Associative algebras and the representation theory of grading-restricted vertex algebras

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Abstract
We introduce an associative algebra $A^\infty(V)$ using infinite matrices with entries in a grading-restricted vertex algebra $V$ such that the associated graded space $Gr(W) = \coprod_{n \in \mathbb{N}} Gr_n(W)$ of a filtration of a lower-bounded generalized $V$-module $W$ is an $A^\infty(V)$-module satisfying additional properties (called a graded $A^\infty(V)$-module). We prove that a lower-bounded generalized $V$-module $W$ is irreducible or completely reducible if and only if the graded $A^\infty(V)$-module $Gr(W)$ is irreducible or completely reducible, respectively. We also prove that the set of equivalence classes of the lower-bounded generalized $V$-modules are in bijection with the set of the equivalence classes of graded $A^\infty(V)$-modules. For $N \in \mathbb{N}$, there is a subalgebra $A^N(V)$ of $A^\infty(V)$ such that the subspace $Gr^N(W) = \coprod_{n=0}^N Gr_n(W)$ of $Gr(W)$ is an $A^N(V)$-module satisfying additional properties (called a graded $A^N(V)$-module). We prove that $A^N(V)$ are finite dimensional when $V$ is of positive energy (CFT type) and $C_2$-cofinite. We prove that the set of the equivalence classes of lower-bounded generalized $V$-modules is in bijection with the set of the equivalence classes of graded $A^N(V)$-modules. In the case that $V$ is a Möbius vertex algebra and the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized $V$-modules are less than or equal to $N \in \mathbb{N}$, we prove that a lower-bounded generalized $V$-module $W$ of finite length is irreducible or completely reducible if and only if the graded $A^N(V)$-module $Gr^N(W)$ is irreducible or completely reducible, respectively.

1 Introduction

In the representation theory of Lie algebras, the universal enveloping algebra of a Lie algebra plays a crucial role because the module categories for a Lie algebra and for its universal enveloping algebra are isomorphic. For a vertex operator algebra, there is also a universal enveloping algebra introduced by Frenkel and Zhu [FZ] such that the module categories for these algebras are isomorphic. Unfortunately, the universal enveloping algebra of a vertex operator algebra is not very useful since it involves certain infinite sums in a suitable topological completion of the tensor algebra of the algebra. On the other hand, the classes of modules that we are interested in the representation theory of vertex operator algebras and two-dimensional conformal field theory do not involve such infinite sums since the vertex operators on these modules are lower truncated when acting on elements of these modules.
Instead, in the representation theory of vertex operator algebras, we have the Zhu algebra $A(V)$ introduced by Zhu in [Z] and its generalizations $A_n(V)$ for $n \in \mathbb{N}$ by Dong, Li and Mason in [DLM] for a vertex operator algebra $V$. These algebras can be used to classify irreducible modules for the vertex operator algebra and to study problems related to different types of modules. But compared with the universal enveloping algebra of a Lie algebra, the role of these associative algebras played in the representation theory of vertex operator algebras is quite limited. For example, the module for one of these associative algebras obtained from a suitable $V$-module in general do not tell us whether the original $V$-module is irreducible or completely reducible.

In the present paper, we introduce an associative algebra $A^\infty(V)$ using infinite matrices with entries in a grading-restricted vertex algebra $V$. The associated graded space $Gr(W) = \bigoplus_{n \in \mathbb{N}} Gr_n(W)$ of a filtration of a lower-bounded generalized $V$-module $W$ is an $A^\infty(V)$-module with an $\mathbb{N}$-grading and some operators having suitable properties (called a graded $A^\infty(V)$-module). In fact, the algebra $A^\infty(V)$ is defined using the associated graded spaces of all lower-bounded generalized $V$-modules. We prove that a lower-bounded generalized $V$-module $W$ is irreducible or completely reducible if and only if the graded $A^\infty(V)$-module $Gr(W)$ is irreducible or completely reducible, respectively. We also prove that the set of the equivalence classes of irreducible lower-bounded generalized $V$-modules is in bijection with the set of the equivalence classes of irreducible graded $A^\infty(V)$-modules.

We show that $A(V)$ in [Z] and $A_n(V)$ [DLM] mentioned above are isomorphic to very special subalgebras of $A^\infty(V)$. This fact gives a conceptual explanation of the role that these associative algebras played in the representation theory of vertex operator algebras.

We then introduce new subalgebras $A^N(V)$ of $A^\infty(V)$ for $N \in \mathbb{N}$. These subalgebras can also be obtained using finite matrices with entries in $V$. In the case that $V$ is of positive energy (or CFT type) and $C_2$-cofinite, we prove that $A^N(V)$ are finite dimensional. The subspace $Gr^N(W) = \bigoplus_{n=0}^{N} Gr_n(W)$ of $Gr(W)$ of a lower-bounded generalized $V$-module $W$ is an $A^N(V)$-module with some operators having suitable properties (called a graded $A^N(V)$-module). We prove that if a lower-bounded generalized $V$-module $W$ is irreducible or completely reducible, then the graded $A^N(V)$-module $Gr^N(W)$ is irreducible or completely reducible, respectively. We also prove that the set of the equivalence classes of irreducible lower-bounded generalized $V$-modules is in bijection with the set of the equivalence classes of irreducible graded $A^N(V)$-modules.

In the case that $V$ is a Möbius vertex algebra so that a lowest weight of a lower-bounded generalized $V$-module is defined, under the assumption that the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized $V$-modules are less than or equal to $N \in \mathbb{N}$, we prove that a lower-bounded generalized $V$-module $W$ of finite length is irreducible or completely reducible if and only if the graded $A^N(V)$-module $Gr^N(W)$ is irreducible or completely reducible, respectively. When $A^N(V)$ for all $N \in \mathbb{N}$ are finite dimensional (for example, when $V$ is of positive energy (or CFT type) and $C_2$-cofinite), we prove that an irreducible lower-bounded generalized $V$-module is an ordinary $V$-module and thus every lower-bounded generalized $V$-module of finite length is grading-restricted. In this case, under the assumptions above on $V$, lowest weights and $N$, a lower-bounded
generalized $V$-module $W$ of finite length or a grading-restricted generalized $V$-module $W$ is a direct sum of irreducible ordinary $V$-modules if and only if the graded $A^N(V)$-module $Gr^N(W)$ is completely reducible.

Many of the main results mentioned above need the construction of universal lower-bounded generalized $V$-modules in [H3] and some results from [H4].

The category of lower-bounded generalized $V$-modules and the category of graded $A^\infty(V)$-modules are not equivalent because of morphisms, but they are “almost” equivalent. We shall study the relations between these categories, the category of lower-bounded generalized $V$-modules of finite lengths and the categories of graded $A^N(V)$-modules for $N \in \mathbb{N}$ in another paper.

This paper is organized as follows: In the next section, we introduce the associative algebra $A^\infty(V)$ associated to a grading-restricted vertex algebra $V$ and prove that the associated graded space $Gr(W)$ of a filtration of a lower-bounded generalized $V$-module $W$ is an $A^\infty(V)$-module. In Section 3, we introduce graded $A^\infty(V)$-modules and prove the results mentioned above on the relations between lower-bounded generalized $V$-modules and graded $A^\infty(V)$-modules. We show that the Zhu algebra and their generalizations by Dong, Li and Mason are isomorphic to subalgebras of $A^\infty(V)$ in Subsection 4.1 and introduce the new subalgebras $A^N(V)$ of $A^\infty(V)$ for $N \in \mathbb{N}$ in Subsection 4.2. We also prove in Subsection 4.2 that when $V$ is of positive energy and $C_2$-cofinite, $A^N(V)$ for $N \in \mathbb{N}$ are finite dimensional. In Section 5, we introduce graded $A^N(V)$-modules and prove the results mentioned above on the relations between lower-bounded generalized $V$-modules, lower-bounded generalized $V$-modules of finite lengths and graded $A^N(V)$-modules.

2 Associative algebra $A^\infty(V)$ and modules

In this paper, we fix a grading-restricted vertex algebra $V$. Most of the constructions and results work and hold for more general vertex algebras, for example, lower-bounded vertex algebras or superalgebras. The constructions and results certainly work and hold for a Möbius vertex algebra or a vertex operator algebra. For some results in Section 5, we shall assume that $V$ is a Möbius vertex algebra.

Let $U^\infty(\mathbb{C})$ be the space of all column-finite infinite matrices with entries in $\mathbb{C}$, but doubly indexed by $\mathbb{N}$ instead of $\mathbb{Z}_+$. In other words, $U^\infty(\mathbb{C})$ is the space of all infinite matrices of the form $[a_{kl}]$ for $a_{kl} \in \mathbb{C}$, $k, l \in \mathbb{N}$ such that for each fixed $l \in \mathbb{N}$, there are only finitely many nonzero $a_{kl}$. Let $I^\infty = [\delta_{kl}]$ be the infinite identity matrix. Then $U^\infty(\mathbb{C})$ is in fact an associative algebra with the identity $I^\infty$. The space $U^\infty(\mathbb{C})$ has a set of linearly independent elements of the form $E_{kl}$ for $k, l \in \mathbb{N}$ with the only nonzero entry being the one in the $k$-th row and $l$-th column. These infinite matrices do not form a basis of $U^\infty(\mathbb{C})$ but form a basis of the subspace $U^\infty_0(\mathbb{C})$ of $U^\infty(\mathbb{C})$ consisting of finitary matrices (matrices with only finitely many nonzero entries). In particular,

$$U^\infty_0(\mathbb{C}) = \coprod_{k,l \in \mathbb{N}} \mathbb{C}E_{kl}.$$
Moreover,

\[ U^\infty(\mathbb{C}) \subset \prod_{k,l \in \mathbb{N}} \mathbb{C}E_{kl}, \]

where \( \prod_{k,l \in \mathbb{N}} \mathbb{C}E_{kl} \) is the algebraic completion of \( U_0^\infty(\mathbb{C}) \) viewed as a graded space. Though elements of \( U^\infty(\mathbb{C}) \) are infinite linear combinations of \( E_{kl} \) for \( k, l \in \mathbb{N} \), any binary product on \( U^\infty(\mathbb{C}) \) satisfying the distribution axioms is still determined completely by the product of \( E_{kl} \) for \( k, l \in \mathbb{N} \). For example, for the usual matrix product, we know that \( E_{kl}E_{mn} = \delta_{lm}E_{kn} \) for \( k, l, m, n \in \mathbb{N} \). Let \( A = \sum_{k,l \in \mathbb{N}} a_{kl}E_{kl} \) and \( B = \sum_{k,l \in \mathbb{N}} b_{kl}E_{kl} \), where \( a_{kl}, b_{kl} \in \mathbb{C} \) for \( k, l \in \mathbb{N} \). Then

\[ AB = \left( \sum_{k,l \in \mathbb{N}} a_{kl}E_{kl} \right) \left( \sum_{m,n \in \mathbb{N}} b_{mn}E_{mn} \right) = \sum_{k,n \in \mathbb{N}} \left( \sum_{m \in \mathbb{N}} a_{km}b_{mn} \right) E_{kn}. \]

So even though \( E_{kl} \) for \( k, l \in \mathbb{N} \) do not form a basis of \( U^\infty(\mathbb{C}) \), all the properties of the associative algebra structure on \( U^\infty(\mathbb{C}) \) can still be derived from the properties these matrices. Thus we can study \( U^\infty(\mathbb{C}) \) using \( E_{kl} \) for \( k, l \in \mathbb{N}, k \leq l \). Also what we are mainly interested is the subalgebra \( \mathbb{C}I^\infty \oplus U_0^\infty(\mathbb{C}) \) of \( U^\infty(\mathbb{C}) \). This subalgebra has a basis \( \{I^\infty\} \cup \{E_{kl}\}_{k,l \in \mathbb{N}} \).

Let \( U^\infty(V) = V \otimes U^\infty(\mathbb{C}) \). Then \( U^\infty(V) \) is the space of column-finite infinite matrices with entries in \( V \), but doubly indexed by \( \mathbb{N} \) instead of \( \mathbb{Z}_+ \). Elements of \( U^\infty(V) \) are of the form \( v = [v_{kl}] \) for \( v_{kl} \in V, k, l \in \mathbb{N} \) such that for each fixed \( l \in \mathbb{N} \), there are only finitely many nonzero \( v_{kl} \). Let \( U_0^\infty(V) \) be the subspace of \( U^\infty(V) \) spanned by elements of the form \( v \otimes E_{kl} \) for \( v \in V \) and \( k, l \in \mathbb{N} \). Then

\[ U_0^\infty(V) = \prod_{k,l \in \mathbb{N}} V \otimes \mathbb{C}E_{kl} \]

and

\[ U^\infty(V) \subset \prod_{k,l \in \mathbb{N}} V \otimes \mathbb{C}E_{kl}. \]

We shall denote \( v \otimes E_{kl} \) simply by \( [v]_{kl} \). Then elements of \( U^\infty(\mathbb{C}) \) can all be written as

\[ \sum_{k,l \in \mathbb{N}} [v_{kl}]_{kl} \]

for \( v_{kl} \in V, k, l \in \mathbb{N} \). As in the case of \( U^\infty(\mathbb{C}) \), we can study any binary product on \( U^\infty(V) \) satisfying the distribution axioms using \([v]_{kl} \) for \( v \in V \) and \( k, l \in \mathbb{N} \). We are also mainly interested in the subspace \( V \otimes I^\infty \oplus U_0^\infty(V) \) of \( U^\infty(V) \). This subspace is spanned by elements of the form \( v \otimes I^\infty \) and \([v]_{kl} \) for \( v \in V \) and \( k, l \in \mathbb{N} \). Because of this reason, though we might give definitions of products and related notions using general elements of \( U^\infty(V) \), we shall study them using only \([v]_{kl} \) for \( v \in V \) and \( k, l \in \mathbb{N} \).

