W_{[m+n]}$ for $m \in P_{\mathfrak{g}} + \mathbb{Z}$ and $n \in \mathbb{C}$, where $\hat{\mathfrak{g}}^{[g]}_{[m]} = \mathfrak{g}^{[\alpha]} \otimes t^m$ for $m \in (P_{\mathfrak{g}} + \mathbb{Z}) \setminus \{0\}$ and $\hat{\mathfrak{g}}^{[g]}_{[0]} = \mathfrak{g} \otimes \mathbb{C}t^0 \oplus \mathbb{C}\mathbf{k}$. A *graded* $\hat{\mathfrak{g}}^{[g]}$*-module of level* $\ell$ is a graded $\hat{\mathfrak{g}}^{[g]}$-module such that $\mathbf{k}$ acts as $\ell \in \mathbb{C}$. A *lower-bounded* $\hat{\mathfrak{g}}^{[g]}$*-module* is a graded $\hat{\mathfrak{g}}^{[g]}$-module $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ such that $W_{[n]} = 0$ when $\Re(n)$ is sufficiently negative. A *grading-restricted* $\hat{\mathfrak{g}}^{[g]}$*-module* is a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ such that $\dim W_{[n]} < \infty$ for $n \in \mathbb{C}$. A $\hat{\mathfrak{g}}^{[g]}$*-module with a compatible action of* $g$ or simply a $\hat{\mathfrak{g}}^{[g]}$*-module with an action of* $g$ is a $\hat{\mathfrak{g}}^{[g]}$-module $W$ with actions of $g$, $\mathcal{S}_g$ and $\mathcal{N}_g$ satisfying the following conditions: (i) $W$ is a direct sum of generalized eigenspaces of $g$. (ii) $g = e^{2\pi i \mathcal{L}_g}$, where $\mathcal{L}_g$ is the operator on $W$ such that $\mathcal{S}_g$ and $\mathcal{N}_g$ on $W$ are the semisimple and nilpotent parts of $\mathcal{L}_g$. (iii) $g(uw) = g(u)g(w)$ for $u \in \hat{\mathfrak{g}}^{[g]}$ and $w \in W$.

In this paper, $\hat{\mathfrak{g}}^{[g]}$-modules are always assumed to be graded and with compatible $g$ actions. So we shall call them simply $\hat{\mathfrak{g}}^{[g]}$-modules. In particular, in this paper, lower-bounded $\hat{\mathfrak{g}}^{[g]}$-modules and grading-restricted $\hat{\mathfrak{g}}^{[g]}$-modules are always with compatible $g$ actions.

# 3 Vertex operator algebras associated to affine Lie algebras and their automorphisms

We recall the vertex operator algebras constructed from suitable modules for the affine Lie algebra $\hat{\mathfrak{g}}$ and their automorphsims in this section.

Let $M$ be a $\mathfrak{g}$-module and let $\ell \in \mathbb{C}$. Let $\hat{\mathfrak{g}}_+$ act on $M$ trivially and let $\mathbf{k}$ act as the scalar multiplication by $\ell$. Then $M$ becomes a $\mathfrak{g} \oplus \mathbb{C}\mathbf{k} \oplus \hat{\mathfrak{g}}_+$-module and we have an induced $\hat{\mathfrak{g}}$-module

$$\widehat{M}_\ell = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}\oplus\mathbb{C}\mathbf{k}\oplus\hat{\mathfrak{g}}_+)} M,$$

Let $M = \mathbb{C}$ and let $\mathfrak{g}$ act on $\mathbb{C}$ trivially. The corresponding $\hat{\mathfrak{g}}$-module $\widehat{\mathbb{C}}_\ell$ is denoted by $M(\ell, 0)$. Let $J(\ell, 0)$ be the maximal proper submodule of $M(\ell, 0)$ and $L(\ell, 0) = M(\ell, 0)/J(\ell, 0)$. Then $L(\ell, 0)$ is the unique irreducible graded $\hat{\mathfrak{g}}$-module such that $\mathbf{k}$ acts as $\ell$ and the space of all elements annihilated by $\hat{\mathfrak{g}}_+$ is isomorphic to the trivial $\mathfrak{g}$-module $\mathbb{C}$.

Frenkel and Zhu [FZ] gave both $M(\ell, 0)$ and $L(\ell, 0)$ natural structures of vertex operator algebras (see also [LL]). In particular, $M(\ell, 0)$ and $L(\ell, 0)$ are grading-restricted vertex algebras. We shall apply the results in [Hua5] to construct lower-bounded and grading-restricted generalized twisted $M(\ell, 0)$- and $L(\ell, 0)$-modules. Since [Hua5] needs the first construction of grading-restricted vertex algebras in [Hua2], we describe the grading-restricted vertex algebra structures on $M(\ell, 0)$ and $L(\ell, 0)$ using the construction in Section 3 in [Hua2].

We discuss $M(\ell, 0)$ first. Note that $U(\hat{\mathfrak{g}}_-)$ is linearly isomorphic to $M(\ell, 0)$. The $\mathbb{Z}_+$-grading on $\hat{\mathfrak{g}}_-$ induces an $\mathbb{N}$-grading on $M(\ell, \lambda)$. We denote the homogeneous subspace of $M(\ell, 0)$ of degree (conformal weight) $n$ by $M_{(n)}(\ell, 0)$ for $n \in \mathbb{N}$. We denote the action of $a \otimes t^n$ on $M(\ell, 0)$ by $a(n)$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$. We also denote $1 \in M(\ell, 0)$ by $\mathbf{1}_{M(\ell,0)}$. Then $M(\ell, 0)$ is spanned by elements of the form $a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}_{M(\ell,0)}$ for $a_1, \dots, a_k \in \mathfrak{g}$ and

$n_1, \dots, n_k \in -\mathbb{Z}_+$. For $a \in \mathfrak{g}$, let $a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$. In particular, $z \mapsto a(z)$ for $z \in \mathbb{C}^\times$ is an analytic map from $\mathbb{C}^\times$ to $\mathrm{Hom}(M(\ell,0), \overline{M(\ell,0)})$.

Let $L_{M(\ell,0)}(0)$ be the operator on $M(\ell,0)$ giving the grading on $M(\ell,0)$, that is, $L_{M(\ell,0)}(0)v = nv$ for $v \in (M_{(n)}(\ell,0))$. We define an operator $L_{M(\ell,0)}(-1)$ on $M(\ell,0)$ by

$$\begin{aligned}
&L_{M(\ell,0)}(-1)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{M(\ell,0)}\\
&\quad = \sum_{i=1}^{k} n_i a_1(-n_1)\cdots a_{i-1}(-n_{i-1})a_i(-n_i-1)a_{i+1}(-n_{i+1})\cdots a_k(-n_k)\mathbf{1}_{M(\ell,0)}.
\end{aligned}$$

It is easy to verify that the series $a(x)$ for $a \in \mathfrak{g}$ and the operators $L_{M(\ell,0)}(0)$ and $L_{M(\ell,0)}(-1)$ have the following properties:

1. For $a \in \mathfrak{g}$, $[L_{M(\ell,0)}(0), a(x)] = x\frac{d}{dx}a(x) + a(x)$.
2. $L_{M(\ell,0)}(-1)\mathbf{1} = 0$, $[L_{M(\ell,0)}(-1), a(x)] = \frac{d}{dz}a(x)$ for $a \in \mathfrak{g}$.
3. For $a \in \mathfrak{g}$, $a(x)\mathbf{1}_{M(\ell,0)} \in M(\ell,0)[[x]]$. Moreover, $\lim_{x\to 0} a(x)\mathbf{1} = a(-1)\mathbf{1}_{M(\ell,0)}$.
4. The vector space $M(\ell,0)$ is spanned by elements of the form $a_1(n_1)\cdots a_k(n_k)\mathbf{1}_{M(\ell,0)}$ for $a_1, \dots, a_k \in \mathfrak{g}$ and $n_1, \dots, n_k \in \mathbb{Z}$.
5. For $a, b \in \mathfrak{g}$,
$$(x_1 - x_2)^2 a(x_1)b(x_2) = (x_1 - x_2)^2 b(x_2)a(x_1).$$

Then by Proposition 3.3 in [Hua2], $\langle v', a_1(z_1)\cdots a_k(z_k)v\rangle$ for $a_1, \dots, a_k \in \mathfrak{g}$, $v \in M(\ell,0)$ and $v' \in M(\ell,0)'$ is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ to a rational function, denoted by $R(\langle v', a_1(z_1)\cdots a_k(z_k)v\rangle)$, in $z_1, \dots, z_k$ with the only possible poles at $z_i = 0$ for $i = 1, \dots k$ and $z_i = z_j$ for $i < j$, $i, j = 1, \dots, k$.

By Theorem 3.5 in [Hua2], the vector space $M(\ell,0)$ equipped with the vertex operator map

$$Y_{M(\ell,0)} : M(\ell,0) \otimes M(\ell,0) \to M(\ell,0)[[x, x^{-1}]]$$

defined by

$$\begin{aligned}
&\langle v', Y_{M(\ell,0)}(a_1(n_1)\cdots a_k(n_k)\mathbf{1}_{M(\ell,0)}, z)v\rangle\\
&\quad = \mathrm{Res}_{\xi_1=0}\cdots \mathrm{Res}_{\xi_k=0}\xi_1^{n_1}\cdots\xi_k^{n_k} R(\langle v', a_1(\xi_1+z)\cdots a_k(\xi_k+z)v\rangle)
\end{aligned}$$

for $z \in \mathbb{C}^\times$, $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}$, $v \in M(\ell,0)$ and $v' \in M(\ell,0)'$ and the vacuum $\mathbf{1}_{M(\ell,0)}$ is a grading-restricted vertex algebra. Moreover, this is the unique grading-restricted vertex algebra structure on $M(\ell,0)$ with the vacuum $\mathbf{1}$ such that $Y(a(-1)\mathbf{1}, x) = a(x)$ for $a \in \mathfrak{g}$. In particular, this grading-restricted vertex algebra structure on $M(\ell,0)$ is the same as the one constructed in [FZ] (see also [LL]), that is, the graded space $M(\ell,0)$, the vertex operator map $Y_{M(\ell,0)}$, the operator $L_{M(\ell,0)}(-1)$ and the vacuum one are equal to the graded space, the vertex operator maps, the operator $L(-1)$ and the vacuum in [FZ].

Since $J(\ell, 0)$ is a $\hat{\mathfrak{g}}$-module, we can define the action of $a(x)$ for $a \in \mathfrak{g}$ on $L(\ell, 0) = M(\ell, 0)/J(\ell, 0)$. Similarly $L_{M(\ell,0)}(0)$ and $L_{M(\ell,0)}(-1)$ induce operators $L_{L(\ell,0)}(0)$ and $L_{L(\ell,0)}(-1)$. We also have an element $\mathbf{1}_{L(\ell,0)} = \mathbf{1} + J(\ell, 0) \in L(\ell, 0)$. It is clear that the space $L(\ell, 0)$, these series, operators and the element also satisfy the five properties above. Thus by Theorem 3.5 in [Hua2], $L(\ell, 0)$ equipped with the vertex operator map $Y_{L(\ell,0)}$ defined by

$$\begin{aligned}\langle v', &Y_{L(\ell,0)}(a_1(n_1)\cdots a_k(n_k)\mathbf{1}_{L(\ell,0)}, z)v\rangle\\ &= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{n_1}\cdots\xi_k^{n_k}R(\langle v', a_1(\xi_1+z)\cdots a_k(\xi_k+z)v\rangle)\end{aligned}$$

for $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}$, $v \in L(\ell, 0)$ and $v' \in L(\ell, 0)'$ and the vacuum $\mathbf{1}_{L(\ell,0)}$ is a grading-restricted vertex algebra. Moreover, this is the unique grading-restricted vertex algebra structure on $L(\ell, 0)$ with the vacuum $\mathbf{1}$ such that $Y(a(-1)\mathbf{1}, x) = a(x)$ for $a \in \mathfrak{g}$. In particular, this grading-restricted vertex algebra structure on $L(\ell, 0)$ is the same as the one constructed first in [FZ] (see also [LL]).

Let $g$ be an automorphism of $\mathfrak{g}$ as is discussed in the preceding section. The actions of $g$, $\mathcal{L}_g$, $\mathcal{S}_g$ and $\mathcal{N}_g$ on $\hat{\mathfrak{g}}$ further induce their actions, still denoted by $g$, $\mathcal{L}_g$, $\mathcal{S}_g$ and $\mathcal{N}_g$, on $M(\ell, 0)$ and $L(\ell, 0)$. Moreover, $g$, $e^{2\pi i\mathcal{S}_g}$ and $e^{2\pi i\mathcal{N}_g}$ are all automorphisms of the grading-restricted vertex algebras $M(\ell, 0)$ and $L(\ell, 0)$.

In the case that $\mathfrak{g}$ is simple, we shall always take $(\cdot, \cdot)$ be the normalized Killing form such that $(\alpha, \alpha) = 2$ for a long root $\alpha$. Let $h^\vee$ be dual Coxeter number of $\mathfrak{g}$. In the case that $\ell + h^\vee \neq 0$, the grading-restricted vertex algebra $M(\ell, 0)$ has a conformal element

$$\omega_{M(\ell,0)} = \frac{1}{2(\ell + h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}}(a^i)'(-1)a^i(-1)\mathbf{1},$$

where $\{a^i\}_{i=1}^{\dim\mathfrak{g}}$ is a basis of $\mathfrak{g}$ and $\{(a^i)'\}_{i=1}^{\dim\mathfrak{g}}$ is its dual basis with respect to $(\cdot, \cdot)$. See [FZ] and also [LL]. The grading-restricted vertex algebra $M(\ell, 0)$ with this conformal element is a vertex operator algebra (or a grading-restricted conformal vertex algebra). Moreover, $L_{M(\ell,0)}(0)$ and $L_{M(\ell,0)}(-1)$ above are in fact the coefficients of $x^{-2}$ and $x^{-1}$ in $Y_{M(\ell,0)}(\omega_{M(\ell,0)}, x)$. Since $\omega_{M(\ell,0)}$ is not in $J(\ell, 0)$, we see that $L(\ell, 0)$ has a conformal element $\omega_{L(\ell,0)} = \omega_{M(\ell,0)} + J(\ell, 0)$. So the grading-restricted vertex algebra $L(\ell, 0)$ with this conformal element is also a vertex operator algebra and $L_{L(\ell,0)}(0)$ and $L_{L(\ell,0)}(-1)$ are in fact the coefficients of $x^{-2}$ and $x^{-1}$ in $Y_{L(\ell,0)}(\omega_{L(\ell,0)}, x)$. Again see [FZ] and also [LL] for details.

Since the Killing form on $\mathfrak{g}$ is invariant under the action of $g$, the conformal element $\omega_{M(\ell,0)}$ and $\omega_{L(\ell,0)}$ are also invariant under $g$. Thus $g$, $e^{2\pi i\mathcal{S}_g}$ and $e^{2\pi i\mathcal{N}_g}$ are in fact automorphisms of the vertex operator algebras $M(\ell, 0)$ and $L(\ell, 0)$. Since $\mathcal{N}_g$ is a nilpotent operator on $\mathfrak{g}$, we must have $\mathcal{N}_g^{\dim\mathfrak{g}} = 0$ on $M(\ell, 0)$ and $L(\ell, 0)$.

The decomposition

$$\hat{\mathfrak{g}} = \coprod_{\alpha\in P_\mathfrak{g}}\hat{\mathfrak{g}}^{[\alpha]}$$

induced from the decomposition

$$\mathfrak{g} = \coprod_{\alpha\in P_\mathfrak{g}}\mathfrak{g}^{[\alpha]}$$

further induces decompositions

$$M(\ell,0)=\coprod_{\alpha\in P_{\mathfrak{g}}} M^{[\alpha]}(\ell,0),$$
$$L(\ell,0)=\coprod_{\alpha\in P_{\mathfrak{g}}} L^{[\alpha]}(\ell,0),$$

where $M^{[\alpha]}(\ell,0)$ and $L^{[\alpha]}(\ell,0)$ for $\alpha\in P_{\mathfrak{g}}$ are the generalized eigenspaces of $g$ (or the eigenspaces of $e^{2\pi i\mathcal{S}_g}$) on $M(\ell,0)$ and $L(\ell,0)$, respectively, with the eigenvalue $e^{2\pi i\alpha}$.

We now choose suitable generating fields $a(x)$ such that Assumption 2.1 in [Hua5] is satisfied for the grading-restricted vertex algebra $M(\ell,0)$. Since $\mathcal{L}_g$ is an operator on a finite-dimensional vector space $\mathfrak{g}$, we can find a Jordan basis $\{a^i\}_{i=1}^{\dim\mathfrak{g}}$ for $g$, that is, a basis $\{a^i\}_{i=1}^{\dim\mathfrak{g}}$ of $\mathfrak{g}$ such that under this basis, the matrix representation of $\mathcal{L}_g$ is a Jordan canonical form. We use $I$ to denote the set $\{1,\dots,\dim\mathfrak{g}\}$. Then the Jordan basis can be written as $\{a^i\}_{i\in I}$. Since $\{a^i\}_{i\in I}$ is a basis of $\mathfrak{g}$ and $M(\ell,0)$ as a grading-restricted vertex algebra is generated by fields of the form $a(x)$ for $a\in\mathfrak{g}$, $M(\ell,0)$ is also generated by the fields $a^i(x)$ for $i\in I$. Since for $i\in I$, $a^i$ is an element of a Jordan basis, there exist an $\alpha_i\in P_{\mathfrak{g}}$ and $n_i\in\mathbb{Z}$ such that $a^i$ is a generalized eigenvector of $\mathcal{L}_g$ with the eigenvalue $\alpha_i+n_i$, or equivalently, a generalized eigenvector of $g$ with the eigenvalue $e^{2\pi i\alpha_i}$. Thus $a^i(-1)\mathbf{1}$ is also a generalized eigenvector of $\mathcal{L}_g$ on $M(\ell,0)$ with the eigenvalue $\alpha_i+n_i$, or equivalently, a generalized eigenvector of $g$ on $M(\ell,0)$ with the eigenvalue $e^{2\pi i\alpha_i}$. Also since $a^i$ is an element of a Jordan basis, either $\mathcal{N}_ga^i=0$ or $\mathcal{N}_ga^i$ is another element in the basis $\{a^i\}_{i\in I}$. Therefore there exists $\mathcal{N}_g(i)\in I$ such that $\mathcal{N}_ga^i=a^{\mathcal{N}_g(i)}$. Thus we also have $\mathcal{N}_ga^i(-1)\mathbf{1}=a^{\mathcal{N}_g(i)}(-1)\mathbf{1}$. These also hold for $L(\ell,0)$. In summary, these discussions give the following:

**Proposition 3.1** *Assumption 2.1 in [Hua5] is satisfied by $M(\ell,0)$ and $L(\ell,0)$ with the set of generating fields $a^i(x)$ for $i\in I$.*

# 4 Lower-bounded and grading-restricted generalized twisted $M(\ell,0)$-modules

In this section, we first construct universal lower-bounded and grading-restricted generalized twisted modules for $M(\ell,0)$ viewed as a grading-restricted vertex algebra in Subsection 4.1. Then in the case that $\mathfrak{g}$ is simple and $\ell+h^\vee\neq 0$, we construct universal lower-bounded and grading-restricted generalized twisted modules for $M(\ell,0)$ viewed as a vertex operator algebra in Subsection 4.2. We also discuss their basic properties such as their universal properties and so on in Subsection 4.3.

