Associative algebras and the representation theory of grading-restricted vertex algebras

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Abstract

We introduce an associative algebra $A^\infty(V)$ using infinite matrices with entries in a grading-restricted vertex algebra V such that the associated graded space $Gr(W) = \coprod_{n\in\mathbb{N}} Gr_n(W)$ of a filtration of a lower-bounded generalized V-module W is an $A^\infty(V)$-module satisfying additional properties (called a nondegenerate graded $A^\infty(V)$-module). We prove that a lower-bounded generalized V-module W is irreducible or completely reducible if and only if the nondegenerate graded $A^\infty(V)$-module $Gr(W)$ is irreducible or completely reducible, respectively. We also prove that the set of equivalence classes of the lower-bounded generalized V-modules are in bijection with the set of the equivalence classes of nondegenerate graded $A^\infty(V)$-modules. For $N \in \mathbb{N}$, there is a subalgebra $A^N(V)$ of $A^\infty(V)$ such that the subspace $Gr^N(W) = \coprod_{n=0}^N Gr_n(W)$ of $Gr(W)$ is an $A^N(V)$-module satisfying additional properties (called a nondegenerate graded $A^N(V)$-module). We prove that $A^N(V)$ are finite dimensional when V is of positive energy (CFT type) and C_2-cofinite. We prove that the set of the equivalence classes of lower-bounded generalized V-modules is in bijection with the set of the equivalence classes of nondegenerate graded $A^N(V)$-modules. In the case that V is a Möbius vertex algebra and the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized V-modules are less than or equal to $N \in \mathbb{N}$, we prove that a lower-bounded generalized V-module W of finite length is irreducible or completely reducible if and only if the nondegenerate graded $A^N(V)$-module $Gr^N(W)$ is irreducible or completely reducible, respectively.

1 Introduction

In the representation theory of Lie algebras, the universal enveloping algebra of a Lie algebra plays a crucial role because the module categories for a Lie algebra and for its universal enveloping algebra are isomorphic. For a vertex operator algebra, there is also a universal enveloping algebra introduced by Frenkel and Zhu [FZ] such that the module categories for these algebras are isomorphic. Unfortunately, the universal enveloping algebra of a vertex operator algebra is not very useful since it involves certain infinite sums in a suitable topological completion of the tensor algebra of the algebra. On the other hand, the classes of modules that we are interested in the representation theory of vertex operator algebras and two-dimensional conformal field theory do not involve such infinite sums since the vertex operators on these modules are lower truncated when acting on elements of these modules.

Instead, in the representation theory of vertex operator algebras, we have the Zhu algebra $A(V)$ introduced by Zhu in [Z] and it's generalizations $A_n(V)$ for $n \in N$ by Dong, Li and Mason in [DLM] for a vertex operator algebra V. These algebras can be used to classify irreducible modules for the vertex operator algebra and to study problems related to different types of modules. But compared with the universal enveloping algebra of a Lie algebra, the role of these associative algebras played in the representation theory of vertex operator algebras is quite limited. For example, the module for one of these associative algebras obtained from a suitable V-module in general do not tell us whether the original V-module is irreducible or completely reducible.

In the present paper, we introduce an associative algebra $A^\infty(V)$ using infinite matrices with entries in a grading-restricted vertex algebra V. The associated graded space $Gr(W) = \coprod_{n\in\mathbb{N}} Gr_n(W)$ of a filtration of a lower-bounded generalized V-module W is an $A^\infty(V)$-module with an $\mathbb{N}$-grading and some operators having suitable properties (called a nondegenerate graded $A^\infty(V)$-module). In fact, the algebra $A^\infty(V)$ is defined using the associated graded spaces of all lower-bounded generalized V-modules. We prove that a lower-bounded generalized V-module W is irreducible or completely reducible if and only if the nondegenerate graded $A^\infty(V)$-module $Gr(W)$ is irreducible or completely reducible, respectively. We also prove that the set of the equivalence classes of irreducible lower-bounded generalized V-modules is in bijection with the set of the equivalence classes of irreducible nondegenerate graded $A^\infty(V)$-modules.

We show that $A(V)$ in [Z] and $A_n(V)$ [DLM] mentioned above are isomorphic to very special subalgebras of $A^\infty(V)$. This fact gives a conceptual explanation of the role that these associative algebras played in the representation theory of vertex operator algebras.

We then introduce new subalgebras $A^N(V)$ of $A^\infty(V)$ for $N \in \mathbb{N}$. These subalgebras can also be obtained using finite matrices with entries in V. In the case that V is of positive energy (or CFT type) and C_2-cofinite, we prove that $A^N(V)$ are finite dimensional. The subspace $Gr^N(W) = \coprod_{n=0}^N Gr_n(W)$ of $Gr(W)$ of a lower-bounded generalized V-module W is an $A^N(V)$-module with some operators having suitable properties (called a nondegenerate graded $A^N(V)$-module). Wer prove that if a lower-bounded generalized V-module W is irreducible or completely reducible, then the nondegenerate graded $A^N(V)$-module $Gr^N(W)$ is irreducible or completely reducible, respectively. We also prove that the set of the equivalence classes of irreducible lower-bounded generalized V-modules is in bijection with the set of the equivalence classes of irreducible nondegenerate graded $A^N(V)$-modules.

In the case that V is a Möbius vertex algebra so that a lowest weight of a lower-bounded generalized V-module is defined, under the assumption that the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized V-modules are less than or equal to $N \in \mathbb{N}$, we prove that a lower-bounded generalized V-module W of finite length is irreducible or completely reducible if and only if the nondegenerate graded $A^N(V)$-module $Gr^N(W)$ is irreducible or completely reducible, respectively. When $A^N(V)$ for all $N \in \mathbb{N}$ are finite dimensional (for example, when V is of positive energy (or CFT type) and C_2-cofinite), we prove that an irreducible lower-bounded generalized V-module is an ordinary V-module and thus every lower-bounded generalized V-module of finite length is grading-restricted. In this case, under the assumptions above on V, lowest weights and N, a lower-bounded generalized V-module W of finite length or a grading-restricted generalized V-module W is a direct sum of irreducible ordinary V-modules if and only if the nondegenerate graded $A^N(V)$-module $Gr^N(W)$ is completely reducible.

Many of the main results mentioned above need the construction of universal lower-bounded generalized V-modules in [H3] and some results from [H4].

The category of lower-bounded generalized V-modules and the category of nondegenerate graded $A^\infty(V)$-modules are not equivalent because of morphisms, but they are "almost" equivalent. We shall study the relations between these categories, the category of lower-bounded generalized V-modules of finite lengths and the categories of graded $A^N(V)$-modules for $N \in \mathbb{N}$ in another paper.

This paper is organized as follows: In the next section, we introduce the associative algebra $A^\infty(V)$ associated to a grading-restricted vertex algebra V and prove that the associated graded space $Gr(W)$ of a filtration of a lower-bounded generalized V-module W is an $A^\infty(V)$-module. In Section 3, we introduce graded $A^\infty(V)$-modules and prove the results mentioned above on the relations between lower-bounded generalized V-modules and nondegenerate graded $A^\infty(V)$-modules. We show that the Zhu algebra and their generalizations by Dong, Li and Mason are isomorphic to subalgebras of $A^\infty(V)$ in Subsection 4.1 and introduce the new subalgebras $A^N(V)$ of $A^\infty(V)$ for $N \in \mathbb{N}$ in Subsection 4.2. We also prove in Subsection 4.2 that when V is of positive energy and C_2-cofinite, $A^N(V)$ for $N \in \mathbb{N}$ are finite dimensional. In Section 5, we introduce graded $A^N(V)$-modules and prove the results mentioned above on the relations between lower-bounded generalized V-modules, lower-bounded generalized V-modules of finite lengths and nondegenerate graded $A^N(V)$-modules.

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2 Associative algebra $A^\infty(V)$ and modules

In this paper, we fix a grading-restricted vertex algebra V. Most of the constructions and results work and hold for more general vertex algebras, for example, lower-bounded vertex algebras or superalgebras. The constructions and results certainly work and hold for a Möbius vertex algebra or a vertex operator algebra. For some results in Section 5, we shall assume that V is a Möbius vertex algebra.

Let $U^\infty(\mathbb{C})$ be the space of all column-finite infinite matrices with entries in $\mathbb{C}$, but doubly indexed by $\mathbb{N}$ instead of $\mathbb{Z}_+$. In other words, $U^\infty(\mathbb{C})$ is the space of all infinite matrices of the form $[a_{kl}]$ for $a_{kl} \in \mathbb{C}$, $k, l \in \mathbb{N}$ such that for each fixed $l \in \mathbb{N}$, there are only finitely many nonzero a_{kl}. Let $I^\infty = [\delta_{kl}]$ be the infinite identity matrix. Then $U^\infty(\mathbb{C})$ is in fact an associative algebra with the identity I^∞. The space $U^\infty(\mathbb{C})$ has a set of linearly independent elements of the form E_{kl} for $k, l \in \mathbb{N}$ with the entry in the k-th row and l-th column equal to 1 and all the other entries equal to 0. These infinite matrices do not form a basis of $U^\infty(\mathbb{C})$ but form a basis of the subspace $U_0^\infty(\mathbb{C})$ of $U^\infty(\mathbb{C})$ consisting of finitary matrices (matrices with only finitely many nonzero entries). In particular,

$$U_0^\infty(\mathbb{C}) = \coprod_{k,l\in\mathbb{N}} \mathbb{C}E_{kl}.$$

Moreover,

$$U^\infty(\mathbb{C}) \subset \prod_{k,l\in\mathbb{N}} \mathbb{C}E_{kl},$$

where $\prod_{k,l\in\mathbb{N}} \mathbb{C}E_{kl}$ is the algebraic completion of $U_0^\infty(\mathbb{C})$ viewed as a graded space. Though elements of $U^\infty(\mathbb{C})$ are infinite linear combinations of E_{kl} for $k, l \in \mathbb{N}$, any binary product on $U^\infty(\mathbb{C})$ satisfying the distribution axioms is still determined completely by the product of E_{kl} for $k, l \in \mathbb{N}$. For example, for the usual matrix product, we know that $E_{kl}E_{mn} = \delta_{lm}E_{kn}$ for $k, l, m, n \in \mathbb{N}$. Let $A = \sum_{k,l\in\mathbb{N}} a_{kl}E_{kl}$ and $B = \sum_{k,l\in\mathbb{N}} b_{kl}E_{kl}$, where $a_{kl}, b_{kl} \in \mathbb{C}$ for $k, l \in \mathbb{N}$. Then

$$AB = \left(\sum_{k,l\in\mathbb{N}} a_{kl}E_{kl}\right)\left(\sum_{m,n\in\mathbb{N}} b_{mn}E_{mn}\right) = \sum_{k,n\in\mathbb{N}}\left(\sum_{m\in\mathbb{N}} a_{km}b_{mn}\right)E_{kn}.$$

So even though E_{kl} for $k, l \in \mathbb{N}$ do not form a basis of $U^\infty(\mathbb{C})$, all the properties of the associative algebra structure on $U^\infty(\mathbb{C})$ can still be derived from the properties these matrices. Thus we can study $U^\infty(\mathbb{C})$ using E_{kl} for $k, l \in \mathbb{N}$, $k \leq l$. Also what we are mainly interested is the subalgebra $\mathbb{C}I^\infty \oplus U_0^\infty(\mathbb{C})$ of $U^\infty(\mathbb{C})$. This subalgebra has a basis $\{I^\infty\} \cup \{E_{kl}\}_{k,l\in\mathbb{N}}$.

Let $U^\infty(V) = V \otimes U^\infty(\mathbb{C})$. Then $U^\infty(V)$ is the space of column-finite infinite matrices with entries in V, but doubly indexed by $\mathbb{N}$ instead of $\mathbb{Z}_+$. Elements of $U^\infty(V)$ are of the form $\mathfrak{v} = [v_{kl}]$ for $v_{kl} \in V$, $k, l \in \mathbb{N}$ such that for each fixed $l \in \mathbb{N}$, there are only finitely many nonzero v_{kl}. Let $U_0^\infty(V)$ be the subspace of $U^\infty(V)$ spanned by elements of the form $v \otimes E_{kl}$ for $v \in V$ and $k, l \in \mathbb{N}$. Then

$$U_0^\infty(V) = \coprod_{k,l\in\mathbb{N}} V \otimes \mathbb{C}E_{kl}$$

and

$$U^\infty(V) \subset \prod_{k,l\in\mathbb{N}} V \otimes \mathbb{C}E_{kl}.$$

We shall denote $v \otimes E_{kl}$ simply by $[v]_{kl}$. Then elements of $U^\infty(\mathbb{C})$ can all be written as

$$\sum_{k,l\in\mathbb{N}} [v_{kl}]_{kl}$$

for $v_{kl} \in V$, $k, l \in \mathbb{N}$. As in the case of $U^\infty(\mathbb{C})$, we can study any binary product on $U^\infty(V)$ satisfying the distribution axioms using $[v]_{kl}$ for $v \in V$ and $k, l \in \mathbb{N}$. We are also mainly interested in the subspace $V\otimes I^{\infty}\oplus U_0^{\infty}(V)$ of $U^{\infty}(V)$. This subspace is spanned by elements of the form $v\otimes I^{\infty}$ and $[v]_{kl}$ for $v\in V$ and $k,l\in\mathbb{N}$. Because of this reason, though we might give definitions of products and related notions using general elements of $U^{\infty}(V)$, we shall study them using only $[v]_{kl}$ for $v\in V$ and $k,l\in\mathbb{N}$.

We also need some particular formal series and polynomials. In this paper, we shall use the convention that a complex power or the integral power of the logarithm of an ordered linear combination of formal variables and a complex number, always means its expansion in nonnegative powers of the formal variables or the complex number that are not the first one in the ordered linear combination. For example, $(x+1)^{-k-1}$ for $k\in\mathbb{N}$ and $(1+x)^{n}$ for $n\in\mathbb{N}$ mean the expansions in nonnegative powers of 1 and in nonnegative powers of x, respectively. For $k,n,l\in\mathbb{N}$, we have

$$\begin{aligned}(x+1)^{-k+n-l-1}&=\sum_{m\in\mathbb{N}}\binom{-k+n-l-1}{m}x^{-k+n-l-m-1}\\ &=T_{k+l+1}((x+1)^{-k+n-l-1})+R_{k+l+1}((x+1)^{-k+n-l-1}),\end{aligned}\tag{2.1}$$

where

$$T_{k+l+1}((x+1)^{-k+n-l-1})=\sum_{m=0}^{n}\binom{-k+n-l-1}{m}x^{-k+n-l-m-1}$$

is the Taylor polynomial in x^{-1} of order $k+l+1$ of $(x+1)^{-k+n-l-1}$ and

$$R_{k+l+1}((x+1)^{-k+n-l-1})=\sum_{m\in n+1+\mathbb{N}}\binom{-k+n-l-1}{m}x^{-k+n-l-m-1}$$

is the remainder of order $k+l+1$.

We define a product $\diamond$ on $U^{\infty}(V)$ by

$$\mathfrak{u}\diamond\mathfrak{v}=[(\mathfrak{u}\diamond\mathfrak{v})_{kl}]$$

for $\mathfrak{u}=[u_{kl}],\mathfrak{v}=[v_{kl}]\in U^{\infty}(V)$, where

$$\begin{aligned}(\mathfrak{u}\diamond\mathfrak{v})_{kl}&=\sum_{n=k}^{l}\mathrm{Res}_xT_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^{l}Y_V((1+x)^{L_V(0)}u_{kn},x)v_{nl}\\ &=\sum_{n=k}^{n}\sum_{m=0}^{l}\binom{-k+n-l-1}{m}\mathrm{Res}_xx^{-k+n-l-m-1}(1+x)^{l}Y_V((1+x)^{L_V(0)}u_{kn},x)v_{nl}\end{aligned}\tag{2.2}$$

for $k,l\in\mathbb{N}$. Then $U^{\infty}(V)$ equipped with $\diamond$ is an algebra but in general is not even associative. Let $O^{\infty}(V)$ be the subspace of $U^{\infty}(V)$ spanned by elements of the form

$$\left[\sum_{n=k}^{l}\mathrm{Res}_xx^{-k-l-p-2}(1+x)^{l}Y_V((1+x)^{L_V(0)}u_{kn},x)v_{nl}\right]$$

for $\mathfrak{u} = [u_{kl}], \mathfrak{v} = [v_{kl}] \in U^\infty(V)$, $p \in \mathbb{N}$ and elements of the form

$$[(L_V(-1) + L_V(0) + l - k)v_{kl}]$$

for $\mathfrak{v} = [v_{kl}] \in U^\infty(V)$.

The product $\diamond$ on $U^\infty(V)$ looks complicated. But as we mentioned above, though $[v]_{kl}$ for $v \in V$ and $k, l \in \mathbb{N}$ does span $U^\infty(V)$, their infinite linear combinations give all the elements of $U^\infty(V)$ and $U^\infty(V)$ can be studied using these elements. In particular, the product $\diamond$ can be studied using these elements. So instead of working with arbitrary matrices in $U^\infty(V)$, we use $[v]_{kl}$ for $v \in V$ and $k, l \in \mathbb{N}$ to write down $\diamond$. For $u, v \in V$ and $k, m, n, l \in \mathbb{N}$, by definition,

$$[u]_{km} \diamond [v]_{nl} = 0$$

when $m \neq n$ and

$$\begin{aligned} [u]_{kn} \diamond [v]_{nl} &= \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_V((1+x)^{L_V(0)}u, x)v\right]_{kl} \\ &= \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_x x^{-k+n-l-m-1}(1+x)^l \left[Y_V((1+x)^{L_V(0)}u, x)v\right]_{kl}. \end{aligned} \tag{2.3}$$

Since $[u]_{km} \diamond [v]_{nl} = 0$ when $m \neq n$, we need only consider $[u]_{kn} \diamond [v]_{nl}$ for $u, v \in V$ and $k, n, l \in \mathbb{N}$. By taking $\mathfrak{u} = [u]_{kn}$ and $\mathfrak{v} = [v]_{nl}$, we see also that the subspace $O^\infty(V)$ is spanned by infinite linear combinations of elements of the form

$$\mathrm{Res}_x x^{-k-l-p-2}(1+x)^l [Y_V((1+x)^{L_V(0)}u, x)v]_{kl}$$

for $u, v \in V$, $k, l, p \in \mathbb{N}$ and elements of the form

$$[(L_V(-1) + L_V(0) + l - k)v]_{kl}$$

for $v \in V$ and $k, l \in \mathbb{N}$, with each pair (k, l) appearing in the linear combinations only finitely many times.

Let $\mathbf{1}^\infty$ be the element of $U^\infty(V)$ with diagonal entries being $\mathbf{1} \in V$ and all the other entries being 0. Then $\mathbf{1}^\infty = \mathbf{1} \otimes I^\infty$.

We shall take a quotient of $U^\infty(V)$ such that the quotient with the product induced from $\diamond$ is an associative algebra and such that the associated graded space of a filtration of every lower-bounded generalized V-module is a module for this associative algebra. To do this, we need to first give an action of the (nonassociative) algebra $U^\infty(V)$ with the product $\diamond$ on a lower-bounded generalized V-module.

We briefly recall the notion of lower-bounded generalized V-module. We refer the reader to Definition 1.2 in [H1], where a lower-bounded generalized V-module is called a lower-truncated generalized V-module. Definition 1.2 in [H1] is for a vertex operator algebra V but the definition applies also to a grading-restricted vertex algebra except that we have to require the existence of operators $L_W(0)$ and $L_W(-1)$ satisfying the same axioms for the corresponding operators coming from the vertex operator of the conformal element of a vertex operator algebra. We also refer the reader to Definition 3.1 in [H2] for this notion in the special case that V is a grading-restricted vertex algebra (not a superalgebra) and the automorphism of V is 1_V. Roughly speaking, a lower-bounded generalized V-module is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ equipped with a vertex operator map $Y_W : V \otimes W \to W[[x, x^{-1}]]$ and operators $L_W(0)$ and $L_W(-1)$ on W satisfying all the axioms for an (ordinary) V-module except that for $n \in \mathbb{C}$, $W_{[n]}$ does not have to be finite dimensional and is the generalized eigenspace with the eigenvalue n of $L_W(0)$ instead of the eigenspace with the eigenvalue n of $L_W(0)$. Module maps between lower-bounded generalized V-modules are defined in the obvious way as in Definition 1.1 in [H1], not those defined in Definition 4.2 in [H4]. On the other hand, if we replace V-module maps in the results below by those in Definition 4.2 in [H4], these results still hold. The notion of generalized V-submodule of a lower-bounded generalized V-module is defined in the obvious way. A generalized V-submodule of a lower-bounded generalized V-module is certainly also lower bounded.