We also need some particular formal series and polynomials. In this paper, we shall use the convention that a complex power or the integral power of the logarithm of an ordered linear combination of formal variables and a complex number, always means its expansion
in nonnegative powers of the formal variables or the complex number that are not the first one in the ordered linear combination. For example, \((x + 1)^{-k-1}\) for \(k \in \mathbb{N}\) and \((1 + x)^n\) for \(n \in \mathbb{N}\) mean the expansions in nonnegative powers of 1 and in nonnegative powers of \(x\), respectively. For \(k, n, l \in \mathbb{N}\), we have

\[
(x + 1)^{-k+n-l-1} = \sum_{m \in \mathbb{N}} \binom{-k+n-l-1}{m} x^{-k+n-l-m-1} = T_{k+l+1}((x + 1)^{-k+n-l-1}) + R_{k+l+1}((x + 1)^{-k+n-l-1}), \tag{2.1}
\]

where

\[
T_{k+l+1}((x + 1)^{-k+n-l-1}) = \sum_{m=0}^{n} \binom{-k+n-l-1}{m} x^{-k+n-l-m-1}
\]

is the Taylor polynomial in \(x^{-1}\) of order \(k + l + 1\) of \((x + 1)^{-k+n-l-1}\) and

\[
R_{k+l+1}((x + 1)^{-k+n-l-1}) = \sum_{m \in n+1+\mathbb{N}} \binom{-k+n-l-1}{m} x^{-k+n-l-m-1}
\]

is the remainder of order \(k + l + 1\).

We define a product \(\diamond\) on \(U^\infty(V)\) by

\[
\mathbf{u} \diamond \mathbf{v} = [\mathbf{u} \diamond \mathbf{v}]_{kl}
\]

for \(\mathbf{u} = [u_{kl}], \mathbf{v} = [v_{kl}] \in U^\infty(V)\), where

\[
(\mathbf{u} \diamond \mathbf{v})_{kl} = \sum_{n=k}^{l} \text{Res}_x T_{k+l+1}((x + 1)^{-k+n-l-1})(1 + x)^l Y_V((1 + x)^L\text{v}_{0}\mathbf{u}_{kn}, x) v_{nl}
\]

\[
= \sum_{n=k}^{l} \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \text{Res}_x x^{-k+n-l-m-1}(1 + x)^l Y_V((1 + x)^L\text{v}_{0}\mathbf{u}_{kn}, x) v_{nl} \tag{2.2}
\]

for \(k, l \in \mathbb{N}\). Then \(U^\infty(V)\) equipped with \(\diamond\) is an algebra but in general is not even associative. Let \(O^\infty(V)\) be the subspace of \(U^\infty(V)\) spanned by elements of the form

\[
\sum_{n=k}^{l} \text{Res}_x x^{-k-l-p-2}(1 + x)^l \left[Y_V((1 + x)^L\text{v}_{0}\mathbf{u}_{kn}, x) v_{nl}\right]
\]

for \(\mathbf{u} = [u_{kl}], \mathbf{v} = [v_{kl}] \in U^\infty(V), p \in \mathbb{N}\) and elements of the form

\[
[(L\text{v}_{-1} + L\text{v}_{0} + l - k)v_{kl}]
\]

for \(\mathbf{v} = [v_{kl}] \in U^\infty(V)\).

The product \(\diamond\) on \(U^\infty(V)\) looks complicated. But as we mentioned above, though \([v]_{kl}\) for \(v \in V\) and \(k, l \in \mathbb{N}\) does span \(U^\infty(V)\), their infinite linear combinations give all the elements
of $U^\infty(V)$ and $U^\infty(V)$ can be studied using these elements. In particular, the product $\diamond$ can be studied using these elements. So instead of working with arbitrary matrices in $U^\infty(V)$, we use $[v]_{kl}$ for $v \in V$ and $k, l \in \mathbb{N}$ to write down $\diamond$. For $u, v \in V$ and $k, m, n, l \in \mathbb{N}$, by definition,
\[ [u]_{km} \diamond [v]_{nl} = 0 \]
when $m \neq n$ and
\[ [u]_{kn} \diamond [v]_{nl} = \text{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l [Y_V((1+x)^{L_V(0)}u,x)v]_{kl} \]
\[ = \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \text{Res}_x x^{-k+n-l-m-1}(1+x)^l [Y_V((1+x)^{L_V(0)}u,x)v]_{kl}. \]
(2.3)

Since $[u]_{km} \diamond [v]_{nl} = 0$ when $m \neq n$, we need only consider $[u]_{kn} \diamond [v]_{nl}$ for $u, v \in V$ and $k, n, l \in \mathbb{N}$. By taking $u = [u]_{kn}$ and $v = [v]_{nl}$, we see also that the subspace $O^\infty(V)$ is spanned by infinite linear combinations of elements of the form
\[ \text{Res}_x x^{-k-l-p-2}(1+x)^l [Y_V((1+x)^{L_V(0)}u,x)v]_{kl} \]
for $u, v \in V, k, l, p \in \mathbb{N}$ and elements of the form
\[ [(L_V(-1) + L_V(0) + l - k)v]_{kl} \]
for $v \in V$ and $k, l \in \mathbb{N}$, with each pair $(k, l)$ appearing in the linear combinations only finitely many times.

Let $1^\infty$ be the element of $U^\infty(V)$ with diagonal entries being $1 \in V$ and all the other entries being 0. Then $1^\infty = 1 \otimes 1^\infty$.

We shall take a quotient of $U^\infty(V)$ such that the quotient with the product induced from $\diamond$ is an associative algebra and such that the associated graded space of a filtration of every lower-bounded generalized $V$-module is a module for this associative algebra. To do this, we need to first give an action of the (nonassociative) algebra $U^\infty(V)$ with the product $\diamond$ on a lower-bounded generalized $V$-module.

We briefly recall the notion of lower-bounded generalized $V$-module. We refer the reader to Definition 1.2 in [H1], where a lower-bounded generalized $V$-module is called a lower-truncated generalized $V$-module. Definition 1.2 in [H1] is for a vertex operator algebra $V$ but the definition applies also to a grading-restricted vertex algebra except that we have to require the existence of operators $L_W(0)$ and $L_W(-1)$ satisfying the same axioms for the corresponding operators coming from the vertex operator of the conformal element of a vertex operator algebra. We also refer the reader to Definition 3.1 in [H2] for this notion in the special case that $V$ is a grading-restricted vertex algebra (not a superalgebra) and the automorphism of $V$ is $1_V$. Roughly speaking, a lower-bounded generalized $V$-module is a $\mathbb{C}$-graded vector space $W = \bigsqcup_{n \in \mathbb{C}} W[n]$ equipped with a vertex operator map $Y_W : V \otimes W \rightarrow W[[x, x^{-1}]]$ and operators $L_W(0)$ and $L_W(-1)$ on $W$ satisfying all the axioms for an (ordinary) $V$-module except that for $n \in \mathbb{C}$, $W[n]$ does not have to be finite dimensional.
and is the generalized eigenspace with the eigenvalue \( n \) of \( L_W(0) \) instead of the eigenspace with the eigenvalue \( n \) of \( L_W(0) \). Module maps between lower-bounded generalized \( V \)-modules are defined in the obvious way as in Definition 1.1 in [H1], not those defined in Definition 4.2 in [H4]. On the other hand, if we replace \( V \)-module maps in the results below by those in Definition 4.2 in [H4], these results still hold. The notion of generalized \( V \)-submodule of a lower-bounded generalized \( V \)-module is defined in the obvious way. A generalized \( V \)-submodule of a lower-bounded generalized \( V \)-module is certainly also lower bounded.

Let \( W \) be a lower-bounded generalized \( V \)-module. For \( n \in \mathbb{N} \), let

\[
\Omega_n(W) = \{ w \in W \mid (Y_W)_k(v)w = 0 \text{ for homogeneous } v \in V, \text{wt } v - k - 1 < -n \}.
\]

Then

\[
\Omega_{n_1}(W) \subset \Omega_{n_2}(W)
\]

for \( n_1 \leq n_2 \) and

\[
W = \bigcup_{n \in \mathbb{N}} \Omega_n(W).
\]

So \( \{ \Omega_n(w) \}_{n \in \mathbb{N}} \) is an ascending filtration of \( W \). Let

\[
Gr(W) = \sum_{n \in \mathbb{N}} Gr_n(W)
\]

be the associated graded space, where

\[
Gr_n(W) = \Omega_n(W)/\Omega_{n-1}(W).
\]

Sometimes we shall use \([w]_n\) to denote the element \( w + \Omega_{n-1}(W) \) of \( Gr_n(W) \), where \( w \in \Omega_n(W) \).

**Lemma 2.1** For \( w \in \Omega_n(W) \) and \( l \in \mathbb{N} \), \( \text{Res}_{x_2}x^{l-1}Y_W(x^{L_V(0)}v, x)w \in \Omega_{n-l}(W) \).

**Proof.** The operator \( \text{Res}_{x_2}x^{l-1}Y_W(x_2^{L_V(0)}v, x_2) \) has weight \(-l\). Then for homogeneous \( u \in V \), \( (Y_W)_p(u)\text{Res}_{x_2}x^{l-1}Y_W(x_2^{L_V(0)}v, x_2) \) has weight \( u - p - 1 - l \). Consider the generalized \( V \)-submodule of \( W \) generated by \( w \). Then \((Y_W)_p(u)\text{Res}_{x_2}x^{l-1}Y_W(x_2^{L_V(0)}v, x_2)w \) is in this generalized \( V \)-submodule. Using the associativity for \( Y_W \), we know that the generalized \( V \)-submodule generated by \( w \) is spanned by elements of the form \((Y_W)_m(\tilde{u})w \) for \( \tilde{u} \in V \). So \((Y_W)_p(u)\text{Res}_{x_2}x^{l-1}Y_W(x_2^{L_V(0)}v, x_2)w \) is a linear combination of such elements. But for homogeneous \( w \), the weight of \((Y_W)_p(u)\text{Res}_{x_2}x^{l-1}Y_W(x_2^{L_V(0)}v, x_2)w \) is \( wt u - p - 1 - l + wt w \). So the elements of the form \((Y_W)_m(\tilde{u})w \) whose linear combination is \((Y_W)_p(u)\text{Res}_{x_2}x^{l-1}Y_W(x_2^{L_V(0)}v, x_2)w \) can also be chosen to be of weight \( wt u - p - 1 - l + wt w \), that is, the weight \( wt \tilde{u} - m - 1 \) of \((Y_W)_m(\tilde{u})w \) is equal to \( wt u - p - 1 - l \). Since \( w \in \Omega_n(W) \), \((Y_W)_m(\tilde{u})w = 0 \) when \( wt \tilde{u} - m - 1 < -n \), or equivalently, \( wt u - p - 1 < -(n - l) \). So we have proved that \((Y_W)_p(u)\text{Res}_{x_2}x^{l-1}Y_W(x_2^{L_V(0)}v, x_2)w = 0 \) when \( wt u - p - 1 < n - l \). This means that \( \text{Res}_{x_2}x^{l-1}Y_W(x^{L_V(0)}v, x)w \in \Omega_{n-l}(W) \). 

\[ \]
By Lemma 2.1, the operator \( \text{Res}_x x^{l-1} Y_W (x^{L_V(0)} v, x) \) in fact induces an operator, still denoted by the same notation, on \( Gr(W) \), which maps \( Gr_n(W) \) to \( Gr_{n-l}(W) \).

For \( \mathbf{v} = [v_{kl}] \in U^\infty(V) \), where \( v_{kl} \in V \) and \( k, l \in \mathbb{N} \), we define an operator \( \vartheta_{Gr(W)}(\mathbf{v}) \) on \( Gr(W) \) as follows: For \( \mathbf{w} \in Gr(W) \), we define

\[
\vartheta_{Gr(W)}(\mathbf{v})\mathbf{w} = \sum_{k,l \in \mathbb{N}} \text{Res}_x x^{l-k-1} Y_W (x^{L_V(0)} v_{kl}, x) \pi_{Gr_l(W)} \mathbf{w},
\]

where \( \pi_{Gr_l(W)} \) is the projection from \( Gr(W) \) to \( Gr_l(W) \). Note that since \( \mathbf{w} \) is a sum of elements of \( Gr_l(W) \) for finitely many \( l \in \mathbb{N} \) and for each \( l \), there are only finitely many nonzero \( v_{kl} \), the sum over \( k \) and \( l \) is finite. So \( \vartheta_{Gr(W)}(\mathbf{v})\mathbf{w} \) is indeed a well defined element of \( Gr(W) \). In the case \( \mathbf{v} = [v]_{kl} \) and \( \mathbf{w} = [w]_n \) for \( v \in V \), \( w \in W \) and \( k, l, n \in \mathbb{N} \), we have

\[
\vartheta_{Gr(W)}([v]_{kl})[w]_n = \delta_{ln} [\text{Res}_x x^{l-k-1} Y_W (x^{L_V(0)} v, x) w]_k.
\]

In the case that \( v \) is homogeneous and \( w \in Gr_l(W) \), we have

\[
\vartheta_{Gr(W)}([v]_{kl})[w]_l = [(Y_W)_{\omega t v+l-k-1}(v) w]_k.
\]

We now have a linear map

\[
\vartheta_{Gr(W)} : U^\infty(V) \to \text{End} \ Gr(W)
\]

\[
v \mapsto \vartheta_{Gr(W)}(v).
\]

Let \( Q^\infty(V) \) be the intersection of \( \ker \vartheta_{Gr(W)} \) for all lower-bounded generalized \( V \)-modules \( W \) and \( A^\infty(V) = U^\infty(V)/Q^\infty(V) \).