## 4.1 The constructions when $M(\ell,0)$ is viewed as a grading-restricted vertex algebra

Before we give constructions of universal lower-bounded and grading-restricted generalized twisted $M(\ell,0)$-modules, we first show that such twisted modules must be a module for the twisted affine Lie algebra $\hat{\mathfrak{g}}^{[g]}$. Let $W$ be a lower-bounded generalized $g$-twisted $M(\ell,0)$-module. For $\alpha\in P_{\mathfrak{g}}$, $a\in\mathfrak{g}^{[\alpha]}$ and $n\in\alpha+\mathbb{Z}$, we write

$$Y_W^g(a(-1)\mathbf{1},x)=\sum_{k=0}^{K}\sum_{n\in\alpha+\mathbb{Z}}(a_W)_{n,k}x^{-n-1}(\log x)^k.$$

From (2.10) in [HY],

$$Y_W^g(a(-1)\mathbf{1},x)=(Y_W^g)_0(x^{\mathcal{N}_g}a(-1)\mathbf{1},x)=\sum_{k=0}^{K}\frac{1}{k!}(Y_W^g)_0((\mathcal{N}_g^k a)(-1)\mathbf{1},x)(\log x)^k,$$

where $(Y_W^g)_0(x^{\mathcal{N}_g}a(-1)\mathbf{1},x)$ is the constant term when $Y_W^g(a(-1)\mathbf{1},x)$ is viewed as a polynomial in $\log x$. So in our notation,

$$(Y_W^g)_0(x^{\mathcal{N}_g}a(-1)\mathbf{1},x)=\sum_{n\in\alpha+\mathbb{Z}}(a_W)_{n,0}x^{-n-1},$$

$$Y_W^g(a(-1)\mathbf{1},x)=\sum_{k=0}^{K}\sum_{n\in\alpha+\mathbb{Z}}\frac{1}{k!}((\mathcal{N}_g^k a)_W)_{n,0}x^{-n-1}(\log x)^k.$$

We shall need the following version ((3.24) in [Hua4]) of the Jacobi identity for $(Y_W^g)_0$ (obtained in [B] and proved to be equivalent to the duality property for $(Y_W^g)$ in [HY]):

$$\begin{aligned}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)&Y_W^g(u,x_1)Y_W^g(v,x_2)-x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_W^g(v,x_2)Y_W^g(u,x_1)\\&=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W^g\left(Y_V\left(\left(\frac{x_2+x_0}{x_1}\right)^{\mathcal{L}_g}u,x_0\right)v,x_2\right).\end{aligned}\tag{4.1}$$

This Jacobi identity holds for lower-bounded generalized twisted modules and even more general types of twisted modules for an arbitrary grading-restricted vertex algebra, including, in particular, $M(\ell,0)$ or $L(\ell,0)$. Using the commutator formula obtained from this Jacobi identity, we have the following result of Bakalov in [B] and, for reader's convenience, we give a proof:

**Proposition 4.1** *Let $W$ be a lower-bounded generalized $g$-twisted $M(\ell,0)$-module. Then $W$, with the action of $\hat{\mathfrak{g}}^{[g]}$ given by $a\otimes t^n\mapsto(a_W)_{n,0}$ and $\mathbf{k}\mapsto\ell 1_W$ for $\alpha\in P_{\mathfrak{g}}$, $a\in\mathfrak{g}^{[\alpha]}$ and $n\in\alpha+\mathbb{Z}$ and with the existing action of $g$, $\mathcal{S}_g$ and $\mathcal{N}_g$ on $W$, is a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module of level $\ell$.*

*Proof.* Taking $\mathrm{Res}_{x_0}$ on both sides of (4.1) and taking $u=a^i(-1)\mathbf{1}$ and $v=a^j(-1)\mathbf{1}$, we obtain the commutator formula

$$\begin{aligned}&Y_W^g(a^i(-1)\mathbf{1},x_1)Y_W^g(a^j(-1)\mathbf{1},x_2)-Y_W^g(a^j(-1)\mathbf{1},x_2)Y_W^g(a^i(-1)\mathbf{1},x_1)\\&\quad=\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W^g\left(Y_{M(\ell,0)}\left(\left(\frac{x_2+x_0}{x_1}\right)^{\mathcal{L}_g}a^i(-1)\mathbf{1},x_0\right)a^j(-1)\mathbf{1},x_2\right).\end{aligned}\tag{4.2}$$

By definition, the left-hand side of (4.2) is equal to

$$\sum_{m\in\alpha_i+\mathbb{Z}}\sum_{k\in\mathbb{N}}\sum_{n\in\alpha_j+\mathbb{Z}}\sum_{l\in\mathbb{N}}\frac{(-1)^k}{k!}\frac{(-1)^l}{l!}((\mathcal{N}_g^k a^i)_W)_{m,0}((\mathcal{N}_g^l a^j)_W)_{n,0}x_1^{-m-1}x_1^{-n-1}(\log x_1)^k(\log x_2)^l$$
$$-\sum_{m\in\alpha_i+\mathbb{Z}}\sum_{k\in\mathbb{N}}\sum_{n\in\alpha_j+\mathbb{Z}}\sum_{l\in\mathbb{N}}\frac{(-1)^k}{k!}\frac{(-1)^l}{l!}((\mathcal{N}_g^l a^j)_W)_{n,0}((\mathcal{N}_g^k a^i)_W)_{m,0}x_1^{-m-1}x_1^{-n-1}(\log x_1)^k(\log x_2)^l. \tag{4.3}$$

On the other hand, by straightforward calculations, we see that the right-hand side of (4.2) is equal to

$$\begin{aligned}&\mathrm{Res}_{x_0}e^{x_0\frac{\partial}{\partial x_2}}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)Y_W^g\left(Y_{M(\ell,0)}\left(\left(\frac{x_2}{x_1}\right)^{\mathcal{L}_g}a^i(-1)\mathbf{1},x_0\right)a^j(-1)\mathbf{1},x_2\right)\\
&\quad=\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\sum_{p\in\alpha_i+\alpha_j+\mathbb{Z}}\frac{(-1)^k}{k!}\frac{(-1)^l}{l!}(([\mathcal{N}_g^k a^i,\mathcal{N}_g^l a^j])_W)_{p,0}x_1^{-n-\alpha_i-1}x_2^{n+\alpha_i-p-1}(\log x_1)^k(\log x_2)^l\\
&\qquad+\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\frac{(-1)^k}{k!}\frac{(-1)^l}{l!}\ell(\mathcal{N}_g^k a^i,\mathcal{N}_g^l a^j)\frac{\partial}{\partial x_2}x_1^{-n-\alpha_i-1}x_2^{n+\alpha_i}(\log x_1)^k(\log x_2)^l.\end{aligned} \tag{4.4}$$

Taking coefficients of $x_1^{-m-1}x_2^{-n-1}(\log x_1)^k(\log x_2)^l$ for $m\in\alpha_i+\mathbb{Z}$, $n\in\alpha_j+\mathbb{Z}$, $k,l\in\mathbb{N}$ in both sides of (4.2), using (4.3) and (4.4), dividing the both results by $\frac{(-1)^k}{k!}\frac{(-1)^l}{l!}$ and then using Corollary 2.5, we obtain

$$\begin{aligned}&[((\mathcal{N}_g^k a^i)_W)_{m,0},((\mathcal{N}_g^l a^j)_W)_{n,0}]\\
&\quad=(([\mathcal{N}_g^k a^i,\mathcal{N}_g^l a^j])_W)_{m+n,0}+m(\mathcal{N}_g^k a^i,\mathcal{N}_g^l a^j)\delta_{m+n,0}\ell+(\mathcal{N}_g^{k+1}a^i,\mathcal{N}_g^l a^j)\delta_{m+n,0}\ell.\end{aligned} \tag{4.5}$$

Let $a=\mathcal{N}_g^k a^i$ and $b=\mathcal{N}_g^l a^j$. Also note that $\mathfrak{g}$ is certainly spanned by such $a$ and $b$. Then (4.5) is exactly what we can also obtain by replacing $a\otimes t^n$ and $\mathbf{k}$ in (2.1) by $(a_W)_{n,0}$ and $\ell 1_W$ for $a\in\mathfrak{g}^{[\alpha]}$ and $n\in\alpha+\mathbb{Z}$. Thus (4.5) gives $W$ a structure of a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module of level $\ell$. ∎

We first construct and identify explicitly universal lower-bounded generalized $g$-twisted $M(\ell,0)$-modules generated by a space annihilated by $\hat{\mathfrak{g}}_+^{[g]}$ using the results in Section 5 of [Hua5] when $M(\ell,0)$ is viewed as a grading-restricted vertex algebra.

Let $M$ be a vector space. Assume that $g$ acts on $M$ and there is an operator $L_M(0)$ on $M$. If $M$ is finite dimensional, then there exist operators $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$ such that on $M$, $g=e^{2\pi i\mathcal{L}_g}$ and $\mathcal{S}_g$ and $\mathcal{N}_g$ are the semisimple and nilpotent, respectively, parts of $\mathcal{L}_g$. In this case, $M$ is also a direct sum of generalized eigenspaces for the operator $L_M(0)$ and $L_M(0)$ can be decomposed as the sum of its semisimple part $L_M(0)_S$ and nilpotent part $L_M(0)_N$. Moreover, the real parts of the eigenvalues of $L_M(0)$ has a lower bound. In the case that $M$ is infinite dimensional, we assume that all of these properties for $g$ and $L_M(0)$ hold. We call the eigenvalue of a generalized eigenvector $w\in M$ for $L_M(0)$ the *(conformal) weight* of $w$

and denote it by $\mathrm{wt}\, w$. We first assume that $M$ is itself a generalized eigenspace of $L_M(0)$ with eigenvalue $h$.

Let $\{w^a\}_{a\in A}$ be a basis of $M$ consisting of vectors homogeneous in $g$-weights (eigenvalues of $g$) such that for $a \in A$, either $L_M(0)_N w^a = 0$ or there exists $L_M(0)_N(a) \in A$ such that $L_M(0)_N w^a = w^{L_M(0)_N(a)}$. For simplicity, when $L_M(0)_N w^a = 0$, we shall use $w^{L_M(0)_N(a)}$ to denote 0. Then for $a \in A$, we always have $L_M(0)_N w^a = w^{L_M(0)_N(a)}$. For $a \in A$, let $\alpha^a \in \mathbb{C}$ such that $\Re(\alpha^a) \in [0, 1)$ and $e^{2\pi i\alpha^a}$ is the eigenvalue of $g$ for the generalized eigenvector $w^a$.

Taking the grading-restricted vertex algebra $V$, the space $M$ and $B \in \mathbb{R}$ in Section 5 of [Hua5] to be $M(\ell, 0)$, the space $M$ above and $\Re(h)$, respectively, we obtain the universal lower-bounded generalized $g$-twisted $M(\ell, 0)$-module $\widehat{M}^{[g]}_h$, which we shall denote by $\widehat{M}^{[g]}_{\ell,h}$ to exhibit explicit the dependence on $\ell$. The twisted generating fields and generator twist fields for $\widehat{M}^{[g]}_{\ell,h}$ are denoted by

$$a^i_{\widehat{M}^{[g]}_{\ell,h}}(x) = \sum_{n\in\alpha^i+\mathbb{Z}} \sum_{k=0}^{K^i} (a^i_{\widehat{M}^{[g]}_{\ell,h}})_{n,k} x^{-n-1}(\log x)^k$$

for $i \in I$ and

$$\psi^a_{\widehat{M}^{[g]}_{\ell,h}}(x) = \sum_{n\in\alpha^i+\mathbb{Z}} \sum_{k=0}^{K^i} (\psi^a_{\widehat{M}^{[g]}_{\ell,h}})_{n,k} x^{-n-1}(\log x)^k$$

for $a \in A$. For simplicity, we shall denote $a^i_{\widehat{M}^{[g]}_{\ell,h}}(x)$ and $(a^i_{\widehat{M}^{[g]}_{\ell,h}})_{n,k}$ by $a^i_{[g],\ell}(x)$, $(a^i_{[g],\ell})_{n,k}$, respectively, since their commutators involve $\ell$ and denote $\psi^a_{\widehat{M}^{[g]}_{\ell,h}}(x)$ and $(\psi^a_{\widehat{M}^{[g]}_{\ell,h}})_{n,k}$ by $\psi^a_{[g]}(x)$ and $(\psi^a_{[g]})_{n,k}$, respectively. For a general element $a \in \mathfrak{g}^{[\alpha]}$ and $w \in M^{[\beta]}$, we shall use the similar notations to denote the twisted and twist fields associated to $a$ and $w$, respectively, and similarly for their components.

The construction above is based on the assumption that $M$ is itself a generalized eigenspace of $L_M(0)$ with eigenvalue $h$. In the general case, $M = \coprod_{h\in Q_M} M_{[h]}$, where $Q_M$ is the set of all eigenvalues of $L_M(0)$ and $M_{[h]}$ is the generalized eigenspace of $L_M(0)$ with the eigenvalue $h$. In this case, we have the lower-bounded generalized $g$-twisted $M(\ell, 0)$-module $\coprod_{h\in Q_M} \widehat{(M_{[h]})}^{[g]}_{\ell,h}$, which we shall denote by $\widehat{M}^{[g]}_\ell$. For $h \in Q_M$, we have a basis $\{w^a\}_{a\in A_h}$ of $M_{[h]}$ satisfying the condition $L_M(0)_N w^a = w^{L_M(0)_N(a)}$ for $a \in A_h$. Let $A = \sqcup_{h\in Q_M} A_h$. Then we have a basis $\{w^a\}_{a\in A}$ of $M$ satisfying the same condition for all $a \in A$.

We now construct a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module that we will prove to be equivalent to $\widehat{M}^{[g]}_\ell$ viewed as a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module. Let $L_{-1}$ be a basis of a one-dimensional vector space $\mathbb{C}L_{-1}$. Let $T(\mathbb{C}L_{-1})$ be the tensor algebra of the one-dimensional space $\mathbb{C}L_{-1}$. Consider the vector space $\Lambda(M) = T(\mathbb{C}L_{-1}) \otimes M$. We define actions of $g$, $\mathcal{L}_g$, $\mathcal{S}_g$ and $\mathcal{N}_g$ on $\Lambda(M)$ by acting only on the second tensor factor $M$. We define an operator $L_{\Lambda(M)}(0)$ on $\Lambda(M)$ by $L_{\Lambda(M)}(0)(L^m_{-1} \otimes w) = m(L^m_{-1} \otimes w) + L^m_{-1} \otimes L_M(0)w$ for $m \in \mathbb{N}$ and $w \in M$. We also define operators $L_{\Lambda(M)}(0)_N$ and $L_{\Lambda(M)}(0)_S$ on $\Lambda(M)$ by $L_{\Lambda(M)}(0)_S(L^m_{-1} \otimes w) = m(L^m_{-1} \otimes w) + L^m_{-1} \otimes L_M(0)_S w$ and $L_{\Lambda(M)}(0)_N(L^m_{-1} \otimes w) = L^m_{-1} \otimes L_M(0)_N w$, respectively, for $m \in \mathbb{N}$ and $w \in M$. Then $L_{\Lambda(M)}(0)_N$ and $L_{\Lambda(M)}(0)_S$ are the semisimple and nilpotent, respectively, parts of $L_{\Lambda(M)}(0)$. The space $\Lambda(M)$ is graded by the eigenvalues of $L_{\Lambda(M)}(0)$. We define another operator $L_{\Lambda(M)}(-1)$ on $\Lambda(M)$ by $L_{\Lambda(M)}(-1)(L_{-1}^{m} \otimes w) = L_{-1}^{m+1} \otimes w$. Then $\Lambda(M)$ is spanned by elements of the form $L_{\Lambda(M)}(-1)^{m}(1 \otimes w)$ for $m \in \mathbb{N}$ and $w \in M$. For simplicity, we shall identify $1 \otimes w$ with $w \in M$ and hence embed $M$ as a subspace of $\Lambda(M)$. Thus $\Lambda(M)$ is spanned by elements of the form $L_{\Lambda(M)}(-1)^{m}w$ for $w \in M$.

Let $\hat{\mathfrak{g}}^{[g]}_{+}$ act on $M$ as 0. We define an action of $\hat{\mathfrak{g}}^{[g]}_{+}$ on $\Lambda(M)$ by the commutator formula

$$[a(m), L_{\Lambda(M)}(-1)] = ma(m-1) + (\mathcal{N}_{g}a)(m-1) \tag{4.6}$$

for $a \in \mathfrak{g}^{[\alpha]}$ and $m \in \alpha + \mathbb{N}$ when $\Re(\alpha) > 0$ and $m \in \alpha + \mathbb{Z}_{+}$ when $\Re(\alpha) = 0$. Let $\mathbf{k}$ act on $\Lambda(M)$ as $\ell$. Then it is clear that $\Lambda(M)$ is a $U(\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k})$-module. and we have the induced lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ (recalling that by a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module we mean a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module with a compatible $g$ action). Using the commutator formula (4.6), we can extend the operator $L_{\Lambda(M)}(-1)$ to an operator on $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$. For simplicity, we shall still denote this extension of $L_{\Lambda(M)}(-1)$ by the same notation $L_{\Lambda(M)}(-1)$. But note that $L_{\Lambda(M)}(-1)$ acts on $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ now.

**Theorem 4.2** *As a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module, $\widehat{M}^{[g]}_{\ell}$ is equivalent to $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$.*

*Proof.* Consider the subspace $\widehat{\Lambda}(M)$ of $\widehat{M}^{[g]}_{\ell}$ spanned by elements of the form

$$L_{\widehat{M}^{[g]}_{\ell}}(-1)^{k}(\psi^{b}_{[g]})_{-1,0}\mathbf{1} \tag{4.7}$$

for $k \in \mathbb{N}$ and $b \in A$. Then we have a linear map $\rho : \Lambda(M) \to \widehat{\Lambda}(M)$ defined by

$$\rho(L_{\Lambda(M)}(-1)^{k}w^{b}) = L_{\widehat{M}^{[g]}_{\ell}}(-1)^{k}(\psi^{b}_{[g]})_{-1,0}\mathbf{1}$$

for $k \in \mathbb{N}$ and $b \in A$. In particular, $\rho(w^{b}) = (\psi^{b}_{[g]})_{-1,0}\mathbf{1}$ for $b \in A$. So $\rho(M)$ is the subspace of $\widehat{M}^{[g]}_{\ell}$ spanned by $(\psi^{b}_{[g]})_{-1,0}\mathbf{1}$ for $b \in A$. From the $\hat{\mathfrak{g}}^{[g]}$-module structure on $\widehat{M}^{[g]}_{\ell}$, we see that $\hat{\mathfrak{g}}^{[g]}_{+}$ acts on $\rho(M)$ as 0. From the commutator formula

$$[L_{\widehat{M}^{[g]}_{\ell,h}}(-1), a_{[g],\ell}(x)] = \frac{d}{dx}a_{[g],\ell}(x),$$

we obtain

$$[L_{\widehat{M}^{[g]}_{\ell,h}}(-1), (a_{[g],\ell})_{m,0}] = m(a_{[g],\ell})_{m-1,0} + ((\mathcal{N}_{g}a)_{[g],\ell})_{m-1,0}$$

for $a \in \mathfrak{g}^{[\alpha]}$ and $m \in \alpha + \mathbb{N}$ when $\Re(\alpha) > 0$ and $m \in \alpha + \mathbb{Z}_{+}$ when $\Re(\alpha) = 0$. Thus we also have an action of $\hat{\mathfrak{g}}^{[g]}_{+}$ on $\widehat{\Lambda}(M)$. From the $\hat{\mathfrak{g}}^{[g]}$-module structure on $\widehat{M}^{[g]}_{\ell}$ again, we see that $\mathbf{k}$ acts on $\widehat{M}^{[g]}_{\ell}$ as $\ell$. These actions give $\widehat{\Lambda}(M)$ a $\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k}$-module structure. From the definitions of $\rho$ and the $\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k}$-module structures on $\Lambda(M)$ and $\widehat{\Lambda}(M)$, we see that

$\rho$ is in fact a $\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k}$-module map. Moreover, by Theorem 2.4 in [Hua6], for $h \in Q_M$, $L_{\widehat{M}^{[g]}_{\ell,h}}(-1)^k(\psi^b_{[g]})_{-1,0}\mathbf{1}$ for $k \in \mathbb{N}$ and $b \in A_h$ are linearly independent and thus form a basis of $\widehat{\Lambda}(M_h)$, which is the subspace of $\widehat{\Lambda}(M)$ spanned by elements of the form (4.7) for $k \in \mathbb{N}$ and $b \in A_h$. Then $L_{\widehat{M}^{[g]}_{\ell,h}}(-1)^k(\psi^b_{[g]})_{-1,0}\mathbf{1}$ for $k \in \mathbb{N}$ and $b \in A$ form a basis of $\widehat{\Lambda}(M)$. So $\rho$ is in fact an equivalence of $\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k}$-modules and commutes with the actions of $L_{\widehat{M}^{[g]}_{\ell}}(-1)$ and $L_{\widehat{M}^{[g]}_{\ell}}(-1)$.

Now by the universal property of the induced module $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$, there exists a unique $\hat{\mathfrak{g}}^{[g]}$-module map

$$\hat{\rho} : U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k})} \Lambda(M) \to \widehat{M}^{[g]}_{\ell}$$

such that $\hat{\rho}|_{\Lambda(M)} = \rho$. Since $\widehat{M}^{[g]}_{\ell}$ as a $\hat{\mathfrak{g}}^{[g]}$-module is generated by $\Lambda(M)$, $\hat{\rho}$ is surjective. We need only prove that $\hat{\rho}$ is injective.

For $h \in Q_M$, by Theorem 2.3 in [Hua6], $\widehat{M}^{[g]}_{\ell,h}$ is spanned by elements of the form

$$(a^{i_1}_{[g],\ell})_{n_1,k_1} \cdots (a^{i_l}_{[g],\ell})_{n_l,k_l} L_{\widehat{M}^{[g]}_{\ell,h}}(-1)^k(\psi^b_{[g]})_{-1,0}\mathbf{1}$$

for $n_j \in \alpha^{i_j} + \mathbb{Z}$, $0 \leq k_j \leq K_j$, $k \in \mathbb{N}$ and $b \in A_h$. On the other hand, from

$$a^i_{[g],\ell}(x) = Y^g_{\widehat{M}^{[g]}_{\ell,h}}(a^i(-1)\mathbf{1}, x) = (Y^g_{\widehat{M}^{[g]}_{\ell,h}})_0(x^{-\mathcal{N}_g}a^i(-1)\mathbf{1}, x),$$

we obtain

$$(a^i_{[g],\ell})_{n_i,k_i} = \frac{(-1)^{k_i}}{k_i!}(\mathcal{N}^{k_i}_g a^i_{[g],\ell})_{n_i,0}.$$

Moreover, $\mathcal{N}^{k_i}_g a^i_{[g],\ell}$ is a linear combination of $a^j$ for $j \in I$ since $a^j$ for $j \in I$ form a basis of $\mathfrak{g}$. Thus $\widehat{M}^{[g]}_{\ell,h}$ is spanned by elements of the form

$$(a^{i_1}_{[g],\ell})_{n_1,0} \cdots (a^{i_l}_{[g],\ell})_{n_l,0} L_{\widehat{M}^{[g]}_{\ell,h}}(-1)^k(\psi^b_{[g]})_{-1,0}\mathbf{1} \tag{4.8}$$

for $i_j \in I$, $n_j \in \alpha^{i_j} + \mathbb{Z}$ for $j = 1, \dots, l$, $k \in \mathbb{N}$ and $b \in A_h$. Therefore $\widehat{M}^{g]}_{\ell}$ is spanned by elements of the form (4.8) with $L_{\widehat{M}^{[g]}_{\ell,h}}(-1)$ replaced by $L_{\widehat{M}^{[g]}_{\ell}}(-1)$ for $i_j \in I$, $n_j \in \alpha^{i_j} + \mathbb{Z}$ for $j = 1, \dots, l$, $k \in \mathbb{N}$ and $b \in A$.