Let W be a lower-bounded generalized V-module. For $n \in \mathbb{N}$, let

$$\Omega_n(W) = \{w \in W \mid (Y_W)_k(v)w = 0 \text{ for homogeneous } v \in V, \text{wt } v - k - 1 < -n\}.$$

Then

$$\Omega_{n_1}(W) \subset \Omega_{n_2}(W)$$

for $n_1 \leq n_2$ and

$$W = \bigcup_{n\in\mathbb{N}} \Omega_n(W).$$

So $\{\Omega_n(w)\}_{n\in\mathbb{N}}$ is an ascending filtration of W. Let

$$Gr(W) = \sum_{n\in\mathbb{N}} Gr_n(W)$$

be the associated graded space, where

$$Gr_n(W) = \Omega_n(W)/\Omega_{n-1}(W).$$

Sometimes we shall use $[w]_n$ to denote the element $w + \Omega_{n-1}(W)$ of $Gr_n(W)$, where $w \in \Omega_n(W)$.

Lemma 2.1 *For $w \in \Omega_n(W)$ and $l \in \mathbb{N}$, $\mathrm{Res}_x x^{l-1} Y_W(x^{L_V(0)}v, x)w \in \Omega_{n-l}(W)$.*

Proof. The operator $\mathrm{Res}_{x_2} x_2^{l-1} Y_W(x_2^{L_V(0)}v, x_2)$ has weight $-l$. Then for homogeneous $u \in V$, $(Y_W)_p(u)\mathrm{Res}_{x_2} x_2^{l-1} Y_W(x_2^{L_V(0)}v, x_2)$ has weight wt $u - p - 1 - l$. Consider the generalized V-submodule of W generated by w. Then $(Y_W)_p(u)\mathrm{Res}_{x_2} x_2^{l-1} Y_W(x_2^{L_V(0)}v, x_2)w$ is in this generalized V-submodule. Using the associativity for Y_W, we know that the generalized V-submodule generated by w is spanned by elements of the form $(Y_W)_m(\tilde{u})w$ for $\tilde{u} \in V$. So $(Y_W)_p(u)\mathrm{Res}_{x_2} x_2^{l-1} Y_W(x_2^{L_V(0)}v, x_2)w$ is a linear combination of such elements. But for homogeneous w, the weight of $(Y_W)_p(u)\mathrm{Res}_{x_2} x_2^{l-1} Y_W(x_2^{L_V(0)}v, x_2)w$ is wt $u -$

$p-1-l+\mathrm{wt}\ w$. So the elements of the form $(Y_W)_m(\tilde{u})w$ whose linear combination is $(Y_W)_p(u)\mathrm{Res}_{x_2}x_2^{l-1}Y_W(x_2^{L_V(0)}v, x_2)w$ can also be chosen to be of weight $\mathrm{wt}\ u-p-1-l+\mathrm{wt}\ w$, that is, the weight $\mathrm{wt}\ \tilde{u}-m-1$ of $(Y_W)_m(\tilde{u})$ is equal to $\mathrm{wt}\ u-p-1-l$. Since $w \in \Omega_n(W)$, $(Y_W)_m(\tilde{u})w = 0$ when $\mathrm{wt}\ \tilde{u}-m-1 < -n$, or equivalently, $\mathrm{wt}\ u-p-1 < -(n-l)$. So we have proved that $(Y_W)_p(u)\mathrm{Res}_{x_2}x_2^{l-1}Y_W(x_2^{L_V(0)}v, x_2)w = 0$ when $\mathrm{wt}\ u-p-1 < n-l$. This means that $\mathrm{Res}_x x^{l-1}Y_W(x^{L_V(0)}v, x)w \in \Omega_{n-l}(W)$. ∎

By Lemma 2.1, the operator $\mathrm{Res}_x x^{l-1}Y_W(x^{L_V(0)}v, x)$ in fact induces an operator, still denoted by the same notation, on $Gr(W)$, which maps $Gr_n(W)$ to $Gr_{n-l}(W)$.

For $\mathfrak{v} = [v_{kl}] \in U^\infty(V)$, where $v_{kl} \in V$ and $k, l \in \mathbb{N}$, we define an operator $\vartheta_{Gr(W)}(\mathfrak{v})$ on $Gr(W)$ as follows: For $\mathfrak{w} \in Gr(W)$, we define

$$\vartheta_{Gr(W)}(\mathfrak{v})\mathfrak{w} = \sum_{k,l\in\mathbb{N}} \mathrm{Res}_x x^{l-k-1}Y_W(x^{L_V(0)}v_{kl}, x)\pi_{Gr_l(W)}\mathfrak{w},$$

where $\pi_{Gr_l(W)}$ is the projection from $Gr(W)$ to $Gr_l(W)$. Note that since $\mathfrak{w}$ is a sum of elements of $Gr_l(W)$ for finitely many $l \in \mathbb{N}$ and for each l, there are only finitely many nonzero v_{kl}, the sum over k and l is finite. So $\vartheta_{Gr(W)}(\mathfrak{v})\mathfrak{w}$ is indeed a well defined element of $Gr(W)$. In the case $\mathfrak{v} = [v]_{kl}$ and $\mathfrak{w} = [w]_n$ for $v \in V$, $w \in W$ and $k, l, n \in \mathbb{N}$, we have

$$\vartheta_{Gr(W)}([v]_{kl})[w]_n = \delta_{ln}[\mathrm{Res}_x x^{l-k-1}Y_W(x^{L_V(0)}v, x)w]_k. \tag{2.4}$$

In the case that v is homogeneous and $w \in Gr_l(W)$, we have

$$\vartheta_{Gr(W)}([v]_{kl})[w]_l = [(Y_W)_{\mathrm{wt}\ v+l-k-1}(v)w]_k. \tag{2.5}$$

We now have a linear map

$$\begin{aligned} \vartheta_{Gr(W)} : U^\infty(V) &\to \mathrm{End}\ Gr(W) \\ \mathfrak{v} &\mapsto \vartheta_{Gr(W)}(\mathfrak{v}). \end{aligned}$$

Let $Q^\infty(V)$ be the intersection of $\ker \vartheta_{Gr(W)}$ for all lower-bounded generalized V-modules W and $A^\infty(V) = U^\infty(V)/Q^\infty(V)$.

We shall need the following lemma:

Lemma 2.2 *For $l \in \mathbb{Z}$, $k \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ and $v \in V$,*

$$\mathrm{Res}_x x^{l-k-1}Y_W\left(x^{L_V(0)}\binom{L_V(-1)+L_V(0)+l}{k+m}v, x\right) = 0. \tag{2.6}$$

In particular, when $k = 0$ and $m = 1$, we have

$$\mathrm{Res}_x x^{l-1}Y_W\left(x^{L_V(0)}(L_V(-1)+L_V(0)+l)v, x\right) = 0. \tag{2.7}$$

For $l \in \mathbb{Z}$ and $v \in V$,

$$\mathrm{Res}_x x^{l-k-1}Y_W\left(x^{L_V(0)}\binom{L_V(-1)+L_V(0)+l}{k}v, x\right) = \mathrm{Res}_x x^{l-k-1}Y_W(x^{L_V(0)}v, x). \tag{2.8}$$

Proof. For $l \in \mathbb{Z}$, $n \in \mathbb{N}$ and $v \in V$, using the $L(-1)$-derivative property for the vertex operator map Y_W repeatedly, we have

$$\frac{1}{n!}\frac{d^n}{dx^n}Y_W(x^{L_V(0)+l}v, x) = Y_W\left(x^{L_V(0)+l-n}\binom{L_V(-1)+L_V(0)+l}{n}v, x\right). \tag{2.9}$$

Multiplying x^p to both sides and then taking Res_x, we obtain

$$\mathrm{Res}_x\frac{x^p}{n!}\frac{d^n}{dx^n}Y_W(x^{L_V(0)+l}v, x) = \mathrm{Res}_x x^{l-n+p}Y_W\left(x^{L_V(0)}\binom{L_V(-1)+L_V(0)+l}{n}v, x\right). \tag{2.10}$$

When $0 \le p \le n-1$, the left-hand side of (2.10) is 0. Thus we obtain

$$\mathrm{Res}_x x^{l-n+p}Y_W\left(x^{L_V(0)}\binom{L_V(-1)+L_V(0)+l}{n}v, x\right) = 0. \tag{2.11}$$

Let $n = k+m$ and $p = m-1$ in (2.11) for $k \in \mathbb{N}$ and $m \in \mathbb{Z}_+$. Then we obtain (2.6).

Let $p = -1$ and $n = k$ in (2.10), we obtain

$$\mathrm{Res}_x\frac{x^{-1}}{k!}\frac{d^k}{dx^k}Y_W(x^{L_V(0)+l}v, x) = \mathrm{Res}_x x^{l-k-1}Y_W\left(x^{L_V(0)}\binom{L_V(-1)+L_V(0)+l}{k}v, x\right). \tag{2.12}$$

Since left-hand side of (2.12) is equal to

$$\mathrm{Res}_x x^{l-k-1}Y_W(x^{L_V(0)}v, x),$$

we obtain (2.8). ∎

Proposition 2.3 *We have $O^\infty(V) \subset Q^\infty(V)$.*

Proof. We need to prove $\vartheta_{Gr(W)}(O^\infty(V)) = 0$ for every lower-bounded generalized V-module W. For

$$\mathrm{Res}_{x_0}x_0^{-k-l-p-2}(1+x_0)^l[Y_W((1+x_0)^{L(0)}v_1, x_0)v_2]_{kl} \in O^\infty(V),$$

where $v_1, v_2 \in V$, $k, l, p \in \mathbb{N}$ and $w \in \Omega_l(W)$, we have

$$\begin{aligned}
&\vartheta_{Gr(W)}(\mathrm{Res}_{x_0}x_0^{-k-l-p-2}(1+x_0)^l[Y_W((1+x_0)^{L(0)}v_1, x_0)v_2]_{kl})[w]_l\\
&\quad = \mathrm{Res}_{x_2}x_2^{l-k-1}\mathrm{Res}_{x_0}x_0^{-k-l-p-2}(1+x_0)^l[Y_W(x_2^{L_V(0)}Y_V((1+x_0)^{L(0)}v_1, x_0)v_2, x_2)\pi_{G_l(W)}w]_k\\
&\quad = \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}x_0^{-k-l-p-2}(1+x_0)^l x_2^{l-k-1}\cdot\\
&\qquad\qquad \cdot [Y_W(Y_V(x_2^{L_V(0)}(1+x_0)^{L(0)}v_1, x_0x_2)x_2^{L_V(0)}v_2, x_2)w]_k\\
&\quad = \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}x_0^{-k-l-p-2}x_2^{-k-1}\mathrm{Res}_{x_1}x_1^l x_1^{-1}\delta\left(\frac{x_2+x_0x_2}{x_1}\right)\cdot\\
&\qquad\qquad \cdot [Y_W(Y_V(x_1^{L_V(0)}v_1, x_0x_2)x_2^{L_V(0)}v_2, x_2)w]_k
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} x_0^{-k-l-p-2} x_2^{-k-1} \mathrm{Res}_{x_1} x_1^{l} x_0^{-1} x_2^{-1} \delta\left(\frac{x_1 - x_2}{x_0 x_2}\right) \cdot \\
&\qquad\qquad \cdot [Y_W(x_1^{L(0)} v_1, x_1) Y_W(x_2^{L_V(0)} v_2, x_2) w]_k \\
&\quad - \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} x_0^{-k-l-p-2} x_2^{-k-1} \mathrm{Res}_{x_1} x_1^{l} x_0^{-1} x_2^{-1} \delta\left(\frac{x_2 - x_1}{-x_0 x_2}\right) \cdot \\
&\qquad\qquad \cdot [Y_W(x_2^{L_V(0)} v_2, x_2) Y_W(x_1^{L(0)} v_1, x_1) w]_k \\
&= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} x_1^{-k-p-2} (1 - x_1^{-1} x_2)^{-k-l-p-2} x_2^{l+p} [Y_W(x_1^{L(0)} v_1, x_1) Y_W(x_2^{L_V(0)} v_2, x_2) w]_k \\
&\quad - \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} (-1 + x_1 x_2^{-1})^{-k-l-p-2} x_1^{l} x_2^{-k-2} [Y_W(x_2^{L_V(0)} v_2, x_2) Y_W(x_1^{L(0)} v_1, x_1) w]_k.
\end{aligned} \tag{2.13}
$$

Since $w \in \Omega_l(W)$ and the series $(1 - x_1^{-1}x_2)^{-k-l-p-2}$ contains only nonnegative powers of x_2,

$$
\mathrm{Res}_{x_2} (1 - x_1^{-1} x_2)^{-k-l-p-2} x_2^{l+p} Y_W(x_2^{L_V(0)} v_2, x_2) w = 0.
$$

So the first term in the right-hand side of (2.13) is 0. Since $w \in \Omega_l(W)$ and the series $(-1 + x_1 x_2^{-1})^{-2k-p-2}$ contains only nonnegative powers of x_1,

$$
\mathrm{Res}_{x_1} (-1 + x_1 x_2^{-1})^{-k-l-p-2} x_1^{l} Y_W(x_1^{L(0)} v_1, x_1) w = 0.
$$

So the second term in the right-hand side of (2.13) is also 0.

Taking l in (2.7) to be $l - k$, we obtain

$$
\begin{aligned}
&\vartheta_{Gr(W)}([(L_V(-1) + L_V(0) + l - k)v]_{kl})[w]_l \\
&\quad = [\mathrm{Res}_x x^{l-k-1} Y_W\left(x^{L_V(0)}(L_V(-1) + L_V(0) + l - k)v, x\right) w]_k \\
&\quad = 0
\end{aligned} \tag{2.14}
$$

for $v \in V$, $k, l \in \mathbb{N}$ and $w \in \Omega_l(W)$. Thus we have $\vartheta_{Gr(W)}(O^\infty(V)) = 0$. ∎

Theorem 2.4 *Let W be a lower-bounded generalized V-module. Then the linear map*

$$
\vartheta_{Gr(W)} : U^\infty(V) \to \mathrm{End}\, Gr(W)
$$

gives a $U^\infty(V)$-module structure on $Gr(W)$ (that is, $\vartheta_{Gr(W)}$ is a homomorphism of (nonassociative) algebras from $U^\infty(V)$ to End $Gr(W)$*). In particular, $U^\infty(V)/\ker \vartheta_{Gr(W)}$ is an associative algebra isomorphic to a subalgebra of* End $Gr(W)$.

Proof. For $u, v \in V$, $k, n, l \in \mathbb{N}$ and $w \in \Omega_l(W)$, using (2.3), we have

$$
\begin{aligned}
&\vartheta_{Gr(W)}([u]_{kn} \diamond [v]_{nl})[w]_l \\
&\quad = \mathrm{Res}_{x_0} T_{k+l+1}((x_0 + 1)^{-k+n-l-1})(1 + x_0)^l \mathrm{Res}_{x_2} x_2^{l-k-1} \cdot \\
&\qquad\qquad \cdot [Y_W(x_2^{L_V(0)} Y_V((1 + x_0)^{L_V(0)} u, x_0) v, x_2) w]_k
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} T_{k+l+1}((x_0+1)^{-k+n-l-1})(1+x_0)^l x_2^{l-k-1}\cdot\\
&\qquad \cdot [Y_W(Y_V((x_2+x_0x_2)^{L_V(0)}u, x_0x_2)x_2^{L_V(0)}v, x_2)w]_k\\
&= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_1^{-1}\delta\left(\frac{x_2+x_0x_2}{x_1}\right) T_{k+l+1}((x_0+1)^{-k+n-l-1})x_1^l x_2^{-k-1}\cdot\\
&\qquad \cdot [Y_W(Y_V(x_1^{L_V(0)}u, x_0x_2)x_2^{L_V(0)}v, x_2)w]_k\\
&= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_0^{-1}x_2^{-1}\delta\left(\frac{x_1-x_2}{x_0x_2}\right) T_{k+l+1}((x_0+1)^{-k+n-l-1})x_1^l x_2^{-k-1}\cdot\\
&\qquad \cdot [Y_W(x_1^{L_V(0)}u, x_1)Y_W(x_2^{L_V(0)}v, x_2)w]_k\\
&\quad - \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_0^{-1}x_2^{-1}\delta\left(\frac{x_2-x_1}{-x_0x_2}\right) T_{k+l+1}((x_0+1)^{-k+n-l-1})x_1^l x_2^{-k-1}\cdot\\
&\qquad \cdot [Y_W(x_2^{L_V(0)}v, x_2)Y_W(x_1^{L_V(0)}u, x_1)w]_k\\
&= \mathrm{Res}_{x_2}\mathrm{Res}_{x_1} T_{k+l+1}((x_0+1)^{-k+n-l-1})\Big|_{x_0=(x_1-x_2)x_2^{-1}} x_1^l x_2^{-k-2}\cdot\\
&\qquad \cdot [Y_W(x_1^{L_V(0)}u, x_1)Y_W(x_2^{L_V(0)}v, x_2)w]_k\\
&\quad - \mathrm{Res}_{x_2}\mathrm{Res}_{x_1} T_{k+l+1}((x_0+1)^{-k+n-l-1})\Big|_{x_0=(-x_2+x_1)x_2^{-1}} x_1^l x_2^{-k-2}\cdot\\
&\qquad \cdot [Y_W(x_2^{L_V(0)}v, x_2)Y_W(x_1^{L_V(0)}u, x_1)w]_k. \qquad (2.15)
\end{aligned}
$$

Since $w \in \Omega_l(W)$, the second term in the right-hand side of (2.15) is 0. Expanding $T_{k+l+1}((x_0+1)^{-k+n-l-1})$ explicitly, we see that the first term in the right-hand side of (2.15) is equal to

$$
\begin{aligned}
&\sum_{m=0}^{n}\binom{-k+n-l-1}{m}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1}(x_1-x_2)^{-k+n-l-m-1}x_2^{k-n+l+m+1}x_1^l x_2^{-k-2}\cdot\\
&\qquad \cdot [Y_W(x_1^{L_V(0)}u, x_1)Y_W(x_2^{L_V(0)}v, x_2)w]_k\\
&= \sum_{m=0}^{n}\sum_{j\in\mathbb{N}}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{j}(-1)^j\mathrm{Res}_{x_2}\mathrm{Res}_{x_1}x_1^{-k+n-m-1-j}\cdot\\
&\qquad \cdot x_2^{-n+l+m-1+j}[Y_W(x_1^{L_V(0)}u, x_1)Y_W(x_2^{L_V(0)}v, x_2)w]_k. \qquad (2.16)
\end{aligned}
$$

In the case $j > n-m$, since $w \in \Omega_l(W)$,

$$
\mathrm{wt}\ v - (\mathrm{wt}\ v - n + l + m - 1 + j) - 1 < -l
$$

when v is homogeneous and hence we have

$$
\mathrm{Res}_{x_2}x_2^{-n+l+m-1+j}Y_W(x_2^{L_V(0)}v, x_2)w = 0.
$$

Hence those terms in the right-hand side of (2.16) with $j > n-m$ is 0. So the right-hand side of (2.16) is equal to

$$\begin{aligned}
&\sum_{m=0}^{n}\sum_{j=0}^{n-m}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{j}(-1)^{j}\cdot\\
&\qquad\qquad \cdot \mathrm{Res}_{x_2}\mathrm{Res}_{x_1}x_1^{-k+n-m-1-j}x_2^{-n+l+m-1+j}[Y_W(x_1^{L_V(0)}u,x_1)Y_W(x_2^{L_V(0)}v,x_2)w]_k\\
&\quad=\sum_{m=0}^{n}\sum_{p=m}^{n}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{p-m}(-1)^{p-m}\cdot\\
&\qquad\qquad \cdot \mathrm{Res}_{x_2}\mathrm{Res}_{x_1}x_1^{-k+n-1-p}x_2^{-n+l-1+p}[Y_W(x_1^{L_V(0)}u,x_1)Y_W(x_2^{L_V(0)}v,x_2)w]_k\\
&\quad=\sum_{p=0}^{n}\left(\sum_{m=0}^{p}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{p-m}(-1)^{p-m}\right)\cdot\\
&\qquad\qquad \cdot \mathrm{Res}_{x_2}\mathrm{Res}_{x_1}x_1^{-k+n-1-p}x_2^{-n+l-1+p}[Y_W(x_1^{L_V(0)}u,x_1)Y_W(x_2^{L_V(0)}v,x_2)w]_k. \qquad (2.17)
\end{aligned}$$

For $p=0,\dots,n$,

$$\begin{aligned}
&\sum_{m=0}^{p}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{p-m}(-1)^{p-m}\\
&\quad=\sum_{m=0}^{p}\frac{(-k+n-l-1)\cdots(-k+n-l-m)}{m!}\cdot\\
&\qquad\qquad\cdot\frac{(-k+n-l-m-1)\cdots(-k+n-l-p)}{(p-m)!}(-1)^{p-m}\\
&\quad=\sum_{m=0}^{p}\frac{(-k+n-l-1)\cdots(-k+n-l-p)}{p!}\frac{p!}{m!(p-m)!}(-1)^{p-m}\\
&\quad=\binom{-k+n-l-1}{p}\sum_{m=0}^{p}\binom{p}{m}(-1)^{p-m}\\
&\quad=\binom{-k+n-l-1}{p}(-1+1)^{p}\\
&\quad=\binom{-k+n-l-1}{p}\delta_{p,0}. \qquad (2.18)
\end{aligned}$$

Using (2.18), we see that the right-hand side of (2.17) is equal to

$$\begin{aligned}
&\mathrm{Res}_{x_2}\mathrm{Res}_{x_1}x_1^{-k+n-1}x_2^{-n+l-1}[Y_W(x_1^{L_V(0)}u,x_1)Y_W(x_2^{L_V(0)}v,x_2)w]_k\\
&\quad=\vartheta_{Gr(W)}([u]_{kn})[\mathrm{Res}_{x_2}x_2^{l-n-1}Y_W(x_2^{L_V(0)}v,x_2)w]_n\\
&\quad=\vartheta_{Gr(W)}([u]_{kn})\vartheta_{Gr(W)}([v]_{nl})[w]_l. \qquad (2.19)
\end{aligned}$$

From (2.15), (2.16), (2.17) and (2.19), we obtain

$$\vartheta_{Gr(W)}([u]_{kn}\diamond[v]_{nl})=\vartheta_{Gr(W)}([u]_{kn})\vartheta_{Gr(W)}([v]_{nl})$$

for $u, v \in V$ and $k, n, l \in \mathbb{N}$. Thus $\vartheta_{Gr(W)}$ gives an $U^{\infty}(V)$-module structure on W. ∎

Lemma 2.5 *Let $L_U(-1)$ and $L_U(0)$ be operators on a vector space U satisfying*

$$[L_U(0), L_U(-1)] = L_U(-1).$$

We have

$$e^{xL_U(-1)}(1+x)^{L_U(0)} = (1+x)^{L_U(-1)+L_U(0)}. \tag{2.20}$$

Proof. This can be proved easily by showing

$$\frac{d}{dx}e^{xL_U(-1)}(1+x)^{L_U(0)}(1+x)^{-(L_U(-1)+L_U(0))} = 0$$

so that it must be independent of x and then setting $x = 0$ to obtain

$$e^{xL_U(-1)}(1+x)^{L_U(0)}(1+x)^{-(L_U(-1)+L_U(0))} = 1_U.$$

∎

We can now write down explicitly the expressions of elements of the form $[v]_{kl} \diamond \mathbf{1}^{\infty}$ for $v \in V$ and $k, l \in \mathbb{N}$ satisfying $k \leq l$.