We shall need the following lemma:

**Lemma 2.2** For \( l \in \mathbb{Z} \), \( k \in \mathbb{N} \) and \( m \in \mathbb{Z}_+ \), and \( v \in V \),

\[
\text{Res}_x x^{l-k-1} Y_W \left( x^{L_V(0)} \left( L_V(-1) + L_V(0) + l \right) \right) v, x \right) = 0.
\]

In particular, when \( k = 0 \) and \( m = 1 \), we have

\[
\text{Res}_x x^{l-1} Y_W \left( x^{L_V(0)} (L_V(-1) + L_V(0) + l) v, x \right) = 0.
\]

For \( l \in \mathbb{Z} \) and \( v \in V \),

\[
\text{Res}_x x^{l-k-1} Y_W \left( x^{L_V(0)} \left( L_V(-1) + L_V(0) + l \right) \right) v, x \right) = \text{Res}_x x^{l-k-1} Y_W \left( x^{L_V(0)+l} v, x \right) \right) = \text{Res}_x x^{l-k-1} Y_W \left( x^{L_V(0)+l} v, x \right)\right).
\]

**Proof.** For \( l \in \mathbb{Z} \), \( n \in \mathbb{N} \) and \( v \in V \), using the \( L(-1) \)-derivative property for the vertex operator map \( Y_W \) repeatedly, we have

\[
\frac{1}{n!} \frac{d^n}{dx^n} Y_W \left( x^{L_V(0)+l} v, x \right) = Y_W \left( x^{L_V(0)+l} \left( \frac{L_V(-1) + L_V(0) + l}{n} \right) v, x \right)\right).
\]
Multiplying $x^p$ to both sides and then taking Res$_x$, we obtain
\[
\text{Res}_x \frac{d^n}{dx^n} Y_W(x^{L_V(0)+l}v, x) = \text{Res}_x x^{l-n+p} Y_W \left( x^{L_V(0)} \left( \frac{L_V(-1) + L_V(0) + l}{n} \right)_v, x \right). \quad (2.10)
\]

When $0 \leq p \leq n - 1$, the left-hand side of (2.10) is 0. Thus we obtain
\[
\text{Res}_x x^{l-n+p} Y_W \left( x^{L_V(0)} \left( \frac{L_V(-1) + L_V(0) + l}{n} \right)_v, x \right) = 0. \quad (2.11)
\]

Let $n = k + m$ and $p = m - 1$ in (2.11) for $k \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then we obtain (2.6).

Let $p = -1$ and $n = k$ in (2.10), we obtain
\[
\text{Res}_x x^{-1} \frac{d^k}{dx^k} Y_W(x^{L_V(0)+l}v, x) = \text{Res}_x x^{l-k-1} Y_W \left( x^{L_V(0)} \left( \frac{L_V(-1) + L_V(0) + l}{k} \right)_v, x \right). \quad (2.12)
\]

Since left-hand side of (2.12) is equal to
\[
\text{Res}_x x^{l-k-1} Y_W(x^{L_V(0)+l}v, x),
\]
we obtain (2.8).

**Proposition 2.3** We have $O^\infty(V) \subset Q^\infty(V)$.

**Proof.** We need to prove $\vartheta_{Gr(W)}(O^\infty(V)) = 0$ for every lower-bounded generalized $V$-module $W$. For
\[
\text{Res}_{x_0} x_0^{-k-l-p-2}(1 + x_0)^l [Y_V((1 + x_0)^{L(0)}v_1, x_0)v_2]_{kl} \in O^\infty(V),
\]
where \( v_1, v_2 \in V, \leq k \leq n \leq l \leq N \) and \( p \in \mathbb{N} \), and \( w \in \Omega_t(W) \), we have

\[
\vartheta_{Gr(W)}(\text{Res}_{x_0} x_0^{-k-l-p-2}(1 + x_0)^{l} [Y_V((1 + x_0)^{L(0)}v_1, x_0)v_2]_l)[w]_l
\]

\[= \text{Res}_{x_2} x_2^{-l-k-1} \text{Res}_{x_0} x_0^{-k-l-p-2}(1 + x_0)^{l} [Y_V(x_2^{L(0)}(1 + x_0)^{L(0)}v_1, x_0)v_2, x_2)\pi_{G_t(W)}(w)]_k \]

\[= \text{Res}_{x_0} \text{Res}_{x_2} x_0^{-k-l-p-2}x_2^{-l-k-1} \cdot [Y_V(Y_V(x_2^{L(0)}(1 + x_0)^{L(0)}v_1, x_0)v_2, x_2)w]_k \]

\[= \text{Res}_{x_0} \text{Res}_{x_2} x_0^{-k-l-p-2}x_2^{-l-k-1} \text{Res}_{x_1} x_1^{-1} \delta \left( \frac{x_2 + x_0x_2}{x_1} \right) \cdot [Y_V(Y_V(x_1^{L(0)}v_1, x_0)x_2^{L(0)}v_2, x_2)w]_k \]

\[= \text{Res}_{x_0} \text{Res}_{x_2} x_0^{-k-l-p-2}x_2^{-l-k-1} \text{Res}_{x_1} x_0^{-1}x_2^{-1} \delta \left( \frac{x_1 - x_2}{x_0x_2} \right) \cdot [Y_V(Y_V(x_1^{L(0)}v_1, x_1)v_2, x_2)w]_k \]

\[= \text{Res}_{x_0} \text{Res}_{x_2} (-1 + x_1x_2^{-1})^{-2k-p-2}x_2^{-l-k-1} [Y_V(x_2^{L(0)}v_2, x_2)Y_V(x_1^{L(0)}v_1, x_1)w]_k. \]

\[(2.13)\]

Since \( w \in \Omega_t(W) \) and the series \((1 - x_1^{-1}x_2)^{-k-l-p-2}\) contains only nonnegative powers of \( x_2 \),

\[
\text{Res}_{x_2} (1 - x_1^{-1}x_2)^{-k-l-p-2}x_2^{l+p}Y_W(x_2^{L(0)}v_2, x_2)w = 0.
\]

So the first term in the right-hand side of (2.13) is 0. Since \( w \in \Omega_t(W) \) and the series \((-1 + x_1x_2^{-1})^{-2k-p-2}\) contains only nonnegative powers of \( x_1 \),

\[
\text{Res}_{x_1} (-1 + x_1x_2^{-1})^{-2k-p-2}x_1x_2^{-k-2} [Y_W(x_2^{L(0)}v_2, x_2)Y_W(x_1^{L(0)}v_1, x_1)w]_k = 0.
\]

So the second term in the right-hand side of (2.13) is also 0.

Taking \( l \) in (2.7) to be \( l - k \), we obtain

\[
\vartheta_{Gr(W)}([(L_V(-1) + L_V(0) + l - k)v]_{kl})[w]_l
\]

\[= [\text{Res}_{x} x^{-l-k-1}Y_W(x^{L_V(0)}(L_V(-1) + L_V(0) + l - k)v, x)w]_k \]

\[= 0 \]

\[(2.14)\]

for \( v \in V, k, l \in \mathbb{N} \) and \( w \in \Omega_t(W) \). Thus we have \( \vartheta_{Gr(W)}(O^\infty(V)) = 0 \).

\[\Box\]

**Theorem 2.4** Let \( W \) be a lower-bounded generalized \( V \)-module. Then the linear map

\[
\vartheta_{Gr(W)} : U^\infty(V) \rightarrow \text{End} \ Gr(W)
\]
gives a $U^\infty(V)$-module structure on $\text{Gr}(W)$ (that is, $\vartheta_{\text{Gr}(W)}$ is a homomorphism of (nonassociative) algebras from $U^\infty(V)$ to $\text{End} \text{Gr}(W)$). In particular, $U^\infty(V)/\ker \vartheta_{\text{Gr}(W)}$ is an associative algebra isomorphic to a subalgebra of $\text{End} \text{Gr}(W)$.

**Proof.** For $u, v \in V$, $k, n, l \in \mathbb{N}$ and $w \in \Omega_i(W)$, using (2.3), we have

$$\vartheta_{\text{Gr}(W)}([u]_{kn} \cdot [v]_{nl})[w] = \text{Res}_{x_0} T_{k+l+1}((x_0 + 1)^{-k-n-l-1}) (1 + x_0)^j \text{Res}_{x_2} x_2^{l-k-1}. $$

$$= \text{Res}_{x_0} \text{Res}_{x_2} T_{k+l+1}((x_0 + 1)^{-k-n-l-1}) (1 + x_0)^j x_2^{l-k-1}. $$

$$= \text{Res}_{x_0} \text{Res}_{x_2} T_{k+l+1}((x_0 + 1)^{-k-n-l-1}) x_2^{l-k-1}. $$

Since $w \in \Omega_i(W)$, the second term in the right-hand side of (2.15) is 0. Expanding $T_{k+l+1}((x_0 + 1)^{-k-n-l-1})$ explicitly, we see that the first term in the right-hand side of (2.15) is equal to

$$\sum_{m=0}^{n} \binom{-k + n - l - 1}{m} \text{Res}_{x_2} \text{Res}_{x_1} (x_1 - x_2)^{-k-n-l-m-1} x_2^{k-n+l+m+1} x_1^{l-k-2}.$$ 

$$= \sum_{m=0}^{n} \sum_{j \in \mathbb{N}} \binom{-k + n - l - 1}{m} \binom{-k + n - l - m - 1}{j} (-1)^j \text{Res}_{x_2} \text{Res}_{x_1} x_1^{-k+n-m-1-j} x_2^{n+l-m-1+j}.$$ 

(2.16)
In the case $j > n - m$, since $w \in \Omega_l(W)$,

$$wt_v - (wt_v - n + l + m - 1 + j) - 1 < -l$$

in the case that $v$ is homogeneous and hence we have

$$\text{Res}_{x_2} x_2^{-n+l+m+1+j} Y_{W}(x_{2}^{Lv(0)}v, x_2)w = 0.$$ 

Hence those terms in the right-hand side of (2.16) with $j > n - m$ is 0 So the right-hand side of (2.16) is equal to

$$\sum_{m=0}^{n} \sum_{j=0}^{n-m} \binom{-k + n - l - 1}{m} \binom{-k + n - l - m - 1}{p - m} (-1)^j.$$

$$\cdot \text{Res}_{x_2} \text{Res}_{x_1} x_1^{-k+n-m-j} x_2^{-n+l+m-1+j} [Y_{W}(x_{1}^{Lv(0)}u, x_1)Y_{W}(x_{2}^{Lv(0)}v, x_2)]_k$$

$$= \sum_{m=0}^{n} \sum_{p=m}^{n} \binom{-k + n - l - 1}{m} \binom{-k + n - l - m - 1}{p - m} (-1)^{p-m}.$$

$$\cdot \text{Res}_{x_2} \text{Res}_{x_1} x_1^{-k+n-1-p} x_2^{-n+l+1+p} [Y_{W}(x_{1}^{Lv(0)}u, x_1)Y_{W}(x_{2}^{Lv(0)}v, x_2)]_k$$

$$= \sum_{p=0}^{n} \binom{-k + n - l - 1}{m} \binom{-k + n - l - m - 1}{p - m} (-1)^{p-m}.$$ 

$$\cdot \text{Res}_{x_2} \text{Res}_{x_1} x_1^{-k+n-1-p} x_2^{-n+l+1+p} [Y_{W}(x_{1}^{Lv(0)}u, x_1)Y_{W}(x_{2}^{Lv(0)}v, x_2)]_k.$$ 

(2.17)

For $p = 0, \ldots, n$,

$$\sum_{m=0}^{p} \binom{-k + n - l - 1}{m} \binom{-k + n - l - m - 1}{p - m} (-1)^{p-m}$$

$$= \sum_{m=0}^{p} \frac{(-k + n - l - 1) \cdots (-k + n - l - m)}{m!} \frac{(-k + n - l - m - 1) \cdots (-k + n - l - p)}{(p - m)!} (-1)^{p-m}$$

$$= \sum_{m=0}^{p} \frac{(-k + n - l - 1) \cdots (-k + n - l - p)}{p!} \frac{m!}{m!(p - m)!} (-1)^{p-m}$$

$$= \frac{(-k + n - l - 1)}{p!} \sum_{m=0}^{p} \binom{p}{m} (-1)^{p-m}$$

$$= \frac{(-k + n - l - 1)}{p} (-1 + 1)^p$$

$$= \frac{(-k + n - l - 1)}{p} \delta_{p,0}.$$ 

(2.18)
Using (2.18), we see that the right-hand side of (2.17) is equal to

\[ \text{Res}_{x_2} \text{Res}_{x_1} x_1^{-k+n-1} x_2^{-n+l-1} [Y_W(x_1^{L_V(0)} u, x_1) Y_W(x_2^{L_V(0)} v, x_2)]_k \]
\[ = \vartheta_{Gr(W)}([u]_{kn}) [\text{Res}_{x_2} x_2^{-n-1} Y_W(x_2^{L_V(0)} v, x_2)]_n \]
\[ = \vartheta_{Gr(W)}([u]_{kn}) \vartheta_{Gr(W)}([v]_{nl}) \] (2.19).

From (2.15), (2.16), (2.17) and (2.19), we obtain
\[ \vartheta_{Gr(W)}([u]_{kn} \circ [v]_{nl}) = \vartheta_{Gr(W)}([u]_{kn}) \vartheta_{Gr(W)}([v]_{nl}) \]
for \( u, v \in V \) and \( k, n, l \in \mathbb{N} \). Thus \( \vartheta_{Gr(W)} \) gives an \( U^\infty(V) \)-module structure on \( W \).

Lemma 2.5 Let \( L_U(-1) \) and \( L_U(0) \) be operators on a vector space \( U \) satisfying
\[ [L_U(0), L_U(-1)] = L_U(-1). \]
We have
\[ e^{xL_U(-1)} (1 + x)^{L_U(0)} = (1 + x)^{L_U(-1)+L_U(0)}. \] (2.20)

Proof. This can be proved easily by showing
\[ \frac{d}{dx} e^{xL_U(-1)} (1 + x)^{L_U(0)} (1 + x)^{-L_U(-1)+L_U(0)} = 0 \]
so that it must be independent of \( x \) and then setting \( x = 0 \) to obtain
\[ e^{xL_U(-1)} (1 + x)^{L_U(0)} (1 + x)^{-L_U(-1)+L_U(0)} = 1_U. \]

We can now write down explicitly the expressions of elements of the form \( [v]_{kl} \diamond 1^\infty \) for \( v \in V \) and \( k, l \in \mathbb{N} \) satisfying \( k \leq l \).