On the other hand, $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ is spanned by elements of the form

$$a^{i_1}(n_1) \cdots a^{i_l}(n_l) L_{\Lambda(M)}(-1)^k w^b \tag{4.9}$$

for $i_j \in I$, $n_j \in \alpha^{i_j} + \mathbb{Z}$ for $j = 1, \dots, l$, $k \in \mathbb{N}$, $b \in A$. Since $\hat{\rho}$ is a $\mathfrak{g}^{[g]}$-module map, we have

$$\hat{\rho}(a^{i_1}(n_1) \cdots a^{i_l}(n_l) L_{\Lambda(M)}(-1)w^b) = (a^{i_1}_{[g],\ell})_{n_1,0} \cdots (a^{i_l}_{[g],\ell})_{n_l,0} L_{\widehat{M}^{[g]}_{\ell}}(-1)(\psi^b_{[g]})_{-1,0}\mathbf{1}$$

for $i_j \in I$, $n_j \in \alpha^{i_j} + \mathbb{Z}$ for $j = 1, \dots, l$, $k \in \mathbb{N}$ and $b \in A$. To prove that $\hat{\rho}$ is injective, we prove that if we replace $a^i(n)$, $L_{\Lambda(M)}(-1)$ and $w^b$ by $(a^i_{[g],\ell})_{n,0}$, $L_{\widehat{M}^{[g]}_\ell}(-1)$ and $(\psi^b_{[g]})_{-1,0}\mathbf{1}$, respectively, for $i \in I$, $n \in \alpha^i + \mathbb{Z}$ and $b \in A$, the relations satisfied by elements of the spanning sets (4.9) of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_+ \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ must be satisfied by elements of the spanning sets (4.8) of $\widehat{M}^{[g]}_\ell$.

To prove this, we first list all the relations satisfied by (4.8). From the construction of $\widehat{M}^{[g]}_{\ell,h}$ for $h \in Q_M$ given in Section 5 of [Hua5] and Theorem 2.4 in [Hua6], we see that the only relations satisfied by elements of the form (4.8) for $i_j \in I$, $n_j \in \alpha^{i_j} + \mathbb{Z}$ for $j = 1, \dots, l$ and $b \in A$ are generated by the following: (i) A homogeneous element of the form (4.8) with $i_j \in I$, $n_j \in \alpha^{i_j} + \mathbb{Z}$ for $j = 1, \dots, l$, $k \in \mathbb{N}$ and $b \in A_h$ satisfying $-n_1 - \cdots - n_l < \Re(h)$ is equal to 0. (ii) The relations induced from the coefficients of the weak commutativity for the generating $g$-twisted fields $a^i_{[g],\ell}(x)$ for $i \in I$. (iii) The commutator relations between $(a^i_{[g],\ell})_{n,0}$ and $L_{\widehat{M}^{[g]}_h}(-1)$ for $a \in \mathfrak{g}^{[\alpha]}$ and $m \in \alpha + \mathbb{N}$ (when $\Re(\alpha) > 0$) and $m \in \alpha + \mathbb{Z}_+$ (when $\Re(\alpha) = 0$). The other relations given in Section 5 of [Hua5] involve elements that are not of the form (4.8).

We need only prove that elements of the form (4.9) also satisfy the relations corresponding to the relations (i), (ii) and (iii). By the definitions of the actions of $a^i(n)$ on $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_+ \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ and the fact that the weights of $w^b$ for $b \in A_h$ are $h$, elements of the form (4.9) satisfy the relations corresponding to (i). Since $a^i(n)$ and $(a^i_{[g],\ell})_{n,0}$ for $i \in I$ and $n \in \alpha^i + \mathbb{Z}$ satisfy the same commutator formula, $a^i(x) = \sum_{n \in \alpha^i + \mathbb{Z}} a^i(n)x^{-n-1}$ and $a^i_{[g],\ell}(x)$ for $i \in I$ also satisfy the same commutator formula. Since weak commutativity follows from the commutator formula for generating twisted fields, we see that elements of the form (4.9) satisfy the relations corresponding to (ii). Since $\rho$ is in fact an equivalence of $\hat{\mathfrak{g}}^{[g]}_+ \oplus \mathbb{C}\mathbf{k}$-modules and commutes with the actions of $L_{\widehat{M}^{[g]}_\ell}(-1)$ and $L_{\widehat{M}^{[g]}_\ell}(-1)$, elements of the form (4.9) satisfy the relations corresponding to (iii). This finishes the proof. ∎

**Remark 4.3** By the Poincaré-Birkhoff-Witt theorem, the induced lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_+ \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ is linearly isomorphic to $U(\hat{\mathfrak{g}}^{[g]}_-) \otimes U(\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}) \otimes \Lambda(M)$. In particular, $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_+ \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ is spanned by elements of the form

$$a^{i_1}(n_1) \cdots a^{i_l}(n_l) a^{j_1}(\alpha^{j_1}) \cdots a^{j_m}(\alpha^{j_m}) L_{\Lambda(M)}(-1)^k w^b$$

for $i_p, j_q \in I$, $n_p \in \alpha^{i_p} - \mathbb{Z}_+$, $\Re(\alpha^{j_q}) = 0$ for $p = 1, \dots, l$, $q = 1, \dots, m$, $k \in \mathbb{N}$ and $b \in A$. Using the commutator formula between $a^j(\alpha^j)$ and $L_{\Lambda(M)}(-1)$ for $j \in I_{\mathbb{I}}$, we see that $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_+ \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ is also spanned by elements of the form

$$a^{i_1}(n_1) \cdots a^{i_l}(n_l) L_{\Lambda(M)}(-1)^k a^{j_1}(\alpha^{j_1}) \cdots a^{j_m}(\alpha^{j_m}) w^b \tag{4.10}$$

for $i_p, j_q \in I$, $n_p \in \alpha^{i_p} - \mathbb{Z}_+$, $\Re(\alpha^{j_q}) = 0$ for $p = 1, \dots, l$, $q = 1, \dots, m$, $k \in \mathbb{N}$ and $b \in A$. By Theorem 4.2, we see that $\widehat{M}^{[g]}_{\ell,h}$ is spanned by elements of the form

$$(a^{i_1}_{[g],\ell})_{n_1,0} \cdots (a^{i_l}_{[g],\ell})_{n_l,0} (a^{j_1}_{[g],\ell})_{\alpha^{j_1},0} \cdots (a^{j_m}_{[g],\ell})_{\alpha^{j_m},0} L_{\widehat{M}^{[g]}_{\ell,h}}(-1)^k (\psi^b_{[g]})_{-1,0}\mathbf{1}$$

for $i_p, j_q \in I$, $n_p \in \alpha^{i_p} - \mathbb{Z}_+$, $\Re(\alpha^{j_q}) = 0$ for $p = 1, \dots, l$, $q = 1, \dots, m$, $k \in \mathbb{N}$ and $b \in A$. Using the commutator formula between $(a^j_{[g],\ell})_{\alpha^j,0}$ and $L_{\widehat{M}^{[g]}_{\ell,h}}(-1)$, we see that $\widehat{M}^{[g]}_{\ell,h}$ is also spanned by elements of the form

$$(a^{i_1}_{[g],\ell})_{n_1,0} \cdots (a^{i_l}_{[g],\ell})_{n_l,0} L_{\widehat{M}^{[g]}_{\ell,h}}(-1)^k (a^{j_1}_{[g],\ell})_{\alpha^{j_1},0} \cdots (a^{j_m}_{[g],\ell})_{\alpha^{j_m},0} (\psi^b_{[g]})_{-1,0}\mathbf{1} \tag{4.11}$$

for $i_p, j_q \in I$, $n_p \in \alpha^{i_p} - \mathbb{Z}_+$, $\Re(\alpha^{j_q}) = 0$ for $p = 1, \dots, l$, $q = 1, \dots, m$, $k \in \mathbb{N}$ and $b \in A$.

Next we construct universal grading-restricted generalized $g$-twisted $M(\ell, 0)$-modules. From (4.11), we see that it is impossible for the homogeneous subspaces of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_+ \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ to be finite dimensional since $a^i(0)$ for $a^i \in \hat{\mathfrak{g}}^{[0]}$ act on $w^b$ generate an infinite-dimensional homogeneous subspace. But if $M$ is a finite-dimensional $\hat{\mathfrak{g}}_{\mathbb{I}}$-module, a quotient of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_+ \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ might be grading restricted. Since $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_+ \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ as a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module is equivalent to $\widehat{M}^{[g]}_\ell$, the same discussion applies to $\widehat{M}^{[g]}_\ell$.

Now we assume that $M$ is in addition a finite-dimensional $\hat{\mathfrak{g}}_{\mathbb{I}}$-module with a compatible action of $g$. Here by $M$ has a compatible action of $g$ we mean $g(a(n)w) = (g(a))(n)g(w)$ for $a \in \mathfrak{g}^{[\alpha]}$ such that $\Re(\alpha) = 0$, $n \in \alpha + \mathbb{Z}$ and $w \in M$. We have a universal lower-bounded generalized $g$-twisted $M(\ell, 0)$-module $\widehat{M}^{[g]}_\ell$. Since $M$ in our case is a $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$-module but the construction of $\widehat{M}^{[g]}_\ell$ above does not use such a structure on $M$, to incorporate such an action on $M$, we need to take a further quotient. Let $I_{\mathbb{I}}$ be the set of elements $\alpha^i$ of $I$ such that $\Re(\alpha^i) = 0$. Then $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$ is spanned by elements of the form $a^i(\alpha^i)$ for $i \in I_{\mathbb{I}}$. Since $\{w^a\}_{a \in A}$ is a basis of $M$, there exist $\lambda^a_{ic} \in \mathbb{C}$ for $i \in I_{\mathbb{I}}$ and $b, c \in A$ such that

$$a^i(\alpha^i)w^b = \sum_{c \in A} \lambda^b_{ic} w^c$$

for $i \in I_{\mathbb{I}}$ and $b \in A$.

Consider the lower-bounded generalized $g$-twisted $M(\ell, 0)$-submodule of $\widehat{M}^{[g]}_{\ell,h}$ generated by elements of the form

$$(a^i_{[g],\ell})_{\alpha^i,0}(\psi^b_{[g],h^b})_{-1,0}\mathbf{1} - \sum_{c \in A} \lambda^b_{ic}(\psi^c_{[g],h^c})_{-1,0}\mathbf{1} \tag{4.12}$$

for $i \in I_{\mathbb{I}}$ and $b \in A$. We denote the quotient of $\widehat{M}^{[g]}_\ell$ by this submodule by $\breve{M}^{[g]}_\ell$. Then $\breve{M}^{[g]}_\ell$ is also a lower-bounded generalized $g$-twisted $M(\ell, 0)$-module. We shall use the same notations for the generating twisted fields, generator twist fields and their coefficients for $\widehat{M}^{[g]}_\ell$ to denotes the corresponding fields and their coefficients for $\breve{M}^{[g]}_\ell$.

On the $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$-module $M$, we define an action of $\hat{\mathfrak{g}}^{[g]}_+$ to be 0. Then we use the commutator formula (4.6) for $a \in \mathfrak{g}^{[\alpha]}$ and $m \in \alpha + \mathbb{N}$ to define an action of $\hat{\mathfrak{g}}^{[g]}_+ \oplus \hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$ on $\Lambda(M)$. Let $\mathbf{k}$ act on $\Lambda(M)$ as $\ell$. Then $\Lambda(M)$ becomes a $\hat{\mathfrak{g}}^{[g]}_+ \oplus \hat{\mathfrak{g}}^{[g]}_0$-module and we have the induced lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}^{[g]}_+ \oplus \hat{\mathfrak{g}}^{[g]}_0} \Lambda(M)$. From the construction, we see that the

$\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]})\otimes_{\hat{\mathfrak{g}}_{+}^{[g]}\oplus\hat{\mathfrak{g}}_{0}^{[g]}}\Lambda(M)$ is in fact the quotient of $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}_{+}^{[g]}\oplus\mathbb{C}\mathbf{k})}\Lambda(M)$ by the submodule generated by elements of the form

$$a^{i}(\alpha^{i})\otimes w^{b}-\sum_{c\in A}\lambda_{ic}^{b}w^{c} \tag{4.13}$$

for $i\in I_{\mathbb{I}}$ and $b\in A$.

**Theorem 4.4** *As a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module, $\breve{M}_{\ell}^{[g]}$ is equivalent to the induced lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]})\otimes_{\hat{\mathfrak{g}}_{+}^{[g]}\oplus\hat{\mathfrak{g}}_{0}^{[g]}}\Lambda(M)$.*

*Proof.* By Theorem 4.2, $\widehat{M}_{\ell}^{[g]}$ as a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module is equivalent to $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]})\otimes_{\hat{\mathfrak{g}}_{+}^{[g]}\oplus\mathbb{C}\mathbf{k}}\Lambda(M)$. It is also clear that the submodule of $\widehat{M}_{\ell}^{[g]}$ generated by elements of the form (4.12) for $i\in I_{\mathbb{I}}$ and $b\in A$ and the submodule of $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]})\otimes_{\hat{\mathfrak{g}}_{+}^{[g]}\oplus\mathbb{C}\mathbf{k}}\Lambda(M)$ generated by elements of the form (4.13) are equivalent under the equivalence from $\widehat{M}_{\ell}^{[g]}$ to $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]})\otimes_{\hat{\mathfrak{g}}_{+}^{[g]}\oplus\mathbb{C}\mathbf{k}}\Lambda(M)$. Thus their quotients $\breve{M}_{\ell}^{[g]}$ and $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]})\otimes_{\hat{\mathfrak{g}}_{+}^{[g]}\oplus\hat{\mathfrak{g}}_{0}^{[g]}}\Lambda(M)$ are equivalent. ∎

**Remark 4.5** From the construction of $U(\hat{\mathfrak{g}}^{[g]})\otimes_{\hat{\mathfrak{g}}_{+}^{[g]}\oplus\hat{\mathfrak{g}}_{0}^{[g]}}\Lambda(M)$, it is spanned by elements of the form

$$a^{i_1}(n_1)\cdots a^{i_l}(n_l)L_{\Lambda(M)}(-1)^{k}w^{b} \tag{4.14}$$

for $i_p\in I$, $n_p\in\alpha^{i_p}-\mathbb{Z}_{+}$ for $p=1,\dots,l$, $k\in\mathbb{N}$ and $b\in A$. Similarly, from the construction of $\breve{M}_{\ell}^{[g]}$, it is spanned by elements of the form

$$(a_{[g],\ell}^{i_1})_{n_1,0}\cdots(a_{[g],\ell}^{i_l})_{n_l,0}L_{\widehat{M}_{\ell,h}^{[g]}}(-1)^{k}(\psi_{[g]}^{b})_{-1,0}\mathbf{1} \tag{4.15}$$

for $i_p,j_q\in I$, $n_p\in\alpha^{i_p}-\mathbb{Z}_{+}$ for $p=1,\dots,l$, $k\in\mathbb{N}$ and $b\in A$.

We are ready to prove that $\breve{M}_{\ell}^{[g]}$ is in fact grading restricted now.

**Theorem 4.6** *The lower-bounded generalized $g$-twisted $M(\ell,0)$-module $\breve{M}_{\ell}^{[g]}$ is in fact grading restricted.*

*Proof.* By Theorem 4.4, we need only prove that $U(\hat{\mathfrak{g}}^{[g]})\otimes_{\hat{\mathfrak{g}}_{+}^{[g]}\oplus\hat{\mathfrak{g}}_{0}^{[g]}}\Lambda(M)$ is grading restricted. By Remark 4.5, $U(\hat{\mathfrak{g}}^{[g]})\otimes_{\hat{\mathfrak{g}}_{+}^{[g]}\oplus\hat{\mathfrak{g}}_{0}^{[g]}}\Lambda(M)$ is spanned by elements of the form (4.14) for $i_p\in I$, $n_p\in\alpha^{i_p}-\mathbb{Z}_{+}$ for $p=1,\dots,l$, $k\in\mathbb{N}$ and $b\in A$. The weight of such an element is $-n_1-\cdots-n_l+k+\mathrm{wt}\,w^{a}$. For fixed $n\in\mathbb{C}$, elements of weight $n$ of the form (4.14) must satisfy $n=-n_1-\cdots-n_l+k+\mathrm{wt}\,w^{a}$. So we have

$$n_1+\cdots+n_l-k=-n+\mathrm{wt}\,w^{a}.$$

Since $M$ is finite dimensional, there are only finitely many $w^a$ and thus finitely many $\mathrm{wt}\, w^a$. Let $N \in \mathbb{R}$ such that $\Re(\mathrm{wt}\, w^a) \geq N$ for $a \in A$. Then

$$\Re(n_1) + \cdots + \Re(n_l) - k = -\Re(n) + \Re(\mathrm{wt}\, w^a) \geq -\Re(n) + N.$$

On the other hand, since $n_j \in \alpha^{i_j} - \mathbb{Z}_+$, $\Re(n_j) < 0$ and we obtain

$$0 > \Re(n_1) + \cdots + \Re(n_l) - k \geq -\Re(n) + N. \tag{4.16}$$

Let $P = \max_{i\in I}\{\Re(\alpha^i) - 1\}$. Then $P \in [-1, 0)$. Since $n_j = \alpha^{i_j} - \mathbb{Z}_+ = \alpha^{i_j} - 1 - \mathbb{N}$, we have $\Re(n_j) \leq \Re(\alpha^{i_j}) - 1 \leq P < 0$. So $\Re(n_1) + \cdots + \Re(n_l) \leq lP$. If $lP < -\Re(n) + N$, we have $\Re(n_1) + \cdots + \Re(n_l) - k < -\Re(n) + N - k \leq -\Re(n) + N$. Contradiction to (4.16). Thus we must have $lP \geq -\Re(n) + N$ or equivalently, $l \leq \frac{1}{P}(-\Re(n) + N)$ (note that $P < 0$). Since $\Re(n_j) < 0$ for $j = 1, \dots, l$, from (4.16) and $-k \leq 0$, we obtain also $\Re(n_j) \geq -\Re(n) + N$ and $-k \geq -\Re(n) + N$. From $0 > \Re(n_j) \geq -\Re(n) + N$ for $j = 1, \dots, l$ and $0 \geq -k \geq -\Re(n) + N$, we see that for fixed $l$, there are only finitely many possible choices of $a^{i_j}$, $n_j$ and $k$. Thus for fixed $n \in \mathbb{C}$, there are only finitely many elements of weight $n$ of the form (4.14). So $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}^{[g]}_+ \oplus \hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}} M$, or equivalently, $\breve{M}^{[g]}_\ell$ is grading restricted. ∎

## 4.2 The constructions when $M(\ell, 0)$ is viewed as a vertex operator algebra

Assume that $\mathfrak{g}$ is simple and $\ell + h^\vee \neq 0$. Then $M(\ell, 0)$ has a conformal vector $\omega_{M(\ell,0)}$ and thus is a vertex operator algebra. Now we want to construct and identify explicitly universal lower-bounded generalized $g$-twisted modules for $M(\ell, 0)$ viewed as a vertex operator algebra. Since in [Hua5], we give only the construction for a grading-restricted vertex algebra or a Möbius vertex algebra, here we first give a construction of universal lower-bounded generalized twisted module for a general vertex operator algebra.

Let $V$ be a vertex operator algebra, that is, a grading-restricted vertex algebra $V$ with a conformal element $\omega$, and $g$ an automorphism of $V$ as a vertex operator algebra (meaning in particular that $g$ fixes $\omega$). Let $M$ be a vector space with actions of $g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_M(0)$, $L_M(0)_S$ and $L_M(0)_N$ and $B$ a real number such that $M$ is a direct sum of generalized eigenspaces of $L_M(0)$ and the real parts of the eigenvalues of $L_M(0)$ are larger than or equal to $B$. From Section 5 of [Hua5], we have a universal lower-bounded generalized $g$-twisted $V$-module $\widehat{M}^{[g]}_B$. Since $g$ fix $\omega$, the coefficients of $Y^g_{\widehat{M}^{[g]}_B}(\omega, x)$ satisfy the Virasoro commutator relations. Note that for a lower-bounded generalized $g$-twisted $V$-module $W$, the operator $L_W(0)$ and $L_W(-1)$ must be equal to the coefficients of $x^{-2}$ and $x^{-1}$, respectively, in the vertex operator $Y_W(\omega, x)$. But $L_{\widehat{M}^{[g]}_B}(0)$ and $L_{\widehat{M}^{[g]}_B}(-1)$ for $\widehat{M}^{[g]}_B$ are not equal to the coefficients of $x^{-2}$ and $x^{-1}$, respectively. To obtain a lower-bounded generalized $g$-twisted module for $V$ viewed as a vertex operator algebra, we have to take the quotient by a submodule generated by the difference of these operators acting on elements of $\widehat{M}^{[g]}_B$.