Lemma 2.6 *For $v \in V$ and $k, l \in \mathbb{N}$,*

$$[v]_{kl} \diamond \mathbf{1}^{\infty} = \sum_{m=0}^{l} \binom{-k-1}{m} \left[\binom{L_V(-1)+L_V(0)+l}{k+m} v\right]_{kl}. \tag{2.21}$$

Proof. By the definition (2.2) of $\diamond$ and the skew-symmetry of Y_V,

$$\begin{aligned}([v]_{kl} \diamond_N \mathbf{1}^{\infty})_{mn} &= \delta_{km}\delta_{ln}\mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^n Y_V((1+x)^{L(0)}v, x)\mathbf{1}\\ &= \delta_{km}\delta_{ln}\mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^n e^{xL_V(-1)}(1+x)^{L_V(0)}v. \end{aligned} \tag{2.22}$$

Thus we obtain

$$[v]_{kl} \diamond \mathbf{1}^{\infty} = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k-1})(1+x)^l [e^{xL_V(-1)}(1+x)^{L_V(0)}v]_{kl}. \tag{2.23}$$

Using (2.20) with $U = V$, expanding the formal series explicitly and then evaluating the formal residue, we see that the right-hand side of (2.23) is equal to

$$\begin{aligned}&\mathrm{Res}_x T_{k+l+1}((x+1)^{-k-1})(1+x)^l[(1+x)^{L_V(0)+L_V(0)}v]_{kl}\\ &\quad= \sum_{m=0}^{l}\binom{-k-1}{m}\mathrm{Res}_x x^{-k-m-1}[(1+x)^{L_V(-1)+L_V(0)+l}v]_{kl}\end{aligned}$$

$$= \sum_{m=0}^{l} \sum_{j\in\mathbb{N}} \binom{-k-1}{m} \mathrm{Res}_x x^{-k-m-1+j} \binom{L_V(-1)+L_V(0)+l}{j} v$$

$$= \sum_{m=0}^{l} \binom{-k-1}{m} \binom{L_V(-1)+L_V(0)+k}{k+m} v, \tag{2.24}$$

proving (2.21). ∎

Proposition 2.7 *For $v \in V$ and $k, l \in \mathbb{N}$, $[v]_{kl} \diamond \mathbf{1}^\infty - [v]_{kl} \in O^\infty(V)$. For $u, v \in V$ and $k, l, n \in \mathbb{N}$, $([v]_{kl} \diamond \mathbf{1}^\infty - [v]_{kl}) \diamond [u]_{ln} \in O^\infty(V)$.*

Proof. For $m \in \mathbb{N}$,

$$\begin{aligned}
\binom{L_V(-1)+L_V(0)+l}{k+m} v &= \binom{(L_V(-1)+L_V(0)+l-k)+k}{k+m} v \\
&= \binom{k}{k+m} v + (L_V(-1)+L_V(0)+l-k)\tilde{v}_m \\
&\equiv \begin{cases} 0 & m \in \mathbb{Z}_+ \\ v & m = 0 \end{cases} \mod O^\infty(V),
\end{aligned} \tag{2.25}$$

where $\tilde{v}_m$ is an element of V depending on m. Thus by (2.21),

$$[v]_{kl} \diamond \mathbf{1}^\infty \equiv v \mod O^\infty(V).$$

By (2.21), (2.25) and (2.30),

$$\begin{aligned}
([v]_{kl} \diamond \mathbf{1}^\infty) \diamond [u]_{ln} &= \sum_{m=0}^{l} \binom{-k-1}{m} \left[\binom{L_V(-1)+L_V(0)+l}{k+m} v \right]_{kl} \diamond [u]_{ln} \\
&= \sum_{m=0}^{l} \binom{-k-1}{m} \binom{k}{k+m} [v]_{kl} \diamond [u]_{ln} \\
&\quad + \sum_{m=0}^{l} \binom{-k-1}{m} [(L_V(-1)+L_V(0)+l-k)\tilde{v}_m]_{kl} \diamond [u]_{ln} \\
&\equiv [v]_{kl} \diamond [u]_{ln} \mod O^\infty(V).
\end{aligned}$$

∎

Theorem 2.8 *The product $\diamond$ on $U^\infty(V)$ induces a product, denoted still by $\diamond$, on $A^\infty(V) = U^\infty(V)/Q^\infty(V)$ such that $A^\infty(V)$ equipped with $\diamond$ is an associative algebra with $\mathbf{1}^\infty + Q^\infty(V)$ as identity. Moreover, the associated graded space $Gr(W)$ of the ascendant filtration $\{\Omega_n(W)\}_{n\in\mathbb{N}}$ of a lower-bounded generalized V-module W is an $A^\infty(V)$-module.*

Proof. Since $\ker \vartheta_{Gr(W)}$ for a lower-bounded generalized V-module W is a two-sided ideal of $U^\infty(V)$, $Q^\infty(V)$ as the intersection of such two-sided ideals is still a two-sided ideal of $U^\infty(V)$. Thus $\diamond$ on $U^\infty(V)$ induces a product on $A^\infty(V)$. Since for each lower-bounded generalized V-module W, the quotient algebra $U^\infty(V)/\ker \vartheta_{Gr(W)}$ is associative, we have

$$\mathfrak{v}_1 \diamond (\mathfrak{v}_2 \diamond \mathfrak{v}_3) - \mathfrak{v}_1 \diamond (\mathfrak{v}_2 \diamond \mathfrak{v}_3) \in \ker \vartheta_{Gr(W)}$$

for $\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3 \in U^\infty(V)$. Then we have

$$\mathfrak{v}_1 \diamond (\mathfrak{v}_2 \diamond \mathfrak{v}_3) - \mathfrak{v}_1 \diamond (\mathfrak{v}_2 \diamond \mathfrak{v}_3) \in \bigcap_W \ker \vartheta_{Gr(W)} = Q^\infty(V)$$

for $\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3 \in U^\infty(V)$. Thus $A^\infty(V) = U^\infty(V)/Q^\infty(V)$ is also associative.

By definition, we have $\mathbf{1}^\infty \diamond [v]_{kl} = [v]_{kl}$. So $\mathbf{1}^\infty$ is in fact a left identity of the algebra $U^\infty(V)$. By Proposition 2.3, $O^\infty(V) \subset Q^\infty(V)$. Then by Proposition 2.7, we have

$$([v]_{kl} + Q^\infty(V)) \diamond (\mathbf{1}^\infty + Q^\infty(V)) = [v]_{kl} + Q^\infty(V).$$

So $\mathbf{1}^\infty + Q^\infty(V)$ is an identity of $A^\infty(V)$.

For a lower-bounded generalized V-module W, by Theorem 2.4, $Gr(W)$ is a module for $U^\infty(V)/\ker \vartheta_{Gr(W)}$. Since $Q^\infty(V)$ is a two-sided subideal of $\ker \vartheta_{Gr(W)}$, $Gr(W)$ is an $A^\infty(V)$-module. ∎

The ideal $Q^\infty(V)$ of $U^\infty(V)$ is defined using all lower-bounded generalized V-modules. From Prposition 3.3 in [H5], it is now known that besides elements of $O^\infty(V)$, $Q^\infty(V)$ also contains elements corresponding to the Jacobi identity for V. We conjecture that $Q^\infty(V)$ is generated by $O^\infty(V)$ and these elements.

The only result above on $O^\infty(V)$ and $Q^\infty(V)$ is Proposition 2.3. Below we give another result on $O^\infty(V)$ (Proposition 2.10). To prove this result, we need the following commutator formula:

Lemma 2.9 *For $v \in V$,*

$$[L_V(-1) + L_V(0), Y_V((1+x)^{L_V(0)}v, x)] = Y_V((1+x)^{L_V(0)}(L_V(-1) + L_V(0))v, x). \tag{2.26}$$

Proof. By the $L(-1)$ and $L(0)$-commutator formula with the vertex operator map Y_V and the fact that the weight of $L_V(-1)$ is 1,

$$\begin{aligned}
&[L_V(-1) + L_V(0), Y_V((1+x)^{L_V(0)}v, x)]\\
&\quad = Y_V(((1+x)L_V(-1) + L_V(0))(1+x)^{L_V(0)}v, x)\\
&\quad = Y_V((1+x)^{L_V(0)}(L_V(-1) + L_V(0))v, x).
\end{aligned}$$

By Theorem 2.4, every lower-bounded generalized V-module is an $A^\infty(V)$-module. ∎

Proposition 2.10 *For $u, v \in V$ and $k, n, l \in V$, both*

$$[(L_V(-1) + L_V(0) + n - k)u]_{kn} \diamond [v]_{nl}$$

and

$$[v]_{kn} \diamond [(L_V(-1) + L_V(0) + l - n)u]_{nl}$$

are in $O^\infty(V)$.

Proof. For $u, v \in V$, $k, l, m \in \mathbb{N}$, by definition,

$$\begin{aligned}
&[(L_V(-1) + L_V(0) + n - k)u]_{kn} \diamond [v]_{nl} \\
&\quad = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l [Y_V((1+x)^{L_V(0)}(L_V(-1) + L_V(0) + n - k)u, x)v]_{kl} \\
&\quad = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^{k-n+l+1} \frac{d}{dx}[Y_V((1+x)^{L_V(0)+n-k}u, x)v]_{kl} \\
&\quad = -\mathrm{Res}_x \left(\frac{d}{dx} T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^{k-n+l+1}\right)[Y_V((1+x)^{L_V(0)+n-k}u, x)v]_{kl} \\
&\quad = -\mathrm{Res}_x \Bigg(\left(\frac{d}{dx} T_{k+l+1}((x+1)^{-k+n-l-1})\right)(1+x)^{k-n+l+1} \\
&\qquad\qquad + (k-n+l+1)T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^{k-n+l}\Bigg) \cdot \\
&\qquad \cdot (1+x)^{-k+n}[Y_V((1+x)^{L_V(0)}u, x)v]_{kl}.
\end{aligned} \tag{2.27}$$

Applying $-\frac{1+x}{k-n+l+1}\frac{d}{dx}$ to bother sides of (2.1), we obtain

$$\begin{aligned}
&(x+1)^{-k+n-l-1} \\
&\quad = -\frac{1+x}{k-n+l+1}\frac{d}{dx}T_{k+l+1}((x+1)^{-k+n-l-1}) - \frac{1+x}{k-n+l+1}\frac{d}{dx}R_{k+l+1}((x+1)^{-k+n-l-1}).
\end{aligned} \tag{2.28}$$

Since the first and second terms in the right-hand side of (2.28) contain only the terms with powers in x^{-1} less than or equal to and larger than, respectively, $k+l+2$, we must have

$$-\frac{1+x}{k-n+l+1}\frac{d}{dx}T_{k+l+1}((x+1)^{-k+n-l-1}) = T_{k+l+2}((x+1)^{-k+n-l-1}),$$

or equivalently,

$$\begin{aligned}
&(1+x)\frac{d}{dx}T_{k+l+1}((x+1)^{-k-1}) \\
&\quad = -(k-n+l+1)T_{k+l+1}((x+1)^{-k+n-l-1}) - (k-n+l+1)\binom{-k+n-l-1}{n+1}x^{-k-l-2},
\end{aligned} \tag{2.29}$$

Using (2.29), the right-hand side of (2.27) becomes

$$(k-n+l+1)\binom{-k+n-l-1}{n+1}\mathrm{Res}_x x^{-k-l-2}(1+x)^l[Y_V((1+x)^{L_V(0)}u,x)v]_{kl}\in O^\infty(V).$$

Thus we obtain

$$\begin{aligned}
&[(L_V(-1)+L_V(0)+n-k)u]_{kn}\diamond[v]_{nl}\\
&\quad=(k-n+l+1)\binom{-k+n-l-1}{n+1}\mathrm{Res}_x x^{-k-l-2}(1+x)^l[Y_V((1+x)^{L_V(0)}u,x)v]_{kl}\\
&\quad\in O^\infty(V).
\end{aligned}\tag{2.30}$$

For $u,v\in V$, $k,l,n\in\mathbb{N}$ satisfying $n\le k\le l$, by the definition, (2.26) and $l-n=(l-k)-(n-k)$

$$\begin{aligned}
&[v]_{kn}\diamond[(L_V(-1)+L_V(0)+l-n)u]_{nl}\\
&\quad=\mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l\cdot\\
&\qquad\quad\cdot[Y_V((1+x)^{L_V(0)}v,x)(L_V(-1)+L_V(0)+l-n)u]_{kl}\\
&\quad=\mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l\cdot\\
&\qquad\quad\cdot[(L_V(-1)+L_V(0)+l-k)Y_V((1+x)^{L_V(0)}v,x)u]_{kl}\\
&\qquad-\mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l\cdot\\
&\qquad\quad\cdot[Y_V((1+x)^{L_V(0)}(L_V(-1)+L_V(0)+n-k)v,x)u]_{kl}.
\end{aligned}\tag{2.31}$$

The first term in the right-hand side of (2.31) is by definition in $O^\infty(V)$. The second term in the right-hand side of (2.31) is equal to $[(L_V(-1)+L_V(0)+n-k)v]_{kn}\diamond[u]_{nl}$, which is also in $O^\infty(V)$ (2.30). So

$$[v]_{kn}\diamond[(L_V(-1)+L_V(0)+l-n)u]_{nl}\in O^\infty(V).$$

∎

Remark 2.11 In [DJ], for $m,n,p\in\mathbb{Z}_+$, a product $*_{m,p}^n$ on V is introduced. In terms of these products, we have $[u]_{kn}\diamond[v]_{nl}=[u*_{l,n}^k v]_{kl}$. For each $m,n\in\mathbb{Z}_+$, a subspace $O'_{m,n}(V)$ is also introduced in [DJ]. In terms of these subspaces, $O^\infty(V)$ can be easily shown to be spanned by infinite linear combinations of elements of the form $[v]_{kl}$ for $v\in O'_{k,l}(V)$ with each pair (k,l) appearing in the linear combinations only finitely many times. For each $m,n\in\mathbb{Z}_+$, a subspace $O_{m,n}(V)$ of V containing in particular $O'_{m,n}(V)$ and associators of the products $*_{r,p}^q$ is further introduced. By taking a suitable subspace of the direct product of the quotient spaces $A_{m,n}(V)=V/O_{m,n}(V)$ for $m,n\in\mathbb{N}$, it is possible to obtain an associative algebra. This associative algebra can be identified as the quotient of the nonassociative algebra $U^\infty(V)$ by the ideal generated by $O^\infty(V)$ and all the associators of the product $\diamond$. Such a construction of associative algebras works for any nonassociative algebra with a subspace. As we have mentioned in the introduction, the algebra $A^{\infty}(V)$ plays the role of universal enveloping algebra of V for the category of lower-bounded generalized V-modules. So we now use an analogy with Lie algebra to compare the associative algebra that one can get from [DJ] and the associative algebra $A^{\infty}(V)$ introduced in this paper. Given a Lie algebra, one can generate a free nonassociative algebra. Then the quotient of this free associative algebra by the ideal generated by associators is an associative algebra isomorphic to the tensor algebra generated by the Lie algebra. To obtain the universal envelpoping algebra of the Lie algebra, we need to further take the quotient by the Jacobi identity and the skew-symmetry relations. The nonassociative algebra $U^{\infty}(V)$ is analogous to the free nonassociative algebra generated by the Lie algebra. The associative algebra that one can get from [DJ] is analogous to the tensor algebra of the Lie algebra. The associative algebra $A^{\infty}(V)$ is analogous to the universal enveloping algebra of the Lie algebra. Certainly $A^{\infty}(V)$ is a quotient of the associative algebra that one can get from [DJ]. But $A^{\infty}(V)$ is not equal to this associative algebra. In fact, Corollary 3.6 in [H5] says that $Q^{\infty}(V)$ contains the elements

$$\begin{aligned}
&\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}}(-1)^{j}\binom{p}{j}[v]_{k,n+p-j}\diamond[u]_{n+p-j,l+p}\\
&\quad-\sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}}(-1)^{p-j}\binom{p}{j}[u]_{k,l-n+k+p-j}\diamond[v]_{l-n+k+p-j,l+p}\\
&\quad-\sum_{j\in\mathbb{N}}\binom{\mathrm{wt}\ v+n-k-1}{j}[(Y_{V})_{p+j}(v)u]_{k,l+p}
\end{aligned}$$

for $k,l,n\in\mathbb{N}$, $p\in\mathbb{Z}$ such that $l+p\in\mathbb{N}$, $u\in V$ and homogeneous $v\in V$, corresponding to elements giving the Jacobi identity. Such elements are in general not in $O_{k,l+p}(V)$ in [DJ]. We conjecture that $Q^{\infty}(V)$ is generated by $O^{\infty}(V)$ and these elements.

3 Lower-bounded generalized V-modules and graded A^{∞}-modules

We study the relations between lower-bounded generalized V-modules and suitable $A^{\infty}(V)$-modules in this section.

Note that W is graded by the generalized eigenspaces of $L_{W}(0)$. Since $\Omega_{n}(W)$ for $n\in\mathbb{N}$ is invariant under $L_{W}(0)$, $L_{W}(0)$ induces an operator on $Gr_{n}(W)=\Omega_{n}(W)/\Omega_{n-1}(W)$ such that $Gr_{n}(W)$ is also graded by the generalized eigenspaces of this operator. These operators on $Gr_{n}(W)$ for $n\in\mathbb{N}$ together define an operator, denoted by $L_{Gr(W)}(0)$, on $Gr(W)$ preserving the $\mathbb{N}$-grading on $Gr(W)$. Then $Gr(W)$ is also graded by the generalized eigenspaces of $L_{Gr(W)}(0)$.