Lemma 2.6 For \( v \in V \) and \( k, l \in \mathbb{N} \),
\[ [v]_{kl} \diamond 1^\infty = \sum_{m=0}^{l} \binom{-k-1}{m} \binom{L_U(-1) + L_U(0) + l}{k+m} v \] (2.21)

Proof. By the definition (2.2) of \( \diamond \) and the skew-symmetry of \( Y_V \),
\[ ([v]_{kl} \diamond 1^\infty)_{mn} = \delta_{km} \delta_{ln} \text{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^{n} Y_V((1+x)^{L_U(0)} v, x) 1 \]
\[ = \delta_{km} \delta_{ln} \text{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^{n} e^{xL_U(-1)}(1+x)^{L_U(0)} v. \] (2.22)

Thus we obtain
\[ [v]_{kl} \diamond 1^\infty = \text{Res}_x T_{k+l+1}((x+1)^{-k-1})(1+x)^l [e^{xL_U(-1)} (1 + x)^{L_U(0)} v]_{kl}. \] (2.23)
Using (2.20) with \( U = V \), expanding the formal series explicitly and then evaluating the formal residue, we see that the right-hand side of (2.23) is equal to

\[
\text{Res} \{ x \} T_{k+l+1}((x+1)^{-k-1})(1+x)^l[(1+x)^{L_V(0)+L_V(0)}v]_{kl}
\]

\[
= \sum_{m=0}^{l} \binom{-k-1}{m} \text{Res} \{ x \} x^{-k-m-1}[(1+x)^{L_V(-1)+L_V(0)+l}v]_{kl}
\]

\[
= \sum_{m=0}^{l} \sum_{j \in \mathbb{N}} \binom{-k-1}{m} \text{Res} \{ x \} x^{-k-m-1+j} \left( \frac{L_V(-1) + L_V(0) + l}{j} \right) v
\]

\[
= \sum_{m=0}^{l} \binom{-k-1}{m} \left( \frac{L_V(-1) + L_V(0) + k}{k+m} \right) v,
\]

(2.24)

proving (2.21).

\[\Box\]

**Proposition 2.7** For \( v \in V \) and \( k,l \in \mathbb{N} \), \([v]_{kl} \diamond 1^\infty - [v]_{kl} \in O^\infty(V)\). For \( u,v \in V \) and \( k,l,n \in \mathbb{N} \),

\([v]_{kl} \diamond 1^\infty - [v]_{kl} \diamond [u]_{ln} \in O^\infty(V)\).

**Proof.** For \( m \in \mathbb{N} \),

\[
\left( \frac{L_V(-1) + L_V(0) + l}{k+m} \right) v = \left( \frac{(L_V(-1) + L_V(0) + l - k) + k}{k+m} \right) v
\]

\[
= \binom{k}{k+m} v + (L_V(-1) + L_V(0) + l - k) \tilde{v}_m
\]

\[
\equiv \begin{cases} 
0 & m \in \mathbb{Z}_+^+ \\
\text{mod } O^\infty(V),
\end{cases}
\]

(2.25)

where \( \tilde{v}_m \) is an element of \( V \) depending on \( m \). Thus by (2.21),

\([v]_{kl} \diamond 1^\infty \equiv v \mod O^\infty(V)\).

By (2.21), (2.25) and (2.30),

\[
([v]_{kl} \diamond 1^\infty) \diamond [u]_{ln} = \sum_{m=0}^{l} \binom{-k-1}{m} \left[ \left( \frac{L_V(-1) + L_V(0) + l}{k+m} \right) v \right]_{kl} \diamond [u]_{ln}
\]

\[
= \sum_{m=0}^{l} \binom{-k-1}{m} \binom{k}{k+m} [v]_{kl} \diamond [u]_{ln}
\]

\[
+ \sum_{m=0}^{l} \binom{-k-1}{m} \left( \frac{(L_V(-1) + L_V(0) + l - k) \tilde{v}_m}{k+m} \right)_{kl} \diamond [u]_{ln}
\]

\[
\equiv [v]_{kl} \diamond [u]_{ln} \mod O^\infty(V).
\]

\[\Box\]
Theorem 2.8 The product \( \odot \) on \( U^\infty(V) \) induces a product, denoted still by \( \odot \), on \( A^\infty(V) = U^\infty(V)/Q^\infty(V) \) such that \( A^\infty(V) \) equipped with \( \odot \) is an associative algebra with \( 1^\infty + Q^\infty(V) \) as identity. Moreover, the associated graded space \( \text{Gr}(W) \) of the ascendant filtration \( \{\Omega_n(W)\}_{n \in \mathbb{N}} \) of a lower-bounded generalized \( V \)-module \( W \) is an \( A^\infty(V) \)-module.

Proof. Since \( \ker \vartheta_{\text{Gr}(W)} \) for a lower-bounded generalized \( V \)-module \( W \) is a two-sided ideal of \( U^\infty(V) \), \( Q^\infty(V) \) as the intersection of such two-sided ideals is still a two-sided ideal of \( U^\infty(V) \). Thus \( \odot \) on \( U^\infty(V) \) induces a product on \( A^\infty(V) \). Since for each lower-bounded generalized \( V \)-module \( W \), the quotient algebra \( U^\infty(V)/\ker \vartheta_{\text{Gr}(W)} \) is associative, we have

\[
v_1 \odot (v_2 \odot v_3) - v_1 \odot (v_2 \odot v_3) \in \ker \vartheta_{\text{Gr}(W)}
\]

for \( v_1, v_2, v_3 \in U^\infty(V) \). Then we have

\[
v_1 \odot (v_2 \odot v_3) - v_1 \odot (v_2 \odot v_3) \in \bigcap_W \ker \vartheta_{\text{Gr}(W)} = Q^\infty(V)
\]

for \( v_1, v_2, v_3 \in U^\infty(V) \). Thus \( A^\infty(V) = U^\infty(V)/Q^\infty(V) \) is also associative.

By definition, we have \( 1^\infty \circ [v]_{kl} = [v]_{kl} \). So \( 1^\infty \) is in fact a left identity of the algebra \( U^\infty(V) \). By Proposition 2.3, \( O^\infty(V) \subset Q^\infty(V) \). Then by Proposition 2.7, we have

\[
([v]_{kl} + Q^\infty(V)) \odot (1^\infty + Q^\infty(V)) = [v]_{kl} + Q^\infty(V).
\]

So \( 1^\infty + Q^\infty(V) \) is an identity of \( A^\infty(V) \).

For a lower-bounded generalized \( V \)-module \( W \), by Theorem 2.4, \( \text{Gr}(W) \) is a module for \( U^\infty(V)/\ker \vartheta_{\text{Gr}(W)} \). Since \( Q^\infty(V) \) is a two-sided subideal of \( \ker \vartheta_{\text{Gr}(W)} \), \( \text{Gr}(W) \) is an \( A^\infty(V) \)-module.

The ideal \( Q^\infty(V) \) of \( U^\infty(V) \) is defined using all lower-bounded generalized \( V \)-modules. It is not easy to find an explicit description of \( Q^\infty(V) \). We even do not know whether \( Q^\infty(V) \) is generated by \( O^\infty(V) \). One research problem is to find an explicit description of \( Q^\infty(V) \), or at least prove some structure theorems about \( A^\infty(V) \).

The only result above on \( Q^\infty(V) \) is Proposition 2.3. Below we give another result (Proposition 2.10). To prove this result, we need the following commutator formula:

Lemma 2.9 For \( v \in V \),

\[
[L_V(-1) + L_V(0), Y_V((1 + x)L_V(0)v, x)] = Y_V((1 + x)^{L_V(0)}(L_V(-1) + L_V(0))v, x).
\]

Proof. By the \( L(-1) \) and \( L(0) \)-commutator formula with the vertex operator map \( Y_V \) and the fact that the weight of \( L_V(-1) \) is 1,

\[
[L_V(-1) + L_V(0), Y_V((1 + x)L_V(0)v, x)]
\]

\[
= Y_V(((1 + x)L_V(-1) + L_V(0))(1 + x)L_V(0)v, x)
\]

\[
= Y_V((1 + x)^{L_V(0)}(L_V(-1) + L_V(0))v, x).
\]

By Theorem 2.4, every lower-bounded generalized \( V \)-module is an \( A^\infty(V) \)-module.
Proposition 2.10 For \( u, v \in V \) and \( k, n, l \in V \), both
\[
[(L_V(-1) + L_V(0) + n - k)u]_{kn} \odot [v]_{nl}
\]
and
\[
[v]_{kn} \odot [(L_V(-1) + L_V(0) + l - n)u]_{nl}
\]
are in \( O^\infty(V) \).

Proof. For \( u, v \in V \), \( k, l, m \in \mathbb{N} \), by definition,
\[
[(L_V(-1) + L_V(0) + n - k)u]_{kn} \odot [v]_{nl}
\]
\[
= \text{Res}_x T_{k+l+1}((x + 1)^{-k+n-l-1})(1 + x) \left[ Y_V((1 + x)^{L_V(0)}(L_V(-1) + L_V(0) + n - k)u, x)v \right]_{kl}
\]
\[
= \text{Res}_x T_{k+l+1}((x + 1)^{-k+n-l-1})(1 + x)^{k-n+l+1} \frac{d}{dx} Y_V((1 + x)^{L_V(0)+n-k}u, x)v \right]_{kl}
\]
\[
= -\text{Res}_x \left( \frac{d}{dx} T_{k+l+1}((x + 1)^{-k+n-l-1})(1 + x)^{k-n+l+1} \right) \left[ Y_V((1 + x)^{L_V(0)+n-k}u, x)v \right]_{kl}
\]
\[
= -\text{Res}_x \left( \frac{d}{dx} T_{k+l+1}((x + 1)^{-k+n-l-1}) \right) (1 + x)^{k-n+l+1}
\]
\[
+ (k - n + l + 1) T_{k+l+1}((x + 1)^{-k+n-l-1})(1 + x)^{k-n+l} \right) \cdot (1 + x)^{-k+n} \left[ Y_V((1 + x)^{L_V(0)}u, x)v \right]_{kl}.
\]
(2.27)

Applying \(-\frac{1+x}{k-n+l+1} \frac{d}{dx}\) to both sides of (2.1), we obtain
\[
(x + 1)^{-k+n-l-1}
\]
\[
= -\frac{1+x}{k-n+l+1} \frac{d}{dx} T_{k+l+1}((x + 1)^{-k+n-l-1}) - \frac{1+x}{k-n+l+1} \frac{d}{dx} R_{k+l+1}((x + 1)^{-k+n-l-1}).
\]
(2.28)

Since the first and second terms in the right-hand side of (2.28) contain only the terms with powers in \( x^{-1} \) less than or equal to and larger than, respectively, \( k + l + 2 \), we must have
\[
-\frac{1+x}{k-n+l+1} \frac{d}{dx} T_{k+l+1}((x + 1)^{-k+n-l-1}) = T_{k+l+2}((x + 1)^{-k+n-l-1}),
\]
or equivalently,
\[
(1 + x) \frac{d}{dx} T_{k+l+1}((x + 1)^{-k-1})
\]
\[
= -(k - n + l + 1) T_{k+l+1}((x + 1)^{-k+n-l-1}) - (k - n + l + 1) \binom{-k+n-l-1}{n+1} x^{-k-l-2},
\]
(2.29)
Using (2.29), the right-hand side of (2.27) becomes
\[(k - n + l + 1) \binom{-k + n - l - 1}{n + 1} \text{Res}_x x^{-l-2}(1 + x)^l [Y_V((1 + x)^{L_V(0)}u, x)v]_{kl} \in O^\infty(V).\]

Thus we obtain
\[[(L_V(-1) + L_V(0) + n - k)u]_{kn} \circ [v]_{nl} = (k - n + l + 1) \binom{-k + n - l - 1}{n + 1} \text{Res}_x x^{-l-2}(1 + x)^l [Y_V((1 + x)^{L_V(0)}u, x)v]_{kl} \in O^\infty(V).\]  
(2.30)

For \(u, v \in V, k, l, n \in \mathbb{N}\) satisfying \(n \leq k \leq l\), by the definition, (2.26) and \(l - n = (l - k) - (n - k)\)
\[\begin{align*}
[v]_{kn} \circ [(L_V(-1) + L_V(0) + l - n)u]_{nl} & = \text{Res}_x T_{k+l+1} ((x + 1)^{-k+n-l-1})(1 + x)^l. \\
& \cdot [Y_V((1 + x)^{L_V(0)}v, x)(L_V(-1) + L_V(0) + l - n)u]_{kl} \\
& = \text{Res}_x T_{k+l+1} ((x + 1)^{-k+n-l-1})(1 + x)^l. \\
& \cdot [(L_V(-1) + L_V(0) + l - k)Y_V((1 + x)^{L_V(0)}v, x)u]_{kl} \\
& \quad - \text{Res}_x T_{k+l+1} ((x + 1)^{-k+n-l-1})(1 + x)^l. \\
& \quad \cdot [Y_V((1 + x)^{L_V(0)}(L_V(-1) + L_V(0) + n - k)v, x)u]_{kl}. \\
\end{align*}\]  
(2.31)

The first term in the right-hand side of (2.31) is by definition in \(O^\infty(V)\). The second term in the right-hand side of (2.31) is equal to \([[(L_V(-1) + L_V(0) + n - k)v]_{kn} \circ [u]_{nl}\), which is also in \(O^\infty(V)\) (2.30). So
\[\begin{align*}
[v]_{kn} \circ [(L_V(-1) + L_V(0) + l - n)u]_{nl} & \in O^\infty(V). \\
\end{align*}\]

\[\square\]

3 Lower-bounded generalized \(V\)-modules and graded \(A^\infty\)-modules

We study the relations between lower-bounded generalized \(V\)-modules and suitable \(A^\infty(V)\)-modules in this section.