Consider the lower-bounded generalized $g$-twisted $V$-submodule of $\widehat{M}_B^{[g]}$ generated by elements of the form

$$L_{\widehat{M}_B^{[g]}}(0)w - \mathrm{Res}_x x Y^g_{\widehat{M}_B^{[g]}}(\omega, x)w,$$
$$L_{\widehat{M}_B^{[g]}}(-1)w - \mathrm{Res}_x Y^g_{\widehat{M}_B^{[g]}}(\omega, x)w$$

for $w \in \widehat{M}_B^{[g]}$. We shall denote the quotient of $\widehat{M}_B^{[g]}$ by this submodule by

$$\widehat{M}_B^{[g]}$$

and call this quotient module the lower-bounded generalized $g$-twisted $V$-module for the vertex operator algebra $V$, not the underlying grading-restricted vertex algebra $V$. By Theorem 5.2 and the construction of $\widehat{M}_B^{[g]}$ in [Hua5], we immediately obtain the following result:

**Theorem 4.7** *Let $V$ be a vertex operator algebra and $(W, Y_W^g)$ a lower-bounded generalized $g$-twisted $V$-module and $M^0$ a subspace of $W$ invariant under the actions of $g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0) = \mathrm{Res}_x x Y_W^g(\omega, x)$, $L_W(0)_S$ and $L_W(0)_N$. Let $B \in \mathbb{R}$ such that $W_{[n]} = 0$ when $\Re(n) < B$. Assume that there is a linear map $f : M \to M^0$ commuting with the actions of $g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)|_{M^0}$ and $L_M(0)$ , $L_W(0)_S|_{M^0}$ and $L_M(0)_S$ and $L_W(0)_N|_{M^0}$ and $L_M(0)_N$. Then there exists a unique module map $\widehat{f} : \widehat{M}_B^{[g]} \to W$ such that $\widehat{f}|_M = f$. If $f$ is surjective and $(W, Y_W^g)$ is generated by the coefficients of $(Y^g)_{WV}^W(w, x)v$ for $w \in M_0$ and $v \in V$, where $(Y^g)_{WV}^W$ is the twist vertex operator map obtained from $Y_W^g$, then $\widehat{f}$ is surjective.*

We now assume that $\mathfrak{g}$ is simple and $\ell + h^\vee \neq 0$. Then $M(\ell, 0)$ is a vertex operator algebra. Take the vertex operator algebra $V$ above to be $M(\ell, 0)$ and $g$ an automorphism of $M(\ell, 0)$ induced from an automorphism of $\mathfrak{g}$ as discussed above. Let $M$, as above, be a vector space with actions of $g$, $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_M(0)$, $L_M(0)_S$ and $L_M(0)_N$. We assume that $M = \coprod_{h \in Q_M} M_{[h]}$ as above, where $M_{[h]}$ is the generalized eigenspace of $L_M(0)$ with eigenvalue $h$ and $Q_M$ is the set of all eigenvalues of $L_M(0)$. For $h \in Q_h$, take $V$, $g$, $M$ and $B$ in the construction above to be $M(\ell, 0)$, $g$, $M_{[h]}$, $\Re(h)$. Then we have a universal lower-bounded generalized $g$-twisted $M(\ell, 0)$-module $\widehat{(M_{[h]})}_h^{[g]}$. To exhibit its dependence on $\ell$ explicitly, we denote it by $\widehat{(M_{[h]})}_{\ell,h}^{[g]}$. Adding them together, we obtain a lower-bounded generalized $g$-twisted $M(\ell, 0)$-module $\coprod_{h \in Q_M} \widehat{(M_{[h]})}_{\ell,h}^{[g]}$, which we shall denote by $\widehat{M}_\ell^{[g]}$. We shall use the same notations $a^i_{[g],\ell}(x)$, $(a^i_{[g],\ell})_{n,k}$, $\psi^b_{[g]}(x)$ and $(\psi^b_{[g]})_{n,k}$ and so on as those for $\widehat{M}_\ell^{[g]}$ and $\breve{M}_\ell^{[g]}$ to denote the generating twisted fields, their coefficients, the generator twist fields and their coefficients for $\widehat{M}_\ell^{[g]}$.

We need to identify $\widehat{M}_\ell^{[g]}$ with a suitable $\hat{\mathfrak{g}}^{[g]}$-module. We first need to identify $L_M(0)$ with the action of an element of $U(\hat{\mathfrak{g}}^{[g]})$. Recall the Jordan basis $\{a^i\}_{i \in I}$ of $\mathfrak{g}$ that we have chosen in the end of the preceding section. Let $\{(a^i)'\}_{i \in I}$ be the dual basis of $\{a^i\}_{i \in I}$ with respect to the nondegenerate bilinear form $(\cdot,\cdot)$. For simplicity (but with an abuse of the notation), we shall denote this dual basis by $\{a^{i'}\}_{i\in I}$. Then

$$(e^{2\pi i\mathcal{S}_g}a^{i'},a^j)=(a^{i'},e^{-2\pi i\mathcal{S}_g}a^j)=(a^{i'},e^{-2\pi i\alpha^j}a^j)=e^{-2\pi i\alpha^j}\delta_{ij}=e^{-2\pi i\alpha^i}\delta_{ij}.$$

This means that

$$e^{2\pi i\mathcal{S}_g}a^{i'}=e^{-2\pi i\alpha^i}a^{i'}.$$

So $a^{i'}\in\mathfrak{g}^{[1-\alpha^i]}$ when $\Re(\alpha^i)>0$ or $i\in I\setminus I_{\mathbb{I}}$ and $a^{i'}\in\mathfrak{g}^{[-\alpha^i]}$ when $\Re(\alpha^i)=0$ or $i\in I_{\mathbb{I}}$. By abuse of notation, let

$$\alpha^{i'}=\begin{cases}1-\alpha^i & i\in I\setminus I_{\mathbb{I}},\\ -\alpha^i & i\in I_{\mathbb{I}}.\end{cases}$$

By definition, the conformal element of $M(\ell,0)$ is

$$\omega_{M(\ell,0)}=\sum_{i\in I}a^{i'}(-1)a^i(-1)\mathbf{1}\in M^{[0]}(\ell,0),$$

where $M^{[0]}(\ell,0)$ is the fixed-point subalgebra of $M(\ell,0)$.

We need to recall the Virasoro operators on $M(\ell,0)$. Since $\omega_{M(\ell,0)}$ is in the fixed-point subalgebra of $M(\ell,0)$, $\mathcal{N}_g\omega_{M(\ell,0)}=0$. Hence

$$Y^g_{\widehat{M}^{[g]}_\ell}(\omega_{M(\ell,0)},x)=(Y^g_{\widehat{M}^{[g]}_\ell})_0(x^{-\mathcal{N}_g}\omega_{M(\ell,0)},x)=(Y^g_{\widehat{M}^{[g]}_\ell})_0(\omega_{M(\ell,0)},x)$$

and from the equivariance property of the twisted vertex operators, $Y^g_{\widehat{(M_{[h]})}^{[g]}_h}(\omega,x)$ or equivalently $(Y^g_{\widehat{(M_{[h]})}^{[g]}_h})_0(\omega,x)$ must have only integral powers of $x$. In particular,

$$Y^g_{\widehat{M}^{[g]}_\ell}(\omega,x)=(Y^g_{\widehat{M}^{[g]}_\ell})_0(\omega,x)=\sum_{n\in\mathbb{Z}}L_{\widehat{M}^{[g]}_\ell}(n)x^{-n-2}.$$

where $L_{\widehat{M}^{[g]}_\ell}(n)$ for $n\in\mathbb{Z}$ are the Virasoro operators on $\widehat{M}^{[g]}_\ell$ satisfying the Virasoro commutator relations with central charge $\frac{\ell\dim\mathfrak{g}}{\ell+h^\vee}$. In particular, we have the operators $L_{\widehat{M}^{[g]}_\ell}(0)$ and $L_{\widehat{M}^{[g]}_\ell}(-1)$.

**Proposition 4.8** *For $n\in\mathbb{Z}$,*

$$\begin{aligned}L_{\widehat{M}^{[g]}_\ell}&(n)\\ =&\sum_{i\in I}\sum_{p\in\alpha^i+\mathbb{Z}_+}\frac{1}{2(\ell+h^\vee)}a^{i'}_{[g],\ell}(-p)a^i_{[g],\ell}(p+n)+\sum_{i\in I}\sum_{p\in\alpha^i-\mathbb{N}}\frac{1}{2(\ell+h^\vee)}a^i_{[g],\ell}(p+n)a^{i'}_{[g],\ell}(-p)\\ &-\sum_{i\in I}\frac{1}{2(\ell+h^\vee)}[(\mathcal{N}_g-\alpha^i)a^{i'},a^i]_{[g],\ell}(n)-\sum_{i\in I}\frac{\ell\delta_{n,0}}{4(\ell+h^\vee)}((\mathcal{N}_g-\alpha^i)(\mathcal{N}_g-\alpha^i-1)a^{i'},a^i).\end{aligned}\tag{4.17}$$

*Proof.* We take $W = \widehat{M}_\ell^{[g]}$, $u = x_1^{\mathcal{N}_g} a^{i'}(-1)\mathbf{1}$ and $v = x_2^{\mathcal{N}_g} a^{i}(-1)\mathbf{1}$ in the Jacobi identity (4.1). Then $Y^g_{\widehat{M}_\ell^{[g]}}(u, x_1) = (Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i'}(-1)\mathbf{1}, x_1)$, $Y^g_{\widehat{M}_\ell^{[g]}}(v, x_2) = (Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i}(-1)\mathbf{1}, x_2)$ and $\mathcal{S}_g a^{i'}(-1)\mathbf{1} = \alpha^{i'} a^{i'}(-1)\mathbf{1}$. Then (4.1) becomes

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i'}(-1)\mathbf{1}, x_1)(Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i}(-1)\mathbf{1}, x_2)\\
&\quad - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)(Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i}(-1)\mathbf{1}, x_2)(Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i'}(-1)\mathbf{1}, x_1)\\
&= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^{-\alpha^i}\cdot\\
&\qquad\cdot (Y^g_{\widehat{M}_\ell^{[g]}})_0\left(Y_{M(\ell,0)}\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{N}_g} a^{i'}(-1)\mathbf{1}, x_0\right) a^{i}(-1)\mathbf{1}, x_2\right),
\end{aligned} \tag{4.18}$$

where we have used

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^{\alpha^{i'}} = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^{-\alpha^i}.$$

Multiplying both sides of (4.18) by $x_1^{-\alpha^i}$ and then take $\mathrm{Res}_{x_1}$, rewriting $(x_2+x_0)^{-\alpha^i}$ as $x_2^{-\alpha^i}\left(1+\frac{x_0}{x_2}\right)^{-\alpha^i}$ and then multiplying both sides by $x_2^{\alpha^i}$, we obtain

$$\begin{aligned}
&\mathrm{Res}_{x_1} x_1^{-\alpha^i} x_2^{\alpha^i} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i'}(-1)\mathbf{1}, x_1)(Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i}(-1)\mathbf{1}, x_2)\\
&\quad - \mathrm{Res}_{x_1} x_1^{-\alpha^i} x_2^{\alpha^i} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)(Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i}(-1)\mathbf{1}, x_2)(Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i'}(-1)\mathbf{1}, x_1)\\
&= \left(1+\frac{x_0}{x_2}\right)^{-\alpha^i}(Y^g_{\widehat{M}_\ell^{[g]}})_0\left(Y_{M(\ell,0)}\left(\left(1+\frac{x_0}{x_2}\right)^{\mathcal{N}_g} a^{i'}(-1)\mathbf{1}, x_0\right) a^{i}(-1)\mathbf{1}, x_2\right).
\end{aligned} \tag{4.19}$$

Using the definition, we have

$$(Y^g_{\widehat{M}_\ell^{[g]}})_0(a(-1)\mathbf{1}, x_1) = (a_{[g],\ell})_0(x),$$

where

$$(a_{[g],\ell})_0(x) = \sum_{n\in\alpha+\mathbb{Z}} (a_{[g],\ell})_{n,0} x^{-n-1}$$

for $a \in \mathfrak{g}^{[\alpha]}$. Then the constant term in $x_0$ (or equivalently, the result of applying $\mathrm{Res}_{x_0} x_0^{-1}$)

of the right-hand side of (4.19) is equal to

$$
\begin{aligned}
&\mathrm{Res}_{x_0} x_0^{-1} (Y^g_{\widehat{M}_\ell^{[g]}})_0 \left( Y_{M(\ell,0)} \left( \left( 1 + \frac{x_0}{x_2} \right)^{\mathcal{N}_g - \alpha^i} a^{i'}(-1)\mathbf{1}, x_0 \right) a^i(-1)\mathbf{1}, x_2 \right) \\
&\quad = \sum_{m\in\mathbb{N}} \mathrm{Res}_{x_0} x_0^{-1} \left( \frac{x_0}{x_2} \right)^m (Y^g_{\widehat{M}_\ell^{[g]}})_0 \left( Y_{M(\ell,0)} \left( \binom{\mathcal{N}_g - \alpha^i}{m} a^{i'}(-1)\mathbf{1}, x_0 \right) a^i(-1)\mathbf{1}, x_2 \right) \\
&\quad = \sum_{m\in\mathbb{N}} \mathrm{Res}_{x_0} x_0^{-1} \left( \frac{x_0}{x_2} \right)^m (Y^g_{\widehat{M}_\ell^{[g]}})_0 \left( \left( \binom{\mathcal{N}_g - \alpha^i}{m} a^{i'} \right)(x_0) a^i(-1)\mathbf{1}, x_2 \right) \\
&\quad = \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{Z}} \mathrm{Res}_{x_0} x_0^{-1} \left( \frac{x_0}{x_2} \right)^m x_0^{-n-1} (Y^g_{\widehat{M}_\ell^{[g]}})_0 \left( \left( \binom{\mathcal{N}_g - \alpha^i}{m} a^{i'} \right)(n) a^i(-1)\mathbf{1}, x_2 \right) \\
&\quad = \sum_{m\in\mathbb{N}} x_2^{-m} (Y^g_{\widehat{M}_\ell^{[g]}})_0 \left( \left( \binom{\mathcal{N}_g - \alpha^i}{m} a^{i'} \right)(m-1) a^i(-1)\mathbf{1}, x_2 \right) \\
&\quad = (Y^g_{\widehat{M}_\ell^{[g]}})_0 (a^{i'}(-1)a^i(-1)\mathbf{1}, x_2) \\
&\qquad + x_2^{-1} (Y^g_{\widehat{M}_\ell^{[g]}})_0 (((\mathcal{N}_g - \alpha^i)a^{i'})(0)a^i(-1)\mathbf{1}, x_2) \\
&\qquad + \frac{x_2^{-2}}{2} (Y^g_{\widehat{M}_\ell^{[g]}})_0 (((\mathcal{N}_g - \alpha^i)(\mathcal{N}_g - \alpha^i - 1)a^{i'})(1)a^i(-1)\mathbf{1}, x_2) \\
&\quad = (Y^g_{\widehat{M}_\ell^{[g]}})_0 (a^{i'}(-1)a^i(-1)\mathbf{1}, x_2) + x_2^{-1} ([(\mathcal{N}_g - \alpha^i)a^{i'}, a^i]_{[g],\ell})_0(x_2) \\
&\qquad + \frac{\ell x_2^{-2}}{2} ((\mathcal{N}_g - \alpha^i)(\mathcal{N}_g - \alpha^i - 1)a^{i'}, a^i). \qquad (4.20)
\end{aligned}
$$

Applying $\mathrm{Res}_{x_0} x_0^{-1}$ to both sides of (4.19), using (4.20), taking sum over $i \in I$ on both sides and dividing both sides by $2(\ell+h^\vee)$, we obtain

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} L_{\widehat{M}_\ell^{[g]}}(n)x_2^{-n-2}\\
&\quad= (Y^g_{\widehat{M}_\ell^{[g]}})_0(\omega, x_2)\\
&\quad= \sum_{i\in I}\frac{1}{2(\ell+h^\vee)}(Y^g_{\widehat{M}_\ell^{[g]}})_0(a^{i'}(-1)a^i(-1)\mathbf{1}, x_2)\\
&\quad= \sum_{i\in I}\frac{1}{2(\ell+h^\vee)}\mathrm{Res}_{x_0}x_0^{-1}\mathrm{Res}_{x_1}x_1^{-\alpha^i}x_2^{\alpha^i}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(a^{i'}_{[g],\ell})_0(x_1)(a^i_{[g],\ell})_0(x_2)\\
&\quad\quad-\sum_{i\in I}\frac{1}{2(\ell+h^\vee)}\mathrm{Res}_{x_0}x_0^{-1}\mathrm{Res}_{x_1}x_1^{-\alpha^i}x_2^{\alpha^i}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)(a^i_{[g],\ell})_0(x_2)(a^{i'}_{[g],\ell})_0(x_1)\\
&\quad\quad-\sum_{i\in I}\frac{x_2^{-1}}{2(\ell+h^\vee)}([(\mathcal{N}_g-\alpha^i)a^{i'}, a^i]_{[g],\ell})_0(x_2)\\
&\quad\quad-\sum_{i\in I}\frac{\ell x_2^{-2}}{4(\ell+h^\vee)}((\mathcal{N}_g-\alpha^i)(\mathcal{N}_g-\alpha^i-1)a^{i'}, a^i)\\
&\quad= \sum_{i\in I}\frac{1}{2(\ell+h^\vee)}\mathrm{Res}_{x_1}x_1^{-\alpha^i}x_2^{\alpha^i}(x_1-x_2)^{-1}(a^{i'}_{[g],\ell})_0(x_1)(a^i_{[g],\ell})_0(x_2)\\
&\quad\quad-\sum_{i\in I}\frac{1}{2(\ell+h^\vee)}\mathrm{Res}_{x_1}x_1^{-\alpha^i}x_2^{\alpha^i}(-x_2+x_1)^{-1}(a^i_{[g],\ell})_0(x_2)(a^{i'}_{[g],\ell})_0(x_1)\\
&\quad\quad-\sum_{i\in I}\frac{x_2^{-1}}{2(\ell+h^\vee)}([(\mathcal{N}_g-\alpha^i)a^{i'}, a^i]_{[g],\ell})_0(x_2)\\
&\quad\quad-\sum_{i\in I}\frac{\ell x_2^{-2}}{4(\ell+h^\vee)}((\mathcal{N}_g-\alpha^i)(\mathcal{N}_g-\alpha^i-1)a^{i'}, a^i).
\end{aligned}\tag{4.21}
$$

Taking the coefficients of $x_2^{-n-2}$ of (4.21), we obtain

$$
\begin{aligned}
&L_{\widehat{M}_\ell^{[g]}}(n)\\
&=\sum_{i\in I}\sum_{m\in\mathbb{Z}_+}\sum_{k\in-\alpha^i+\mathbb{Z}}\sum_{l\in\alpha^i+\mathbb{Z}}\frac{1}{2(\ell+h^\vee)}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{-\alpha^i-m-k-2}x_2^{\alpha^i+m+n-l}a^{i'}_{[g],\ell}(k)a^i_{[g],\ell}(l)\\
&\quad+\sum_{i\in I}\sum_{m\in\mathbb{Z}_+}\sum_{k\in-\alpha^i+\mathbb{Z}}\sum_{l\in\alpha^i+\mathbb{Z}}\frac{1}{2(\ell+h^\vee)}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{-\alpha^i+m-k}x_2^{\alpha^i+n-m-l-1}a^i_{[g],\ell}(l)a^{i'}_{[g],\ell}(k)\\
&\quad-\sum_{i\in I}\frac{1}{2(\ell+h^\vee)}[(\mathcal{N}_g-\alpha^i)a^{i'},a^i]_{[g],\ell}(n)-\sum_{i\in I}\frac{\ell\delta_{n,0}}{4(\ell+h^\vee)}((\mathcal{N}_g-\alpha^i)(\mathcal{N}_g-\alpha^i-1)a^{i'},a^i)\\
&=\sum_{i\in I}\sum_{p\in\alpha^i+\mathbb{Z}_+}\frac{1}{2(\ell+h^\vee)}a^{i'}_{[g],\ell}(-p)a^i_{[g],\ell}(p+n)+\sum_{i\in I}\sum_{p\in\alpha^i-\mathbb{N}}\frac{1}{2(\ell+h^\vee)}a^i_{[g],\ell}(p+n)a^{i'}_{[g],\ell}(-p)\\
&\quad-\sum_{i\in I}\frac{1}{2(\ell+h^\vee)}[(\mathcal{N}_g-\alpha^i)a^{i'},a^i]_{[g],\ell}(n)-\sum_{i\in I}\frac{\ell\delta_{n,0}}{4(\ell+h^\vee)}((\mathcal{N}_g-\alpha^i)(\mathcal{N}_g-\alpha^i-1)a^{i'},a^i),
\end{aligned}
$$

proving (4.17). ∎

From (4.17), we obtain

$$
\begin{aligned}
&L_{\widehat{M}_\ell^{[g]}}(0)\\
&=\sum_{i\in I}\sum_{p\in\alpha^i+\mathbb{Z}_+}\frac{1}{2(\ell+h^\vee)}a^{i'}_{[g],\ell}(-p)a^i_{[g],\ell}(p)+\sum_{i\in I}\sum_{p\in\alpha^i-\mathbb{N}}\frac{1}{2(\ell+h^\vee)}a^i_{[g],\ell}(p)a^{i'}_{[g],\ell}(-p)\\
&\quad-\sum_{i\in I}\frac{1}{2(\ell+h^\vee)}[(\mathcal{N}_g-\alpha^i)a^{i'},a^i]_{[g],\ell}(0)-\sum_{i\in I}\frac{\ell}{4(\ell+h^\vee)}((\mathcal{N}_g-\alpha^i)(\mathcal{N}_g-\alpha^i-1)a^{i'},a^i)
\end{aligned}
\tag{4.22}
$$

and

$$
\begin{aligned}
&L_{\widehat{M}_\ell^{[g]}}(-1)\\
&=\sum_{i\in I}\sum_{p\in\alpha^i+\mathbb{Z}_+}\frac{1}{2(\ell+h^\vee)}a^{i'}_{[g],\ell}(-p)a^i_{[g],\ell}(p-1)+\sum_{i\in I}\sum_{p\in\alpha^i-\mathbb{N}}\frac{1}{2(\ell+h^\vee)}a^i_{[g],\ell}(p-1)a^{i'}_{[g],\ell}(-p)\\
&\quad-\sum_{i\in I}\frac{1}{2(\ell+h^\vee)}[(\mathcal{N}_g-\alpha^i)a^{i'},a^i]_{[g],\ell}(-1).
\end{aligned}
\tag{4.23}
$$

Note that for $b\in A$,

$$\begin{aligned}L_{\widehat{M}_\ell^{[g]}}(0)(\psi^b_{[g]})_{-1,0}\mathbf{1}=&\sum_{i\in I_{\mathbb{I}}}\frac{1}{2(\ell+h^\vee)}a^i_{[g],\ell}(\alpha^i)a^{i'}_{[g],\ell}(-\alpha^i)(\psi^b_{[g]})_{-1,0}\mathbf{1}\\&-\sum_{i\in I}\frac{1}{2(\ell+h^\vee)}[(\mathcal{N}_g-\alpha^i)a^{i'},a^i]_{[g],\ell}(0)(\psi^b_{[g]})_{-1,0}\mathbf{1}\\&-\sum_{i\in I}\frac{\ell}{4(\ell+h^\vee)}((\mathcal{N}_g-\alpha^i)(\mathcal{N}_g-\alpha^i-1)a^{i'},a^i)(\psi^b_{[g]})_{-1,0}\mathbf{1},\end{aligned}\tag{4.24}$$

that is, as an operator on the subspace of $\widehat{M}_\ell^{[g]}$ spanned by $(\psi^b_{[g]})_{-1,0}\mathbf{1}$, $L_{\widehat{M}_\ell^{[g]}}(0)$ is equal to

$$\begin{aligned}&\sum_{i\in I_{\mathbb{I}}}\frac{1}{2(\ell+h^\vee)}a^i_{[g],\ell}(\alpha^i)a^{i'}_{[g],\ell}(-\alpha^i)-\sum_{i\in I}\frac{1}{2(\ell+h^\vee)}[(\mathcal{N}_g-\alpha^i)a^{i'},a^i]_{[g],\ell}(0)\\&\quad-\sum_{i\in I}\frac{\ell}{4(\ell+h^\vee)}((\mathcal{N}_g-\alpha^i)(\mathcal{N}_g-\alpha^i-1)a^{i'},a^i).\end{aligned}$$