For $v\in V$, $k\in\mathbb{Z}$ and $w\in\Omega_{n}(W)$, by the $L(-1)$-commutator formula,

$$(Y_{W})_{k}(v)L_{W}(-1)w=L_{W}(-1)(Y_{W})_{k}(v)w+k(Y_{W})_{k-1}(v)w.$$

When wt $v-k-1<-(n+1)$, we have wt $v-k-1<-n$ and wt $v-(k-1)-1<-n$. So in this case, $(Y_W)_k(v)w=(Y_W)_{k-1}(v)w=0$ since $w\in\Omega_n(W)$. Thus $(Y_W)_k(v)L_W(-1)w=0$ when wt $v-k-1<-(n+1)$. This means that $L_W(-1)w\in\Omega_{n+1}(W)$. In particular, $L_W(-1)$ induces a linear map from $Gr_n(W)$ to $Gr_{n+1}(W)$ for $n\in\mathbb{N}$. These maps for $n\in\mathbb{N}$ together define an operator, denoted by $L_{Gr(W)}(-1)$, on $Gr(W)$.

The operators $L_{Gr(W)}(0)$, $L_{Gr(W)}(-1)$ and $\vartheta_{Gr(W)}([v]_{kl})$ satisfy the same commutator formulas as those between $L_W(0)$, $L_W(-1)$ and $\mathrm{Res}_x x^{l-k-1}Y_W(x^{L_V(0)}v,x)$ for $v\in V$ and $k,l\in\mathbb{N}$. These structures on $Gr(W)$ motivates the following definition:

Definition 3.1 Let G be an $A^\infty(V)$-module with the $A^\infty(V)$-module structure on G given by a homomorphism $\vartheta_G : A^\infty(V)\to \mathrm{End}\ G$ of associative algebras. We say that G is a *graded $A^\infty(V)$-module* if the following conditions are satisfied:

1. G is graded by $\mathbb{N}$, that is, $G=\coprod_{n\in\mathbb{N}}G_n$, and for $v\in V$, $k,l\in\mathbb{N}$, $\vartheta_G([v]_{kl}+Q^\infty(V))$ maps G_n to 0 when $n\neq l$ and to G_k when $n=l$.

2. G is a direct sum of generalized eigenspaces of an operator $L_G(0)$ on G, G_n for $n\in\mathbb{N}$ are invariant under $L_G(0)$ and the real parts of the eigenvalues of $L_G(0)$ have a lower bound.

3. There is an operator $L_G(-1)$ on G mapping G_n to G_{n+1} for $n\in\mathbb{N}$.

4. The commutator relations

$$\begin{aligned}
[L_G(0),L_G(-1)]&=L_G(-1),\\
[L_G(0),\vartheta_G([v]_{kl}+Q^\infty(V))]&=(k-l)\vartheta_G([v]_{kl}+Q^\infty(V)),\\
[L_G(-1),\vartheta_G([v]_{kl}+Q^\infty(V))]&=\vartheta_G([L_V(-1)v]_{(k+1)l}+Q^\infty(V))
\end{aligned}$$

   hold for $v\in V$ and $k,l\in\mathbb{N}$

A graded $A^\infty(V)$-algebra G is said to be *nondegenerate* if it satisfies in addition the following condition: For $g\in G_l$, if $\vartheta_G([v]_{0l}+Q^\infty(V))g=0$ for all $v\in V$, then $g=0$. Let G_1 and G_2 be graded $A^\infty(V)$-modules. A *graded $A_\infty(V)$-module map* from G_1 to G_2 is an $A^N(V)$-module map $f: G_1\to G_2$ such that $f((G_1)_n)\subset (G_2)_n$, $f\circ L_{G_1}(0)=L_{G_2}(0)\circ f$ and $f\circ L_{G_1}(-1)=L_{G_2}(-1)\circ f$. A *graded $A^\infty(V)$-submodule* of a graded $A^\infty(V)$-module G is an $A^\infty(V)$-submodule of G that is also an $\mathbb{N}$-graded subspace of G and invariant under the operators $L_G(0)$ and $L_G(-1)$. A graded $A^\infty(V)$-module G is said to be *generated by a subset S* if G is equal to the smallest graded $A^\infty(V)$-submodule containing S, or equivalently, G is spanned by homogeneous elements with respect to the $\mathbb{N}$-grading and the grading given by $L_G(0)$ obtained by applying elements of $A^\infty(V)$, $L_G(0)$ and $L_G(-1)$ to homogeneous summands of elements of S. A graded $A^\infty(V)$-module is said to be *irreducible* if it has no nonzero proper graded $A^\infty(V)$-submodules. A graded $A^\infty(V)$-module is said to be *completely reducible* if it is a direct sum of irreducible graded $A^\infty(V)$-modules.

From Theorem 2.8 and the properties of a lower-bounded generalized V-module W and its associated graded space $Gr(W)$, we obtain immediately:

Theorem 3.2 *For a lower-bounded generalized V-module W, $Gr(W)$ is a nondegenerate graded $A^{\infty}(V)$-module. Let W_1 and W_2 be lower-bounded generalized V-modules and $f : W_1 \to W_2$ a V-module map. Then f induces a graded $A^{\infty}(V)$-module map $Gr(f) : Gr(W_1) \to Gr(W_2)$.*

We now give a direct and explicit description of $Gr(W)$ for a completely reducible lower-bounded generalized V-module W. In this case,

$$W = \coprod_{\mu \in \mathcal{M}} W^{\mu},$$

where $\mathcal{M}$ is an index set and W^{μ} for $\mu \in \mathcal{M}$ are irreducible lower-bounded generalized V-modules. For $\mu \in \mathcal{M}$, since W^{μ} is irreducible, there exists $h^{\mu} \in \mathbb{C}$ such that

$$W^{\mu} = \coprod_{n \in \mathbb{N}} W^{\mu}_{[h^{\mu}+n]},$$

where as usual, $W^{\mu}_{[h^{\mu}+n]}$ for $n \in \mathbb{N}$ is the subspace of W^{μ} of weight $h^{\mu}+n$, and $W^{\mu}_{[h^{\mu}]} \neq 0$. For $n \in \mathbb{N}$, let

$$G_n(W) = \coprod_{\mu \in \mathcal{M}} W_{[h^{\mu}+n]}.$$

Then

$$W = \coprod_{n \in \mathbb{N}} G_n(W).$$

For $n \in \mathbb{N}$, let

$$T_n(W) = \coprod_{m=0}^{n} G_m(W).$$

It is clear that $T_n(W) \subset \Omega_n(W)$. In particular, $G_n(W) \subset \Omega_N(W)$ for $n \leq N$. Let $e_W : W \to Gr(W)$ be defined by $e_W(w) = w + \Omega_{n-1}(W)$ for $w \in G_n(W)$ and $n \in \mathbb{N}$. Then e_W preserves the $\mathbb{N}$-grading. We also define a map $\vartheta_W : U^{\infty}(V) \to \mathrm{End}\, W$ by

$$\vartheta_W(\mathfrak{v})w = \sum_{k,l \in \mathbb{N}} \mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)} v_{kl}, x) \pi_{G_l(W)} w$$

for $\mathfrak{v} \in U^{\infty}$ and $w \in W$, where $\pi_{G_l(W)}$ is the projection from W to $G_l(W)$. In the case $\mathfrak{v} = [v]_{kl}$ and $w \in G_n(W)$ for $v \in V$ and $k, l, n \in \mathbb{N}$, we have

$$\vartheta_W([v]_{kl})w = \delta_{ln} \mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)} v, x) w. \tag{3.1}$$

Proposition 3.3 *Let W be a completely reducible lower-bounded generalized V-module. Then $\Omega_n(W) = T_n(W)$ for $n \in \mathbb{N}$. Moreover, W equipped with ϑ_W is a nondegenerate graded $A^{\infty}(V)$-module and $e_W : W \to Gr(W)$ is an isomorphism of graded $A^{\infty}(V)$-modules.*

Proof. If $T_n(W) \neq \Omega_n(W)$, then there exists homogeneous $w \in \Omega_n(W)$ but $w \notin T_n(W)$. Then $w = \sum_{\mu\in\mathcal{M}} w^\mu$, where $w_\mu \in W^\mu$ for $\mu \in \mathcal{M}$ and only finitely many w^μ is not 0. Since w is homogeneous, we can assume that w^μ for $\mu \in \mathcal{M}$ are homogeneous. Since $w \in \Omega_n(W)$ but $w \notin T_n(W)$, there is at least one w^μ such that $w^\mu \in \Omega_n(W^\mu)$ but $w^\mu \notin T_n(W^\mu) = \coprod_{m=0}^n W^\mu_{[h^\mu+m]}$. Let W_0^μ be the generalized V-submodule of W^μ generated by such a w^μ. Since $w^\mu \notin T_n(W^\mu)$, $w^\mu \neq 0$ and hence $W_0^\mu \neq 0$. But W^μ is irreducible. So $W_0^\mu = W^\mu$. Since w^μ is homogeneous, there is $m \in \mathbb{N}$ such that wt $w^\mu \in W^\mu_{[h^\mu+m]}$. Since $w^\mu \notin T_n(W^\mu)$, we must have $m > n$. Since $W^\mu = W_0^\mu$, W^μ is spanned by elements of the form $(Y_W)_k(v)w^\mu$ for $v \in V$ and $k \in \mathbb{Z}$. Since $w^\mu \in \Omega_n(W^\mu)$, $(Y_W)_k(v)w^\mu = 0$ for homogeneous $v \in V$ and $k \in \mathbb{Z}$ satisfying wt $v - k - 1 < -n$. Thus the homogeneous subspaces of $W^\mu_{[h^\mu+m-n-p]} = 0$ for $p \in \mathbb{Z}_+$. But for $p = m - n \in \mathbb{Z}_+$, $W^\mu_{[h^\mu+m-n-p]} = W^\mu_{[h^\mu]} \neq 0$. Contradiction. Thus $T_n(W) = \Omega_n(W)$.

For $n \in \mathbb{N}$, we have $Gr_n(W) = \Omega_n(W)/\Omega_{n-1}(W) = T_n(W)/T_{n-1}(W)$. Then $e_W\big|_{G_n(W)}$ is clearly a linear isomorphism from $G_n(W)$ to $T_n(W)/T_{n-1}(W) = Gr_n(W)$. This shows that e_W is an isomorphism of graded spaces. For $v \in V$, $k, l \in \mathbb{N}$ and $w \in G_l(W)$,

$$\begin{aligned}
e_W(\vartheta_W([v]_{kl})w) &= e_W(\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w) \\
&= \mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w + T_{k-1}(W) \\
&= \vartheta_{Gr(W)}([v]_{kl})e_W(w).
\end{aligned}$$

Thus we have $e_W \circ \vartheta_W = \vartheta_{Gr(W)} \circ e_W$. In particular, the $A^\infty(V)$-module structure on $Gr(W)$ given by $\vartheta_{Gr(W)}$ is transported to W by e_W so that W equipped with ϑ_W is an $A^\infty(V)$-module and $e_W : W \to Gr(W)$ is an isomorphism of $A^\infty(V)$-modules. ∎

Theorem 3.4 *A lower-bounded generalized V-module W is irreducible or completely reducible if and only if the nondegenerate graded $A^\infty(V)$-module $Gr(W)$ is irreducible or completely reducible, respectively.*

Proof. Let W be an irreducible lower-bounded generalized V-module. By Theorem 3.3, W is a nondegenerate graded $A^\infty(V)$-module isomorphic to $Gr(W)$. Let W_0 be a nonzero graded $A^\infty(V)$-submodule of the graded $A^\infty(V)$-module W. For a homogeneous element $v \in V$, $n \in \mathbb{Z}$ and $w \in W_0$,

$$\mathrm{Res}_x x^n Y_W(v, x)w = \sum_{l\in\mathbb{N}} \vartheta_W([v]_{(\mathrm{wt}\ v-n-1+l)l})\pi_{G_l(W)}w \in W_0.$$

This means that W_0 is invariant under the action of the vertex operators on W. By the definition of graded $A^\infty(V)$-submodule, W_0 is invariant under the actions of $L_W(0)$ and $L_W(-1)$ and is the direct sum of generalized eigenspaces of $L_W(0)\big|_{W_0}$. Thus W_0 is also a nonzero lower-bounded generalized V-submodule of W. Since W is an irreducible lower-bounded generalized V-module, $W_0 = W$. So as a graded $A^\infty(V)$-module, W is also irreducible. Since as a graded $A^\infty(V)$-module, $Gr(W)$ is equivalent to W, we see that $Gr(W)$ is irreducible.

Conversely, assume that for a lower-bounded generalized V-module W, the nondegenerate graded $A^\infty(V)$-module $Gr(W)$ is irreducible. Let W_0 be a nonzero generalized V-submodule of W. Then $\Omega_{n-1}(W_0) \subset \Omega_{n-1}(W)$ for $n \in \mathbb{N}$ (when $n = 0$, $\Omega_{-1}(W) = 0$). We have a map from $Gr(W_0)$ to $Gr(W)$ given by $w_0 + \Omega_{n-1}(W_0) \mapsto w_0 + \Omega_{n-1}(W)$ for $n \in \mathbb{N}$ and $w_0 \in \Omega_n(W_0)$. This map is an injective graded $A^\infty(V)$-module map. So the image of $Gr(W_0)$ under this map is a graded $A^\infty(V)$-submodule of $Gr(W)$. Since W_0 is nonzero, $Gr(W_0)$ is nonzero. Since $Gr(W)$ is irreducible, the image of $Gr(W_0)$ under this map is equal to $Gr(W)$. Now it is easy to derive $W_0 = W$. In fact, for $n \in \mathbb{N}$, the image of $Gr_n(W_0)$ under the map from $Gr(W_0)$ to $Gr(W)$ above is $\{w_0 + \Omega_n(W) \mid w_0 \in \Omega_n(W_0)\}$. So $Gr_n(W) = \{w_0 + \Omega_{n-1}(W) \mid w_0 \in \Omega_n(W_0)\}$. For $n = 0$, we obtain $\Omega_0(W) = Gr_0(W) = Gr(W_0) = \Omega_0(W_0)$. Assume that $\Omega_{n-1}(W) = \Omega_{n-1}(W_0)$. Given $w \in \Omega_n(W)$, $w + \Omega_{n-1}(W) \in Gr_n(W)$. By $Gr_n(W) = \{w_0 + \Omega_{n-1}(W) \mid w_0 \in \Omega_n(W_0)\}$, there exists $w_0 \in \Omega_n(W_0)$ such that $w + \Omega_{n-1}(W) = w_0 + \Omega_{n-1}(W)$, or equivalently, $w - w_0 \in \Omega_{n-1}(W) = \Omega_{n-1}(W_0)$. Thus $w \in \Omega_n(W_0)$. This shows $\Omega_n(W) = \Omega_n(W_0)$ for $n \in \mathbb{N}$. Then we have $W = \cup_{n\in\mathbb{N}}\Omega_n(W) = \cup_{n\in\mathbb{N}}\Omega_n(W_0) = W_0$. So W is irreducible.

Assume that a lower-bounded generalized V-module W is completely reducible. Then $W = \coprod_{\mu\in\mathcal{M}} W^\mu$, where W^μ for $\mu \in \mu$ are irreducible generalized V-modules. From what we have proved above, W^μ for $\mu \in \mathcal{M}$ as graded $A^\infty(V)$-modules are also irreducible. So W as a graded $A^\infty(V)$-module is completely reducible. But $Gr(W)$ is equivalent to W as a graded $A^\infty(V)$-module by Proposition 3.3. So $Gr(W)$ is also completely reducible. Conversely, assume that for a lower-bounded generalized V-module W, the graded $A^\infty(V)$-module $Gr(W)$ is completely reducible. Then $Gr(W) = \coprod_{\mu\in\mathcal{M}} G^\mu$, where G^μ for $\mu \in \mathcal{M}$ are irreducible nondegenerate graded $A^\infty(V)$-submodules of $Gr(W)$. For $\mu \in \mathcal{M}$, since G^μ is a nondegenerate graded $A^\infty(V)$-submodule of $Gr(W)$, we have $G^\mu_n \subset Gr_n(W) = \Omega_n(W)/\Omega_{n-1}(W)$. Let W^μ be the subspace of W consisting of elements of the form $w^\mu \in \Omega_n(W)$ such that $w^\mu + \Omega_{n-1}(W) \in G^\mu_n$ for $n \in \mathbb{N}$. Since G^μ is a nondegenerate graded $A^\infty(V)$-submodule of $Gr(W)$, for $v \in V$, $k, l \in \mathbb{N}$ and $w^\mu \in \Omega_l(W)$ such that $w^\mu + \Omega_{l-1}(W) \in G^\mu_l$,

$$\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w^\mu + \Omega_{k-1}(W) \in G^\mu_k.$$

By the definition of W^μ, we obtain $\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w^\mu \in W^\mu$. Since $w^\mu \in \Omega_l(W)$, $\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w^\mu = 0$ for $k \in -\mathbb{Z}_+$. Thus $\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w^\mu \in W^\mu$ for $k \in \mathbb{N}$ are all the nonzero coefficients of $Y_W(v, x)w^\mu$. This means that W^μ is closed under the action of the vertex operators on W. Since G^μ is invariant under the actions of $L_{Gr(W)}(0)$ and $L_{Gr(W)}(-1)$ and is a direct sum of generalized eigenspaces of $L_{Gr(W)}(0)$, W^μ is invariant under the actions of $L_W(0)$ and $L_W(-1)$ and is a direct sum of generalized eigenspaces of $L_W(0)$. Thus W^μ is a generalized V-submodule of W.

Let $w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu)$, where $n \in \mathbb{N}$ and $w^\mu \in \Omega_n(W^\mu) \subset \Omega_n(W)$. By the definition of W^μ, we see that since w^μ is an element of W^μ, $w^\mu + \Omega_{n-1}(W) \in G^\mu_n$. So we obtain a linear map from $Gr_n(W^\mu)$ to G^μ_n given by $w^\mu + \Omega_{n-1}(W^\mu) \mapsto w^\mu + \Omega_{n-1}(W)$ for $w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu)$. These maps for $n \in \mathbb{N}$ give a map from $Gr(W^\mu)$ to G^μ. It is clear that this map is a graded $A^\infty(V)$-module map. If the image $w^\mu + \Omega_{n-1}(W)$ of $w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu)$ under this map is 0 in G^μ, then $w^\mu \in \Omega_{n-1}(W)$. But

$w^\mu \in \Omega_n(W^\mu) \subset W^\mu$. So $w^\mu \in \Omega_{n-1}(W^\mu)$ and $w^\mu + \Omega_{n-1}(W^\mu)$ is 0 in $Gr(W^\mu)$. This means that this graded $A^\infty(V)$-module map is injective. In particular, the image of $Gr(W^\mu)$ under this map is a nonzero nondegenerate graded $A^\infty(V)$-submodule of G^μ. But G^μ is irreducible. So $Gr(W^\mu)$ must be equivalent to G^μ and is therefore also irreducible. From what we have proved above, since $Gr(W^\mu)$ is irreducible, W^μ is irreducible. This shows that W is complete reducible. ∎

Theorem 3.4 implies that there is a map from the set of the equivalence classes of irreducible lower-bounded generalized V-modules to the set of equivalence classes of irreducible nondegenerate graded $A^\infty(V)$-modules. This map is in fact a bijection. To prove this, we need to construct a lower-bounded generalized V-module $S(G)$ from a nondegenerate graded $A^\infty(V)$-module G. We use the construction in Section 5 of [H3]. Take the generating fields for the grading-restricted vertex algebra V to be $Y_V(v,x)$ for $v \in V$. By definition, G is a direct sum of generalized eigenspaces of $L_G(0)$ and the real parts of the eigenvalues of $L_G(0)$ has a lower bound $B \in \mathbb{R}$. We take M and B in Section 5 of [H3] to be G and the lower bound B above. Using the construction in Section 5 of [H3], we obtain a universal lower-bounded generalized V-module $\widehat{G}_B^{[1_V]}$. For simplicity, we shall denote it simply by $\widehat{G}$.