Note that \(W\) is graded by the generalized eigenspaces of \(L_W(0)\). Since \(\Omega_n(W)\) for \(n \in \mathbb{N}\) is invariant under \(L_W(0)\), \(L_W(0)\) induces an operator on \(Gr_n(W) = \Omega_n(W)/\Omega_{n-1}(W)\) such that \(Gr_n(W)\) is also graded by the generalized eigenspaces of this operator. These operators on \(Gr_n(W)\) for \(n \in \mathbb{N}\) together define an operator, denoted by \(L_{Gr(W)}(0)\), on \(Gr(W)\) preserving
the \( \mathbb{N} \)-grading on \( Gr(W) \). Then \( Gr(W) \) is also graded by the generalized eigenspaces of \( L_{Gr(W)}(0) \).

For \( v \in V, \ k \in \mathbb{Z} \) and \( w \in \Omega_n(W) \), by the \( L(-1) \)-commutator formula,

\[
(Y_W)_k(v)L_W(-1)w = L_W(-1)(Y_W)_k(v)w + k(Y_W)_{k-1}(v)w.
\]

When \( wt \ v - k - 1 < -(n + 1) \), we have \( wt \ v - k - 1 < -n \) and \( wt \ v - (k - 1) - 1 < -n \). So in this case, \( (Y_W)_k(v)w = (Y_W)_{k-1}(v)w = 0 \) since \( w \in \Omega_n(W) \). Thus \( (Y_W)_k(v)L_W(-1)w = 0 \) when \( wt \ v - k - 1 < -(n + 1) \). This means that \( L_W(-1)w \in \Omega_{n+1}(W) \). In particular, \( L_W(1) \) induces a linear map from \( Gr_n(W) \) to \( Gr_{n+1}(W) \) for \( n \in \mathbb{N} \). These maps for \( n \in \mathbb{N} \) together define an operator, denoted by \( L_{Gr(W)}(-1) \), on \( Gr(W) \).

The operators \( L_{Gr(W)}(0) \), \( L_{Gr(W)}(-1) \) and \( \vartheta_{Gr(W)}([v]_{kl}) \) satisfy the same commutator formulas as those between \( L_W(0) \), \( L_W(1) \) and \( \text{Res}_x x^{l-k-1}Y_W(x^{L_W(0)}v, x) \) for \( v \in V \) and \( k, l \in \mathbb{N} \). These structures on \( Gr(W) \) motivates the following definition:

**Definition 3.1** Let \( G \) be an \( A^\infty(V) \)-module with the \( A^\infty(V) \)-module structure on \( G \) given by a homomorphism \( \vartheta : A^\infty(V) \rightarrow \text{End} \ G \) of associative algebras. We say that \( G \) is a \textit{graded} \( A^\infty(V) \)-module if the following conditions are satisfied:

1. \( G \) is graded by \( \mathbb{N} \), that is, \( G = \coprod_{n \in \mathbb{N}} G_n \), and for \( v \in V, k, l \in \mathbb{N} \), \( \vartheta_G([v]_{kl} + Q^\infty(V)) \) maps \( G_n \) to 0 when \( n \neq l \) and to \( G_k \) when \( n = l \).

2. For \( g \in G_l \), if \( \vartheta_G([v]_{kl} + Q^\infty(V))g = 0 \) for all \( v \in V \), then \( g = 0 \).

3. \( G \) is a direct sum of generalized eigenspaces of an operator \( L_G(0) \) on \( G \) and the real parts of the eigenvalues of \( L_G(0) \) has a lower bound.

4. There is an operator \( L_G(-1) \) on \( G \) mapping \( G_n \) to \( G_{n+1} \) for \( n \in \mathbb{N} \).

5. The commutator relations

\[
[L_G(0), L_G(-1)] = L_G(-1),
\]

\[
[L_G(0), \vartheta_G([v]_{kl} + Q^\infty(V))] = (k - l) \vartheta_G([v]_{kl} + Q^\infty(V)),
\]

\[
[L_G(-1), \vartheta_G([v]_{kl} + Q^\infty(V))] = \vartheta_G([L_V(-1)v]_{(k+1)l} + Q^\infty(V))
\]

hold for \( v \in V \) and \( k, l \in \mathbb{N} \).

Let \( G_1 \) and \( G_2 \) be graded \( A^\infty(V) \)-modules. A \textit{graded} \( A^\infty(V) \)-module map from \( G_1 \) to \( G_2 \) is an \( A^\mathbb{N}(V) \)-module map \( f : G_1 \rightarrow G_2 \) such that \( f((G_1)_n) \subset (G_2)_n \), \( f \circ L_{G_1}(0) = L_{G_2}(0) \circ f \) and \( f \circ L_{G_1}(-1) = L_{G_2}(-1) \circ f \). A \textit{graded} \( A^\infty(V) \)-submodule of a graded \( A^\infty(V) \)-module \( G \) is an \( A^\infty(V) \)-submodule of \( G \) that is also an \( \mathbb{N} \)-graded subspace of \( G \) and invariant under the operators \( L_G(0) \) and \( L_G(-1) \). A graded \( A^\infty(V) \)-module \( G \) is said to be \textit{generated by a subset} \( S \) if \( G \) is equal to the smallest graded \( A^\infty(V) \)-submodule containing \( S \), or equivalently, \( G \) is spanned by homogeneous elements with respect to the \( \mathbb{N} \)-grading and the grading given by \( L_G(0) \) obtained by applying elements of \( A^\infty(V) \), \( L_G(0) \) and \( L_G(-1) \) to homogeneous summands of elements of \( S \). A graded \( A^\infty(V) \)-module is said to be \textit{irreducible} if it has no nonzero proper graded \( A^\infty(V) \)-submodules. A graded \( A^\infty(V) \)-module is said to be \textit{completely reducible} if it is a direct sum of irreducible graded \( A^\infty(V) \)-modules.
From Theorem 2.8 and the properties of a lower-bounded generalized $V$-module $W$ and its associated graded space $Gr(W)$, we obtain immediately:

**Theorem 3.2** For a lower-bounded generalized $V$-module $W$, $Gr(W)$ is a graded $A^\infty$-module. Let $W_1$ and $W_2$ be lower-bounded generalized $V$-modules and $f : W_1 \to W_2$ a $V$-module map. Then $f$ induces a graded $A^\infty(V)$-module map $Gr(f) : Gr(W_1) \to Gr(W_2)$.

We now give a direct and explicit description of $Gr(W)$ for a completely reducible lower-bounded generalized $V$-module $W$. In this case,

$$W = \prod_{\mu \in \mathcal{M}} W^\mu,$$

where $\mathcal{M}$ is an index set and $W^\mu$ for $\mu \in \mathcal{M}$ are irreducible lower-bounded generalized $V$-modules. For $\mu \in \mathcal{M}$, since $W^\mu$ is irreducible, there exists $h^\mu \in \mathbb{C}$ such that

$$W^\mu = \bigoplus_{n \in \mathbb{N}} W^\mu_{[h^\mu + n]}.$$

where as usual, $W^\mu_{[h^\mu + n]}$ for $n \in \mathbb{N}$ is the subspace of $W^\mu$ of weight $h^\mu + n$, and $W^\mu_{[h^\mu]} \neq 0$. For $n \in \mathbb{N}$, let

$$G_n(W) = \prod_{\mu \in \mathcal{M}} W_{[h^\mu + n]}.$$

Then

$$W = \bigoplus_{n \in \mathbb{N}} G_n(W).$$

For $n \in \mathbb{N}$, let

$$T_n(W) = \prod_{m=0}^{n} G_m(W).$$

It is clear that $T_n(W) \subset \Omega_n(W)$. In particular, $G_n(W) \subset \Omega_N(W)$ for $n \leq N$. Let $e_W : W \to Gr(W)$ be defined by $e_W(w) = w + \Omega_{n-1}(W)$ for $w \in G_n(W)$ and $n \in \mathbb{N}$. Then $e_W$ preserves the $\mathbb{N}$-grading. We also define a map $\vartheta_W : U^\infty(V) \to \text{End } W$ by

$$\vartheta_W(v)w = \sum_{k,l \in \mathbb{N}} \text{Res}_x x^{l-k-1} Y_W(x^{LV(0)}v_{kl}, x) \pi_{G_l(W)}w$$

for $v \in U^\infty$ and $w \in W$, where $\pi_{G_l(W)}$ is the projection from $W$ to $G_l(W)$. In the case $v = [v]_{kl}$ and $w \in G_n(W)$ for $v \in V$ and $k, l, n \in \mathbb{N}$, we have

$$\vartheta_W([v]_{kl})w = \delta_{h^\mu} \text{Res}_x x^{l-k-1} Y_W(x^{LV(0)}v, x)w. \quad (3.1)$$

**Proposition 3.3** Let $W$ be a completely reducible lower-bounded generalized $V$-module. Then $\Omega_n(W) = T_n(W)$ for $n \in \mathbb{N}$. Moreover, $W$ equipped with $\vartheta_W$ is a graded $A^\infty(V)$-module and $e_W : W \to Gr(W)$ is an isomorphism of graded $A^\infty(V)$-modules.
Proof. If $T_n(W) \neq \Omega_n(W)$, then there exists homogeneous $w \in \Omega_n(W)$ but $w \not\in T_n(W)$. Then $w = \sum_{\mu \in \mathcal{M}} w^\mu$, where $w^\mu \in W^\mu$ for $\mu \in \mathcal{M}$ and only finitely many $w^\mu$ is not 0. Since $w$ is homogeneous, we can assume that $w^\mu$ for $\mu \in \mathcal{M}$ are homogeneous. Since $w \in \Omega_n(W)$ but $w \not\in T_n(W)$, there is at least one $w^\mu$ such that $w^\mu \in \Omega_n(W^\mu)$ but $w^\mu \not\in T_n(W^\mu) = \bigcap_{m=0}^n W^\mu_{[\mu + m]}$. Let $W_0^\mu$ be the generalized $V$-submodule of $W^\mu$ generated by such a $w^\mu$. Since $w^\mu \not\in T_n(W^\mu)$, $w^\mu \neq 0$ and hence $W_0^\mu \neq 0$. But $W^\mu$ is irreducible. So $W_0^\mu = W^\mu$. Since $w^\mu$ is homogeneous, there is $m \in \mathbb{N}$ such that wt $w^\mu \in W^\mu_{[\mu + m]}$. Since $w^\mu \not\in T_n(W^\mu)$, we must have $m > n$. Since $W^\mu = W_0^\mu$, $W^\mu$ is spanned by elements of the form $(Y_W)_k(v)w^\mu$ for $v \in V$ and $k \in \mathbb{Z}$. Since $w^\mu \in \Omega_n(W^\mu)$, $(Y_W)_k(v)w^\mu = 0$ for homogeneous $v \in V$ and $k \in \mathbb{Z}$ satisfying wt $v - k - 1 < -n$. Thus the homogeneous subspaces of $W^\mu_{[\mu + m - n]} = 0$ for $\mu \in \mathbb{Z}_+$. But for $p = m - n \in \mathbb{Z}_+$, $W^\mu_{[\mu + m - n - p]} = W^\mu_{[\mu + p]} \neq 0$. Contradiction. Thus $T_n(W) = \Omega_n(W)$.

For $n \in \mathbb{N}$, we have $Gr_n(W) = \Omega_n(W)/\Omega_{n-1}(W) = T_n(W)/T_{n-1}(W)$. Then $e_W|_{Gr_n(W)}$ is clearly a linear isomorphism from $G_n(W)$ to $T_n(W)/T_{n-1}(W) = Gr_n(W)$. This shows that $e_W$ is an isomorphism of graded spaces. For $v \in V, k, l \in \mathbb{N}$ and $w \in G_l(W)$,

$$e_W(\vartheta_W([v]_{kl})w = e_W(\text{Res}_x x^{l-k-1}Y_W(x^{L_v(0)}v, x)w)
= \text{Res}_x x^{l-k-1}Y_W(x^{L_v(0)}v, x)w + T_{k-1}(W)
= \vartheta_{Gr(W)}([v]_{kl})e_W(w).$$

Thus we have $e_W \circ \vartheta_W = \vartheta_{Gr(W)} \circ e_W$. In particular, the $A^\infty(V)$-module structure on $Gr(W)$ given by $\vartheta_{Gr(W)}$ is transported to $W$ by $e_W$ so that $W$ equipped with $\vartheta_W$ is an $A^\infty(V)$-module and $e_W : W \rightarrow Gr(W)$ is an isomorphism of $A^\infty(V)$-modules.

Theorem 3.4 A lower-bounded generalized $V$-module $W$ is irreducible or completely reducible if and only if the graded $A^\infty(V)$-module $Gr(W)$ is irreducible or completely reducible, respectively.