For $h\in Q_M$ and $b\in A_h$, by definition,

$$L_{\widehat{M}_\ell^{[g]}}(0)(\psi^b_{[g]})_{-1,0}\mathbf{1}=h(\psi^b_{[g]})_{-1,0}\mathbf{1}.$$

Together with (4.24), we obtain the relation

$$\begin{aligned}h(\psi^b_{[g]})_{-1,0}\mathbf{1}=&\sum_{i\in I_{\mathbb{I}}}\frac{1}{2(\ell+h^\vee)}a^i_{[g],\ell}(\alpha^i)a^{i'}_{[g],\ell}(-\alpha^i)(\psi^b_{[g]})_{-1,0}\mathbf{1}\\&-\sum_{i\in I}\frac{1}{2(\ell+h^\vee)}[(\mathcal{N}_g-\alpha^i)a^{i'},a^i]_{[g],\ell}(0)(\psi^b_{[g]})_{-1,0}\mathbf{1}\\&-\sum_{i\in I}\frac{\ell}{4(\ell+h^\vee)}((\mathcal{N}_g-\alpha^i)(\mathcal{N}_g-\alpha^i-1)a^{i'},a^i)(\psi^b_{[g]})_{-1,0}\mathbf{1}.\end{aligned}\tag{4.25}$$

Let

$$\begin{aligned}\Omega^{[g]}=&\sum_{i\in I_{\mathbb{I}}}a^i(\alpha^i)a^{i'}(-\alpha^i)-\sum_{i\in I}[(\mathcal{N}_g-\alpha^i)a^{i'},a^i](0)\\&-\sum_{i\in I}\frac{\ell}{2}((\mathcal{N}_g-\alpha^i)(\mathcal{N}_g-\alpha^i-1)a^{i'},a^i)\\&\in U(\hat{\mathfrak{g}}^{[g]})^{[0]},\end{aligned}\tag{4.26}$$

where $U(\hat{\mathfrak{g}}^{[g]})^{[0]}$ is the fixed-point subspace of $U(\hat{\mathfrak{g}}^{[g]})$ under $g$. Let $\Omega^{[g]}$, $\hat{\mathfrak{g}}^{[g]}_+$ and $\mathbf{k}$ act on $M$ as $L_M(0)$, 0 and $\ell$, respectively. Let $G(\Omega^{[g]},\hat{\mathfrak{g}}^{[g]}_+,\mathbf{k})$ be the subalgebra of $U(\hat{\mathfrak{g}}^{[g]})$ generated by $\Omega^{[g]}$, $\hat{\mathfrak{g}}^{[g]}_+$ and $\mathbf{k}$. Then we have the induced lower-bounded $\hat{\mathfrak{g}}^{[g]}_+$-module $U(\hat{\mathfrak{g}}^{[g]})\otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}^{[g]}_+,\mathbf{k})}M$. Note that this induced $\hat{\mathfrak{g}}^{[g]}_+$-module is a quotient of $U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_+\oplus\mathbb{C}\mathbf{k})}M$.

**Theorem 4.9** *The universal lower-bounded generalized $g$-twisted $M(\ell,0)$-module $\widehat{M}_\ell^{[g]}$ is equivalent as a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module to $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}_+^{[g]},\mathbf{k})} M$.*

*Proof.* We know that $\widehat{M}_\ell^{[g]}$ as a quotient of $\widehat{M}_\ell^{[g]}$ is also spanned by elements of the form (4.8) for $i_j \in I$, $n_j \in \alpha^{i_j} + \mathbb{Z}$ for $j = 1, \dots, l$, $k \in \mathbb{N}$ and $b \in A$. Using (4.23), we see that elements of the form (4.8) in $\widehat{M}_\ell^{[g]}$ can be written as linear combinations of elements of the form

$$(a^{i_1}_{[g],\ell})_{n_1,0} \cdots (a^{i_l}_{[g],\ell})_{n_l,0} (\psi^b_{[g]})_{-1,0}\mathbf{1} \tag{4.27}$$

for $i_j \in I$, $n_j \in \alpha^{i_j} + \mathbb{Z}$ for $j = 1, \dots, l$ and $b \in A$. On the other hand, by definition, $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}_+^{[g]},\mathbf{k})} M$ is spanned by elements of the form

$$a^{i_1}(n_1) \cdots a^{i_l}(n_l) w^b \tag{4.28}$$

for $i_j \in I$, $n_j \in \alpha^{i_j} + \mathbb{Z}$ for $j = 1, \dots, l$ and $b \in A$.

By Theorem 4.2, we have an invertible $\hat{\mathfrak{g}}^{[g]}$-module map $\hat{\rho} : U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \mathbb{C}\mathbf{k})} \Lambda(M) \to \widehat{M}_\ell^{[g]}$ such that $\hat{\rho}$ maps the element (4.8) to the element (4.9). In particular, $\hat{\rho}$ maps the element of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ of the same form as (4.28) to the element (4.27). We want to use the map $\hat{\rho}$ restricted to elements of the same form as (4.28) to obtain an invertible $\hat{\mathfrak{g}}^{[g]}$-module map from $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}_+^{[g]},\mathbf{k})} M$ to $\widehat{M}_\ell^{[g]}$.

To do this, we need only prove that the relations among elements of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ of the same form as (4.28) and the relations among elements of the form (4.27) in $\widehat{M}_\ell^{[g]}$ are the same. The relations among elements of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ of the same form as (4.28) are generated by the following two types: The first type of relations are induced from the relations in $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$. The second type is the additional relations $\frac{1}{2(\ell+h^\vee)}\Omega^{[g]} w^b = L_M(0) w^b$ for $b \in A$. For $h \in Q_M$ and $b \in A_h$, this additional relations become

$$\begin{aligned} hw^b = &\sum_{i \in I_{\mathbb{I}}} \frac{1}{2(\ell+h^\vee)} a^i(\alpha^i) a^{i'}(-\alpha^i) w^b - \sum_{i \in I} \frac{1}{2(\ell+h^\vee)} [(\mathcal{N}_g - \alpha^i) a^{i'}, a^i](0) w^b \\ &- \sum_{i \in I} \frac{\ell}{4(\ell+h^\vee)} ((\mathcal{N}_g - \alpha^i)(\mathcal{N}_g - \alpha^i - 1) a^{i'}, a^i) w^b. \end{aligned} \tag{4.29}$$

The first type of relations are the same as the corresponding type of relations in $\widehat{M}_\ell^{[g]}$ by Theorem 4.2. The second type of relations (4.29) correspond exactly to the relations (4.24) in $\widehat{M}_\ell^{[g]}$. The relations (4.24) are also the only relations in $\widehat{M}_\ell^{[g]}$ in addition to the relations induced from the relations in $\widehat{M}_\ell^{[g]}$. Thus the theorem is proved. ∎

**Remark 4.10** We have seen in the proof of Theorem 4.9 that $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}^{[g]}_{+}, \mathbf{k})} M$ is spanned by elements of the form (4.28). Using the commutator relations for $a^i(n)$ for $i \in I$ and $n \in \alpha^i + \mathbb{Z}$, we see that it is in fact spanned by elements of the form

$$a^{i_1}(n_1) \cdots a^{i_l}(n_l) a^{j_1}(\alpha^{j_1}) \cdots a^{j_m}(\alpha^{j_m}) w^b \tag{4.30}$$

for $i_p \in I$, $n_p \in \alpha^{i_p} - \mathbb{Z}_+$ for $p = 1, \dots, l$, $j_q \in I_{\mathbb{I}}$ for $q = 1, \dots, m$ and $b \in A$. By Theorem 4.2, we also see that $\widehat{M}^{[g]}_{\ell}$ is spanned by elements of the form

$$(a^{i_1}_{[g],\ell})_{n_1,0} \cdots (a^{i_l}_{[g],\ell})_{n_l,0} (a^{j_1}_{[g],\ell})_{\alpha^{j_1},0} \cdots (a^{j_m}_{[g],\ell})_{\alpha^{j_m},0} (\psi^b_{[g],h})_{-1,0} \mathbf{1} \tag{4.31}$$

for $i_p \in I$, $n_p \in \alpha^{i_p} - \mathbb{Z}_+$ for $p = 1, \dots, l$, $j_q \in I_{\mathbb{I}}$ for $q = 1, \dots, m$ and $b \in A$.

We now construct and identify explicitly grading-restricted generalized $g$-twisted modules for $M(\ell, 0)$ viewed as a vertex operator algebra. We assume that $M$ is in addition a finite-dimensional $\hat{\mathfrak{g}}_{\mathbb{I}}$-module with a compatible action of $g$ such that the action of $\Omega^{[g]}$, or equivalently, the operator $L_M(0) = \frac{1}{2(\ell + h^\vee)} \Omega^{[g]}$, on $M$ is induced from this $\hat{\mathfrak{g}}_{\mathbb{I}}$-module structure. In particular, $M$ is a direct sum of generalized eigenspaces of $L_M(0)$ as above. We have a universal lower-bounded generalized $g$-twisted $M(\ell, 0)$-module $\widehat{M}^{[g]}_{\ell}$. As in the preceding subsection, since $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$ is spanned by elements of the form $a^i(\alpha^i)$ for $i \in I_{\mathbb{I}}$ and $\{w^a\}_{a \in A}$ is a basis of $M$, there exist $\lambda^a_{ic} \in \mathbb{C}$ for $i \in I_{\mathbb{I}}$ and $b, c \in A$ such that

$$a^i(\alpha^i) w^b = \sum_{c \in A} \lambda^b_{ic} w^c$$

for $i \in I_{\mathbb{I}}$ and $b \in A$. Consider the lower-bounded generalized $g$-twisted $M(\ell, 0)$-submodule of $\widehat{M}^{[g]}_{\ell}$ generated by elements of the form

$$(a^i_{[g],\ell})_{\alpha^i,0} (\psi^b_{[g]})_{-1,0} \mathbf{1} - \sum_{c \in A} \lambda^b_{ic} (\psi^c_{[g]})_{-1,0} \mathbf{1}$$

for $i \in I_{\mathbb{I}}$ and $b \in A$. We denote the quotient of $\widehat{M}^{[g]}_{\ell}$ by this submodule by $\widetilde{M}^{[g]}_{\ell}$. Then $\widetilde{M}^{[g]}_{\ell}$ is also a lower-bounded generalized $g$-twisted $M(\ell, 0)$-module. We shall use the same notations for the generating twisted fields, generator twist fields and their coefficients for $\widehat{M}^{[g]}_{\ell}$ to denote the corresponding fields and coefficients for $\widetilde{M}^{[g]}_{\ell}$.

On the $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$-module $M$, we define actions of $\hat{\mathfrak{g}}^{[g]}_{+}$ and $\mathbf{k}$ to be 0 and $\ell$, respectively. Then we have the induced lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+} \oplus \hat{\mathfrak{g}}^{[g]}_{0})} M$.

**Theorem 4.11** *As a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module, $\widetilde{M}^{[g]}_{\ell}$ is equivalent to the induced lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+} \oplus \hat{\mathfrak{g}}^{[g]}_{0})} M$.*

*Proof.* By Theorem 4.9, we have an invertible $\hat{\mathfrak{g}}^{[g]}$-module map $\hat{\rho}$ from $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}^{[g]}_{+}, \mathbf{k})} M$ to $\widehat{M}^{[g]}_{\ell}$ which maps (4.30) to (4.31). Since $\widetilde{M}^{[g]}_{\ell}$ is a quotient of $\widehat{M}^{[g]}_{\ell}$, we have a surjective $\hat{\mathfrak{g}}^{[g]}$-module map $\tilde{\varrho}$ from $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}^{[g]}_{+}, \mathbf{k})} M$ to $\widetilde{M}^{[g]}_{\ell}$.

Note that by Poincaré-Birkhoff-Witt theorem, $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_0^{[g]})} M$ as a graded vector space is isomorphic to $U(\hat{\mathfrak{g}}_-^{[g]}) \otimes M$. In particular, the $\hat{\mathfrak{g}}^{[g]}$-module $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_0^{[g]})} M$ is spanned by elements of the form

$$a^{i_1}(n_1) \cdots a^{i_l}(n_l) w^b \tag{4.32}$$

for $i_p \in I$, $n_p \in \alpha^{i_p} - \mathbb{Z}_+$ for $p = 1, \dots, l$ and $b \in A$.

By Remark 4.10, $\widehat{M}_\ell^{[g]}$ is spanned by elements of the form (4.31) for $i_p \in I$, $n_p \in \alpha^{i_p} - \mathbb{Z}_+$ for $p = 1, \dots, l$, $j_q \in I_{\mathbb{I}}$ for $q = 1, \dots, m$ and $b \in A$. Then $\widetilde{M}_\ell^{[g]}$ is spanned by elements of the form

$$(a_{[g],\ell}^{i_1})_{n_1,0} \cdots (a_{[g],\ell}^{i_l})_{n_l,0} (\psi_{[g]}^b)_{-1,0} \mathbf{1} \tag{4.33}$$

for $i_p \in I$, $n_p \in \alpha^{i_p} - \mathbb{Z}_+$ for $p = 1, \dots, l$ and $b \in A$.

Since elements of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}_+^{[g]}, \mathbf{k})} M$ of the same form as (4.32) are sent under $\hat{\rho}$ to elements of $\widehat{M}_\ell^{[g]}$ of the same form as (4.33), $\tilde{\varrho}$ maps elements of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}_+^{[g]}, \mathbf{k})} M$ of the same form as (4.32) to elements of $\widetilde{M}_\ell^{[g]}$ of the form (4.33). But the only relations among elements of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}_+^{[g]}, \mathbf{k})} M$ of the same form as (4.32) are generated by the commutator relations for $a^i(n)$ for $i \in I$ and $n \in \alpha^i - \mathbb{Z}_+$. Since $(a_{[g],\ell}^i)_{n,0}$ for $i \in I$ and $n \in \alpha^i - \mathbb{Z}_+$ satisfy the same commutator relations as $a^i(n)$ and the only relations among elements of $\widetilde{M}_\ell^{[g]}$ of the form (4.33) are generated by these commutator relations, the surjective $\hat{\mathfrak{g}}^{[g]}$-module map $\tilde{\varrho}$ induces a bijective $\hat{\mathfrak{g}}^{[g]}$-module map $\tilde{\rho}$ from $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_0^{[g]})} M$ to $\widetilde{M}_\ell^{[g]}$. Thus $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_0^{[g]})} M$ is equivalent to $\widetilde{M}_\ell^{[g]}$. ∎

**Remark 4.12** From the proof of Theorem 4.11, we see that $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_{\mathbb{I}}^{[g]})} M$ and $\widetilde{M}_\ell^{[g]}$ are spanned by elements of the form (4.32) and (4.33), respectively, for $i_p \in I$, $n_p \in \alpha^{i_p} - \mathbb{Z}_+$ for $p = 1, \dots, l$ and $b \in A$.

**Theorem 4.13** *The lower-bounded generalized $g$-twisted $M(\ell, 0)$-module $\widetilde{M}_\ell^{[g]}$ is in fact grading restricted.*

*Proof.* By Theorem 4.11, we need only prove that $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_0^{[g]})} M$ is grading restricted. But $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_0^{[g]})} M$ is a graded subspace of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_0^{[g]})} \Lambda(M)$. By Theorem 4.6, $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_0^{[g]})} \Lambda(M)$ is grading restricted, $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_0^{[g]})} M$ is also grading restricted. ∎

## 4.3 Basic properties

The lower-bounded or grading-restricted $g$-twisted $M(\ell, 0)$-modules $\widehat{M}_\ell^{[g]}$, $\breve{M}_\ell^{[g]}$, $\widehat{M}_\ell^{[g]}$, $\widetilde{M}_\ell^{[g]}$ constructed above all have their own universal properties and other basic properties. We first give the universal properties of $\widehat{M}_\ell^{[g]}$ and $\widehat{M}_\ell^{[g]}$.

**Theorem 4.14** *Let $(W, Y_W^g)$ be a lower-bounded generalized $g$-twisted module for $M(\ell, 0)$ viewed as a grading-restricted vertex algebra (vertex operator algebra when $\mathfrak{g}$ is simple and $\ell + h^\vee \neq 0$) and $M^0$ a subspace of $W$ invariant under the actions of $g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$. Assume that $\hat{\mathfrak{g}}_+^{[g]}$ acts on $M^0$ as $0$. If there is a linear map $f : M \to M^0$ commuting with the actions of $g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)|_{M^0}$ and $L_M(0)$, $L_W(0)_S|_{M^0}$ and $L_M(0)_S$ and $L_W(0)_N|_{M^0}$ and $L_M(0)_N$, then there exists a unique module map $\hat{f} : \widehat{M}_\ell^{[g]} \to W$ ($\overset{\frown}{f} : \overset{\frown}{M}{}_\ell^{[g]} \to W$) such that $\hat{f}|_M = f$ ($\overset{\frown}{f}|_M = f$). If $f$ is surjective and $(W, Y_W^g)$ is generated by the coefficients of $(Y^g)_{WM(\ell,0)}^W(w, x)\mathbf{1}$ for $w \in M^0$, where $(Y^g)_{WM(\ell,0)}^W$ is the twist vertex operator map obtained from $Y_W^g$ (see [Hua4]), then $\hat{f}$ ($\overset{\frown}{f}$) is surjective.*

*Proof.* Since $f$ commutes with the action of $L_W(0)|_{M^0}$ and $L_{\widehat{M}_\ell^{[g]}}(0)$, we have $f(M_{[h]}) \subset M_{[h]}^0$ for $h \in Q_M$. Since $\hat{\mathfrak{g}}_+^{[g]}$ acts on $M_{[h]}^0$ as $0$, no nonzero elements of the submodule of $W$ generated by $M_{[h]}^0$ have weights less than $h$. Then by the universal property of $\widehat{(M_{[h]})}_{\ell,h}^{[g]}$ (given by Theorem 5.2 in [Hua5]), there exists a unique module map $\hat{f}_h : \widehat{(M_{[h]})}_{\ell,h}^{[g]} \to W$ such that $\hat{f}_h|_{M_{[h]}} = f|_{M_{[h]}}$. Let $\hat{f} : \widehat{M}_\ell^{[g]} \to W$ be defined to be $\hat{f}_h$ on $\widehat{(M_{[h]})}_{\ell,h}^{[g]}$. Then $\hat{f}|_M = f$. The uniqueness of $\hat{f}$ follows from the uniqueness of $\hat{f}_h$ for $h \in Q_M$. The second conclusion also follows from the property of $\hat{f}_h$ (see Theorem 5.2 in [Hua5]) and the fact that $\widehat{(M_{[h]})}_{\ell,h}^{[g]}$ is generated by the subspace spanned by $(\psi_{[g]}^b)_{-1,0}$ (see Theorem 2.3 in [Hua6]).

In the case that $M(\ell, 0)$ is viewed as a vertex operator algebra, we have a module map $\hat{f} : \widehat{M}_\ell^{[g]} \to W$. Since on $W$, $L_W(0) = \mathrm{Res}_x x Y_W^g(\omega_{M(\ell,0)}, x)$ and $L_W(-1) = \mathrm{Res}_x Y_W^g(\omega_{M(\ell,0)}, x)$ and $\overset{\frown}{f}|_M = f$ is obtained from $\widehat{M}_\ell^{[g]}$ by taking the quotient by a submodule generated by exactly these relations, we see that $\hat{f}$ induces a module map $\overset{\frown}{f} : \overset{\frown}{M}{}_\ell^{[g]} \to W$. The other conclusions follow from the properties of $\hat{f}$ which we have proved. ∎

We have the following immediate consequence whose proof is omitted:

**Corollary 4.15** *Let $W$ be a lower-bounded generalized $g$-twisted module for $M(\ell, 0)$ viewed as a grading-restricted vertex algebra (vertex operator algebra when $\mathfrak{g}$ is simple and $\ell + h^\vee \neq 0$) generated by a subspace $M$ invariant under $g$, $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$ and annihilated by $\hat{\mathfrak{g}}_+^{[g]}$. Then $W$ as a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module is equivalent to a quotient of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ ($U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}_+^{[g]}, \mathbf{k})} M$). Conversely, let $M$ be a vector space with actions of $g$, $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$ satisfying the conditions given above. Then a quotient module of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{U(\hat{\mathfrak{g}}_+^{[g]} \oplus \mathbb{C}\mathbf{k})} \Lambda(M)$ ($U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}_+^{[g]}, \mathbf{k})} M$) has a natural structure of a lower-bounded generalized $g$-twisted $M(\ell, 0)$-module when $M(\ell, 0)$ is viewed as a grading-restricted vertex algebra (vertex operator algebra when $\mathfrak{g}$ is simple and $\ell + h^\vee \neq 0$).*

Now we discuss the universal properties of $\breve{M}_\ell^{[g]}$ and $\widetilde{M}_\ell^{[g]}$.