By Theorem 3.3 in [H4] and the construction in Section 5 of [H3] and by identifying elements of the form $(\psi_{\widehat{G}}^a)_{-1,0}$ with basis elements $g^a \in G$ for $a \in A$ for a basis $\{g^a\}_{a\in A}$ of G, we see that $\widehat{G}$ is generated by G (in the sense of Definition 3.1 in [H4]). Moreover, after identifying $(\psi_{\widehat{G}}^a)_{-1,0}$ with basis elements $w^a \in G$ for $a \in A$, Theorems 3.3 and 3.4 in [H4] in fact say that elements of the form $L_{\widehat{G}}(-1)^p w^a$ for $p \in \mathbb{N}$ and $a \in A$ are linearly independent and $\widehat{G}$ is spanned by elements obtained by applying the components of the vertex operators to these elements. In particular, G can be embedded into $\widehat{G}$ as a subspace. So from now on, we shall view G as a subspace of $\widehat{G}$. Let J_G be the generalized V-submodule of $\widehat{G}$ generated by elements of the forms

$$\mathrm{Res}_x x^{l-k-1} Y_{\widehat{G}}(x^{L_V(0)}v, x)g \tag{3.2}$$

for $l \in \mathbb{N}$, $k \in -\mathbb{Z}_+$ and $g \in G_l$,

$$\mathrm{Res}_x x^{l-k-1} Y_{\widehat{M}}(x^{L_V(0)}v, x)g - \vartheta_G([v]_{kl} + Q^\infty(V))g \tag{3.3}$$

for $v \in V$, $k, l \in \mathbb{N}$, $g \in G_l$ and

$$L_{\widehat{G}}(-1)g - L_G(-1)g \tag{3.4}$$

for $l \in \mathbb{N}$, $g \in G_l$.

Let $S(G) = \widehat{G}/J_G$. Then $S(G)$ is a lower-bounded generalized V-module. Let $\pi_{S(G)}$ be the projection from $\widehat{G}$ to $S(G)$. Since $\widehat{G}$ is generated by G (in the sense of Definition 3.1 in [H4]), $S(G)$ is generated by $\pi_{S(G)}(G)$ (in the same sense). In particular, $S(G)$ is spanned by elements of the form

$$\mathrm{Res}_x x^{(l+p)-n-1} Y_{S(G)}(x^{L_{S(G)}(0)}v, x) L_{S(G)}(-1)^p \pi_{S(G)}(g) \tag{3.5}$$

for $v \in V$, $n, l, p \in \mathbb{N}$ and $g \in G_l$. For $n \in \mathbb{N}$, let $G_n(S(G))$ be the subspace of $S(G)$ spanned by elements of the form (3.5) for $v \in V$, $l, p \in \mathbb{N}$ and $g \in G_l$.

Proposition 3.5 *Let G be a graded $A^\infty(V)$-module.*

1. *For $n \in \mathbb{N}$, $G_n(S(G)) = \pi_{S(G)}(G_n)$ and for $n_1 \neq n_2$, $G_{n_1}(S(G)) \cap G_{n_2}(S(G)) = 0$. Moreover, $S(G) = \pi_{S(G)}(G) = \coprod_{n\in\mathbb{N}} G_n(S(G))$.*

2. *If G is nondegenerate, then for $n \in \mathbb{N}$, $\Omega_n(S(G)) = \coprod_{j=0}^{n} \pi_{S(G)}(G_j) = \coprod_{j=0}^{n} G_j(S(G))$.*

3. *If G is nondegenerate, then $Gr(S(G))$ is equivalent to G as a graded $A^\infty(V)$-module.*

Proof. Since elements of the forms (3.3) and (3.4) are in J_G, for $n \in \mathbb{N}$, the element (3.5) for $v \in V$ for $l, p \in \mathbb{N}$ and $g \in G_l$ is in fact equal to

$$\pi_{S(G)}(\vartheta_G([v]_{n(l+p)} + Q^\infty(V))L_G(-1)^p g). \tag{3.6}$$

Since $\vartheta_G([v]_{n(l+p)} + Q^\infty(V))L_G(-1)^p g$ for $l, p \in \mathbb{N}$ and $g \in G_l$ certainly span G_n and elements of the form (3.5) for $v \in V$ for $l, p \in \mathbb{N}$ and $g \in G_l$ span $G_n(S(G))$, elements of the form (3.6) for $v \in V$ for $l, p \in \mathbb{N}$ and $g \in G_l$ also span $G_n(S(G))$. Thus $G_n(S(G)) = \pi_{S(G)}(G_n)$. When $n_1 \neq n_2$, we know $G_{n_1} \cap G_{n_2} = 0$. Then $G_{n_1}(S(G)) \cap G_{n_2}(S(G)) = \pi_{S(G)}(G_{n_1} \cap G_{n_2}) = 0$. As is mentioned above, $S(G)$ is spanned by elements of the form (3.5) for $v \in V$, $k, l, p \in \mathbb{N}$ and $g \in G_l$. But we already see that (3.5) is in fact equal to (3.6). Thus $S(G) = \pi_{S(G)}(G)$. Since $G_n(S(G)) = \pi_{S(G)}(G_n)$ and $G_{n_1}(S(G)) \cap G_{n_2}(S(G)) = 0$, we have $S(G) = \pi_{S(G)}(G) = \coprod_{n\in\mathbb{N}} G_n(S(G))$.

By definition, for $j \leq n$, $G_j(S(G)) \subset \Omega_n(S(G))$. Then for $j = 0, \dots, n$, $\pi_{S(G)}(G_j) \subset \Omega_j(S(G)) \subset \Omega_n(S(G))$. So we obtain $\pi_{S(G)}(\coprod_{j=0}^n G_j) \subset \Omega_n(S(G))$. If G is nondegenerate, nonzero elements of G_j for $j > n$ are not in $\Omega_n(\widehat{G})$. From the construction of $\widehat{G}$, nonzero elements of the form (3.2), (3.3) or (3.4) are not in $G \subset \widehat{G}$. In particular, the intersection of $J(G)$ with G is 0. So $\pi_{S(G)}\big|_G : G \to S(G)$ is injective. Since $\pi_{S(G)}\big|_G$ is injective, we conclude that nonzero elements of $\pi_{S(G)}(G_j)$ for $j > n$ are not in $\Omega_n(S(G))$. So we have

$$\Omega_n(S(G)) = \pi_{S(G)}\left(\coprod_{j=0}^n G_j\right) = \coprod_{j=0}^n \pi_{S(G)}(G_j) = \coprod_{j=0}^n G_j(S(G)).$$

Since $\Omega_n(S(G)) = \coprod_{j=0}^n G_j(S(G))$ for $n \in \mathbb{N}$ when G is nondegenerate, we see that as a $\mathbb{N}$-graded space, $Gr(S(G))$ is isomorphic to $\coprod_{n\in\mathbb{N}} G_n(S(G)) = \pi_{S(G)}(G)$. We use f_G to denote the isomorphism from $Gr(S(G))$ to $\pi_{S(G)}(G)$. Then we have

$$f_G \circ \vartheta_{Gr(S(G))}([v]_{kl} + Q^\infty(V)) = \mathrm{Res}_x x^{l-k-1} Y_{S(G)}(x^{L_{S(G)}(0)} v, x) \circ f_G$$

for $v \in V$, $k, l \in \mathbb{N}$, $f_G \circ L_{Gr(S(G))}(0) = L_{S(G)}(0) \circ f_G$ and $f_G \circ L_{Gr(S(G))}(-1) = L_{S(G)}(-1) \circ f_G$. We have proved that $\pi_{S(G)}\big|_G$ is injective and surjective and preserves the $\mathbb{N}$-gradings. So it is an isomorphism of graded spaces from G to $S(G)$. From the fact that $\pi_{S(G)}$ is a V-module map and on $G \subset \widehat{G}$, $L_{\widehat{G}}(0) = L_G(0)$ and $L_{\widehat{G}}(-1) = L_G(-1)$, we have

$$\pi_{S(G)}\big|_G \circ \vartheta_G([v]_{kl} + Q^\infty(V)) = \mathrm{Res}_x x^{l-k-1} Y_{S(G)}(x^{L_{S(G)}(0)}v, x) \circ \pi_{S(G)}\big|_G$$

for $v \in V$, $k, l \in \mathbb{N}$, $\pi_{S(G)}\big|_G \circ L_G(0) = L_{S(G)}(0) \circ \pi_{S(G)}\big|_G$ and $\pi_{S(G)}\big|_G \circ L_G(-1) = L_{S(G)}(-1) \circ \pi_{S(G)}\big|_G$. Then by the properties of f_G and $\pi_{S(G)}\big|_G$ above, we see that $(\pi_{S(G)}\big|_G)^{-1} \circ f_G$ is an equivalence of graded $A^\infty(V)$-modules from $Gr(S(G))$ to G. ∎

Remark 3.6 Note that our construction of the lower-bounded generalized V-module $\widehat{G}$ seems to depend on the lower bound B of the real parts of the eigenvalues of $L_G(0)$. But by Proposition 3.5, $S(G)$ depends only on G, not on B.

Theorem 3.7 *The set of the equivalence classes of irreducible lower-bounded generalized V-modules is in bijection with the set of the equivalence classes of irreducible nondegenerate graded $A^\infty(V)$-modules.*

Proof. Let $[\mathfrak{W}]_{\mathrm{irr}}$ be the set of the equivalence classes of irreducible lower-bounded generalized V-modules and $[\mathfrak{G}]_{\mathrm{irr}}$ the set of the equivalence classes of irreducible nondegenerate graded $A^\infty(V)$-modules. Given an irreducible lower-bounded generalized V-module W, by Theorem 3.4, $Gr(W)$ is an irreducible nondegenerate graded $A^N(V)$-module. Thus we obtain a map $f : [\mathfrak{W}]_{\mathrm{irr}} \to [\mathfrak{G}]_{\mathrm{irr}}$ given by $f([W]) = [Gr(W)]$, where $[W] \in [\mathfrak{W}]_{\mathrm{irr}}$ is the equivalence class containing the irreducible lower-bounded generalized V-module W and $[Gr(W)] \in [\mathfrak{G}]_{\mathrm{irr}}$ is the equivalence class containing the irreducible nondegenerate graded $A^N(V)$-module $Gr(W)$. By Proposition 3.3, $[Gr(W)] = [W]$ in $[\mathfrak{G}]_{\mathrm{irr}}$, where W is viewed as a nondegenerate graded $A^\infty(V)$-module.

Given an irreducible nondegenerate graded $A^\infty(V)$-module G, we have a lower-bounded generalized V-module $S(G)$. By Proposition 3.5, $Gr(S(G))$ is equivalent to G. Since G is irreduible, $Gr(S(G))$ is also irreducible. Then by Theorem 3.4, $S(G)$ is an irreducible lower-bounded generalized V-module. Thus we obtain a map $g : [\mathfrak{G}]_{\mathrm{irr}} \to [\mathfrak{W}]_{\mathrm{irr}}$ given by $g([G]) = [S(G)]$.

We still need to show that f and g are inverse to each other. By Proposition 3.5, $Gr(S(G))$ is equivalent to G for an irreducible nondegenerate graded $A^\infty(V)$-module G. We obtain $[Gr(S(G))] = [G]$. This means $f(g([G])) = [G]$. So we have $f \circ g = 1_{[\mathfrak{G}]_{\mathrm{irr}}}$.

Let W be an irreducible lower-bounded generalized V-module. By Theorem 3.4, $Gr(W)$ is an irreducible nondegenerate graded $A^\infty(V)$-module. We then have a lower-bounded generalized V-module $S(Gr(W))$. By Proposition 3.5, $Gr(S(Gr(W)))$ is equivalent to $Gr(W)$ as a graded $A^\infty(V)$-module. Since $Gr(W)$ is irreducible, $Gr(S(Gr(W)))$ is also irreducible. By Theorem 3.4, $S(Gr(W))$ is an irreducible lower-bounded generalized V-module. Since both W and $S(Gr(W))$ are irreducible, by Proposition 3.3, W and $S(Gr(W))$ are nondegenerate graded $A^\infty(V)$-modules and are equivalent to $Gr(W)$ and $Gr(S(Gr(W)))$, respectively. But we already know that $Gr(S(Gr(W)))$ is equivalent to $Gr(W)$ as a graded $A^\infty(V)$-module.

So both W and $S(Gr(W))$ are equivalent to $Gr(W)$ as graded $A^\infty(V)$-modules. Since vertex operators on W and $S(Gr(W))$ can be expressed using the actions of elements of $A^\infty(V)$, we see that W and $S(Gr(W))$ are also equivalent as lower-bounded generalized V-modules. Thus $[S(Gr(W))] = [W]$, or $g(f([W])) = [W]$. So $g \circ f = 1_{[\mathfrak{W}]_{\mathrm{irr}}}$. ∎

4 Subalgebras of $A^\infty(V)$

We give some very special subalgebras of $A^\infty(V)$ and prove that they are isomorphic to the Zhu algebra $A(V)$ [Z] and its generalizations $A_N(V)$ for $N \in \mathbb{N}$ by Dong-Li-Mason [DLM] in Subsection 4.1. Then we introduce the main interesting and new subalgebras $A^N(V)$ for $N \in \mathbb{N}$ of $A^\infty(V)$ in Subsection 4.2. Note that we use the superscript N instead of the subscript N to distinguish this algebra from $A_N(V)$ in [DLM].

4.1 Zhu algebra and the generalizations by Dong-Li-Mason

Let

$$U_{00}(V) = \{[v]_{00} \mid v \in V\} \subset U^\infty(V).$$

Then $U_{00}(V)$ can be canonically identified with V through the map $i_{00} : U_{00}(V) \to V$ given by $i_{00}([v]_{00}) = v$ for $v \in V$. Since by (2.3),

$$[u]_{00} \diamond [v]_{00} = \mathrm{Res}_x x^{-1} \left[Y_V((1+x)^{L(0)}u, x)v\right]_{00},$$

$U_{00}(V)$ is closed under the product $\diamond$. Let

$$A_{00}(V) = \{[v]_{00} + Q^\infty(V) \mid v \in V\}.$$

Theorem 4.1 *The subspace $A_{00}(V)$ of $A^\infty(V)$ is closed under $\diamond$ and is thus a subalgebra of $A^\infty(V)$ with $[\mathbf{1}]_{00} + Q^\infty(V)$ as its identity. The associative algebra $A_{00}(V)$ is isomorphic to the Zhu algebra $A(V)$ in [Z] and, in particular, $[\omega]_{00} + Q^\infty(V)$ is in the center of $A_{00}(V)$ if V is a vertex operator algebra with the conformal vector ω.*

Since this result is a special case of the result on the generalizations $A_N(V)$ in [DLM], we will not give a proof. The proof is the special case $N = 0$ of the proof of Theorem 4.2 below for $A_N(V)$.

Let

$$U^{NN} = \left\{ \sum_{k=0}^{N} [v]_{kk} \,\middle|\, v \in V \right\} \subset U^\infty(V).$$

By the definition of $\diamond$,

$$\left(\sum_{k=0}^{N} [u]_{kk}\right) \diamond \left(\sum_{k=0}^{N} [v]_{kk}\right) = \sum_{k,l=0}^{N} [u]_{kk} \diamond [v]_{ll}$$

$$= \sum_{k=0}^{N} [u]_{kk} \diamond [v]_{kk}$$
$$= \sum_{k=0}^{N} \mathrm{Res}_x T_{2k+1}((x+1)^{-k-1})(1+x)^k \left[Y_V((1+x)^{L(0)}u, x)v\right]_{kk}.$$

Let
$$A^{NN}(V) = \left\{ \sum_{k=0}^{N} [v]_{kk} + Q^{\infty}(V) \;\middle|\; v \in V \right\} \subset A^{\infty}(V).$$
Note that $A^{00}(V) = A_{00}(V)$. Also let
$$\mathbf{1}^N = \sum_{k=0}^{N} [\mathbf{1}]_{kk},$$
$$\omega^N = \sum_{k=0}^{N} [\omega]_{kk},$$

Theorem 4.2 *The subspace $A^{NN}(V)$ of $A^{\infty}(V)$ is closed under $\diamond$ and is thus a subalgebra of $A^{\infty}(V)$ with $\mathbf{1}^N + Q^{\infty}(V)$ as the identity. The associative algebra $A^{NN}(V)$ is isomorphic to the associative algebra $A_N(V)$ of Dong, Li and Mason in [DLM] and, in particular, $\omega^N + Q^{\infty}(V)$ is in the center of $A^{NN}(V)$ if V is a vertex operator algebra with the conformal vector ω.*

Proof. For $u, v \in V$, we have
$$\left(\sum_{k=0}^{N} [u]_{kk}\right) \diamond \left(\sum_{k=0}^{N} [v]_{kk}\right) = \sum_{k=0}^{N} \mathrm{Res}_x T_{2k+1}((x+1)^{-k-1})(1+x)^k \left[Y_V((1+x)^{L(0)}u, x)v\right]_{kk}$$
$$= \sum_{k=0}^{N} \mathrm{Res}_x \sum_{m=0}^{k} \binom{-k-1}{m} x^{-k-m-1}(1+x)^k \left[Y_V((1+x)^{L(0)}u, x)v\right]_{kk}$$
$$= \sum_{k=0}^{N} [u *_k v]_{kk}$$
$$\simeq \sum_{k=0}^{N} [u *_N v]_{kk} \mod Q^{\infty}(V),$$

where in the last step, we have used the result obtained in the proof of Proposition 2.4 in [DLM] that $u *_k v$ is equal to $u *_N v$ modulo $O_k(V)$ for $k = 0, \ldots, N$ and the fact that $[O_k(V)]_{kk} \in O^{\infty}(V) \subset Q^{\infty}(V)$. This calculation shows that $A^{NN}(V)$ is closed under $\diamond$ and is thus a subalgebra of $A^{\infty}(V)$. Let $f^N : U^{NN}(V) \to A_N(V)$ be defined by
$$f^N\left(\sum_{k=0}^{N} [v]_{kk}\right) = v + O_N(V)$$

for $v \in V$.

We now view $A_N(V)$ as an $A_N(V)$-module. We construct a lower-bounded generalized V-module $S(A_N(V))$ from $A_N(V)$ using the construction in Section 5 of [H3] as follows: Take the generating fields for the grading-restricted vertex algebra V to be $Y_V(v, x)$ for $v \in V$. Take M in Section 5 of [H3] to be $A_N(V)$. We define the operator $L_M(0)$ on M to be the multiplication by the scalar N. So M itself is an eigenspace of $L_M(0)$ with eigenvalue N. Take the automorphism g of V in Section 5 of [H3] to be 1_V since we are interested only in untwisted modules. Take B in Section 5 of [H3] to be 0. Then we obtain a lower-bounded generalized V-module $\widehat{M}_0^{[1_V]}$, which shall be denoted by $\widehat{A_N(V)}$ here. By Theorem 3.3 in [H4] and the construction in Section 5 of [H3], $\widehat{A_N(V)}$ is spanned by elements of the form

$$(Y_{\widehat{A_N(V)}})_n(u)L_{\widehat{A_N(V)}}(-1)^p(v + O_N(V))$$

for homogeneous $u, v \in V$, $p \in \mathbb{N}$ and $n \in \mathrm{wt}\ u + N + p - 1 - \mathbb{N}$. Let J be the generalized V-submodule of $\widehat{A_N(V)}$ generated by elements of the form

$$(Y_{\widehat{A_N(V)}})_{\mathrm{wt}\ u-1}(u)(v + O_N(V)) - u *_N v + O_N(V)$$

for $u, v \in V$. Let $S(A_N(V)) = \widehat{A_N(V)}/J$. Then

$$S(A_N(V)) = \coprod_{n\in\mathbb{N}} (S(A_N(V)))_{[n]}$$

is a lower-bounded generalized V-module such that $(S(A_N(V)))_{[N]} = A_N(V)$. From the construction in Section 5 of [H3] and the definition of $S(A_N(V))$ above, elements of the form

$$(Y_{S(A_N(V))})_{\mathrm{wt}\ u-1+N}(u)(v + O_N(V))$$

for homogeneous nonzero $u \in V$ and $v \in V \setminus O_N(V)$ are not 0. Thus for $v \in V \setminus O_N(V)$, $v + O_N(V) \in A_N(V)$ is not in $\Omega_{N-1}(S(A_N(V)))$. In other words, if $v + O_N(V) \in \Omega_{N-1}(S(A_N(V)))$, then $v \in O_N(V)$. On the other hand, we know that $A_N(V) = (S(A_N(V)))_{[N]} \subset \Omega_N(S(A_N(V)))$.