Proof. Let $W$ be an irreducible lower-bounded generalized $V$-module. By Theorem 3.3, $W$ is a graded $A^\infty(V)$-module isomorphic to $Gr(W)$. Let $W_0$ be a nonzero graded $A^\infty(V)$-submodule of the graded $A^\infty(V)$-module $W$. For a homogeneous element $v \in V, n \in \mathbb{Z}$ and $w \in W_0$,

$$\text{Res}_x x^n Y_W(v, x)w = \sum_{l \in \mathbb{N}} \vartheta_W([v]_{[\text{wt } v - n - 1 + l]_{l}}) \pi_{G_l(W)} w \in W_0.$$  

This means that $W_0$ is invariant under the action of the vertex operators on $W$. By the definition of graded $A^\infty(V)$-submodule, $W_0$ is invariant under the actions of $L_W(0)$ and $L_W(-1)$ and is the direct sum of generalized eigenspaces of $L_W(0)|_{W_0}$. Thus $W_0$ is also a nonzero lower-bounded generalized $V$-submodule of $W$. Since $W$ is an irreducible lower-bounded generalized $V$-module, $W_0 = W$. So as a graded $A^\infty(V)$-module, $W$ is also irreducible. Since as a graded $A^\infty(V)$-module, $Gr(W)$ is equivalent to $W$, we see that $Gr(W)$ is irreducible.
Conversely, assume that for a lower-bounded generalized $V$-module $W$, the graded $A^\infty(V)$-module $Gr(W)$ is irreducible. Let $W_0$ be a nonzero generalized $V$-submodule of $W$. Then $\Omega_{n-1}(W_0) \subset \Omega_{n-1}(W)$ for $n \in \mathbb{N}$ (when $n = 0$, $\Omega_{-1}(W) = 0$). We have a map from $Gr(W_0)$ to $Gr(W)$ given by $w_0 + \Omega_{n-1}(W_0) \mapsto w_0 + \Omega_{n-1}(W)$ for $n \in \mathbb{N}$ and $w_0 \in \Omega_n(W_0)$. This map is an injective graded $A^\infty(V)$-module map. So the image of $Gr(W_0)$ under this map is a graded $A^\infty(V)$-submodule of $Gr(W)$. Since $W_0$ is nonzero, $Gr(W_0)$ is nonzero. Since $Gr(W)$ is irreducible, the image of $Gr(W_0)$ under this map is equal to $Gr(W)$. Now it is easy to derive $W_0 = W$. In fact, for $n \in \mathbb{N}$, the image of $Gr_n(W_0)$ under the map from $Gr(W_0)$ to $Gr(W)$ above is $\{w_0 + \Omega_n(W) \mid w_0 \in \Omega_n(W_0)\}$. So $Gr_n(W) = \{w_0 + \Omega_{n-1}(W) \mid w_0 \in \Omega_n(W_0)\}$. For $n = 0$, we obtain $\Omega_0(W) = Gr_0(W) = \Omega_0(W_0)$. Assume that $\Omega_{n-1}(W) = \Omega_{n-1}(W_0)$. Given $w \in \Omega_n(W)$, $w + \Omega_{n-1}(W) \in Gr_n(W)$. By $Gr_n(W) = \{w_0 + \Omega_{n-1}(W) \mid w_0 \in \Omega_n(W_0)\}$, there exists $w_0 \in \Omega_n(W_0)$ such that $w + \Omega_{n-1}(W) = w_0 + \Omega_{n-1}(W)$, or equivalently, $w - w_0 \in \Omega_{n-1}(W) = \Omega_{n-1}(W_0)$. Thus $w \in \Omega_n(W_0)$. This shows $\Omega_n(W) = \Omega_n(W_0)$ for $n \in \mathbb{N}$. Then we have $W = \bigcup_{n \in \mathbb{N}} \Omega_n(W) = \bigcup_{n \in \mathbb{N}} \Omega_n(W_0) = W_0$. So $W$ is irreducible.

Assume that a lower-bounded generalized $V$-module $W$ is completely reducible. Then $W = \bigsqcup_{\mu \in \mathcal{M}} W^\mu$, where $W^\mu$ for $\mu \in \mathcal{M}$ are irreducible generalized $V$-modules. From what we have proved above, $W^\mu$ for $\mu \in \mathcal{M}$ as graded $A^\infty(V)$-modules are also irreducible. So $W$ as a graded $A^\infty(V)$-module is completely reducible. But $Gr(W)$ is equivalent to $W$ as a graded $A^\infty(V)$-module by Proposition 3.3. So $Gr(W)$ is also completely reducible. Conversely, assume that for a lower-bounded generalized $V$-module $W$, the graded $A^\infty(V)$-module $Gr(W)$ is completely reducible. Then $Gr(W) = \bigsqcup_{\mu \in \mathcal{M}} G^\mu$, where $G^\mu$ for $\mu \in \mathcal{M}$ are irreducible graded $A^\infty(V)$-submodules of $Gr(W)$. For $\mu \in \mathcal{M}$, since $G^\mu$ is a graded $A^\infty(V)$-submodule of $Gr(W)$, we have $G^\mu \subset Gr_n(W) = \Omega_n(W)/\Omega_{n-1}(W)$. Let $W^\mu$ be the subspace of $W$ consisting of elements of the form $w^\mu \in \Omega_n(W)$ such that $w^\mu + \Omega_{n-1}(W) \in G^\mu_n$ for $n \in \mathbb{N}$. Since $G^\mu$ is a graded $A^\infty(V)$-submodule of $Gr(W)$, for $v \in V$, $k, l \in \mathbb{N}$ and $w^\mu \in \Omega_k(W)$ such that $w^\mu + \Omega_{l-1}(W) \in G^\mu_k$, \[Res_x x^{-k-1} Y_W(x^{L_W(0)} v, x) w^\mu + \Omega_{k-1}(W) \in G^\mu_k.\]

By the definition of $W^\mu$, we obtain \[Res_x x^{-k-1} Y_W(x^{L_W(0)} v, x) w^\mu \in W^\mu.\] Since $w^\mu \in \Omega_l(W)$, \[Res_x x^{-k-1} Y_W(x^{L_W(0)} v, x) w^\mu = 0 \text{ for } k \in -Z_+.\] Thus \[Res_x x^{-k-1} Y_W(x^{L_W(0)} v, x) w^\mu \in W^\mu\] for $k \in \mathbb{N}$ are all the nonzero coefficients of $Y_W(v, x) w^\mu$. This means that $W^\mu$ is closed under the action of the vertex operators on $W$. Since $G^\mu$ is invariant under the actions of $L_{Gr(W)(0)}$ and $L_{Gr(W)(-1)}$ and is a direct sum of generalized eigenspaces of $L_{Gr(W)(0)}$, $W^\mu$ is invariant under the actions of $L_W(0)$ and $L_W(-1)$ and is a direct sum of generalized eigenspaces of $L_W(0)$. Thus $W^\mu$ is a generalized $V$-submodule of $W$.

Let $w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu)$, where $n \in \mathbb{N}$ and $w^\mu \in \Omega_n(W^\mu) \subset \Omega_n(W)$. By the definition of $w^\mu$, we see that since $w^\mu$ is an element of $W^\mu$, $w^\mu + \Omega_{n-1}(W) \in G^\mu_n$. So we obtain a linear map from $Gr_n(W^\mu)$ to $G^\mu_n$ given by $w^\mu + \Omega_{n-1}(W^\mu) \mapsto w^\mu + \Omega_{n-1}(W)$ for $w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu)$. These maps for $n \in \mathbb{N}$ give a map from $Gr(W)$ to $G^\mu$. It is clear that this map is a graded $A^\infty(V)$-module map. If the image $w^\mu + \Omega_{n-1}(W)$ of $w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu)$ under this map is 0 in $G^\mu$, then $w^\mu \in \Omega_{n-1}(W)$. But $w^\mu \in \Omega_n(W^\mu) \subset W^\mu$. So $w^\mu \in \Omega_n(W^\mu)$ and $w^\mu + \Omega_{n-1}(W^\mu)$ is 0 in $Gr(W^\mu)$. This means
that this graded $A^\infty(V)$-module map is injective. In particular, the image of $Gr(W^\mu)$ under this map is a nonzero graded $A^\infty(V)$-submodule of $G^\mu$. But $G^\mu$ is irreducible. So $Gr(W^\mu)$ must be equivalent to $G^\mu$ and is therefore also irreducible. From what we have proved above, since $Gr(W^\mu)$ is irreducible, $W^\mu$ is irreducible. This shows that $W$ is complete reducible.

Theorem 3.4 implies that there is a map from the set of the equivalence classes of lower-bounded generalized $V$-modules to the set of equivalence classes of graded $A^\infty(V)$-modules. This map is in fact a bijection. To prove this, we need to construct a lower-bounded generalized $V$-module $S(G)$ from a graded $A^\infty(V)$-module $G$. We use the construction in Section 5 of [H3]. Take the generating fields for the grading-restricted vertex algebra $V$ to be $Y_V(v,x)$ for $v \in V$. By definition, $G$ is a direct sum of generalized eigenspaces of $L_G(0)$ and the real parts of the eigenvalues of $L_G(0)$ has a lower bound $B \in \mathbb{R}$. We take $M$ and $B$ in Section 5 of [H3] to be $G$ and the lower bound $B$ above. Using the construction in Section 5 of [H3], we obtain a universal lower-bounded generalized $V$-module $\hat{G}_B^{[\nu]}$. For simplicity, we shall denote it simply by $\hat{G}$.

By Theorem 3.3 in [H4] and the construction in Section 5 of [H3] and by identifying elements of the form $(\psi^a_G)_{-1,0}$ with basis elements $g^a \in G$ for $a \in A$ for a basis $\{g^a\}_{a \in A}$ of $G$, we see that $\hat{G}$ is generated by $G$ (in the sense of Definition 3.1 in [H4]). Moreover, after identifying $(\psi^a_G)_{-1,0}$ with basis elements $w^a \in G$ for $a \in A$, Theorems 3.3 and 3.4 in [H4] in fact say that elements of the form $L_{\hat{G}}(-1)^p w^a$ for $p \in \mathbb{N}$ and $a \in A$ are linearly independent and $\hat{G}$ is spanned by elements obtained by applying the components of the vertex operators to these elements. In particular, $\hat{G}$ can be embedded into $\hat{G}$ as a subspace. So from now on, we shall view $G$ as a subspace of $\hat{G}$. Let $J_G$ be the generalized $V$-submodule of $\hat{G}$ generated by elements of the forms

$$\text{Res}_x x^{l-k-1} Y_G(x^{L_V(0)} v, x) g$$

for $l \in \mathbb{N}$, $k \in -\mathbb{Z}_+$ and $g \in G_I$,

$$\text{Res}_x x^{l-k-1} Y_M(x^{L_V(0)} v, x) g - \vartheta_G([v]_{kl} + Q^\infty(V)) g$$

for $v \in V$, $k, l \in \mathbb{N}$, $g \in G_I$ and

$$L_{\hat{G}}(-1) g - L_G(-1) g$$

for $l \in \mathbb{N}$, $g \in G_I$.

Let $S(G) = \hat{G}/J_G$. Then $S(G)$ is a lower-bounded generalized $V$-module. Let $\pi_{S(G)}$ be the projection from $\hat{G}$ to $S(G)$. Since $\hat{G}$ is generated by $G$ (in the sense of Definition 3.1 in [H4]), $S(G)$ is generated by $\pi_{S(G)}(G)$ (in the same sense). In particular, $S(G)$ is spanned by elements of the form

$$\text{Res}_x x^{(l+p)-n-1} Y_{S(G)}(x^{L_{S(G)}(0)} v, x) L_{S(G)}(-1)^p \pi_{S(G)}(g)$$

for $v \in V$, $n, l, p \in \mathbb{N}$ and $g \in G_I$. For $n \in \mathbb{N}$, let $G_n(S(G))$ be the subspace of $S(G)$ spanned by elements of the form (3.5) for $v \in V$, $l, p \in \mathbb{N}$ and $g \in G_I$.

**Proposition 3.5** Let $G$ be a graded $A^\infty(V)$-module.
1. For \( n \in \mathbb{N} \), \( G_n(S(G)) = \pi_{S(G)}(G_n) \) and for \( n_1 \neq n_2 \), \( G_{n_1}(S(G)) \cap G_{n_2}(S(G)) = 0 \). Moreover, \( S(G) = \pi_{S(G)}(G) = \prod_{n \in \mathbb{N}} G_n(S(G)) \).

2. For \( n \in \mathbb{N} \), \( \Omega_{n}(S(G)) = \prod_{j=0}^{n} \pi_{S(G)}(G_j) = \prod_{j=0}^{n} G_j(S(G)) \).

3. \( \text{Gr}(S(G)) \) is equivalent to \( G \) as a graded \( \mathbb{A}^\infty(V) \)-module.

Proof. Since elements of the forms (3.3) and (3.4) are in \( J_G \), for \( n \in \mathbb{N} \), the element (3.5) for \( v \in V \) for \( l,p \in \mathbb{N} \) and \( g \in G_I \) is in fact equal to

\[
\pi_{S(G)}(\partial_G([v]_{n(l+p)} + Q^\infty(V))L_G(-1)^p g). \tag{3.6}
\]

Since \( \partial_G([v]_{n(l+p)} + Q^\infty(V))L_G(-1)^p g \) for \( l,p \in \mathbb{N} \) and \( g \in G_I \) certainly span \( G_n \) and elements of the form (3.5) for \( v \in V \) for \( l,p \in \mathbb{N} \) and \( g \in G_I \) span \( G_n(S(G)) \), elements of the form (3.6) for \( v \in V \) for \( l,p \in \mathbb{N} \) and \( g \in G_I \) also span \( G_n(S(G)) \). Thus \( G_n(S(G)) = \pi_{S(G)}(G_n) \). When \( n_1 \neq n_2 \), we know \( G_{n_1} \cap G_{n_2} = 0 \). Then \( G_{n_1}(S(G)) \cap G_{n_2}(S(G)) = \pi_{S(G)}(G_{n_1} \cap G_{n_2}) = 0 \).

As is mentioned above, \( S(G) \) is spanned by elements of the form (3.5) for \( v \in V \), \( k,l \in \mathbb{N} \) and \( g \in G_I \). But we already see that (3.5) is in fact equal to (3.6). Thus \( S(G) = \pi_{S(G)}(G) \). Since \( G_n(S(G)) = \pi_{S(G)}(G_n) \) and \( G_{n_1}(S(G)) \cap G_{n_2}(S(G)) = 0 \), we have \( S(G) = \pi_{S(G)}(G) = \prod_{n \in \mathbb{N}} G_n(S(G)) \).