**Theorem 4.16** *Let $(W, Y_W^g)$ be a lower-bounded generalized $g$-twisted module for $M(\ell, 0)$ viewed as a grading-restricted vertex algebra (vertex operator algebra when $\mathfrak{g}$ is simple and $\ell + h^\vee \neq 0$). Let $M^0$ a finite-dimensional $\hat{\mathfrak{g}}_{\mathbb{I}}^{[g]}$-submodule of $W$ invariant also under the actions of $g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$ and annihilated by $\hat{\mathfrak{g}}_{+}^{[g]}$. Assume that there is a $\hat{\mathfrak{g}}_{\mathbb{I}}^{[g]}$-module map $f : M \to M^0$ commuting with the actions of $g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)|_{M^0}$ and $L_M(0)$, $L_W(0)_S|_{M^0}$ and $L_M(0)_S$ and $L_W(0)_N|_{M^0}$ and $L_M(0)_N$. Then there exists a unique module map $\breve{f} : \breve{M}_\ell^{[g]} \to W$ ($\tilde{f} : \widetilde{M}_\ell^{[g]} \to W$) such that $\breve{f}|_M = f$ ($\tilde{f}|_M = f$). If $f$ is surjective and $(W, Y_W^g)$ is generated by the coefficients of $(Y^g)_{WM(\ell,0)}^W(w, x)\mathbf{1}$ for $w \in M^0$, where $(Y^g)_{WM(\ell,0)}^W$ is the twist vertex operator map obtained from $Y_W^g$, then $\breve{f}$ ($\tilde{f}$) is surjective and thus $W$ is grading restricted.*

*Proof.* By Theorem 4.14, we have a unique module map $\hat{f} : \widehat{M}_\ell^{[g]} \to W$ such that $\hat{f}|_M = f$. Since $f$ is a $\hat{\mathfrak{g}}_{\mathbb{I}}^{[g]}$-module map, we have

$$a_W^i(\alpha^i) f(w^b) - \sum_{c \in A} \lambda_{ic}^b f(w^c) = f\left(a^i(\alpha^i) w^b - \sum_{c \in A} \lambda_{ic}^b w^c\right) = f(0) = 0$$

for $i \in I_{\mathbb{I}}$, $b \in A$. Since $\hat{f}$ is a module map, we have

$$\hat{f}((\psi_{[g],h^b}^b)_{-1,0}\mathbf{1}) = \hat{f}(((Y^g)_{\widehat{M}_\ell^{[g]} M(\ell,0)}^{\widehat{M}_\ell^{[g]}})_{-1,0}(w^b)\mathbf{1}) = ((Y^g)_{WM(\ell,0)}^W)_{-1,0}(f(w^b))\mathbf{1} = f(w^b)$$

for $b \in A$. Thus we obtain

$$\hat{f}\left((a_{[g],\ell}^i)_{\alpha^i,0}(\psi_{[g],h^b}^b)_{-1,0}\mathbf{1} - \sum_{c \in A} \lambda_{ic}^b (\psi_{[g],h^c}^c)_{-1,0}\mathbf{1}\right) = a_W^i(\alpha^i) f(w^b) - \sum_{c \in A} \lambda_{ic}^b f(w^c) = 0.$$

So

$$(a_{[g],\ell}^i)_{\alpha^i,0}(\psi_{[g],h^b}^b)_{-1,0}\mathbf{1} - \sum_{c \in A} \lambda_{ic}^b (\psi_{[g],h^c}^c)_{-1,0}\mathbf{1}$$

is in the kernel of $\hat{f}$. In particular, we have a module map $\breve{f} : \breve{M}_{\ell,h}^{[g]} \to W$. The uniqueness of $\breve{f}$ and the surjectivity of $\breve{f}$ when $f$ is surjective follow from the uniqueness of $\hat{f}$ and the surjectivity of $\hat{f}$. Since $\breve{M}_\ell^{[g]}$ is grading restricted, $W$ is grading restricted when $\breve{f}$ is surjective.

The proof for $\widetilde{M}_\ell^{[g]}$ is the same except that we use $\widehat{M}_\ell^{[g]}$ instead of $\widehat{M}_\ell^{[g]}$. ∎

We also have the following immediate consequence whose proof is also omitted:

**Corollary 4.17** *Let $W$ be a lower-bounded generalized $g$-twisted module for $M(\ell, 0)$ viewed as a grading-restricted vertex algebra (vertex operator algebra when $\mathfrak{g}$ is simple and $\ell + h^\vee \neq 0$) generated by a finite-dimensional $\hat{\mathfrak{g}}_{I_{\mathbb{I}}}$-submodule $M$ invariant under $g$, $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$ and annihilated by $\hat{\mathfrak{g}}_{+}^{[g]}$. Then $W$ as a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module is* *equivalent to a quotient of* $U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+}\oplus\hat{\mathfrak{g}}^{[g]}_{0})}\Lambda(M)$ $(U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+}\oplus\hat{\mathfrak{g}}^{[g]}_{0})}M)$ *and, in particular,* $W$ *is grading restricted. Conversely, let* $M$ *be a finite-dimensional* $\hat{\mathfrak{g}}_{I_{\mathbb{I}}}$*-module with compatible actions of* $g$, $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_N$ *and* $L_W(0)_N$ *satisfying the conditions we discussed above. Then a quotient module of* $U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+}\oplus\hat{\mathfrak{g}}^{[g]}_{0})}\Lambda(M)$ $(U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+}\oplus\hat{\mathfrak{g}}^{[g]}_{0})}M)$ *has a natural structure of a grading-restricted generalized* $g$*-twisted* $M(\ell,0)$*-module when* $M(\ell,0)$ *is viewed as a grading-restricted vertex algebra (vertex operator algebra when* $\mathfrak{g}$ *is simple and* $\ell+h^{\vee}\neq 0$*).*

**Remark 4.18** In [Hua6], only the existence of a grading-restricted generalized $g$-twisted $V$-module for a grading-restricted vertex algebra $V$ and an automorphism $g$ of $V$ is proved under suitable conditions. But no construction is given in that paper. Theorems 4.6, 4.13 and Corollary 4.17 give explicit constructions of grading-restricted generalized $g$-twisted $M(\ell,0)$-modules.

# 5 Lower-bounded and grading-restricted generalized twisted $L(\ell,0)$-modules

In this section, we construct lower-bounded and grading-restricted generalized $g$-twisted $L(\ell,0)$-modules. We shall mainly discuss the case that $\mathfrak{g}$ is simple, $\ell\in\mathbb{Z}_+$ and $L(\ell,0)$ is viewed as a vertex operator algebra.

We first give some straightforward general results.

**Proposition 5.1** *Let* $W$ *be a lower-bounded generalized* $g$*-twisted module for* $L(\ell,0)$ *viewed as a grading-restricted vertex algebra (vertex operator algebra when* $\mathfrak{g}$ *is simple and* $\ell+h^{\vee}\neq 0$*) generated by a subspace* $M$ *invariant under* $g$, $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_N$ *and* $L_W(0)_N$ *and annihilated by* $\hat{\mathfrak{g}}^{[g]}_{+}$*. Then* $W$ *is a lower-bounded generalized* $g$*-twisted module for* $M(\ell,0)$ *viewed as a grading-restricted vertex algebra (vertex operator algebra). In particular,* $W$ *is a lower bounded* $\hat{\mathfrak{g}}^{[g]}$*-module and is a quotient of* $U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+}\oplus\mathbb{C}\mathbf{k})}\Lambda(M)$ $(U(\hat{\mathfrak{g}}^{[g]})\otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}^{[g]}_{+},\mathbf{k})}M)$*. If* $M$ *is in addition a finite-dimensional* $\hat{\mathfrak{g}}_{\mathbb{I}}$*-module, then* $W$ *is a quotient of* $U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+}\oplus\hat{\mathfrak{g}}^{[g]}_{0})}\Lambda(M)$ $(U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+}\oplus\hat{\mathfrak{g}}^{[g]}_{0})}M)$ *and in particular,* $W$ *is grading restricted.*

*Proof.* Since $L(\ell,0)$ is a quotient of $M(\ell,0)$, $W$ must be a lower-bounded generalized $g$-twisted $M(\ell,0)$-module. By Proposition 4.1, $W$ is a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module. By Corollary 4.15, $W$ as a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module is a quotient of $U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+}\oplus\mathbb{C}\mathbf{k})}\Lambda(M)$ $(U(\hat{\mathfrak{g}}^{[g]})\otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}^{[g]}_{+},\mathbf{k})}M)$.

If $M$ is in addition a finite-dimensional $\hat{\mathfrak{g}}_{\mathbb{I}}$-module, then by Corollary 4.17, $W$ as a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module is a quotient of $U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+}\oplus\hat{\mathfrak{g}}^{[g]}_{0})}\Lambda(M)$ $(U(\hat{\mathfrak{g}}^{[g]})\otimes_{U(\hat{\mathfrak{g}}^{[g]}_{+}\oplus\hat{\mathfrak{g}}^{[g]}_{0})}M)$ and, in particular, $W$ is grading restricted. ∎

Using the structure of $L(\ell, 0)$ as a $\hat{\mathfrak{g}}$-module and properties of lower-bounded or grading-restricted generalized $g$-twisted $M(\ell, 0)$-modules, we have the following result:

**Proposition 5.2** *Assume that $\mathfrak{g}$ is simple with a given Cartan subalgebra and a given set of simple roots and $\ell \in \mathbb{Z}_+$. Let $(W, Y_W^g)$ be a lower-bounded (grading-restricted) generalized $g$-twisted $M(\ell, 0)$-module. Then $(W, Y_W^g)$ is a lower-bounded (grading-restricted) generalized $g$-twisted $L(\ell, 0)$-module if and only if $Y_W^g(e_\theta(-1)^{\ell+1}\mathbf{1}, x) = 0$ , where $\theta$ is the highest root of $\mathfrak{g}$ and $e_\theta \in \mathfrak{g}_\theta \setminus \{0\}$.*

*Proof.* From [K] and Proposition 6.6.17 in [LL], we know that $L(\ell, 0) = M(\ell, 0)/I(\ell, 0)$ where $I(\ell, 0) = U(\hat{\mathfrak{g}})e_\theta(-1)^{\ell+1}\mathbf{1}$. Then $W$ is a lower-bounded (grading-restricted) generalized $g$-twisted $L(\ell, 0)$-module if and only if $Y_W^g(I(\ell, 0), x) = 0$. We need only prove that $Y_W^g(I(\ell, 0), x) = 0$ if and only if $Y_W^g(e_\theta(-1)^{\ell+1}\mathbf{1}, x) = 0$.

If $Y_W^g(I(\ell, 0), x) = 0$, then certainly $Y_W^g(e_\theta(-1)^{\ell+1}\mathbf{1}, x) = 0$. If $Y_W^g(e_\theta(-1)^{\ell+1}\mathbf{1}, x) = 0$, then $Y_W^g(e_\theta(-1)^{\ell+1}\mathbf{1}, x)w = 0$ for $w \in W$. Therefore we have $(Y^g)^W_{WM(\ell,0)}(w, x)e_\theta(-1)^{\ell+1}\mathbf{1} = 0$ for $w \in W$, where $(Y^g)^W_{WM(\ell,0)}$ is the twist vertex operator map introduced and studied in [Hua4]. But by Corollary 4.3 in [Hua4] (commutativity for twisted and twist vertex operators), for $v \in M(\ell, 0)$ and $w' \in W'$,

$$\begin{aligned}
&F^p(\langle w', (Y^g)^W_{WM(\ell,0)}(w, z_2)Y_{M(\ell,0)}(v, z_1)e_\theta(-1)^{\ell+1}\mathbf{1}\rangle) \\
&\quad = F^p(\langle w', Y_W^g(v, z_1)(Y^g)^W_{WM(\ell,0)}(w, z_2)e_\theta(-1)^{\ell+1}\mathbf{1}\rangle) \\
&\quad = 0,
\end{aligned}$$

where the first two lines are different expressions of the $p$-th branch of a multivalued analytic function that

$$\langle w', Y_W^g(v, z_1)(Y^g)^W_{WM(\ell,0)}(w, z_2)e_\theta(-1)^{\ell+1}\mathbf{1}\rangle$$

and

$$\langle w', (Y^g)^W_{WM(\ell,0)}(w, z_2)Y_{M(\ell,0)}(v, z_1)e_\theta(-1)^{\ell+1}\mathbf{1}\rangle$$

converge to in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively. Thus we have

$$(Y^g)^W_{WM(\ell,0)}(w, x)Y_{M(\ell,0)}(v, x_1)e_\theta(-1)^{\ell+1}\mathbf{1} = 0$$

for $w \in W$. From the definition of the twist vertex operator map $(Y^g)^W_{WV}$, we obtain

$$Y_W^g(Y_{M(\ell,0)}(v, x_1)e_\theta(-1)^{\ell+1}\mathbf{1}, x) = 0.$$

But the coefficients of $Y_{M(\ell,0)}(v, x_1)$ span $U(\hat{\mathfrak{g}})$. So we obtain $Y_W^g(I(\ell, 0), x) = 0$. ∎

Our goal is to give and identify explicitly universal lower-bounded and grading-restricted generalized $g$-twisted $L(\ell, 0)$-modules using the results we obtained in the preceding section and some conditions on the corresponding $\hat{\mathfrak{g}}^{[g]}$-modules. To do this, we first prove some results on $\mathcal{S}_g$ and $\mathcal{N}_g$.

For $\mathcal{S}_g$, or equivalently, the semisimple automorphism $\sigma = e^{2\pi i \mathcal{S}_g}$, we have the following generalization of Proposition 8.1 in [K] on automorphisms of finite orders of a simple Lie algebra:

**Proposition 5.3** *Assume that $\mathfrak{g}$ is simple with a given Cartan subalgebra $\mathfrak{h}$ and a given set $\Delta$ of simple roots. Let $\sigma = e^{2\pi i \mathcal{S}_g}$ be the semisimple part of $g$. Then there exists an automorphism $\tau_\sigma$ of $\mathfrak{g}$ such that $\sigma = \tau_\sigma e^{\mathrm{ad}\, h}\mu\tau_\sigma^{-1}$, where $\mu$ is a diagram automorphism of $\mathfrak{g}$ preserving $\mathfrak{h}$ and $\Delta$ and $h$ is an element of the fixed-point subspace $\mathfrak{h}^{[0]}$ of $\mathfrak{h}$.*

*Proof.* Let $\tilde{\mathfrak{h}}^{[0]} \subset \mathfrak{g}^{[0]}$ be a maximal toral subalgebra (that is, maximal ad-diagonalizable subalgebra) of $\mathfrak{g}^{[0]}$ and $C_{\mathfrak{g}}(\tilde{\mathfrak{h}}^{[0]})$ the centralizer of $\tilde{\mathfrak{h}}^{[0]}$ in $\mathfrak{g}$. Extend $\tilde{\mathfrak{h}}^{[0]}$ to a maximal toral subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$. Then $\tilde{\mathfrak{h}}$ is a Cartan subalgebra of $\mathfrak{g}$ and $C_{\mathfrak{g}}(\tilde{\mathfrak{h}}^{[0]}) = \tilde{\mathfrak{h}} + \sum_\xi \tilde{\mathfrak{g}}_\xi$, where the sum is over the roots $\xi$ in $\tilde{\mathfrak{h}}$ such that $\xi$ restricted to $\tilde{\mathfrak{h}}^{[0]}$ is 0 and $\tilde{\mathfrak{g}}_\xi$ is the root space associated to $\xi$. We first prove that $C_{\mathfrak{g}}(\tilde{\mathfrak{h}}^{[0]}) = \tilde{\mathfrak{h}}$.

Let $\mathfrak{s} = \sum_\xi \tilde{\mathfrak{g}}_\xi$. By definition, $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ invariant under $g$ such that $\mathfrak{s}\cap\mathfrak{g}^{[0]} = 0$. Moreover, the restriction of the bilinear form $(\cdot,\cdot)$ to $\mathfrak{s}$ is nondegenerate because $\tilde{\mathfrak{g}}_{\xi_1}$ and $\tilde{\mathfrak{g}}_{\xi_2}$ are orthogonal if $\xi_1+\xi_2 \neq 0$ and the restriction of $(\cdot,\cdot)$ to $\tilde{\mathfrak{g}}_\xi \times \tilde{\mathfrak{g}}_{-\xi}$ is nondegenerate. Since $\mathfrak{s}$ is invariant under $g$, we have $\mathfrak{s} = \coprod_{\alpha\in P_{\mathfrak{s}}} \mathfrak{s}^{[\alpha]}$ where $P_{\mathfrak{s}}$ is the set of $\alpha \in [0,1)+i\mathbb{R}$ such that $e^{2\pi i\alpha}$ is an eigenvalue of $\sigma$ (or equivalently, of $g$) and for $\alpha \in \mathfrak{s}$, $\mathfrak{s}^{[\alpha]} = \mathfrak{s} \cap \mathfrak{g}^{[\alpha]}$ is the eigenspace of $\sigma$ (or equivalently, the generalized eigenspace of $g$) in $\mathfrak{s}$ with the eigenvalue $e^{2\pi i\alpha}$. For $\alpha \in ([0,1)+i\mathbb{R}) \setminus P_{\mathfrak{s}}$, let $\mathfrak{s}^{[\alpha]} = 0$. Then we have $\mathfrak{s} = \coprod_{\alpha\in[0,1)+i\mathbb{R}} \mathfrak{s}^{[\alpha]}$. Moreover, by Lemma 2.1 and the fact that $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$, we have $[\mathfrak{s}^{[\alpha]}, \mathfrak{s}^{[\beta]}] \subset \mathfrak{s}^{[s(\alpha,\beta)]}$ for $\alpha,\beta \in [0,1)+i\mathbb{R}$ (recall $s(\alpha,\beta)$ defined before Lemma 2.1). We need only prove that $\mathfrak{s} = 0$.

Since $\mathfrak{g}$ is finite dimensional, $\mathfrak{s}$ is finite dimensional and hence $P_{\mathfrak{s}}$ is a finite set. We use induction on the finitely many real parts of the elements of $P_{\mathfrak{s}}$. First, we know that $\mathfrak{s}^{[0]} = \mathfrak{s}\cap\mathfrak{g}^{[0]} = 0$. For $\alpha \in (P_{\mathfrak{s}} \setminus \{0\}) \cap (i\mathbb{R})$ and $a \in \mathfrak{s}^{[\alpha]}$, we know that $(\mathrm{ad}\ a)^r \mathfrak{s}^{[\beta]} \in \mathfrak{s}^{[r\alpha+\beta]}$ for $\beta \in P_g$. Since $P_{\mathfrak{s}}$ is a finite set, $\mathfrak{s}^{[r\alpha+\beta]}$ must be 0 when $r$ is sufficiently large. So ad $a$ is nilpotent on $\mathfrak{g}$. Applying Lemma 2.3 to $\mathfrak{s}$ and the restriction of $\sigma$ to $\mathfrak{s}$, we see that $(\cdot,\cdot)$ restricted to $\mathfrak{s}^{[\alpha]} \times \mathfrak{s}^{[-\alpha]}$ is nondegenerate. In particular, if $\mathfrak{s}^{[\alpha]} \neq 0$, then $\mathfrak{s}^{[-\alpha]} \neq 0$. If $\mathfrak{s}^{[\alpha]} \neq 0$, let $a \in \mathfrak{s}^{[\alpha]} \setminus \{0\}$ and $b \in \mathfrak{s}^{[-\alpha]} \setminus \{0\}$. Then both ad $a$ and ad $b$ are nilpotent on $\mathfrak{s}$. Therefore the eigenvalues of ad $a$ and ad $b$ are all 0. Since $[\mathfrak{s}^{[\alpha]}, \mathfrak{s}^{[-\alpha]}] \subset \mathfrak{s}^{[0]} = 0$, ad $a$ and ad $b$ commute and hence can be diagonalized simultaneously. In particular, the trace of (ad $a$)(ad $b$) is 0. But this contradicts the nondegneracy of $(\cdot,\cdot)$ restricted to $\mathfrak{s}^{[\alpha]} \times \mathfrak{s}^{[-\alpha]}$ because $(a,b)$ is proportional to this trace. Thus $\mathfrak{s}^{[\alpha]} = 0$.