Let W be a lower-bounded generalized V-module. Then $\ker \vartheta_{Gr(W)}$ is a two-sided ideal of $U^\infty(V)$. So $\ker \vartheta_{Gr(W)} \cap U_{NN}(V)$ is a two-sided ideal of $U_{NN}(V)$. From [DLM], the map $o_W : V \to \mathrm{End}\ \Omega_N(W)$ defined by $o_W(v) = (Y_W)_{\mathrm{wt}\ v-1}(v) = \mathrm{Res}_x x^{-1} Y_W(x^{L_V(0)}v, x)$ gives $\Omega_N(W)$ an $A_N(V)$-module structure. In particular, $o_W(O_N(V)) = 0$. So $O_N(V) \subset \ker o_W$. We take $W = S(A_N(V))$. Then $A_N(V)$ is an $A_N(V)$-submodule of $\Omega_N(S(A_N(V)))$. We use $o_{A_N(V)}$ to denote the corresponding map from V to $\mathrm{End}\ A_N(V)$. By the definition of $o_{A_N(V)}$, we have

$$o_{A_N(V)}(u)(v + O_N(V)) = u *_N v + O_N(V)$$

for $u, v \in V$. For $u \in \ker o_{A_N(V)}$, we have

$$o_{A_N(V)}(u)(v + O_N(V)) = 0$$

for $v \in V$. So we have $u *_N v + O_N(V) = 0$ or $u *_N v \in O_N(V)$. In particular, for $v = \mathbf{1}$, we have $u *_N \mathbf{1} \in O(V)$. But modulo $O_N(V)$, $u *_N \mathbf{1}$ is equal to u. So $u \in O_N(V)$. This means $\ker o_{S_N(A_N(V))} \subset O_N(V)$ and thus $\ker o_{A_N(V)} = O_N(V)$.

For $v \in V$, we have shown that $v + O_N(V) \in \Omega_{N-1}(S(A_N(V)))$ implies $v \in O_N(V)$ or equivalently $v + O_N(V) = O_N(V)$. So using our notation above, we see that

$$[v + O_N(V)]_N = (v + O_N(V)) + \Omega_{N-1}(S(A_N(V)))$$

is an element of $Gr_N(S(A_N(V)))$ and if it is equal to $0 \in Gr_N(S(A_N(V)))$, then $v + O_N(V) = O_N(V)$. By definition, for $u, v \in V$,

$$\begin{aligned}\vartheta_{Gr_N(S(A_N(V)))}([u]_{NN})[(v + O_N(V))]_N &= [\mathrm{Res}_x x^{-1} Y_{S(A_N(V))}(x^{L_V(0)}u, x)(v + O_N(V))]_N \\ &= [o_{A_N(V)}(u)(v + O_N(V))]_N.\end{aligned}$$

Then $\vartheta_{Gr_N(S(A_N(V)))}([u]_{NN})[(v + O_N(V))]_N = 0$ if and only if $o_{A_N(V)}(u)(v + O_N(V)) \in \Omega_{N-1}(S(A_N(V)))$.

For $\sum_{k=0}^N [u]_{kk} \in Q^\infty(V)$,

$$\begin{aligned}\vartheta_{S_N(A_N(V))}([u]_{NN})[(v + O_N(V))]_N &= \vartheta_{S_N(A_N(V))}\left(\sum_{k=0}^N [u]_{kk}\right)[(v + O_N(V))]_N \\ &= 0\end{aligned}$$

for all $v \in V$. So $o_{A_N(V)}(u)(v + O_N(V)) \in \Omega_{N-1}(S(A_N(V)))$ or equivalently $[o_{A_N(V)}(u)(v + O_N(V))]_N$ is equal to $0 \in Gr_N(S_N(A_N(V)))$ for all $v \in V$. Thus $o_{S_N(A_N(V))}(u)(v + O_N(V)) = O_N(V)$ or equivalently $o_{S_N(A_N(V))}(u)(v + O_N(V))$ is equal to $0 \in A_N(V)$. Then we have $u \in \ker o_{S_N(A_N(V))} = O_N(V)$.

We have proved that $\sum_{k=0}^N [u]_{kk} \in Q^\infty(V)$ implies $u \in \ker o_{S_N(A_N(V))} = O_N(V)$. On the other hand, since $O_N(V) \subset O_k(V)$ for $k = 0, \dots, N$ and $[O_k(V)]_{kk} \subset O^\infty(V) \subset Q^\infty(V)$, we have $\sum_{k=0}^N [u]_{kk} \in Q^\infty(V)$ for $u \in O_N(V)$. Thus $\sum_{k=0}^N [u]_{kk} \in Q^\infty(V)$ if and only if $u \in O_N(V)$. By this result, we obtain $\ker f^N = U^{NN} \cap Q^\infty(V)$. In particular, f^N induces a linear isomorphism $\hat{f}^N : A^{NN}(V) \to A_N(V)$.

For $u, v \in V$, using the calculation above, we have

$$\begin{aligned}&\hat{f}^N\left(\left(\sum_{k=0}^N [u]_{kk} + Q^\infty(V)\right) \diamond \left(\sum_{k=0}^N [v]_{kk} + Q^\infty(V)\right)\right) \\ &= \hat{f}^N\left(\left(\sum_{k=0}^N [u]_{kk}\right) \diamond \left(\sum_{k=0}^N [v]_{kk}\right) + Q^\infty(V)\right) \\ &= \hat{f}^N\left(\sum_{k=0}^N [u *_N v]_{kk} + Q^\infty(V)\right) \\ &= u *_N v + O_N(V) \\ &= (u + O_N(V)) *_N (v + O_N(V)).\end{aligned}$$

Therefore $\hat{f}^N$ is an isomorphism of associative algebras.

Since $\mathbf{1} + O_N(V)$ is the identity of $A_N(V)$, $\mathbf{1}^N + O^\infty(V)$ is the identity of $A^{NN}(V)$. If V is a vertex operator algebra with the conformal vector ω, since $\omega + O_N(V)$ is in the center of $A_N(V)$, $\omega^N + O^\infty(V)$ is in the center of $A^{NN}(V)$. ∎

4.2 Associative algebras from finite matrices

We now introduce new subalgebras of $A^\infty(V)$. For $N \in \mathbb{N}$, let $U^N(V)$ be the space of all $(N+1) \times (N+1)$ matrices with entries in V. It is clear that $U^N(V)$ can be canonically embedded into $U_0^\infty(V)$ as a subspace. We shall view $U^N(V)$ as a subspace of $U_0^\infty(V)$ in this paper. As a subspace of $U_0^\infty(V)$, $U^N(V)$ consists of infinite matrices in $U^\infty(V)$ whose (k,l)-th entries for $k > N$ or $l > N$ are all 0 and is spanned by elements of the form $[v]_{kl}$ for $v \in V$, $k, l = 0, \dots, N$.

Recall the element

$$\mathbf{1}^N = \sum_{k=0}^{N} [\mathbf{1}]_{kk},$$

that is, $\mathbf{1}^N$ is the element of $U^N(V)$ with the only nonzero entries to be equal to $\mathbf{1}$ at the diagonal (k,k)-th entries for $k = 0, \dots, N$. By (2.2), we have

$$\mathbf{1}^N \diamond [v]_{kl} = \mathrm{Res}_x T_{k+l+1}((x+1)^{-l-1})(1+x)^l [Y_V((1+x)^{L(0)}\mathbf{1}, x)v]_{kl} = [v]_{kl}$$

for $v \in V$ and $k, l = 0, \dots, N$. So $\mathbf{1}^N$ is a left identity of $U^N(V)$ with respect to the product $\diamond$. Note that for $v \in V$ and $k, l = 0, \dots, N$,

$$[v]_{kl} \diamond \mathbf{1}^N = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k-1})(1+x)^l [Y_V((1+x)^{L(0)}v, x)\mathbf{1}]_{kl} = [v]_{kl} \diamond \mathbf{1}^\infty.$$

This formula together with (2.21) immediately gives

$$[v]_{kl} \diamond \mathbf{1}^N = \sum_{m=0}^{k} \binom{-k-1}{m} \left[\binom{L_V(-1) + L_V(0) + l}{k+m} v \right]_{kl} \tag{4.1}$$

for $v \in V$ and $k, l = 0, \dots, N$.

By (2.3), for $u, v \in V$ and $k, n, l = 0, \dots, N$,

$$[u]_{kn} \diamond [v]_{nl} = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_V((1+x)^{L(0)}u, x)v \right]_{kl} \in U^N(V). \tag{4.2}$$

So $U^N(V)$ is closed under the product $\diamond$. Let

$$A^N(V) = \{\mathfrak{v} + Q^\infty(V) \mid \mathfrak{v} \in U^N(V)\} = \pi_{A^\infty(V)}(U^N(V)),$$

where $\pi_{A^\infty(V)}$ is the projection from $U^\infty(V)$ to $A^\infty(V)$. Then $A^N(V)$ is spanned by elements of the form $[v]_{kl} + Q^\infty(V)$ for $v \in V$ and $k, l = 0, \dots, N$.

Proposition 4.3 *The subspace $A^N(V)$ is closed under $\diamond$ and is thus a subalgebra of $A^\infty(V)$ with the identity $\mathbf{1}^N + Q^\infty(V)$.*

Proof. By (4.2), we have

$$\begin{aligned}&([u]_{kn} + Q^\infty(V)) \diamond ([v]_{nl} + Q^\infty(V))\\ &\quad = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_V((1+x)^{L(0)}u, x)v\right]_{kl} + Q^\infty(V)\\ &\quad \in A^N(V)\end{aligned}$$

for $u, v \in V$ and $k, n, l = 0, \ldots, N$. Thus $A^N(V)$ is closed under $\diamond$ and is thus a subalgebra of $A^\infty(V)$.

Since $\mathbf{1}^N$ is a left identity of $U^N(V)$ with respect to the product $\diamond$, $\mathbf{1}^N + Q^\infty(V)$ is a left identity of $A^N(V)$. Since

$$[v]_{kl} \diamond \mathbf{1}^N = [v]_{kl} \diamond \mathbf{1}^\infty \equiv [v]_{kl} \mod Q^\infty(V),$$

$\mathbf{1}^N + Q^\infty(V)$ is also a right identity of $A^N(V)$. In particular, it is the identity of $A^N(V)$. ∎

Remark 4.4 We have derived $A^N(V)$ as a subalgebra of $A^\infty(V)$. One can certainly obtain $A^N(V)$ directly starting with the space $U^N(V)$ of $(N+1) \times (N+1)$ matrices with entries in V.

Remark 4.5 It is clear from the definition that $A^{nn}(V)$ for $n = 0, \ldots, N$ are subalgebras of $A^N(V)$. In particular, the Zhu algebra $A(V)$ in [Z] and its generalizations $A_n(V)$ for $n = 0, \ldots, N$ by Dong, Li and Mason in [DLM] can be viewed as subalgebras of $A^N(V)$. In the case $N = 0$, A^0 is equal to $A^{00} = A_{00}(V)$ and is thus isomorphic to the Zhu algebra $A(V)$ by Theorem 4.1.

We say that V is of positive energy if $V = \coprod_{n \in \mathbb{N}} V_{(n)}$ and $V_{(0)} = \mathbb{C}\mathbf{1}$. (In some papers, V being of positive energy is said to be of CFT type.) We recall that for $n \in \mathbb{N}$, V is C_n-cofinite if $\dim V/C_n(V) < \infty$, where $C_n(V)$ is the subspace of V spanned by elements of the form $(Y_V)_{-n}(u)v$ for $u, v \in V$.

Theorem 4.6 *Assume that V is of positive energy and C_2-cofinite. Then $A^N(V)$ is finite dimensional.*

Proof. By Theorem 11 in [GN] (see Proposition 5.5 in [AN]), V is also C_n-cofinite for $n \geq 2$. In particular, V is C_{k+l+2}-cofinite for $k, l = 0, \ldots, N$. By definition, $C_{k+l+2}(V)$ are spanned by elements of the form $(Y_V)_{-k-l-2}(u)v$ for $u, v \in V$. Since V is C_{k+l+2}-cofinite, there exists a finite dimensional subspace X_{k+l} of V such that $X + C_{k+l+2}(V) = V$. Let $U^N(X)$ be the subspace of $U^N(V)$ consisting matrices in $U^N(V)$ whose (k,l)-th entries are in X for $k, l = 0, \ldots, N$. Since X_{k+l} for $k, l = 0, \ldots, N$ are finite dimensional, $U^N(X)$ is also finite dimensional. We now prove $U^N(X) + (O^\infty(V) \cap U^N(V)) = U^N(V)$. To prove this, we need only prove that every element of $U^N(V)$ of the form $[v]_{kl}$ for $v \in V$ and $0 \leq k, l \leq N$, can be written as $[v]_{kl} = [v_1]_{kl} + [v_2]_{kl}$, where $v_1 \in X_{k+l}$ and $v_2 \in V$ such that $[v_2]_{kl} \in O^\infty(V)$. We shall denote the subspace of V consisting of elements v such that $[v]_{kl} \in O^\infty(V)$ by $O^\infty_{kl}(V)$. Then what we need to prove is $V = X_{k+l} + O^\infty_{kl}(V)$.

We can always take X_{k+l} to be a subspace of V containing $\mathbf{1}$. We use induction on the weight of v. When wt $v = 0$, v is proportional to $\mathbf{1}$ and can indeed be written as $v = v + 0$, where $v \in X$ and $0 \in O^\infty_{kl}(V)$.

Assume that when wt $v = p < q$, $v = v_1 + v_2$, where $v_1 \in X_{k+l}$ and $v_2 \in O^\infty_{kl}(V)$. Then since V is C_{k+l+2}-cofinite, for $v \in V_{(q)}$, there exists homogeneous $u_1 \in X_{k+l}$ and homogeneous $u^i, v^i \in V$ for $i = 1, \dots, m$ such that $v = u_1 + \sum_{i=1}^m u^i_{-k-l-2}v^i$. Moreover, we can always find such u_1 and $u^i, v^i \in V$ for $i = 1, \dots, m$ such that wt $u_1 =$ wt $u^i_{-k-l-2}v^i =$ wt $v = q$. Since

$$\mathrm{wt}\ u^i_{n-k-l-2}v^i < \mathrm{wt}\ u^i_{-k-l-2}v^i = \mathrm{wt}\ v = q$$

for $i = 1, \dots, m$ and $n \in \mathbb{Z}_+$, by induction assumption, $u^i_{n-k-l-2}v^i \in X_{k+l} + O^\infty_{kl}(V)$ for $i = 1, \dots, m$ and $k \in \mathbb{Z}_+$. Thus

$$\begin{aligned} v &= u_1 + \sum_{i=1}^m u^i_{-k-l-2}v^i \\ &= u_1 + \sum_{i=1}^m \mathrm{Res}_x x^{-k-l-2}(1+x)^l Y((1+x)^{L(0)}u^i, x)v^i \\ &\quad - \sum_{i=1}^m \sum_{n \in \mathbb{Z}_+} \binom{\mathrm{wt}\ u^i + l}{n} u^i_{n-k-l-2}v^i. \end{aligned}$$

By definition,

$$[\mathrm{Res}_x x^{-k-l-2}(1+x)^l Y((1+x)^{L(0)}u^i, x)v^i]_{kl} \in O^\infty(V).$$

Thus

$$\mathrm{Res}_x x^{-k-l-2}(1+x)^l Y((1+x)^{L(0)}u^i, x)v^i \in O^\infty_{kl}(V).$$

Thus we have $v = v_1 + v_2$, where $v_1 \in X_{k+l}$ and $v_2 \in O^\infty_{kl}(V)$. By induction principle, we have $V = X_{k+l} + O^\infty_{kl}(V)$.

We now have proved $U^N(X) + (O^\infty(V) \cap U^N(V)) = U^N(V)$. Since $O^\infty(V) \cap U^N(V) \subset Q^\infty(V) \cap U^N(V)$, we also have $U^N(X) + (Q^\infty(V) \cap U^N(V)) = U^N(V)$. Since $U^N(X)$ is finite dimensional, $A^N(V)$ is finite dimensional. ∎

5 Lower-bounded generalized V-modules and graded $A^N(V)$-modules

By Theorem 2.8, the associated graded space $Gr(W)$ of a filtration of a lower-bounded generalized V-module W is a nondegenerate graded $A^\infty(V)$-module. In this section, for $N \in \mathbb{N}$, we give an $A^N(V)$-module structure to a subspace of $Gr(W)$ and use it to study W.

Let $N \in \mathbb{N}$. Let W be a lower-bounded generalized V-module. Since $A^N(V)$ is a subalgebra of $A^\infty(V)$, $Gr(W)$ as an $A^\infty(V)$-module is also an $A^N(V)$-module. Let

$$Gr^N(W) = \coprod_{n=0}^{N} Gr_n(W) \subset Gr(W).$$

By the definition of $\vartheta_{Gr(W)}$, we see that for $\mathfrak{v} \in A^N(V)$ and $[w]_n \in Gr^N(W)$, $\vartheta_{Gr(W)}(\mathfrak{v})[w]_n \in Gr^N(W)$. Thus $Gr^N(W)$ is an $A^N(W)$-submodule of $Gr(W)$. But $Gr^N(W)$ has some additional structures and properties and we are only interested in those $A^N(W)$-modules having these additional structures and properties. Similar to Definition 3.1, we have the following notion:

Definition 5.1 Let M be an $A^N(V)$-module M with the $A^N(V)$-module structure on M given by $\vartheta_M : A^N(V) \to \mathrm{End}\, M$. We say that M is a *graded $A^N(V)$-module* if the following conditions are satisfied:

1. $M = \coprod_{n=0}^{N} G_n(M)$ such that for $v \in V$ and $k, l = 0, \dots, N$, $\vartheta_M([v]_{kl} + Q^\infty(V))$ maps $G_n(M)$ for $0 \leq n \leq N$ to 0 when $n \neq l$ and to $G_k(M)$ when $n = l$.

2. M is a direct sum of generalized eigenspaces of of an operator $L_M(0)$ on M. $G_n(M)$ for $n \in \mathbb{N}$ are invariant under $L_M(0)$ and the real parts of the eigenvalues of $L_M(0)$ has a lower bound.

3. There is a linear map $L_M(-1) : \coprod_{n=0}^{N-1} G_{n-1}(M) \to \coprod_{n=1}^{N} G_n(M)$ mapping $G_n(M)$ to $G_{n+1}(M)$ for $n = 0, \dots, N-1$.

4. The commutator relations

$$\begin{aligned} [L_M(0), L_M(-1)] &= L_M(-1), \\ [L_M(0), \vartheta_M([v]_{kl} + Q^\infty(V))] &= (k-l)\vartheta_M([v]_{kl} + Q^\infty(V)), \\ [L_M(-1), \vartheta_M([v]_{pl} + Q^\infty(V))] &= \vartheta_M([L_V(-1)v]_{(p+1)l} + Q^\infty(V)) \end{aligned}$$

   hold for $v \in V$, $k, l = 0, \dots, N$ and $p = 0, \dots, N-1$.

A graded $A^N(V)$-module M is said to be *nondegenerate* if the following additional condition holds: For $w \in G_l(M)$, if $\vartheta_M([v]_{0l} + Q^\infty(V))w = 0$ for all $v \in V$, then $w = 0$. Let M_1 and M_2 be graded $A^N(V)$-modules. An *graded $A^N(V)$-module map* from M_2 to M_2 is an $A^N(V)$-module map $f : M_1 \to M_2$ such that $f(G_n(M_1)) \subset G_n(M_2)$ for $n = 0 \dots, N$, $f \circ L_{M_1}(0) = L_{M_2}(0) \circ f$ and $f \circ L_{M_1}(-1) = L_{M_2}(-1) \circ f$. A *graded $A^N(V)$-submodule* of a graded $A^N(V)$-module M is an $A^N(V)$-submodule M_0 of M such that with the $A^N(V)$-module structure, the $\mathbb{N}$-grading induced from M and the operators $L_M(0)\big|_{M_0}$ and $L_M(-1)\big|_{M_0}$, M_0 is a graded $A^N(V)$-module. A graded $A^\infty(V)$-module M is said to be *generated by a subset S* if M is equal to the smallest graded $A^N(V)$-submodule containing S, or equivalently, M is spanned by homogeneous elements obtained by applying elements of $A^N(V)$, $L_M(0)$ and $L_M(-1)$ to homogeneous summands of elements of S. A graded $A^N(V)$-module is said to be *irreducible* if it has no nonzero proper graded $A^N(V)$-modules. A graded $A^N(V)$-module is said to be *completely reducible* if it is a direct sum of irreducible graded $A^N(V)$-modules.

From the discussion above and the property of $Gr^N(W)$, we obtain immediately:

Proposition 5.2 *For a lower-bounded generalized V-module W, $Gr^N(W)$ is a nondegenerate graded $A^N(V)$-module. Let W_1 and W_2 be lower-bounded generalized V-modules and $f: W_1 \to W_2$ a V-module map. Then f induces a graded $A^N(V)$-module map $Gr^N(f): Gr^N(W_1) \to Gr^N(W_2)$.*

We have the following results on irreducible and completely reducible lower-bounded generalized V-modules without additional conditions:

Proposition 5.3 *Let W be a lower-bounded generalized V-module. If W is irreducible or completely reducible, then $Gr^N(W)$ is equivalent to $T_N(W)$ as an $A^N(V)$-module and is also irreducible or completely reducible, respectively.*

Proof. Let W be irreducible. By Proposition 3.3, $\Omega_n(W) = T_n(W)$ for $n = 0, \dots, N$. Then $T_N(W)$ is a nondegenerate graded $A^N(V)$-module equivalent to $Gr^N(W)$. We need to prove that the nondegenerate graded $A^N(V)$-module $T_N(W)$ is irreducible.