By definition, for \( j \leq n \), \( G_j(S(G)) \subset \Omega_{n}(S(G)) \). Then for \( j = 0, \ldots, n \), \( \pi_{S(G)}(G_j) \subset \Omega_{j}(S(G)) \subset \Omega_{n}(S(G)) \). So we obtain \( \pi_{S(G)}(\bigcap_{j=0}^{n} G_j) \subset \Omega_{n}(S(G)) \). By Condition 2 in the definition of graded \( \mathbb{A}^\infty(V) \)-module, nonzero elements of \( G_j \) for \( j > n \) are not in \( \Omega_{n}(S(G)) \). From the construction of \( \hat{G} \), nonzero elements of the form (3.2), (3.3) or (3.4) are not in \( G \subset \hat{G} \). In particular, the intersection of \( J(G) \) with \( G \) is 0. So \( \pi_{S(G)}|_G : G \to S(G) \) is injective. Since \( \pi_{S(G)}|_G \) is injective, we conclude that nonzero elements of \( \pi_{S(G)}(G_j) \) for \( j > n \) are not in \( \Omega_{n}(S(G)) \). So we have

\[
\Omega_{n}(S(G)) = \pi_{S(G)} \left( \prod_{j=0}^{n} G_j \right) = \prod_{j=0}^{n} \pi_{S(G)}(G_j) = \prod_{j=0}^{n} G_j(S(G)).
\]

Since \( \Omega_{n}(S(G)) = \prod_{j=0}^{n} G_j(S(G)) \) for \( n \in \mathbb{N} \), we see that as a \( \mathbb{N} \)-graded space, \( \text{Gr}(S(G)) \) is isomorphic to \( \prod_{n \in \mathbb{N}} G_n(S(G)) = \pi_{S(G)}(G) \). We use \( f_G \) to denote the isomorphism from \( \text{Gr}(S(G)) \) to \( \pi_{S(G)}(G) \). Then we have

\[
f_G \circ \partial_{Gr(S(G))}([v]_{k,l} + Q^\infty(V)) = \text{Res}_{x} x^{l-k-1} Y_{S(G)}(x^{L_{S(G)}(0)}v, x) \circ f_G
\]

for \( v \in V, k,l \in \mathbb{N} \), \( f_G \circ L_{Gr(S(G))}(0) = L_{S(G)}(0) \circ f_G \) and \( f_G \circ L_{Gr(S(G))}(-1) = L_{S(G)}(-1) \circ f_G \). We have proved that \( \pi_{S(G)}|_G \) is injective and surjective and preserves the \( \mathbb{N} \)-gradings. So it is an isomorphism of graded spaces from \( G \) to \( S(G) \). From the fact that \( \pi_{S(G)} \) is a \( V \)-module map and on \( G \subset \hat{G} \), \( L_G(0) = L_{\hat{G}}(0) \) and \( L_G(-1) = L_{\hat{G}}(-1) \), we have

\[
\pi_{S(G)}|_G \circ \partial_G([v]_{k,l} + Q^\infty(V)) = \text{Res}_{x} x^{l-k-1} Y_{S(G)}(x^{L_{S(G)}(0)}v, x) \circ \pi_{S(G)}|_G
\]
for \( v \in V, k, l \in \mathbb{N}, \pi_{S(G)}|_G \circ L_G(0) = L_{S(G)}(0) \circ \pi_{S(G)}|_G \) and \( \pi_{S(G)}|_G \circ L_G(-1) = L_{S(G)}(-1) \circ \pi_{S(G)}|_G \). Then by the properties of \( f_G \) and \( \pi_{S(G)}|_G \) above, we see that \( (\pi_{S(G)}|_G)^{-1} \circ f_G \) is an equivalence of graded \( A^\infty(V) \)-modules from \( \text{Gr}(\hat{S}(G)) \) to \( G \).

**Remark 3.6** Note that our construction of the lower-bounded generalized \( V \)-module \( \hat{G} \) seems to depend on the lower bound \( B \) of the real parts of the eigenvalues of \( L_G(0) \). But by Proposition 3.5, \( S(G) \) depends only on \( G \), not on \( B \).

**Theorem 3.7** The set of the equivalence classes of irreducible lower-bounded generalized \( V \)-modules is in bijection with the set of the equivalence classes of irreducible graded \( A^\infty(V) \)-modules.

**Proof.** Let \([\mathfrak{M}]_{\text{irr}}\) be the set of the equivalence classes of irreducible lower-bounded generalized \( V \)-modules and \([\mathfrak{S}]_{\text{irr}}\) the set of the equivalence classes of irreducible graded \( A^\infty(V) \)-modules. Given an irreducible lower-bounded generalized \( V \)-module \( W \), by Theorem 3.4, \( \text{Gr}(W) \) is an irreducible graded \( A^N(V) \)-module. Thus we obtain a map \( f : [\mathfrak{M}]_{\text{irr}} \to [\mathfrak{S}]_{\text{irr}} \) given by \( f([W]) = [\text{Gr}(W)] \), where \([W] \in [\mathfrak{M}]_{\text{irr}}\) is the equivalence class containing the irreducible lower-bounded generalized \( V \)-module \( W \) and \([\text{Gr}(W)] \in [\mathfrak{S}]_{\text{irr}}\) is the equivalence class containing the irreducible graded \( A^N(V) \)-module \( \text{Gr}(W) \). By Proposition 3.3, \([\text{Gr}(W)] = [W] \) in \([\mathfrak{S}]_{\text{irr}}\), where \( W \) is viewed as a graded \( A^\infty(V) \)-module.

Given an irreducible graded \( A^\infty(V) \)-module \( G \), we have a lower-bounded generalized \( V \)-module \( S(G) \). By Proposition 3.5, \( \text{Gr}(S(G)) \) is equivalent to \( G \). Since \( G \) is irreducible, \( \text{Gr}(S(G)) \) is also irreducible. Then by Theorem 3.4, \( S(G) \) is an irreducible lower-bounded generalized \( V \)-module. Thus we obtain a map \( g : [\mathfrak{S}]_{\text{irr}} \to [\mathfrak{M}]_{\text{irr}} \) given by \( g([G]) = [S(G)] \).

We still need to show that \( f \) and \( g \) are inverse to each other. By Proposition 3.5, \( \text{Gr}(S(G)) \) is equivalent to \( G \) for an irreducible graded \( A^\infty(V) \)-module \( G \). We obtain \([\text{Gr}(S(G))] = [G] \). This means \( f(g([G])) = [G] \). So we have \( f \circ g = 1_{[\mathfrak{S}]_{\text{irr}}} \).

Let \( W \) be an irreducible lower-bounded generalized \( V \)-module. By Theorem 3.4, \( \text{Gr}(W) \) is an irreducible graded \( A^\infty(V) \)-module. We then have a lower-bounded generalized \( V \)-module \( S(\text{Gr}(W)) \). By Proposition 3.5, \( \text{Gr}(S(\text{Gr}(W))) \) is equivalent to \( \text{Gr}(W) \) as a graded \( A^\infty(V) \)-module. Since \( \text{Gr}(W) \) is irreducible, \( \text{Gr}(S(\text{Gr}(W))) \) is also irreducible. By Theorem 3.4, \( S(\text{Gr}(W)) \) is an irreducible lower-bounded generalized \( V \)-module. Since both \( W \) and \( S(\text{Gr}(W)) \) are irreducible, by Proposition 3.3, \( W \) and \( S(\text{Gr}(W)) \) are graded \( A^\infty(V) \)-modules and are equivalent to \( \text{Gr}(W) \) and \( \text{Gr}(S(\text{Gr}(W))) \), respectively. But we already know that \( \text{Gr}(S(\text{Gr}(W))) \) is equivalent to \( \text{Gr}(W) \) as a graded \( A^\infty(V) \)-module. So both \( W \) and \( S(\text{Gr}(W)) \) are equivalent to \( \text{Gr}(W) \) as graded \( A^\infty(V) \)-modules. Since vertex operators on \( W \) and \( S(\text{Gr}(W)) \) can be expressed using the actions of elements of \( A^\infty(V) \), we see that \( W \) and \( S(\text{Gr}(W)) \) are also equivalent as lower-bounded generalized \( V \)-modules. Thus \([S(\text{Gr}(W))] = [W] \), or \( g(f([W])) = [W] \). So \( g \circ f = 1_{[\mathfrak{M}]_{\text{irr}}} \).
4 Subalgebras of $A^\infty(V)$

We give some very special subalgebras of $A^\infty(V)$ and prove that they are isomorphic to the Zhu algebra $A(V)$ [Z] and its generalizations $A_N(V)$ for $N \in \mathbb{N}$ by Dong-Li-Mason [DLM] in Subsection 4.1. Then we introduce the main interesting and new subalgebras $A_N(V)$ for $N \in \mathbb{N}$ of $A^\infty(V)$ in Subsection 4.2. Note that we use the superscript $N$ instead of the subscript $N$ to distinguish this algebra from $A_N(V)$ in [DLM].

4.1 Zhu algebra and the generalizations by Dong-Li-Mason

Let $U_{00}(V) = \{[v]_{00} \mid v \in V\} \subset U^\infty(V)$.

Then $U_{00}(V)$ can be canonically identified with $V$ through the map $i_{00} : U_{00}(V) \to V$ given by $i_{00}([v]_{00}) = v$ for $v \in V$. Since by (2.3),

$$[u]_{00} \diamond [v]_{00} = \text{Res}_x x^{-1} \left[ Y_V(1 + x)^L u, x^N v \right]_{00},$$

$U_{00}(V)$ is closed under the product $\diamond$. Let

$$A_{00}(V) = \{[v]_{00} + Q^\infty(V) \mid v \in V\}.$$

**Theorem 4.1** The subspace $A_{00}(V)$ of $A^\infty(V)$ is closed under $\diamond$ and is thus a subalgebra of $A^\infty(V)$ with $[1]_{00} + Q^\infty(V)$ as its identity. The associative algebra $A_{00}(V)$ is isomorphic to the Zhu algebra $A(V)$ in [Z] and, in particular, $[\omega]_{00} + Q^\infty(V)$ is in the center of $A_{00}(V)$ if $V$ is a vertex operator algebra with the conformal vector $\omega$.

Since this result is a special case of the result on the generalizations $A_N(V)$ in [DLM], we will not give a proof. The proof is the special case $N = 0$ of the proof of Theorem 4.2 below for $A_N(V)$.

Fix $N \in \mathbb{N}$. Let

$$U_{NN} = \{[v]_{NN} \mid v \in V\} \subset U^\infty(V).$$

By (2.3),

$$[u]_{NN} \diamond [v]_{NN} = \text{Res}_x T_{2N+1}((x + 1)^{-N-1})(1 + x)^N \left[ Y_V(1 + x)^L u, x^N v \right]_{NN}$$

for $u, v \in V$. So $U_{NN}(V)$ is closed under the product $\diamond$. Let

$$A_{NN}(V) = \{[v]_{NN} + Q^\infty(V) \mid v \in V\} \subset A^\infty(V).$$

**Theorem 4.2** The subspace $A_{NN}(V)$ of $A^\infty(V)$ is closed under $\diamond$ and is thus a subalgebra of $A^\infty(V)$ with $[1]_{NN} + Q^\infty(V)$ as its identity. The associative algebra $A_{NN}(V)$ is isomorphic to the associative algebra $A_N(V)$ of Dong, Li and Mason in [DLM] and, in particular, $[\omega]_{NN} + Q^\infty(V)$ is in the center of $A_{NN}(V)$ if $V$ is a vertex operator algebra with the conformal vector $\omega$. 

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Proof. By (2.3), we have

\[(u)_{NN} + Q^\infty(v)) \circ ([v]_{NN} + Q^\infty(v))
= \text{Res}_x T_{2N+1}((x + 1)^{N-1}) (1 + x)^N [Y_V((1 + x)^L(0)u, x)v]_{NN} + Q^\infty(v)
\in A_{NN}(V)
\]

for \( u, v \in V \). Thus \( A_{NN}(V) \) is closed under \( \circ \) and is a subalgebra of \( A^\infty(V) \). Let \( f_{NN} : U_{NN}(V) \to A_N(V) \) be defined by \( f_{NN}([v]_{NN}) = v + O_N(V) \) for \( v \in V \).

We now view \( A_N(V) \) as an \( A_N(V) \)-module. We construct a lower-bounded generalized \( V \)-module \( S(A_N(V)) \) from \( A_N(V) \) using the construction in Section 5 of [H3] as follows: Take the generating fields for the grading-restricted vertex algebra \( V \) to be \( Y_V(v, x) \) for \( v \in V \). Take \( M \) in Section 5 of [H3] to be \( A_N(V) \). We define the operator \( L_M(0) \) on \( M \) to be the multiplication by the scalar \( N \). So \( M \) itself is an eigenspace of \( L_M(0) \) with eigenvalue \( N \). Take \( u, v \in V \). Let \( S(A_N(V)) = A_N(V) \) be the generalized \( V \)-submodule of \( A_N(V) \) generated by elements of the form

\[ (Y_{A_N(V)})_{n} ((u)_{A_N(V)})(-1)^p(v + O_N(V)) \]

for homogeneous \( u, v \in V, p \in N \) and \( n \in \text{wt} u + N + p - 1 - N \). Let \( L \) be the generalized \( V \)-submodule of \( A_N(V) \) generated by elements of the form

\[ (Y_{A_N(V)})_{\text{wt} u - 1}(u)(v + O_N(V)) - u \ast_N v + O_N(V) \]

for \( u, v \in V \). Let \( S(A_N(V)) = A_N(V)/L \). Then

\[ S(A_N(V)) = \prod_{n \in N} (S(A_N(V)))_{[n]} \]

is a lower-bounded generalized \( V \)-module such that \( (S(A_N(V)))_{[N]} = A_N(V) \). From the construction in Section 5 of [H3] and the definition of \( S(A_N(V)) \) above, elements of the form

\[ (Y_{S(A_N(V))})_{\text{wt} u - 1 + N}(u)(v + O_N(V)) \]

for homogeneous nonzero \( u \in V \) and nonzero \( v \in V \) are not 0. Thus for nonzero \( v \in V \), \( v + O_N(V) \in A_N(V) \) is not in \( \Omega_{N-1}(S(A_N(V))) \). In other words, if \( v + O_N(V) \in \Omega_{N-1}(S(A_N(V))) \), then \( v = 0 \). On the other hand, we know that \( A_N(V) = (S(A_N(V)))_{[N]} \subset \Omega_N(S(A_N(V))) \).

Let \( W \) be a lower-bounded generalized \( V \)-module. Then \( \ker \varphi_{Gr(W)} \) is a two-sided ideal of \( U^\infty(V) \). So \( \ker \varphi_{Gr(W)} \cap U_{NN}(V) \) is a two-sided ideal of \( U_{NN}(V) \). From [DLM], the map \( \sigma_W : V \to \text{End} \Omega_N(W) \) defined by \( \sigma_W(v) = (Y_W)_{\text{wt} v - 1}(v) = \text{Res}_x x^{-1} Y_W(x^{L(0)}v, x) \) gives
\( \Omega_N(W) \) an \( A_N(V) \)-module structure. In particular, \( o_W(O_N(V)) = 0 \). So \( O_N(V) \subset \ker o_W \).