Note that for $\alpha \in P_{\mathfrak{s}}$ such that $\Re(\alpha) > 0$, if $\mathfrak{s}^{[\alpha]} \neq 0$, then $\mathfrak{s}^{[1-\alpha]} \neq 0$ since by Lemma 2.3, the restriction of $(\cdot,\cdot)$ to $\mathfrak{s}$ is nondegenerate and $\mathfrak{s}^{[\alpha]}$ is orthogonal to $\mathfrak{s}^{[\beta]}$ for $\beta \neq 1-\alpha$. We now assume that for $\alpha \in P_{\mathfrak{s}}$ with $\Re(\alpha) > 0$, $\mathfrak{s}^{[\alpha']} = 0$ for $\alpha' \in P_g$ and $\Re(\alpha') < \Re(\alpha)$ and for $\alpha' \in P_g$ and $\Re(\alpha') > \Re(1-\alpha)$. Then for $a \in \mathfrak{s}^{[\alpha]}$, $(\mathrm{ad}\ a)^r\mathfrak{s}^{[\beta]} \in \mathfrak{s}^{[s(\alpha,r,\beta)]}$, where $s(\alpha,r,\beta) \equiv r\alpha+\beta \mod \mathbb{Z}$ satisfying $0 \leq \Re(s(\alpha,r,\beta)) < 1$. Since $0 \leq \beta < 1$, there exists $r \in \mathbb{Z}_+$ such that $\Re((r-1)\alpha+\beta) < 1$ but $\Re(r\alpha+\beta) \geq 1$. From $\Re((r-1)\alpha+\beta) < 1$, we obtain $0 \leq s(\alpha,r,\beta) = \Re(r\alpha+\beta) - 1 < \Re(\alpha)$. By the induction assumption, $\mathfrak{s}^{[s(\alpha,r,\beta)]} = 0$. So we obtain $(\mathrm{ad}\ a)^r\mathfrak{s}^{[\beta]} = 0$ for $\beta \in P_{\mathfrak{s}}$. Thus ad $a$ is nilpotent on $\mathfrak{s}$. Similarly, for $b \in \mathfrak{s}^{[1-\alpha]}$, $(\mathrm{ad}\ b)^r\mathfrak{s}^{[\beta]} \in \mathfrak{s}^{[s(1-\alpha,r,\beta)]}$, where $s(1-\alpha,r,\beta) \equiv r(1-\alpha)+\beta \mod \mathbb{Z}$ satisfying $0 \leq \Re(s(1-\alpha,r,\beta)) < 1$. When $\Re(\beta) = 0$, since we have proved $\mathfrak{s}^{[\beta]} = 0$, $(\mathrm{ad}\ b)^r\mathfrak{s}^{[\beta]} = 0$ for $r \in \mathbb{Z}_+$. When $\Re(\beta) \neq 0$, there exists $r \in \mathbb{Z}_+$ such that $\Re(-r\alpha+\beta) \geq -1$ but $\Re(-(r+1)\alpha+\beta) < -1$. Then $\Re(r(1-\alpha)+\beta) \geq r-1$ but $\Re((r-1)(1-\alpha)+\beta) < r$. Since $0 < \Re(\beta) < 1$, we obtain $0 < \Re(1-\alpha) < \Re(1-\alpha+\beta) = \Re(s(1-\alpha,r,\beta)) < 1$. By the induction assumption, $\mathfrak{s}^{[s(1-\alpha,r,\beta)]} = 0$. So we obtain $(\mathrm{ad}\ b)^r\mathfrak{s}^{[\beta]} = 0$. Thus ad $b$ is also nilpotent on $\mathfrak{s}$. If $\mathfrak{s}^{[\alpha]} \neq 0$, let $a \in \mathfrak{s}^{[\alpha]} \setminus \{0\}$ and $b \in \mathfrak{s}^{[1-\alpha]} \setminus \{0\}$. Then we have proved that both ad $a$ and ad $b$ are nilpotent on $\mathfrak{s}$. Therefore the eigenvalues of ad $a$ and ad $b$ are all 0. Since $[\mathfrak{s}^{[\alpha]}, \mathfrak{s}^{[1-\alpha]}] \subset \mathfrak{s}^{[0]} = 0$, ad $a$ and ad $b$ commute and hence can be diagonalized simultaneously. In particular, the trace of $(\mathrm{ad}\ a)(\mathrm{ad}\ b)$ is 0. Contradiction. Thus $\mathfrak{s}^{[\alpha]} = 0$. This proves $\mathfrak{s} = 0$.

We have proved that $C_{\mathfrak{g}}(\tilde{\mathfrak{h}}^{[0]}) = \tilde{\mathfrak{h}}$. Now choose $a \in \tilde{\mathfrak{h}}^{[0]}$ such that the centralizer $C_{\mathfrak{g}}(a)$ of $a$ in $\mathfrak{g}$ is minimal among the collection of all centralizer $C_{\mathfrak{g}}(b)$ of $b$ in $\mathfrak{g}$ for $b \in \tilde{\mathfrak{h}}^{[0]}$. Note that since elements of $\tilde{\mathfrak{h}}^{[0]}$ are all semisimple or ad-diagonalizable, $C_{\mathfrak{g}}(b)$ for $b \in \tilde{\mathfrak{h}}^{[0]}$ is equal to the space of all elements of $\mathfrak{g}$ on which ad $b$ acts nilpotently. Since $\tilde{\mathfrak{h}}^{[0]} \subset C_{\mathfrak{g}}(a)$, by Lemma A in Subsection 15.2 in [Hum], we have $C_{\mathfrak{g}}(a) \subset C_{\mathfrak{g}}(b)$ for $b \in \tilde{\mathfrak{h}}^{[0]}$. But $C_{\mathfrak{g}}(\tilde{\mathfrak{h}}^{[0]}) = \cap_{b \in \tilde{\mathfrak{h}}^{[0]}} C_{\mathfrak{g}}(b)$. So $C_{\mathfrak{g}}(a) \subset C_{\mathfrak{g}}(\tilde{\mathfrak{h}}^{[0]}) = \tilde{\mathfrak{h}}$. But $\tilde{\mathfrak{h}} \subset C_{\mathfrak{g}}(a)$. So we must have $\tilde{\mathfrak{h}} = C_{\mathfrak{g}}(a)$, that is, $a \in \tilde{\mathfrak{h}}^{[0]}$ is a regular semisimple element. As the centralizer of a fixed point of $g$, $\tilde{\mathfrak{h}}$ is a Cartan subalgebra of $\mathfrak{g}$ invariant under $\sigma$. In particular, we have a root system $\widetilde{\Phi}$ obtained from $\tilde{\mathfrak{h}}$.

The regular semisimple element $a$ cannot be orthogonal to any root $\xi$. Otherwise $[a, e_\xi] = (\xi, a)e_\xi = 0$ for $e_\xi \in \mathfrak{g}_\xi$ so that $e_\xi \in C_{\mathfrak{g}}(a) = \tilde{\mathfrak{h}}$, which is impossible. Thus if we let $\widetilde{\Phi}^+ = \{\xi \in \widetilde{\Phi} \mid (\xi, a) > 0\}$, then $\widetilde{\Phi} = \widetilde{\Phi}^+ - \widetilde{\Phi}^+$. By Theorem′ in Subsection 10.1 in [Hum], the set $\widetilde{\Delta}$ of all indecomposable roots in $\widetilde{\Phi}^+$ is a set of simple roots of $\widetilde{\Phi}$ and $\widetilde{\Phi}^+$ is the set of positive roots. Since $a$ is fixed by $\sigma$, $\sigma$ induces an automorphism of $\widetilde{\Phi}^+$. Choose $e_\xi \in \mathfrak{g}_\xi \setminus \{0\}$ for $\xi \in \widetilde{\Delta}$. Let $\tilde{\mu}$ be the diagram automorphism of $\mathfrak{g}$ corresponding to this automorphism of $\widetilde{\Phi}^+$ and $e_\xi$ for $\xi \in \widetilde{\Delta}$. Then $\sigma\tilde{\mu}^{-1}$ fix every element of $\tilde{\mathfrak{h}}$. In particular, $\sigma\tilde{\mu}^{-1}$ commutes with ad $\tilde{a}$ for all $\tilde{a} \in \tilde{\mathfrak{h}}$ and thus can be diagonalized simultaneously together with ad $\tilde{a}$. The root space decomposition $\mathfrak{g} = \tilde{\mathfrak{h}} \oplus \coprod_{\xi \in \widetilde{\Phi}} \tilde{\mathfrak{g}}_\xi$ gives a diagonalization of ad $\tilde{a}$. Also $\tilde{\mathfrak{g}}_\xi$ for $\xi \in \widetilde{\Phi}$ are all one dimensional and $\sigma\tilde{\mu}^{-1}$ acts as the identity on $\tilde{\mathfrak{h}}$, we see the this root space decomposition also give a diagonalization of $\sigma\tilde{\mu}^{-1}$. So on each root space $\tilde{\mathfrak{g}}_\xi$, it must act as a scalar multiplication by $\lambda_\xi \in \mathbb{C}^\times$. Let $l_0(\lambda_\xi) = \log|\lambda_\xi| + i \arg \lambda_\xi$, where $0 \leq \arg \lambda_\xi < 2\pi$. Let $\tilde{h} \in \tilde{\mathfrak{h}}$ be defined by $(\xi, \tilde{h}) = \frac{1}{2\pi i} l_0(\lambda_\xi)$ for $\xi \in \widetilde{\Delta}$. Then $\lambda_\xi e_\xi = e^{2\pi i(\xi, \tilde{h})} e_\xi = e^{2\pi i(\mathrm{ad}\ \tilde{h})} e_\xi$. Thus we obtain $\sigma\tilde{\mu}^{-1} = e^{2\pi i(\mathrm{ad}\ \tilde{h})}$, or equivalently, $\sigma = e^{2\pi i(\mathrm{ad}\ \tilde{h})}\tilde{\mu}$. Since $\sigma\tilde{\mu}^{-1}$ fix every element of $\tilde{\mathfrak{h}}$, $\tilde{h}$ must be in $\tilde{\mathfrak{h}}$.

Since any two Cartan subalgebras are conjugate to each other, there exists an automorphism $\nu$ of $\mathfrak{g}$ such that $\nu(\mathfrak{h}) = \tilde{\mathfrak{h}}$ and $\nu(\Delta) = \widetilde{\Delta}$. Let $\mu = \nu\tilde{\mu}\nu^{-1}$ and $\breve{h} = \nu^{-1}(\tilde{h}) \in \mathfrak{h}$. Then it is clear that $\mu$ is a diagram automorphism of $\mathfrak{g}$ preserving $\mathfrak{h}$ and $\Delta$ and we have

$$\sigma = e^{2\pi i(\mathrm{ad}\ \nu(\breve{h}))}\nu\mu\nu^{-1} = \nu e^{2\pi i(\mathrm{ad}\ \breve{h})}\mu\nu^{-1}.$$

But $\breve{h}$ might not be fixed by $\mu$. We need to find another automorphism such that after the conjugation by this automorphism, we have $h \in \mathfrak{h}^{[0]}$ (that is, fixed by $\mu$). This argument was in fact given by the proof of Lemma 8.3 in [EMS]: Let $r$ be the order of $\mu$ (in fact $r = 1, 2$ or 3), $h = \frac{1}{m}\sum_{k=1}^{r-1} \mu^k \breve{h}$ and $\eta = e^{\frac{2\pi i}{r}\sum_{k=0}^{r-1} k(\mathrm{ad}\ \mu^k \breve{h})}$. Then $\mu h = h$ and $h \in \mathfrak{h}$ since $\mathfrak{h}$ is invariant under $\mu$. Moreover,

$$\begin{aligned}\eta e^{2\pi i(\mathrm{ad}\,\check{h})}\mu\eta^{-1} &= e^{\frac{2\pi i}{r}\sum_{k=1}^{r-1}k(\mathrm{ad}\,\mu^k\check{h})}e^{2\pi i(\mathrm{ad}\,\check{h})}\mu e^{-\frac{2\pi i}{r}\sum_{k=1}^{r-1}k(\mathrm{ad}\,\mu^k\check{h})}\\
&= e^{\frac{2\pi i}{r}\sum_{k=1}^{r-1}k(\mathrm{ad}\,\mu^k\check{h})}e^{2\pi i(\mathrm{ad}\,\check{h})}e^{-\frac{2\pi i}{r}\sum_{k=1}^{r-1}k(\mathrm{ad}\,\mu^{k+1}\check{h})}\mu\\
&= e^{2\pi i\frac{1}{r}\sum_{k=1}^{r-1}(\mathrm{ad}\,\mu^k\check{h})}\mu\\
&= e^{2\pi i(\mathrm{ad}\,h)}\mu.\end{aligned}$$

Let $\tau_\sigma = \eta\nu^{-1}$. Then

$$\sigma = \nu e^{2\pi i(\mathrm{ad}\,\check{h})}\mu\nu^{-1} = \nu\eta^{-1}e^{2\pi i(\mathrm{ad}\,h)}\mu\eta\nu^{-1} = \tau_\sigma e^{2\pi i(\mathrm{ad}\,h)}\mu\tau_\sigma^{-1}.$$

∎

For $\mathcal{N}_g$, we have the following result:

**Proposition 5.4** *Assume that $\mathfrak{g}$ is semisimple. Then we have the following:*

1. *There exists $a_{\mathcal{N}_g} \in \mathfrak{g}^{[0]}$ such that $\mathcal{N}_g b = [a_{\mathcal{N}_g}, b]$ for $b \in \mathfrak{g}$, that is, $\mathcal{N}_g = \mathrm{ad}\, a_{\mathcal{N}_g}$.*
2. *On $\hat{\mathfrak{g}}$, $\mathcal{N}_g(b \otimes t^m) = [a_{\mathcal{N}_g} \otimes t^0, b \otimes t^m] = [a_{\mathcal{N}_g}, b] \otimes t^m$ for $b \in \mathfrak{g}$, $m \in \mathbb{Z}$ and $\mathcal{N}_g\mathbf{k} = [a_{\mathcal{N}_g} \otimes t^0, \mathbf{k}] = 0$.*
3. *On $\hat{\mathfrak{g}}^{[g]}$, $\mathcal{N}_g(b \otimes t^m) = [a_{\mathcal{N}_g} \otimes t^0, b \otimes t^m] = [a_{\mathcal{N}_g}, b] \otimes t^m$ for $b \in \mathfrak{g}^{[\beta]}$ and $m \in \beta + \mathbb{Z}$ and $\mathcal{N}_g\mathbf{k} = [a_{\mathcal{N}_g} \otimes t^0, \mathbf{k}] = 0$.*
4. *On $M(\ell, 0)$ or $L(\ell, 0)$, $\mathcal{N}_g = a_{\mathcal{N}_g}(0)$.*

*Proof.* By Corollary 2.2, $\mathcal{N}_g$ is a derivation of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, we know that every derivation of $\mathfrak{g}$ is inner. So there exists $a_{\mathcal{N}_g} \in \mathfrak{g}$ such that $\mathcal{N}_g b = [a_{\mathcal{N}_g}, b]$ for $b \in \mathfrak{g}$. Since $\mathcal{S}_g$ commutes with $\mathcal{N}_g$, for $b \in \mathfrak{g}^{[\beta]}$, $\mathcal{S}_g[a_{\mathcal{N}_g}, b] = \mathcal{S}_g\mathcal{N}_g b = \mathcal{N}_g\mathcal{S}_g b = \beta\mathcal{N}_g b = \beta[a_{\mathcal{N}_g}, b]$. So $[a_{\mathcal{N}_g}, b] \in \mathfrak{g}^{[\beta]}$. Thus $a_{\mathcal{N}_g} \in \mathfrak{g}^{[0]}$. This finishes the proof of Conclusion 1.

Conclusions 2, 3 and 4 follow immediately from the definitions of the actions of $\mathcal{N}_g$ on $\hat{\mathfrak{g}}$, $\hat{\mathfrak{g}}^{[g]}$, $M(\ell, 0)$ and $L(\ell, 0)$. ∎

In the rest of this section, we assume that $\mathfrak{g}$ is simple with a Cartan subalgebra $\mathfrak{h}$ and a set $\Delta$ of simple roots which gives a root system $\Phi$. For $a \in \mathfrak{g}$, $a = \sum_{\alpha \in P_g} a^\alpha$, where $a^\alpha \in \mathfrak{g}^{[\alpha]}$. Given a $\hat{\mathfrak{g}}$-module $W$, we have introduced $a^\alpha(x) = \sum_{n \in \alpha + \mathbb{Z}} a(n)x^{-n-1}$ for $\alpha \in P_{\mathfrak{g}}$ above.

We shall need the following result later:

**Proposition 5.5** *Assume that $\mathfrak{g}$ is simple with a given Cartan subalgebra $\mathfrak{h}$ and a given set $\Delta$ of simple roots. Let $g$, $\mu$, $h$ and $\tau_\sigma$ be the same as in Proposition 5.3. Let $\theta$ be the highest root of $\mathfrak{g}$ and $W$ a $\hat{\mathfrak{g}}^{[g]}$-module. Then there exists $r_\theta \in \mathbb{Z}$ such that $e_\theta \in \mathfrak{g}^{[(\theta,h)+r_\theta]}$ and*

$$[(\tau_\sigma e_\theta)(x_1), (\tau_\sigma e_\theta)(x_2)] = 0. \tag{5.1}$$

*Proof.* Since $(\mathrm{ad}\ h)e_\theta = [h,\theta] = (\theta,h)e_\theta$, $e^{2\pi i(\mathrm{ad}\ h)}e_\theta = e^{2\pi i(\theta,h)}e_\theta$. Then

$$e^{2\pi i(\mathrm{ad}\ \tau_\sigma h)}\tau_\sigma e_\theta = \tau_\sigma e^{2\pi i(\mathrm{ad}\ h)}e_\theta = e^{2\pi i(\theta,h)}\tau_\sigma e_\theta.$$

Since $\theta$ is the highest root and $\mu$ is an automorphism of $\Phi_+$, $\theta$ is fixed under $\mu$ by the definition and by the uniqueness of the highest root. Thus $e_\theta$ is also fixed under $\mu$. So $\tau_\sigma\mu\tau_\sigma^{-1}$ fixes $\tau_\sigma e_\theta$. Then by Proposition 5.3,

$$\sigma\tau_\sigma e_\theta = \tau_\sigma e^{2\pi i(\mathrm{ad}\ h)}\mu e_\theta = e^{2\pi i(\theta,h)}\tau_\sigma e_\theta. \tag{5.2}$$

From $\sigma = e^{2\pi i\mathcal{S}_g}$ and (5.2), there exists $r_\theta \in \mathbb{Z}$ such that $(\theta,h)+r_\theta \in P_{\mathfrak{g}}$ and

$$\mathcal{S}_g\tau_\sigma e_\theta = ((\theta,h)+r_\theta)\tau_\sigma e_\theta. \tag{5.3}$$

Thus $e_\theta \in \mathfrak{g}^{[(\theta,h)+r_\theta]}$.

To prove (5.1), first we have

$$[\tau_\sigma e_\theta, \tau_\sigma e_\theta] = 0. \tag{5.4}$$

Since $\theta+\theta\neq 0$, $(e_\theta,e_\theta)=0$. Then by the invariance of the bilinear form $(\cdot,\cdot)$,

$$(\tau_\sigma e_\theta, \tau_\sigma e_\theta) = 0. \tag{5.5}$$

Using the invariance of the bilinear form $(\cdot,\cdot)$ and $\mathcal{N}_g = \mathrm{ad}\ a_{\mathcal{N}_g}$ (Part 1 of Proposition 5.4), we obtain

$$(\mathcal{N}_g\tau_\sigma e_\theta, \tau_\sigma e_\theta) = ([a_{\mathcal{N}_g}, \tau_\sigma e_\theta], \tau_\sigma e_\theta) = (a_{\mathcal{N}_g}, [\tau_\sigma e_\theta, \tau_\sigma e_\theta]) = 0. \tag{5.6}$$

Let $\gamma = (\theta,h)+r_\theta$. Then $e_\theta \in \mathfrak{g}^{[\gamma]}$. Using (5.4), (5.5) and (5.6), we have

$$\begin{aligned}
&[(\tau_\sigma e_\theta)(x_1), (\tau_\sigma e_\theta)(x_2)]\\
&\quad= \sum_{m\in\gamma+\mathbb{Z}}\sum_{n\in\gamma+\mathbb{Z}}[(\tau_\sigma e_\theta)(m), (\tau_\sigma e_\theta)(n)]\\
&\quad= \sum_{m\in\gamma+\mathbb{Z}}\sum_{n\in\gamma+\mathbb{Z}}\Big([(\tau_\sigma e_\theta), (\tau_\sigma e_\theta)](m+n)\\
&\qquad\qquad + m((\tau_\sigma e_\theta), (\tau_\sigma e_\theta))\delta_{m+n,0}\ell + (\mathcal{N}_g(\tau_\sigma e_\theta), (\tau_\sigma e_\theta))\delta_{m+n,0}\ell\Big)\\
&\quad= 0.
\end{aligned} \tag{5.7}$$

∎

We also need the following general lemma:

**Lemma 5.6** *Let $V$ be a grading-restricted vertex algebra (or a vertex operator algebra), $g$ an automorphism of $V$ and $W$ a lower-bounded generalized $g$-twisted $V$-module. Assume that for some $u, v \in V$,*

$$(Y_W^g)_0(u,x_1)(Y_W^g)_0(v,x_2) = (Y_W^g)_0(v,x_2)(Y_W^g)_0(u,x_1),$$

*where* $(Y_W^g)_0(v,x)$ *for* $v\in V$ *is the constant term of* $Y_W^g(v,x)$ *viewed as a power series in* $\log x$*. Then* $(Y_W^g)_0(u,x)(Y_W^g)_0(v,x)$ *is well defined and*

$$(Y_W^g)_0((Y_V)_{-1}(u)v,x)) = (Y_W^g)_0(u,x)(Y_W^g)_0(v,x). \tag{5.8}$$

*Proof.* For $w\in W$,

$$Y_W^g(u,x_1)Y_W^g(v,x_2)w = Y_W^g(v,x_2)Y_W^g(u,x_1)w$$

has only finitely many negative power terms in both $x_1$ and $x_2$. In particular, we can let $x_1=x_2=x$ to obtain a well defined formal series $(Y_W^g)_0(u,x)(Y_W^g)_0(v,x)$.