Let M be a nonzero graded $A^N(V)$-submodule of $T_N(W)$. We use the construction in Section 5 of [H3] to construct a universal lower-bounded generalized V-module $\widehat{M}$ from M. We take the generating fields for the grading-restricted vertex algebra V to be $Y_V(v, x)$ for $v \in V$. By definition, M is a direct sum of generalized eigenspaces of $L_M(0)$ and the real parts of the eigenvalues of $L_M(0)$ have a lower bound $B \in \mathbb{R}$. We take M and B in Section 5 of [H3] to be the given nondegenerate graded $A^N(V)$-module M and the lower bound B above. Using the construction in Section 5 of [H3], we obtain a universal lower-bounded generalized V-module $\widehat{M}_B^{[1_V]}$. For simplicity, we shall denote it simply by $\widehat{M}$. By the universal property of $\widehat{M}$ (Theorem 5.2 in [H3]), for the embedding map $e_M : M \to T_N(W)$, there is a unique V-module map $\widehat{e_M} : \widehat{M} \to W$ such that $\widehat{e_M}\big|_M = e$. Then $\widehat{e_M}(\widehat{M})$ is a generalized V-submodule of W generated by M. It is nonzero since $M \subset \widehat{e_M}(\widehat{M})$. Since W is irreducible, it must be W. Then W is generated by M. In particular, $T_N(W)$ is obtained by applying the components of the vertex operators on W, $L_W(0)$ and $L_W(-1)$ to elements of M. Since the components of the vertex operators on W and the operators $L_W(0)$ and $L_W(-1)$ preserving $T_N(W)$ are by definition the actions of elements of $A^N(V)$, $L_W(0)$ and $L_M(-1)$ preserving $T_N(W)$, we see that as a graded $A^N(V)$-module, $T_N(W)$ is generated by M. But M itself is an $A^N(V)$-submodule of $T_N(W)$. So we have $M = T_N(W)$. Thus $T_N(W)$ as a graded A^N-module is irreducible.

If W is completely reducible, by Proposition 3.3 again, $\Omega_n(W) = T_n(W)$ for $n = 0, \dots, N$. Then $T_N(W)$ is a nondegenerate graded $A^N(V)$-module equivalent to $Gr^N(W)$. Since W is completely reducible, $W = \coprod_{\mu \in \mathcal{M}} W^\mu$, where W^μ for $\mu \in \mathcal{M}$ are irreducible lower-bounded generalized V-modules. By the definition of $T_N(W)$, we have $T_N(W) = \coprod_{\mu \in \mathcal{M}} T_N(W^\mu)$. From what we have proved above, for $\mu \in \mathcal{M}$, $T_N(W^\mu)$ is an irreducible graded $A^N(V)$-module. Thus we see that $T_N(W)$ is completely reducible. ∎

Let M be a graded $A^N(V)$-module given by a linear map $\vartheta_M : A^N(V) \to \mathrm{End}\, M$ and operators $L_M(0)$ and $L_M(-1)$. We now construct a lower-bounded generalized V-module

$S^N(M)$ from M. We use the construction in Section 5 of [H3]. We take the generating fields for the grading-restricted vertex algebra V to be $Y_V(v, x)$ for $v \in V$. By definition, M is a direct sum of generalized eigenspaces of $L_M(0)$ and the real parts of the eigenvalues of $L_M(0)$ has a lower bound $B \in \mathbb{R}$. We take M and B in Section 5 of [H3] to be the given graded $A^N(V)$-module M and the lower bound B above. Using the construction in Section 5 of [H3], we obtain a universal lower-bounded generalized V-module $\widehat{M}_B^{[1_V]}$. For simplicity, we shall denote it simply by $\widehat{M}$.

By Theorem 3.4 in [H4] and the construction in Section 5 of [H3] and by identifying elements of the form $(\psi^a_{\widehat{M}})_{-1,0}$ with basis elements $w^a \in M$ for $a \in A$ for a basis $\{w^a\}_{a\in A}$ of M, we see that $\widehat{M}$ is generated by M (in the sense of Definition 3.1 in [H4]). Moreover, Theorems 3.3 and 3.4 in [H4] state that elements of the form $L_{\widehat{M}}(-1)^p w^a$ for $p \in \mathbb{N}$ and $a \in A$ are linearly independent and $\widehat{M}$ is spanned by elements obtained by applying the components of the vertex operators to these elements. In particular, we identify M as a subspace of $\widehat{M}$. Let J_M be the generalized V-submodule of $\widehat{M}$ generated by elements of the forms

$$\mathrm{Res}_x x^{l-k-1} Y_{\widehat{M}}(x^{L_V(0)}v, x)w \tag{5.1}$$

for $l = 0, \dots, N$, $k \in -\mathbb{Z}_+$ and $w \in G_l(M)$,

$$\mathrm{Res}_x x^{l-k-1} Y_{\widehat{M}}(x^{L_V(0)}v, x)w - \vartheta_M([v]_{kl})w \tag{5.2}$$

for $v \in V$, $k, l = 0, \dots, N$ and $w \in G_l(M)$ and

$$L_{\widehat{M}}(-1)w - L_M(-1)w \tag{5.3}$$

for $w \in \coprod_{n=0}^{N-1} G_n(M)$.

Let $S^N(M) = \widehat{M}/J_M$. Then $S^N(M)$ is a lower-bounded generalized V-module. Let $\pi_{S^N(M)}$ be the projection from $\widehat{M}$ to $S^N(M)$. Since $\widehat{M}$ is generated by M (in the sense of Definition 3.1 in [H4]), $S^N(M)$ is generated $\pi_{S^N(M)}(M)$ (in the same sense). In particular, $S^N(M)$ is spanned by elements of the form

$$\mathrm{Res}_x x^{(l+p)-n-1} Y_{S^N(M)}(x^{L_V(0)}v, x) L_{S^N(M)}(-1)^p \pi_{S^N(M)}(w) \tag{5.4}$$

for $v \in V$, $l = 0, \dots, N$, $n, p \in \mathbb{N}$ and $w \in G_l(M)$. For $n \in \mathbb{N}$, let $G_n(S^N(M))$ be the subspace of $S^N(M)$ spanned by elements of the form (5.4) for $v \in V$, $l = 0, \dots, N$, $p \in \mathbb{N}$ and $w \in G_l(M)$.

Proposition 5.4 *Let M be a graded $A^N(V)$-module.*

1. *For $0 \le n \le N$, $G_n(S^N(M)) = \pi_{S^N(M)}(G_n(M))$ and for $0 \le n_1, n_2 \le N$, $n_1 \ne n_2$, $G_{n_1}(S^N(M)) \cap G_{n_2}(S^N(M)) = 0$. Moreover, $S^N(M) = \coprod_{n\in\mathbb{N}} G_n(S^N(M))$ and $\pi_{S^N(M)}(M) = \coprod_{n=0}^{N} G_n(S^N(M))$.*

2. *For $n = 0, \ldots, N$,*

$$\pi_{S^N(M)}\left(\coprod_{j=0}^{n} G_j(M)\right) = \coprod_{j=0}^{n} G_j(S^N(M)) \subset \Omega_n(S^N(M)) \tag{5.5}$$

and in the case that M is nondegenerate,

$$\pi_{S^N(M)}\left(\coprod_{j=n}^{N} G_j(M)\right) \cap \Omega_n(S^N(M)) = \left(\coprod_{j=n}^{N} G_j(S^N(M))\right) \cap \Omega_n(S^N(M)) = 0. \tag{5.6}$$

3. *In the case that M is nondegenerate, M is equivalent to a graded $A^N(V)$-submodule of $Gr^N(S^N(M))$.*

Proof. By definition, $G_n(S^N(M))$ for $0 \leq n \leq N$ is spanned by elements of the form (5.4) for $v \in V$, $l = 0, \ldots, N$, $p \in \mathbb{N}$ and $w \in G_l(M)$. Using the $L(-1)$-commutator formula for the vertex operator map $Y_{S^N(M)}$, we see that it is also spanned by elements of the form

$$\mathrm{Res}_x x^{l-k-1} L_{S^N(M)}(-1)^p Y_{S^N(M)}(x^{L_V(0)}v, x)\pi_{S^N(M)}(w) \tag{5.7}$$

for $v \in V$, $l, k = 0, \ldots, N$, $p = 0, \ldots, n-k$ and $w \in G_l(M)$. Since elements of the forms (5.2) and (5.3) are in J_M, we see that (5.7) is in fact equal to

$$\pi_{S^N(M)}(L_M(-1)^p \vartheta_M([v]_{kl} + Q^\infty(V))w) \in \pi_{S^N(M)}(G_n(M)). \tag{5.8}$$

Since $L_M(-1)^p\vartheta_M([v]_{k(l+p)} + Q^\infty(V))w$ for $v \in V$, $l, k = 0, \ldots, N$, $p = 0, \ldots, n-k$ and $w \in G_l(M)$ certainly span $G_n(M)$ (in fact, we need only $v = \mathbf{1}$, $k = l = n$, $p = 0$ and $w \in G_n(W)$) and elements of the form (5.7) for $v \in V$, $l, k = 0, \ldots, N$, $p = 0, \ldots, n-k$ and $w \in G_l(M)$ span $G_n(S(G))$ for $0 \leq n \leq N$, we see that elements of the form (5.8) for $v \in V$, $l, k = 0, \ldots, N$, $p = 0, \ldots, n-k$ and $w \in G_l(M)$ also span $G_n(S^N(M))$. Thus we obtain $G_n(S^N(M)) = \pi_{S^N(M)}(G_n(M))$ for $n = 0, \ldots, N$. When $n_1 \neq n_2$, we know $G_{n_1}(M) \cap G_{n_2}(M) = 0$. Then $G_{n_1}(S^N(M)) \cap G_{n_2}(S^N(M)) = \pi_{S(G)}(G_{n_1}(M) \cap G_{n_2}(M)) = 0$. Since $S^N(M)$ is spanned by elements of the form (5.4) for $v \in V$, $l = 0, \ldots, N$, $n, p \in \mathbb{N}$ and $w \in G_l(M)$, by the definition of $G_n(S^N(M))$, we have $S^N(M) = \coprod_{n \in \mathbb{N}} G_n(S^N(M))$. Since $G_n(S^N(M)) = \pi_{S^N(M)}(G_n(M))$ for $n = 0, \ldots, N$, we have

$$\pi_{S^N(M)}(M) = \coprod_{n=0}^{N} \pi_{S^N(M)}(G_n(M)) = \coprod_{n=0}^{N} G_n(S^N(M)).$$

By definition, for $0 \leq j \leq n \leq N$, $G_j(S^N(M)) \subset \Omega_n(S^N(M))$. Then for $j = 0, \ldots, n$,

$$\pi_{S^N(M)}(G_j(M)) = G_j(S^N(M)) \subset \Omega_j(S^N(M)) \subset \Omega_n(S^N(M)).$$

So we obtain (5.5). By the nondegeneracy of M, nonzero elements of $G_j(M)$ for $N \geq j > n$ are not in $\Omega_n(\widehat{M})$. From the construction of $\widehat{M}$, nonzero elements of the form (5.1), (5.2) or (5.3) are not in $M \subset \widehat{M}$. In particular, we see that the intersection of $J(M)$ with M is 0. So $\pi_{S^N(M)}\big|_M$ is injective. Since $\pi_{S^N(M)}\big|_M$ is injective, we see that nonzero elements of $G_j(S^N(M)) = \pi_{S^N(M)}(G_j(M))$ for $N \geq j > n$ are not in $\Omega_n(S^N(M))$. Thus we obtain (5.6).

For $0 \leq n \leq N$ and $w \in G_n(M)$, we define $f_M(w) = \pi_{S^N(M)}(w) + \Omega_{n-1}(S^N(M))$. Since $\pi_{S^N(M)}(w) \in \Omega_n(S^N(M))$, $f_M(w) \in Gr_n(S^N(M))$. Therefore we obtain a linear map $f_M : M \to Gr^N(S^N(M))$. It is clear from the definition that f_M is in fact a graded $A^N(V)$-module map. If for some $0 \leq n \leq N$ and $w \in G_n(M)$, $f_M(w) = 0$, then $\pi_{S^N(M)}(w) \in \Omega_{n-1}(S^N(M))$. But we have proved above that nonzero elements of $\pi_{S^N(M)}(G_n(M))$ are not in $\Omega_{n-1}(S^N(M))$. So $\pi_{S^N(M)}(w) = 0$. Since $\pi_{S^N(M)}\big|_M$ is injective, we obtain $w = 0$. So f_M is injective. Thus M is equivalent to the nondegenerate graded $A^N(V)$-submodule $f_M(M)$ of $Gr^N(S^N(M))$. ∎

Remark 5.5 As in the case of $S(G)$ in Section 3, our construction of the lower-bounded generalized V-module $\widehat{M}$ depends on the lower bound B of the real parts of the eigenvalues of $L_M(0)$. But by Proposition 5.4, $S^N(M)$ depends only on M, not on B.

Theorem 5.6 *For $N \in \mathbb{N}$, the set of the equivalence classes of irreducible lower-bounded generalized V-modules is in bijection with the set of the equivalence classes of irreducible nondegenerate graded $A^N(V)$-modules.*

Proof. Recall the set $[\mathfrak{W}]_{\mathrm{irr}}$ of the equivalence classes of irreducible lower-bounded generalized V-modules in the proof of Theorem 3.7. Let $[\mathfrak{M}^N]_{\mathrm{irr}}$ be the set of the equivalence classes of irreducible nondegenerate graded $A^N(V)$-modules. Given an irreducible lower-bounded generalized V-module W, by Theorem 5.3, $Gr^N(W) = T_N(W)$ is an irreducible nondegenerate graded $A^N(V)$-module. Thus we obtain a map $f : [\mathfrak{W}]_{\mathrm{irr}} \to [\mathfrak{M}^N]_{\mathrm{irr}}$ given by $f([W]) = [T_N(W)]$, where $[W] \in [\mathfrak{W}]_{\mathrm{irr}}$ is the equivalence class containing the irreducible lower-bounded generalized V-module W and $[T_N(W)] \in [\mathfrak{M}^N]_{\mathrm{irr}}$ is the equivalence class containing the irreducible nondegenerate graded $A^N(V)$-module $T_N(W)$.

Given an irreducible nondegenerate graded $A^N(V)$-module M, we have the lower-bounded generalized V-module $S^N(M)$ generated by $\pi_{S^N(M)}(M)$. The main difference of the proof here and the the proof of Theorem 3.7 is that we do not know whether $S^N(M)$ is irreducible. So we need to take a quotient of $S^N(M)$. Since M is an irreducible nondegenerate graded $A^N(V)$-module, it is generated by any nonzero element. Since $S^N(M)$ is generated by $\pi_{S^N(M)}(M)$, it is also generated by any element $w_0 \in \pi_{S^N(M)}(M)$. Then by Theorem 4.7 in [H4], there is a maximal generalized V-submodule $J_{\pi_{S^N(M)}(M),w_0}$ of $S^N(M)$ such that $J_{\pi_{S^N(M)}(M),w_0}$ does not contain w_0 and $S^N(M)/J_{\pi_{S^N(M)}(M),w_0}$ is irreducible. The maximal generalized V-submodule $J_{\pi_{S^N(M)}(M),w_0}$ is in fact independent of $w_0 \in \pi_{S^N(M)}(M)$. We prove this fact by proving that no nonzero element of $\pi_{S^N(M)}(M)$ is in $J_{\pi_{S^N(M)}(M),w_0}$. In fact, if a nonzero $w \in \pi_{S^N(M)}(M)$ is also in $J_{\pi_{S^N(M)}(M),w_0}$, since the actions of components of vertex operators on w are equal to the actions of elements of $A^N(V)$ and M is generated also by w, we see that w_0 must also be in $J_{\pi_{S^N(M)}(M),w_0}$. Contradiction. Thus $J_{\pi_{S^N(M)}(M),w_0}$ is in fact the maximal generalized V-submodule of $S^N(M)$ such that it does not contain nonzero elements of M. We denote it by $\widetilde{J}_M$, which depends only on $\pi_{S^N(M)}(M)$, or equivalently, M. Thus we obtain a map $g : [\mathfrak{M}^N]_{\rm irr} \to [\mathfrak{W}]_{\rm irr}$ given by $g([M]) = [S^N(M)/\widetilde{J}_M]$.

We still need to show that the two maps above are inverses of each other. Let M be an irreducible nondegenerate graded $A^N(V)$-module. Since $S^N(M)/\widetilde{J}_M$ is irreducible, by Proposition 3.3, $Gr_N(S^N(M)/\widetilde{J}_M)$ is an irreducible nondegenerate graded $A^N(V)$-module. By Proposition 5.4, M is equivalent to a nondegenerate graded $A^N(V)$-submodule of $Gr_N(S^N(M))$. As in the proof of Proposition 5.4, we denote this equivalence by f_M. Let $\pi_{\widetilde{J}_M} : S^N(M) \to S^N(M)/\widetilde{J}_M$ be the projection map. Since $\widetilde{J}_M \cap \pi_{S^N(M)}(M) = 0$, $\pi_{\widetilde{J}_M}\big|_{\pi_{S^N(M)}(M)}$ is injective and in particular, is not 0. The V-module map $\pi_{\widetilde{J}_M}$ induces a graded $A^N(V)$-module map $Gr^N(\pi_{\widetilde{J}_M}) : Gr_N(S^N(M)) \to Gr_N(S^N(M)/\widetilde{J}_M)$. Since $\pi_{\widetilde{J}_M}\big|_{\pi_{S^N(M)}(M)}$ is not 0, the restriction $Gr^N(\pi_{\widetilde{J}_M})\big|_{f_M(M)}$ of $Gr^N(\pi_{\widetilde{J}_M})$ to the image of M under f_M is also not 0. Consider the $A^N(V)$-module map $Gr^N(\pi_{\widetilde{J}_M}) \circ f_M : M \to Gr_N(S^N(M)/\widetilde{J}_M)$. Since f_M is injective and $Gr^N(\pi_{\widetilde{J}_M})\big|_{f_M(M)} \neq 0$, $Gr^N(\pi_{\widetilde{J}_M}) \circ f_M$ is not 0. But both M and $Gr_N(S^N(M)/\widetilde{J}_M)$ are irreducible. So $Gr^N(\pi_{\widetilde{J}_M}) \circ f_M$ must be an equivalence of graded $A^N(V)$-modules. Moreover, by Proposition 5.3, $Gr_N(S^N(M)/\widetilde{J}_M)$ is equivalent to $T_N(S^N(M)/\widetilde{J}_M)$. So M is equivalent to $T_N(S^N(M)/\widetilde{J}_M)$. Thus $[M] = [T_N(\widehat{M}/\widetilde{J}_M)]$. This means $f(g([M])) = [M]$. So we obtain $f \circ g = 1_{[\mathfrak{M}^N]_{\rm irr}}$.