To prove (5.8), we use the the Jacobi identity (4.1). Using $Y_W^g(u,x)=(Y_W^g)_0(x^{-\mathcal{N}_g}u,x)$ ((2.10) in [HY]) for $u\in V$ and $x_2^{\mathcal{N}_g}Y_V(u,x_0)=Y_V(x_2^{\mathcal{N}_g}u,x)x_2^{\mathcal{N}_g}$ ((2.5) in [Hua4]), and replacing $u$ and $v$ in (4.1) by $x_1^{\mathcal{N}_g}u$ and $x_2^{\mathcal{N}_g}v$, respectively, we see that that (4.1) becomes

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(Y_W^g)_0(u,x_1)(Y_W^g)_0(v,x_2) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)(Y_W^g)_0(v,x_2)(Y_W^g)_0(u,x_1)\\
&\quad = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y_W^g)_0\left(Y_V\left(\left(\frac{x_2}{x_1}\right)^{\mathcal{S}_g}\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}_g}u,x_0\right)v,x_2\right).
\end{aligned} \tag{5.9}$$

By the assumption, the left-hand side of (5.9) is equal to

$$\begin{aligned}
&\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\right)(Y_W^g)_0(u,x_1)(Y_W^g)_0(v,x_2)\\
&\quad = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y_W^g)_0(u,x_1)(Y_W^g)_0(v,x_2).
\end{aligned}$$

Thus from (5.9), we obtain

$$\begin{aligned}
&x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y_W^g)_0\left(Y_V\left(\left(\frac{x_2}{x_1}\right)^{\mathcal{S}_g}\left(1+\frac{x_0}{x_2}\right)^{\mathcal{L}_g}u,x_0\right)v,x_2\right)\\
&\quad = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W^g(u,x_1)Y_W^g(v,x_2).
\end{aligned} \tag{5.10}$$

Replacing $u$ in (5.10) by $\left(1+\frac{x_0}{x_2}\right)^{-\mathcal{L}_g}\left(\frac{x_2}{x_1}\right)^{-\mathcal{S}_g}u$, we obtain

$$\begin{aligned}
&x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y_W^g)_0\left(Y_V\left(u,x_0\right)v,x_2\right)\\
&\quad = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y_W^g)_0\left(\left(1+\frac{x_0}{x_2}\right)^{-\mathcal{L}_g}\left(\frac{x_2}{x_1}\right)^{-\mathcal{S}_g}u,x_1\right)(Y_W^g)_0(v,x_2).
\end{aligned} \tag{5.11}$$

Since $V = \coprod_{\alpha \in P_V} V^{[\alpha]}$, we have $u = \sum_{\alpha \in P_V} u^\alpha$, where $u^\alpha \in V^{[\alpha]}$ for $\alpha \in P_V$. Also note that $(Y_W^g)_0(u^\alpha, x) \in x^{-\alpha}(\mathrm{End}\, W)[[x, x^{-1}]]$. Then we have

$$\begin{aligned}
&x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y_W^g)_0\left(\left(1+\frac{x_0}{x_2}\right)^{-\mathcal{L}_g}\left(\frac{x_2}{x_1}\right)^{-\mathcal{S}_g}u, x_1\right)\\
&\quad= \sum_{\alpha \in P_V} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)x_1^{\alpha}(Y_W^g)_0\left(\left(1+\frac{x_0}{x_2}\right)^{-\mathcal{L}_g}x_2^{-\mathcal{S}_g}u^\alpha, x_1\right)\\
&\quad= \sum_{\alpha \in P_V} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(x_2+x_0)^{\alpha}(Y_W^g)_0\left(\left(1+\frac{x_0}{x_2}\right)^{-\mathcal{L}_g}x_2^{-\mathcal{S}_g}u^\alpha, x_2+x_0\right)\\
&\quad= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y_W^g)_0\left(\left(1+\frac{x_0}{x_2}\right)^{-\mathcal{L}_g}\left(\frac{x_2}{x_2+x_0}\right)^{-\mathcal{S}_g}u, x_2+x_0\right)\\
&\quad= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y_W^g)_0\left(\left(1+\frac{x_0}{x_2}\right)^{-\mathcal{N}_g}u, x_2+x_0\right).
\end{aligned}\tag{5.12}$$

Using (5.12), we see that the right-hand side of (5.11) is equal to

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(Y_W^g)_0\left(\left(1+\frac{x_0}{x_2}\right)^{-\mathcal{N}_g}u, x_2+x_0\right)(Y_W^g)_0(v, x_2).\tag{5.13}$$

Then $\mathrm{Res}_{x_1}$ of the left-hand side of (5.11) and (5.13) are also equal, that is,

$$(Y_W^g)_0\left(Y_V(u, x_0)v, x_2\right) = (Y_W^g)_0\left(\left(1+\frac{x_0}{x_2}\right)^{-\mathcal{N}_g}u, x_2+x_0\right)Y_W^g(v, x_2).\tag{5.14}$$

Taking the constant terms in $x_0$ in both sides of (5.14) and then replacing $x_2$ by $x$, we obtain (5.8). ∎

Applying Proposition 5.5 and Lemma 5.6 to the lower-bounded (grading-restricted) generalized $g$-twisted $M(\ell, 0)$-module $\widehat{M}_\ell^{[g]}$ ($\widetilde{M}_\ell^{[g]}$) and using Theorems 4.9 and 4.11, we have the following consequence:

**Corollary 5.7** *On the lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module $\widehat{M}_\ell^{[g]}$ (the grading-restricted $\hat{\mathfrak{g}}^{[g]}$-module $\widetilde{M}_\ell^{[g]}$), $(\tau_\sigma e_\theta)(x)^m$ for $m \in \mathbb{N}$ are well defined and*

$$Y^g_{\widehat{M}_\ell^{[g]}}((\tau_\sigma e_\theta)(-1)^m\mathbf{1}, x) = (\tau_\sigma e_\theta)(x)^m\tag{5.15}$$

$$\left(Y^g_{\widetilde{M}_\ell^{[g]}}((\tau_\sigma e_\theta)(-1)^m\mathbf{1}, x) = (\tau_\sigma e_\theta)(x)^m\right).\tag{5.16}$$

*In particular, on $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}_+^{[g]}, \mathbf{k})} M$ and $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}_+^{[g]} \oplus \hat{\mathfrak{g}}_0^{[g]}} M$, $(\tau_\sigma e_\theta)(x)^m$ for $m \in \mathbb{N}$ are well defined.*

*Proof.* By the definition of $\hat{\mathfrak{g}}^{[g]}$-module structure on $\widehat{M}_{\ell}^{[g]}$ (see Proposition 4.1), we have

$$(Y^g_{\widehat{M}_{\ell}^{[g]}})_0((\tau_\sigma e_\theta)(-1)\mathbf{1}, x) = (\tau_\sigma e_\theta)(x).$$

Then from (5.1), we have

$$\begin{aligned}(Y^g_{\widehat{M}_{\ell}^{[g]}})_0&((\tau_\sigma e_\theta)(-1)\mathbf{1}, x_1)(Y^g_{\widehat{M}_{\ell}^{[g]}})_0((\tau_\sigma e_\theta)(-1)\mathbf{1}, x_2)\\ &= (Y^g_{\widehat{M}_{\ell}^{[g]}})_0((\tau_\sigma e_\theta)(-1)\mathbf{1}, x_2)(Y^g_{\widehat{M}_{\ell}^{[g]}})_0((\tau_\sigma e_\theta)(-1)\mathbf{1}, x_1).\end{aligned} \tag{5.17}$$

By (5.17), we can use Lemma 5.6 for $u = v = (\tau_\sigma e_\theta)(-1)$. So we have

$$Y^g_{\widehat{M}_{\ell}^{[g]}}((\tau_\sigma e_\theta)(-1)^2\mathbf{1}, x) = (\tau_\sigma e_\theta)(x)^2,$$

where the right-hand side is well defined. From (5.1), we obtain

$$[(\tau_\sigma e_\theta)(x), (\tau_\sigma e_\theta)(x)^m] = \sum_{i=1}^{m}(\tau_\sigma e_\theta)(x)^{i-1}[(\tau_\sigma e_\theta)(x), (\tau_\sigma e_\theta)(x)](\tau_\sigma e_\theta)(x)^{m-i} = 0.$$

Using induction and Lemma 5.6, we see that for $m \in \mathbb{N}$, $(\tau_\sigma e_\theta)(x)^m$ is well defined and (5.15) holds. The proof for $\widetilde{M}^{[g]}_{L(\ell,0)}$ is completely the same.

By Theorems 4.9 and 4.11, we see that $(\tau_\sigma e_\theta)(x)^m$ for $m \in \mathbb{N}$ are also well defined on $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}^{[g]}_+, \mathbf{k})} M$ and $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}^{[g]}_+ \oplus \hat{\mathfrak{g}}^{[g]}_0} M$. ∎

Let $V$ be a grading-restricted vertex algebra (or a vertex operator algebra), $g$ and $h$ automorphisms of $V$ and $(W, Y^g_W)$ a lower-bounded (grading-restricted) generalized $g$-twisted $V$-module. Recall the lower-bounded (grading-restricted) $hgh^{-1}$-twisted $V$-module $(W, \phi_h(Y^g))$ (see Proposition 3.2 in [Hua3]), where

$$\begin{aligned}\phi_h(Y^g_W) : V \times W &\to W\{x\}[\log x]\\ v \otimes w &\mapsto \phi_h(Y^g_W)(v, x)w\end{aligned}$$

is the linear map defined by $\phi_h(Y^g_W)(v, x)w = Y^g_W(h^{-1}v, x)w$. The $hgh^{-1}$-twisted $V$-module $(W, \phi_h(Y^g_W))$ is also denoted by $\phi_h(W)$.

Let $M$, as in Subsection 4.2, be a vector space with actions of $g$, $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_M(0)$, $L_M(0)_S$ and $L_M(0)_N$ such that $M = \coprod_{h\in Q_M} M_{[h]}$, where $Q_M$ is the set of all eigenvalues of $L_M(0)$ and $M_{[h]}$ is the generalized eigenspace of $L_M(0)$ with eigenvalue $h \in Q_M$. Then just as in the construction of $\widehat{M}_{\ell}^{[g]}$ in Subsection 4.2 for the vertex operator algebra $M(\ell, 0)$, for each $h \in Q_M$, we have a universal lower-bounded generalized $g$-twisted $L(\ell, 0)$-module generated by $M_h$ (see Theorem 4.7 for its universal property). We shall denote them by $\widehat{M}^{[g]}_{L(\ell,0),h}$ for $h \in Q_M$. Let $\widehat{M}^{[g]}_{L(\ell,0)} = \coprod_{h\in Q_M} \widehat{M}^{[g]}_{L(\ell,0),h}$. This is the universal lower-bounded generalized $g$-twisted $L(\ell, 0)$-module generated by a subspace $M$ annihilated by $\hat{\mathfrak{g}}_+$. If $M$ is in addition a finite-dimensional $\hat{\mathfrak{g}}_{\mathbb{I}}$-module such that the action of $g$ is compatible, then also as in Subsection 4.2, we have a quotient of $\widehat{M}^{[g]}_{L(\ell,0),h}$. We shall denote this quotient by $\widetilde{M}^{[g]}_{L(\ell,0)}$.

From the definitions of $\widehat{M}^{[g]}_{L(\ell,0)}$ and $\widetilde{M}^{[g]}_{L(\ell,0)}$, we have the following universal properties for them:

**Theorem 5.8** *Let $(W, Y^g_W)$ be a lower-bounded generalized $g$-twisted $L(\ell,0)$-module (when $L(\ell,0)$ is viewed as a vertex operator algebra). Let $M^0$ a subspace (finite-dimensional $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$-submodule) of $W$ invariant under the actions of $g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$ and annihilated by $\hat{\mathfrak{g}}^{[g]}_+$. Assume that there is a linear ($\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$-module map) $f : M \to M^0$ commuting with the actions of $g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)|_{M^0}$ and $L_M(0)$, $L_W(0)_S|_{M^0}$ and $L_M(0)_S$ and $L_W(0)_N|_{M^0}$ and $L_M(0)_N$. Then there exists a unique module map $\hat{f} : \widehat{M}^{[g]}_{L(\ell,0)} \to W$ ($\tilde{f} : \widetilde{M}^{[g]}_{L(\ell,0)} \to W$) such that $\hat{f}|_M = f$ ($\tilde{f}|_M = f$). If $f$ is surjective and $(W, Y^g_W)$ is generated by the coefficients of $(Y^g)^W_{WL(\ell,0)}(w,x)\mathbf{1}$ for $w \in M^0$, where $(Y^g)^W_{WL(\ell,0)}$ is the twist vertex operator map obtained from $Y^g_W$, then $\hat{f}$ ($\tilde{f}$) is surjective.*

*Proof.* For $h \in Q_M$, by the universal property of $\widehat{M}^{[g]}_{L(\ell,0),h}$ (see the construction of $\widehat{M}^{[g]}_{L(\ell,0),h}$ and Theorem 4.7, ), there is a unique $L(\ell,0)$-module map $\hat{f}_h$ from $\widehat{M}^{[g]}_{L(\ell,0),h}$ to the submodule of $W$ generated by $f(M^0)$. Let $\hat{f} : \widehat{M}^{[g]}_{L(\ell,0)} \to W$ be defined by $\hat{f}(w) = \hat{f}_h(w)$ for $w \in \widehat{M}^{[g]}_{L(\ell,0),h}$. Then $\hat{f}$ is clearly a module map. The uniqueness of $\hat{f}$ follows from the uniqueness of $\hat{f}_h$ for $h \in Q_M$. It is also clear that the second conclusion holds.

In the case that $M^0$ is a finite-dimensional $\hat{\mathfrak{g}}^{[g]}_{\mathbb{I}}$-submodule of $W$, the proof is the same as that of Theorem 4.16 except that we should use $\widehat{M}^{[g]}_{L(\ell,0)}$ instead of $\widehat{M}^{[g]}_{\ell}$. ∎

Let $\hat{I}^{[g]}_{L(\ell,0)}$ ($\widetilde{I}^{[g]}_{L(\ell,0)}$) be the submodules of $U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}^{[g]}_+, \mathbf{k})} M$ ($U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}^{[g]}_+ \oplus \hat{\mathfrak{g}}^{[g]}_0} M$) generated by the coefficients of $(\tau_\sigma e_\theta)(x)^{\ell+1} w$ for $w \in U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}^{[g]}_+, \mathbf{k})} M$ ($w \in U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}^{[g]}_+ \oplus \hat{\mathfrak{g}}^{[g]}_0} M$). Then on $(U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}^{[g]}_+, \mathbf{k})} M)/\hat{I}^{[g]}_{L(\ell,0)}$, $(U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}^{[g]}_+ \oplus \hat{\mathfrak{g}}^{[g]}_0} M)/\widetilde{I}^{[g]}_{L(\ell,0)}$ and their quotients,

$$(\tau_\sigma e_\theta)(x)^{\ell+1} = 0. \tag{5.18}$$

**Theorem 5.9** *Assume that $\mathfrak{g}$ is simple and $\ell \in \mathbb{Z}_+$. The universal lower-bounded (grading-restricted) generalized $g$-twisted $L(\ell,0)$-module $\widehat{M}^{[g]}_{L(\ell,0)}$ ($\widetilde{M}^{[g]}_{L(\ell,0)}$) is equivalent as a lower-bounded (grading-restricted) $\hat{\mathfrak{g}}^{[g]}$-module to $(U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}^{[g]}_+, \mathbf{k})} M)/\hat{I}^{[g]}_{L(\ell,0)}$ ($(U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}^{[g]}_+ \oplus \hat{\mathfrak{g}}^{[g]}_0} M)/\widetilde{I}^{[g]}_{L(\ell,0)}$). In particular, the lower-bounded generalized $g$-twisted $L(\ell,0)$-module $\widetilde{M}^{[g]}_{L(\ell,0)}$ is in fact grading restricted.*

*Proof.* Note that the automorphism $\tau_\sigma$ of $\mathfrak{g}$ induces automorphisms, denoted still by $\tau_\sigma$, of the vertex operator algebras $M(\ell,0)$ and $L(\ell,0)$. Then $\tau_\sigma(\widehat{M}^{[g]}_{L(\ell,0)})$ is a lower-bounded generalized $\tau_\sigma^{-1}g\tau_\sigma$-twisted $L(\ell,0)$-module. By Proposition 5.1, $\tau_\sigma(\widehat{M}^{[g]}_{L(\ell,0)})$ is also a lower-bounded generalized $\tau_\sigma^{-1}g\tau_\sigma$-twisted $M(\ell,0)$-module. By Proposition 5.2, We have

$$\phi_{\tau_\sigma}\left(Y^g_{\widehat{M}^{[g]}_{L(\ell,0)}}\right)(e_\theta(-1)^{\ell+1}\mathbf{1},x)=0,$$

or equivalently,

$$Y^g_{\widehat{M}^{[g]}_{L(\ell,0)}}(\tau_\sigma e_\theta(-1)^{\ell+1}\mathbf{1},x)=0.$$

Since $\tau_\sigma$ is an automorphism of $L(\ell,0)$ induced from the automorphism $\tau_\sigma$ of $\mathfrak{g}$, we have $\tau_\sigma(e_\theta(-1)^{\ell+1})\mathbf{1}=(\tau_\sigma e_\theta)(-1)^{\ell+1}\mathbf{1}$. Hence we have

$$Y^g_{\widehat{M}^{[g]}_{L(\ell,0)}}((\tau_\sigma e_\theta)(-1)^{\ell+1}\mathbf{1},x)=0. \tag{5.19}$$

From (5.19) and (5.15), we see that (5.18) holds on the $\hat{\mathfrak{g}}^{[g]}$-module $\widehat{M}^{[g]}_{L(\ell,0)}$. By Proposition 5.1, $\widehat{M}^{[g]}_{L(\ell,0)}$ is also a lower-bounded generalized $g$-twisted $M(\ell,0)$-module generated by $M$. Then by Corollary 4.15, $\widehat{M}^{[g]}_{L(\ell,0)}$ is equivalent to a quotient of $U(\hat{\mathfrak{g}}^{[g]})\otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}^{[g]}_+,\mathbf{k})}M$. Since on $\widehat{M}^{[g]}_{L(\ell,0)}$ (5.18) holds, we see that $\widehat{M}^{[g]}_{L(\ell,0)}$ is equivalent to a quotient of $(U(\hat{\mathfrak{g}}^{[g]})\otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}^{[g]}_+,\mathbf{k})}M)/\hat{I}^{[g]}_{L(\ell,0)}$.

On the other hand, by Theorem 4.9, $(U(\hat{\mathfrak{g}}^{[g]})\otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}^{[g]}_+,\mathbf{k})}M)/\hat{I}^{[g]}_{L(\ell,0)}$ is equivalent as a $\hat{\mathfrak{g}}$-module to the lower-bounded generalized $g$-twisted $M(\ell,0)$-module $\widehat{M}^{[g]}_\ell$. Then by Corollary 4.15, $W=(U(\hat{\mathfrak{g}}^{[g]})\otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}^{[g]}_+,\mathbf{k})}M)/\hat{I}^{[g]}_{L(\ell,0)}$ is also a lower-bounded generalized $g$-twisted $M(\ell,0)$-module. But by definition, (5.18) holds on $W$. So we have

$$\begin{aligned}\phi_{\tau_\sigma}(Y^g_W)(e_\theta(-1)^{\ell+1}\mathbf{1},x)&=Y^g_W(\tau_\sigma e_\theta(-1)^{\ell+1}\mathbf{1},x)\\&=Y^g_W((\tau_\sigma e_\theta)(-1)^{\ell+1}\mathbf{1},x)\\&=(\tau_\sigma e_\theta)(x)^{\ell+1}\\&=0.\end{aligned}$$

By Proposition 5.2, $\phi_{\tau_\sigma}(W)$ is a lower-bounded generalized $\tau_\sigma^{-1}g\tau_\sigma$-twisted $L(\ell,0)$-module and thus $W$ is a lower-bounded generalized $g$-twisted $L(\ell,0)$-module. Also $M$ can be viewed as a subspace of $W$ invariant under the actions of $g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$ and with $\hat{\mathfrak{g}}^{[g]}_+$ acting on $M$ as 0 and we have the identity map from $M$ to itself. Thus by Theorem 5.8, there exists a unique surjective $L(\ell,0)$-module map from $\widehat{M}^{[g]}_{L(\ell,0)}$ to $W$. In particular, this surjective $L(\ell,0)$-module map is a surjective $\hat{\mathfrak{g}}$-module map. Thus we have a surjective $\hat{\mathfrak{g}}$-module map from $\widehat{M}^{[g]}_{L(\ell,0)}$ to $W$. Since we have proved that $\widehat{M}^{[g]}_{L(\ell,0)}$ is a quotient of $W$, the existence of such a surjective $\hat{\mathfrak{g}}$-module map means that $\widehat{M}^{[g]}_{L(\ell,0)}$ is equivalent to $W=(U(\hat{\mathfrak{g}}^{[g]})\otimes_{G(\Omega^{[g]},\hat{\mathfrak{g}}^{[g]}_+,\mathbf{k})}M)/\hat{I}^{[g]}_{L(\ell,0)}$.

The proof for $\widetilde{M}^{[g]}_{L(\ell,0)}$ is completely the same except that we use the results in Subsections 4.2 and 4.3 on $\widetilde{M}^{[g]}_{\ell}$ instead of $\widehat{M}^{[g]}_{\ell}$. Since $(U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}^{[g]}_{+} \oplus \hat{\mathfrak{g}}^{[g]}_{0}} M)/\widetilde{I}^{[g]}_{L(\ell,0)}$ is grading-restricted, we see that $\widetilde{M}^{[g]}_{L(\ell,0)}$ is grading restricted. ∎

We also have the following immediate consequence whose proof is also omitted:

**Corollary 5.10** *Let $W$ be a lower-bounded generalized $g$-twisted $L(\ell,0)$-module (when $L(\ell,0)$ is viewed as a vertex operator algebra) generated by a subspace (finite-dimensional $\hat{\mathfrak{g}}_{\mathbb{I}}$-submodule) $M$ invariant under $g$, $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$ and annihilated by $\hat{\mathfrak{g}}^{[g]}_{+}$. Then $W$ as a lower-bounded $\hat{\mathfrak{g}}^{[g]}$-module is equivalent to a quotient of $(U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}^{[g]}_{+}, \mathbf{k})} M)/\widehat{I}^{[g]}_{L(\ell,0)}$ ($(U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}^{[g]}_{+} \oplus \hat{\mathfrak{g}}^{[g]}_{0}} M)/\widetilde{I}^{[g]}_{L(\ell,0)}$ and, in particular, $W$ is grading restricted). Conversely, let $M$ be a vector space (finite-dimensional $\hat{\mathfrak{g}}_{I_{\mathbb{I}}}$-module) with (compatible) actions of $g$, $\mathcal{L}_g$, $\mathcal{S}_g$, $\mathcal{N}_g$, $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$ satisfying the conditions discussed in Section 4. Then a quotient module of $(U(\hat{\mathfrak{g}}^{[g]}) \otimes_{G(\Omega^{[g]}, \hat{\mathfrak{g}}^{[g]}_{+}, \mathbf{k})} M)/\widehat{I}^{[g]}_{L(\ell,0)}$ ($(U(\hat{\mathfrak{g}}^{[g]}) \otimes_{\hat{\mathfrak{g}}^{[g]}_{+} \oplus \hat{\mathfrak{g}}^{[g]}_{0}} M)/\widetilde{I}^{[g]}_{L(\ell,0)}$) has a natural structure of a grading-restricted generalized $g$-twisted $L(\ell,0)$-module (when $L(\ell,0)$ is viewed as a vertex operator algebra).*

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