Let W be an irreducible lower-bounded generalized V-module. By Theorem 5.3, $T_N(W)$ is an irreducible $A^N(V)$-module. We then have a lower-bounded generalized V-module $S^N(T_N(W))$. By the universal property of $\widehat{T_N(W)}$, there is a unique V-module map $\widehat{1_{T_N(W)}} : \widehat{T_N(W)} \to W$ such that $\widehat{1_{T_N(W)})}\big|_{T_N(W)} = 1_{T_N(W)}$, where $1_{T_N(W)}$ is the identity operator on $T_N(W)$. Since W is irreducible, the image of $\widehat{T_N(W)}$ under $\widehat{1_{T_N(W)}}$ is either 0 or W. Since $\widehat{1_{T_N(W)})}\big|_{T_N(W)} = 1_{T_N(W)}$, the image of $\widehat{T_N(W)}$ under $\widehat{1_{T_N(W)}}$ cannot be 0 and thus must be W. In particular, $\widehat{1_{T_N(W)}}$ is surjective. Moreover, since $J_{T_N(W)}$ is generated by (5.1), (5.2) and (5.3) with $M = T_N(W)$, the image of $J_{T_N(W)}$ under $\widehat{1_{T_N(W)}}$ is 0, that is, $J_{T_N(W)} \in \ker \widehat{1_{T_N(W)}}$. In particular, $\widehat{1_{T_N(W)}}$ induces a surjective V-module map $f_{T_N(W)}$: $S^N(T_N(W)) = \widehat{T_N(W)}/J_{T_N(W)} \to W$. Since $J_{T_N(W)} \cap T_N(W) = 0$, $f_{T_N(W)}(T_N(W)) = T_N(W)$. We have a maximal generalized V-submodule $\widetilde{J}_{T_N(W)}$ of $S^N(T_N(W))$ as in the construction above such that $T_N(W) \cap \widetilde{J}_{T_N(W)} = 0$ and $S^N(T_N(W))/\widetilde{J}_{T_N(W)}$ is irreducible. Since $f_{T_N(W)}(T_N(W)) = T_N(W)$, $\ker f_{T_N(W)}$ is a generalized V-submodule of $S^N(T_N(W))$ that does not contain nonzero elements of M. Hence $\ker f_{T_N(W)} \subset \widetilde{J}_{T_N(W)}$. Thus we obtain a surjective V-module map from $S^N(T_N(W))/\widetilde{J}_{T_N(W)}$ to W. Since both $S^N(T_N(W))/\widetilde{J}_{T_N(W)}$ and W are irreducible, this surjective V-module map must be an equivalence. So we obtain $[S^N(T_N(W))/\widetilde{J}_{T_N(W)}] = [W]$, that is, $g(f([W])) = [W]$. So we obtain $g \circ f = 1_{[\mathfrak{W}]_{\rm irr}}$. This finishes the proof that $[\mathfrak{W}]_{\rm irr}$ is in bijection with $[\mathfrak{M}^N]_{\rm irr}$. ∎

Corollary 5.7 *For $N_1, N_2 \in \mathbb{N}$ or equal to ∞, the set of the equivalence classes of irreducible* *nondegenerate graded $A^{N_1}(V)$-modules is in bijection with the set of the equivalence classes of irreducible nondegenerate graded $A^{N_2}(V)$-modules.*

We now assume that V is a Möbius vertex algebra, that is, a grading-restricted vertex algebra equipped with an operator $L_V(1)$ such that $L_V(1)$, $L_V(0)$ and $L_V(-1)$ satisfying the usually commutator relations for the standard basis of $\mathfrak{sl}_2$ and the usual commutator formula between $L_V(1)$ and vertex operators for a vertex operator algebra. See, for example, Definition 7.1 in [H4] for the precise definition. In this case, a lower-bounded generalized V-module should also have an operator $L_W(1)$ satisfying the same relations as $L_V(1)$. We assume that V is a grading-restricted Möbius vertex algebra in the remaining part of the paper because in this case, a lowest weight of a lower-bounded generalized V-module is well defined. See Remark 7.3 in [H4].

Proposition 5.8 *Let V be a Möbius vertex algebra. Assume that $A^N(V)$ for all $N \in \mathbb{N}$ are finite dimensional (for example, when V is C_2-cofinite and of positive energy by Theorem 4.6). Then every irreducible lower-bounded generalized V-module is an ordinary V-module and every lower-bounded generalized V-module of finite length is grading restricted.*

Proof. Since for $N \in \mathbb{N}$, $A^N(V)$ is finite dimensional, there are only finitely many irreducible $A^N(V)$-modules. By Theorem 5.6, there are also finitely many irreducible lower-bounded generalized V-modules. For an irreducible lower-bounded generalized V-module W with lowest weight h_W and $N \in \mathbb{N}$, $T_N(W)$ is an irreducible nondegenerate graded $A^N(V)$-module by Proposition 5.3. Since $A^N(V)$ is finite dimensional, $T_N(W)$ is also finite dimensional. Thus $G_N(W) = W_{[h_W+N]} \subset T_N(W)$ is also finite dimensional. Since this is true for $N \in \mathbb{N}$, we see that W is grading restricted. Since W is irreducible, $L_W(0)$ must act semisimply on W. So W is an irreducible ordinary V-module.

Since as a graded vector space, a lower-bounded generalized V module W of finite length is a finite sum of irreducible lower-bounded generalized V-modules, which are all ordinary V-modules from what we have proved above. Then W must be grading restricted. ∎

Since V is a Möbius verex algebra, the associative algebras $A^\infty(V)$ and $A^N(V)$ for $N \in \mathbb{N}$ have an additional operator $L_V(1)$ induced from the operator $L_V(1)$ acting on V. For a lower-bounded generalized V-module W, there is also an operator $L_{Gr(W)}(1)$ on the $A^\infty(V)$-module $Gr(W)$ induced from $L_W(1)$ on W such that $L_{Gr(W)}(1)$ maps $Gr_n(W)$ to $Gr_{n-1}(W)$. Restricting $L_{Gr(W)}(1)$ to $Gr^N(W)$, we obtain an operator $L_{Gr^N(W)}(1)$ on $Gr^N(W)$.

Definition 5.9 Let V be a Möbius vertex algebra. A *graded $A^N(V)$-module* is a graded $A^N(V)$-module M when V is viewed as a grading-restricted vertex algebra together with an operator $L_M(1)$ satisfying the following conditions:

1. $L_M(1)$ maps $G_n(M)$ to $G_{n-1}(M)$ for $n = 0, \ldots, N$, where $G_{-1}(M) = 0$.

2. The operators $L_M(1)$ satisfies the commutator relations

$$[L_M(0), L_M(1)] = -L_M(1),$$

$$[L_M(1), L_M(-1)] = 2L_M(0),$$
$$[L_M(1), \vartheta_M([v]_{kl} + Q^\infty(V))] = \vartheta_M([(L_V(1) + 2L_V(0) + L_V(-1))v]_{(k-1)l} + Q^\infty(V)).$$

An graded $A^N(V)$-module M is said to be *nondegenerate* if M is nondegenerate when V is viewed as a grading-restricted vertex algebra. Let M_1 and M_2 be graded $A^N(V)$-modules. An *graded $A^N(V)$-module map* from M_1 to M_2 is an $A^N(V)$-module map $f : M_1 \to M_2$ such that $f(G_n(M_1)) \subset G_n(M)$ for $n = 0 \dots, N$, $f \circ L_{M_1}(1) = L_{M_2}(1) \circ f$, $f \circ L_{M_1}(0) = L_{M_2}(0) \circ f$ and $f \circ L_{M_1}(-1) = L_{M_2}(-1) \circ f$. A *graded $A^N(V)$-submodule* of a graded $A^N(V)$-module M is an $A^N(V)$-submodule M_0 of M such that with the $A^N(V)$-module structure and the $\mathbb{N}$-grading induced from M and the operators $L_M(1)\big|_{M_0}$, $L_M(0)\big|_{M_0}$ and $L_M(-1)\big|_{M_0}$, M_0 is a graded $A^N(V)$-module. A graded $A^\infty(V)$-module M is said to be *generated by a subset S* if M is equal to the smallest graded $A^N(V)$-submodule containing S, or equivalently, M is spanned by homogeneous elements obtained by applying elements of $A^N(V)$, $L_M(1)$ and $L_M(-1)$ to homogeneous summands of elements of S. *Irreducible* and *completely reducible* graded $A^N(V)$-module are defined in the same way as in the case that V is a grading-restricted vertex algebra.

From Proposition 5.2 and the property of $L_W(1)$, we immediately obtain the following:

Proposition 5.10 *Let V be a Möbius vertex algebra. For a lower-bounded generalized V-module W, $Gr^N(W)$ is nondegenerate graded $A^N(V)$-module. Let W_1 and W_2 be lower-bounded generalized V-modules and $f : W_1 \to W_2$ a V-module map. Then f induces a graded $A^N(V)$-module map $Gr^N(f) : Gr^N(W_1) \to Gr^N(W_2)$.*

As is mentioned above, in the remaining part of this paper, we assume that V is a Möbius vertex algebra. We shall not repeat this assumption except in the statements of propositions, theorems, corollaries and so on. Lower-bounded generalized V-modules and graded $A^N(V)$-modules always mean those for V as a Möbius vertex algebra, not as a grading-restricted vertex algebra. All the results that we have obtained above certainly still hold.

We recall the notion of lower-bounded generalized V-module of finite length. A lower-bounded generalized V-module W is said to be *of fnite length* if there is a composition series $W = W_0 \supset \cdots \supset W_{l+1} = 0$ of lower-bounded generalized V-modules such that W_i/W_{i+1} for $i = 0, \dots, l$ are irreducible lower-bounded generalized V-modules.

Proposition 5.11 *Let V be a Möbius vertex algebra. Assume that the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized V-modules are all less than or equal to $N \in \mathbb{N}$. Then a lower-bounded generalized V-module W of finite length is generated by*

$$\coprod_{\Re(h_W) \leq \Re(n) \leq \Re(h_W)+N} W_{[n]} \subset \Omega_N(W),$$

where h_W is a lowest weight of W.

Proof. Let $W = W_0 \supset W_1 \supset \cdots \supset W_{l+1} = 0$ be a finite composition series such that W_i/W_{i+1} for $i = 0, \dots, l$ are irreducible lower-bounded generalized V-modules. As a graded vector space, W is isomorphic to $\coprod_{i=0}^{l} W_i/W_{i+1}$. In particular, the lowest weight of one of the irreducible lower-bounded generalized V-modules W_i/W_{i+1} for $i = 0, \dots, l$ is a lowest weight h_W of W.

Let $w_i \in W_i$ be homogeneous for $i = 0, \dots, l$ such that $w_i + W_{i+1}$ is a lowest weight vector of W_i/W_{i+1}. Then by assumption, the differences between the real parts of the lowest weights of W_i/W_{i+1} for $i = 0, \dots, l$ are less than or equal to N. Since one of these lowest weights is a lowest weight h_W of W, we see that the the differences between the real parts of the lowest weights of W_i/W_{i+1} for $i = 0, \dots, l$ and $\Re(h_W)$ are less than or equal to N. In particular $w_i \in \coprod_{\Re(h_W) \leq \Re(n) \leq \Re(h_W)+N} W_{[n]}$. Since for each i, W_i/W_{i+1} is generated by $w_i + W_{i+1}$, W_i is generated by w_i and W_{i+1}. Thus W is generated by w_i for $i = 0, \dots, l$. Since $w_i \in \coprod_{\Re(h_W) \leq \Re(n) \leq \Re(h_W)+N} W_{[n]}$, W is generated by $\coprod_{\Re(h_W) \leq \Re(n) \leq \Re(h_W)+N} W_{[n]}$. It is clear that $\coprod_{\Re(h_W) \leq \Re(n) \leq \Re(h_W)+N} W_{[n]}$ is a subspace of $\Omega_N(W)$. ∎

Corollary 5.12 *Let V be a Möbius vertex algebra. Assume that $A_{N'}(V)$ for all $N' \in \mathbb{N}$ are finite dimensional (for example, when V is C_2-cofinite and of positive energy by Theorem 4.6). Let $N \in \mathbb{N}$ such that the differences between the real parts of the lowest weights of the finitely many (inequivalent) irreducible ordinary V-modules are less than or equal to N. Then a lower-bounded generalized V-module W of finite length or a grading-restricted generalized V-module W is generated by*

$$\coprod_{\Re(h_W) \leq \Re(n) \leq \Re(h_W)+N} W_{[n]} \subset \Omega_N(W).$$

Proof. Since by Proposition 5.8, the finitely many (inequivalent) irreducible lower-bounded generalized V-modules are all ordinary V-modules, the condition in Proposition 5.11 is satisfied. Also, by Corollary 3.16 in [H1], every grading-restricted generalized V-module is of finite length. Thus W is generated by $\coprod_{\Re(h_W) \leq \Re(n) \leq \Re(h_W)+N} W_{[n]}$. ∎

Theorem 5.13 *Let V be a Möbius vertex algebra. Assume that the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized V-modules are all less than or equal to $N \in \mathbb{N}$. Then a lower-bounded generalized V-module W of finite length is irreducible or completely reducible if and only if the nondegenerate graded $A^N(V)$-module $Gr^N(W)$ is irreducible or completely reducible, respectively.*

Proof. By Proposition 5.3, we already know that if W is irreducible, $Gr^N(W) = T_N(W)$ is irreducibile. Conversely, assume that the nondegenerate graded $A^N(V)$-module $Gr^N(W)$ is irreducible. Let W_0 be a nonzero generalized V-submodule of W. Let $e_{W_0} : W_0 \to W$ be the embedding map. Then we have a graded $A^N(V)$-moudle map $Gr(e_{W_0}) : Gr^N(W_0) \to Gr^N(W)$ given by $(Gr(e_{W_0}))(w_0 + \Omega_{n-1}(W_0)) = w_0 + \Omega_{n-1}(W)$ for $n = 0, \dots, N$ and $w_0 \in \Omega_n(W_0)$. Since e_W is injective, $Gr(e_{W_0})$ is also injective . So $(Gr(e_{W_0}))(Gr^N(W_0))$ is a nondegenerate graded $A^N(V)$-submodule of $Gr^N(W)$. Since W_0 is nonzero, $Gr^N(W_0)$ is nonzero. Since $Gr^N(W)$ is irreducible and $Gr(e_{W_0})$ is injective, $(Gr(e_{W_0}))(Gr^N(W_0))$ is equal to $Gr^N(W)$. We now prove $W_0 = W$. In fact, for $n = 0, \dots, N$, $(Gr(e_{W_0}))(Gr_n(W_0)) = \{w_0 + \Omega_n(W) \mid w_0 \in \Omega_n(W_0)\}$. So $Gr_n(W) = \{w_0 + \Omega_{n-1}(W) \mid w_0 \in \Omega_n(W_0)\}$. For $n = 0$, we obtain $\Omega_0(W) = Gr_0(W) = Gr(W_0) = \Omega_0(W_0)$. Assume that $\Omega_{n-1}(W) = \Omega_{n-1}(W_0)$ for $n < N$. Given $w \in \Omega_n(W)$, $w + \Omega_{n-1}(W) \in Gr_n(W)$. By $Gr_n(W) = \{w_0 + \Omega_{n-1}(W) \mid w_0 \in \Omega_n(W_0)\}$, there exists $w_0 \in \Omega_n(W_0)$ such that $w + \Omega_{n-1}(W) = w_0 + \Omega_{n-1}(W)$, or equivalently, $w - w_0 \in \Omega_{n-1}(W) = \Omega_{n-1}(W_0)$. Thus $w \in \Omega_n(W_0)$. This shows $\Omega_n(W) = \Omega_n(W_0)$ for $n = 0, \dots, N$. In particular, $\Omega_N(W) = \Omega_N(W_0)$. But by Proposition 5.11, W_0 and W are generated by $\Omega_N(W_0)$ and $\Omega_N(W)$, respectively. Since $\Omega_N(W) = \Omega_N(W_0)$, we must have $W = W_0$. So W is irreducible.

If W is completely reducible, then by Proposition 5.3, $Gr^N(W) = T_N(W)$ is completely reducible. Conversely, assume that the nondegenerate graded $A^N(V)$-module $Gr^N(W)$ is completely reducible. Then $Gr^N(W) = \coprod_{\mu \in \mathcal{M}} M^\mu$, where M^μ for $\mu \in \mathcal{M}$ are irreducible nondegenerate graded $A^N(V)$-submodules of $Gr^N(W)$. For $\mu \in \mathcal{M}$, since M^μ is a nondegenerate graded $A^N(V)$-submodule of $Gr^N(W)$, we have $M^\mu_n \subset Gr_n(W) = \Omega_n(W)/\Omega_{n-1}(W)$ for $n = 0, \dots, N$. Let W^μ be the generalized V-submodule of W generated by the set of elements of the form $w^\mu \in \Omega_n(W)$ such that $w^\mu + \Omega_{n-1}(W) \in M^\mu_n$ for for $n = 0, \dots, N$. Since W^μ is a generalized V-submodule of W, for $v \in V$, $k, l \in \mathbb{N}$ and $w^\mu \in \Omega_l(W)$ such that $w^\mu + \Omega_{l-1}(W) \in M^\mu_l$,

$$\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w^\mu + \Omega_{k-1}(W) \in Gr_k(W^\mu).$$

By the definition of W^μ, we see that $\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w^\mu \in W^\mu$. Since $w^\mu \in \Omega_l(W)$, $\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w^\mu = 0$ for $k \in -\mathbb{Z}_+$. Therefore $\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w^\mu \in W^\mu$ for $k \in \mathbb{N}$ are all the nonzero coefficients of $Y_W(v, x)w^\mu$. So W^μ is closed under the action of the vertex operators on W. Since M^μ is invariant under the actions of $L_{Gr(W)}(0)$ and $L_{Gr(W)}(-1)$ and is a direct sum of generalized eigenspaces of $L_{Gr(W)}(0)$, W^μ is invariant under the actions of $L_W(0)$ and $L_W(-1)$ and is a direct sum of generalized eigenspaces of $L_W(0)$. Thus W^μ is a generalized V-submodule of W.

Let $w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu)$, where $0 \leq n \leq N$ and $w^\mu \in \Omega_n(W^\mu) \subset \Omega_n(W)$. By the definition of W^μ, we see that since w^μ is an element of W^μ, $w^\mu + \Omega_{n-1}(W) \in G_n(M^\mu)$. So we obtain a linear map from $Gr_n(W^\mu)$ to $G_n(M^\mu)$ given by $w^\mu + \Omega_{n-1}(W^\mu) \mapsto w^\mu + \Omega_{n-1}(W)$ for $w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu)$. These maps for $n = 0, \dots, N$ give a map from $Gr^N(W^\mu)$ to M^μ. It is clear that this map is a graded $A^N(V)$-module map. If for $0 \leq n \leq N$, the image $w^\mu + \Omega_{n-1}(W)$ of $w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu)$ under this map is 0 in M^μ, then $w^\mu \in \Omega_{n-1}(W)$. But $w^\mu \in \Omega_n(W^\mu) \subset W^\mu$. So $w^\mu \in \Omega_{n-1}(W^\mu)$ and $w^\mu + \Omega_{n-1}(W^\mu)$ is 0 in $Gr^N(W^\mu)$. This means that this graded $A^N(V)$-module map is injective. In particular, the image of $Gr^N(W^\mu)$ under this map is a nonzero nondegenerate graded $A^N(V)$-submodule of M^μ. But M^μ is irreducible. So $Gr^N(W^\mu)$ must be equivalent to M^μ and is therefore also irreducible. From what we have proved above, since $Gr^N(W^\mu)$ is irreducible, W^μ is irreducible. Thus W is complete reducible. ∎

From Corollary 5.12 and Theorem 5.13, we obtain the following result:

Corollary 5.14 *Let V be a Möbius vertex algebra. Assume that $A_{N'}(V)$ for all $N' \in \mathbb{N}$ are finite dimensional (for example, when V is C_2-cofinite and of positive energy by Theorem 4.6). Let $N \in \mathbb{N}$ such that the differences between the real parts of the lowest weights of the finitely many (inequivalent) irreducible ordinary V-modules are less than or equal to N. Then a lower-bounded generalized V-module W of finite length or a grading-restricted generalized V-module W is a direct sum of irreducible ordinary V-modules if and only if the nondegenerate graded $A^N(V)$-module $Gr^N(W)$ is completely reducible.*

Proof. Since $A^{N'}(V)$ is finite dimensional, there are only finitely many (inequivalent) irreducible nondegenerate graded $A^{N'}(V)$-modules. By Theorem 5.6, there are finitely many irreducible lower-bounded generalized V-modules. By Corollary 5.12, these finitely many irreducible lower-bounded generalized V-modules are all irreducible ordinary V-modules. There exists $N \in \mathbb{N}$ such that the differences between the real parts of the lowest weights of the finitely many irreducible ordinary V-modules are less than or equal to N. For such N, the condition in Theorem 5.13 holds. So by Theorem 5.13, a lower-bounded generalized V-module W of finite length is a direct sum of irreducible ordinary V-modules if and only if $Gr^N(W)$ is completely reducible as a nondegenerate graded $A^N(V)$-module.

By Corollary 3.16 in [H1], every grading-restricted generalized V-module is of finite length. Thus the conclusion holds also for a grading-restricted generalized V-module W. ∎

Remark 5.15 Note that the assumption or condition on the lowest weights of irreducible V-modules in Proposition 5.11, Corollary 5.12, Theorem 5.13 and Corollary 5.14 can be weakened to the assumption or condition that the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized V-modules appearing as a quotient in a composition series of W are all less than or equal to $N \in \mathbb{N}$. This is because the proofs used only this weaker assumption or condition. For the study of some particular lower-bounded generalized V-modules of finite length or some grading-restricted generalized V-modules, this weaker assumption or condition is certainly easier to verify than the more general ones in the statements of these results.

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