We take \( W = S(A_N(V)) \). By the definition of \( o_{S_N(A_N(V))} \), we have

\[
o_{S_N(A_N(V))}(u)(v + O_N(V)) = u *_N v + O_N(V)
\]

for \( u, v \in V \). For \( u \in \ker o_{S_N(A_N(V))} \), we have

\[
o_{S_N(A_N(V))}(u)(v + O_N(V)) = 0
\]

for \( v \in V \). So we have \( u *_N v + O_N(V) = 0 \) or \( u *_N v \in O_N(V) \). In particular, for \( v = 1 \), we have \( u *_N 1 \in O(V) \). But modulo \( O_N(V) \), \( u *_N 1 \) is equal to \( u \). So \( u \in O_N(V) \). This means \( \ker o_{S_N(A_N(V))} \subset O_N(V) \) and thus \( \ker o_{S_N(A_N(V))} = O_N(V) \).

For \( v \in V \), we have shown that \( v + O_N(V) \in \Omega_N(S(A_N(V))) \) and that \( v + O_N(V) \in \Omega_{N-1}(S(A_N(V))) \) implies \( v = 0 \). So using our notation above, we see that

\[
[v + O_N(V)]_N = (v + O_N(V)) + \Omega_{N-1}(S(A_N(V)))
\]

is an element of \( Gr_N(S_N(A_N(V))) \) and if it is equal to \( 0 \) in \( Gr_N(S_N(A_N(V))) \), then \( v = 0 \). By definition, for \( u, v \in V \),

\[
\partial_{Gr_N(S_N(A_N(V)))}([u]_{NN})([v + O_N(V)])_N = [\text{Res}_x x^{-1} Y_{S_N(A_N(V))}(x^{L_0(0)} u, x)v + O_N(V)]_N
\]

Then \( \partial_{Gr_N(S_N(A_N(V)))}([u]_{NN})([v + O_N(V)])_N = 0 \) if and only if \( o_{S_N(A_N(V))}(u)v = 0 \). If \( [u]_{NN} \in Q^\infty(V) \), then \( \partial_{S_N(A_N(V))}([u]_{NN})([v + O_N(V)])_N = 0 \) for all \( v \in V \). So \( o_{S_N(A_N(V))}(u)v = 0 \) for all \( v \in V \). Thus \( o_{S_N(A_N(V))}(u) = 0 \) and \( u \in \ker o_{S_N(A_N(V))} = O_N(V) \). Then \( f_{NN}([u]_{NN}) = u + O_N(V) = 0 + O_N(V) \) or in other words, \( [u]_{NN} \in \ker f_{NN} \). On the other hand, if \( [u]_{NN} \in \ker f_{NN} \), that is, \( f_{NN}([u]_{NN}) = 0 + O_N(V) \), then \( u \in O_N(V) \). From the definitions of \( O_N(V) \) and \( O^\infty(V) \), we have \( [u]_{NN} \in O^\infty(V) \subset Q^\infty(V) \). This shows that \( \ker f_{NN} = Q^\infty(V) \cap U_{NN} \). It is clear that \( f_{NN} \) is surjective. In particular, \( f_{NN} \) induces a linear isomorphism, still denoted by \( f_{NN} \), from \( A_{NN}(V) \) to \( A(V) \).

For \( u, v \in V \),

\[
f_{NN}([u]_{NN} \diamond [v]_{NN})
= \text{Res}_x T_{2N+1}((x + 1)^{-N-1})(1 + x)^N f_{NN}([Y_V((1 + x)^{L_0(0)} u, x)v]_{NN})
= \text{Res}_x T_{2N+1}((x + 1)^{-N-1})(1 + x)^N Y_V((1 + x)^{L_0(0)} u, x)v + O_N(V)
= \text{Res}_x \sum_{m=0}^{N} \left( \frac{-N - 1}{m} \right) x^{-N-m-1}(1 + x)^N Y_V((1 + x)^{L_0(0)} u, x)v + O_N(V)
= u *_N v + O_N(V)
= (u + O_N(V)) *_N (v + O_N(V)).
\]

Therefore \( f_{NN} \) is an isomorphism of associative algebra. Since \( 1 + O_N(V) \) is the identity of \( A_N(V) \), \( [1]_{NN} + O^\infty(V) \) is the identity of \( A_{NN}(V) \). If \( V \) is a vertex operator algebra with the conformal vector \( \omega \), since \( \omega + O_N(V) \) is in the center of \( A_N(V) \), \( [\omega]_{NN} + O^\infty(V) \) is in the center of \( A_{NN}(V) \).
4.2 Associative algebras from finite matrices

We now introduce new subalgebras of $A^\infty(V)$. For $N \in \mathbb{N}$, let $U_N^N(V)$ be the space of all $(N + 1) \times (N + 1)$ matrices with entries in $V$. It is clear that $U_N^N(V)$ can be canonically embedded into $U_0^\infty(V)$ as a subspace. We shall view $U_N^N(V)$ as a subspace of $U_0^\infty(V)$ in this paper. As a subspace of $U_0^\infty(V)$, $U_N^N(V)$ consists of infinite matrices in $U^\infty(V)$ whose $(k, l)$-th entries for $k > N$ or $l > N$ are all 0 and is spanned by elements of the form $[v]_{kl}$ for $v \in V, k, l = 0, \ldots, N$.

Let

$$1^N = \sum_{k=0}^{N} [1]_{kk},$$

that is, $1^N$ is the element of $U_N^N(V)$ with the only nonzero entries to be equal to 1 at the diagonal $(k, k)$-th entries for $k = 0, \ldots, N$. By (2.2), we have

$$1^N \odot [v]_{kl} = \text{Res}_x T_{k+l+1}((x + 1)^{-l-1})(1 + x)^l [Y_V((1 + x)L^{(0)}1, x)v]_{kl} = [v]_{kl}$$

for $v \in V$ and $k, l = 0, \ldots, N$. So $1^N$ is a left identity of $U_N^N(V)$ with respect to the product $\odot$. Note that for $v \in V$ and $k, l = 0, \ldots, N$,

$$[v]_{kl} \odot 1^N = \text{Res}_x T_{k+l+1}((x + 1)^{-k-1})(1 + x)^l [Y_V((1 + x)L^{(0)}v, x)1]_{kl} = [v]_{kl} \odot 1^\infty.$$

This formula together with (2.21) immediately gives

$$[v]_{kl} \odot 1^N = \sum_{m=0}^{N} \binom{-k-1}{m} \left( L_V(-1) + L_V(0) + I \right)_v^{kl}$$

(4.1)

for $v \in V$ and $k, l = 0, \ldots, N$.

By (2.3), for $u, v \in V$ and $k, n, l = 0, \ldots, N$,

$$[u]_{kn} \odot [v]_{nl} = \text{Res}_x T_{k+l+1}((x + 1)^{-k+n-l-1})(1 + x)^l [Y_V((1 + x)L^{(0)}u, x)v]_{kl} \in U_N^N(V).$$

(4.2)

So $U_N^N(V)$ is closed under the product $\odot$. Let

$$A_N^N(V) = \{ v + Q^\infty(V) \mid v \in U_N^N(V) \} = \pi_{A^\infty(V)}(U_N^N(V)),$$

where $\pi_{A^\infty(V)}$ is the projection from $U^\infty(V)$ to $A^\infty(V)$. Then $A_N^N(V)$ is spanned by elements of the form $[v]_{kl} + Q^\infty(V)$ for $v \in V$ and $k, l = 0, \ldots, N$.

**Proposition 4.3** The subspace $A_N^N(V)$ is closed under $\odot$ and is thus a subalgebra of $A^\infty(V)$ with the identity $1^N + Q^\infty(V)$.

**Proof.** By (4.2), we have

$$([u]_{kn} + Q^\infty(V)) \odot ([v]_{nl} + Q^\infty(V))$$

$$= \text{Res}_x T_{k+l+1}((x + 1)^{-k+n-l-1})(1 + x)^l [Y_V((1 + x)L^{(0)}u, x)v]_{kl} + Q^\infty(V)$$

$$\in A_N^N(V)$$
for \( u, v \in V \) and \( k, n, l = 0, \ldots, N \). Thus \( A^N(V) \) is closed under \( \diamond \) and is thus a subalgebra of \( A^\infty(V) \).

Since \( 1^N \) is a left identity of \( U^N(V) \) with respect to the product \( \diamond \), \( 1^N + Q^\infty(V) \) is a left identity of \( A^N(V) \). Since
\[
[v]_{kl} \diamond 1^N = [v]_{kl} \diamond 1^\infty \equiv [v]_{kl} \mod Q^\infty(V),
\]
\( 1^N + Q^\infty(V) \) is also a right identity of \( A^N(V) \). In particular, it is the identity of \( A^N(V) \). $\blacksquare$

**Remark 4.4** We have derived \( A^N(V) \) as a subalgebra of \( A^\infty(V) \). One can certainly obtain \( A^N(V) \) directly starting with the space \( U^N(V) \) of \( (N + 1) \times (N + 1) \) matrices with entries in \( V \).

**Remark 4.5** It is clear from the definition that \( A_{nn}(V) \) for \( n = 0, \ldots, N \) are subalgebras of \( A^N(V) \). In particular, the Zhu algebra \( A(V) \) in \([Z]\) and its generalizations \( A_n(V) \) for \( n = 0, \ldots, N \) by Dong, Li and Mason in \([DLM]\) can be viewed as subalgebras of \( A^N(V) \). In the case \( N = 0 \), \( A^0 \) is equal to \( A_{00}(V) \) and is thus isomorphic to the Zhu algebra \( A(V) \) by Theorem 4.1.

We say that \( V \) is of positive energy if \( V = \bigsqcup_{n \in \mathbb{N}} V_n \) and \( V(0) = 0 \). (In some papers, \( V \) is of positive energy is said to be of CFT type.) We recall that for \( n \in \mathbb{N} \), \( V \) is \( C_n \)-cofinite if \( \dim V/C_n(V) < \infty \), where \( C_n(V) \) is the subspace of \( V \) spanned by elements of the form \((Y_v)_n(u) v \) for \( u, v \in V \).

**Theorem 4.6** Assume that \( V \) is of positive energy and \( C_2 \)-cofinite. Then \( A^N(V) \) is finite dimensional.

**Proof.** By Theorem 11 in \([GN]\) (see Proposition 5.5 in \([AN]\)), \( V \) is also \( C_n \)-cofinite for \( n \geq 2 \). In particular, \( V \) is \( C_{k+l+2} \)-cofinite for \( k, l = 0, \ldots, N \). By definition, \( C_{k+l+2}(V) \) are spanned by elements of the form \((Y_v)_{-k-l-2}(u) v \) for \( u, v \in V \). Since \( V \) is \( C_{k+l+2} \)-cofinite, there exists a finite dimensional subspace \( X_{k+l} \) of \( V \) such that \( X + C_{k+l+2}(V) = V \). Let \( U^N(X) \) be the subspace of \( U^N(V) \) consisting matrices in \( U^N(V) \) whose entries are in \( X \). Since \( X \) is finite dimensional, \( U^N(X) \) is also finite dimensional. We now prove \( U^N(X) + (O^\infty(V) \cap U^N(V)) = U^N(V) \). To prove this, we need only prove that every element of \( U^N(V) \) of the form \([v]_{kl} \) for \( v \in V \) and \( 0 \leq k, l \leq N \), can be written as \([v]_{kl} = [v_1]_{kl} + [v_2]_{kl} \), where \( v_1 \in X_{k+l} \) and \( v_2 \in V \) such that \([v_2]_{kl} \in O^\infty(V) \). We shall denote the subspace of \( V \) consisting of elements \( v \) such that \([v]_{kl} \in O^\infty(V) \) by \( O^\infty_{kl}(V) \). Then what we need to prove is \( V = X_{k+l} + O^\infty_{kl}(V) \).

We can always take \( X_{k+l} \) to be a subspace of \( V \) containing \( 1 \). We use induction on the weight of \( v \). When \( wt v = 0 \), \( v \) is proportional to \( 1 \) and can indeed be written as \( v = v + 0 \), where \( v \in X \) and \( 0 \in O^\infty_{kl}(V) \).

Assume that when \( wt v = p < q \), \( v = v_1 + v_2 \), where \( v_1 \in X_{k+l} \) and \( v_2 \in O^\infty_{kl}(V) \). Then since \( V \) is \( C_{k+l+2} \)-cofinite, for \( v \in V(q) \), there exists homogeneous \( u_1 \in X_{k+l} \) and homogeneous

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$u^i, v^i \in V$ for $i = 1, \ldots, m$ such that $v = u_1 + \sum_{i=1}^m u_{i-k-l-2}^i v^i$. Moreover, we can always find such $u_1$ and $u^i, v^i \in V$ for $i = 1, \ldots, m$ such that $\text{wt } u_1 = \text{wt } u_{i-k-l-2}^i v^i = \text{wt } v = q$. Since

$$\text{wt } u_{n-k-l-2}^i v^i < \text{wt } u_{i-k-l-2}^i v^i = \text{wt } v = q$$

for $i = 1, \ldots, m$ and $n \in \mathbb{Z}_+$, by induction assumption, $u_{n-k-l-2}^i v^i \in X_{k+l} + O_{kl}^\infty(V)$ for $i = 1, \ldots, m$ and $k \in \mathbb{Z}_+$. Thus

$$v = u_1 + \sum_{i=1}^m u_{i-k-l-2}^i v^i$$

$$= u_1 + \sum_{i=1}^m \text{Res}_x x^{-k-l-2} (1 + x)^l Y((1 + x)^L(0) u^i, x)v^i$$

$$- \sum_{i=1}^m \sum_{n \in \mathbb{Z}_+} \left( \text{wt } u^i + l \right) u_{n-k-l-2}^i v^i.$$ 

By definition,

$$[\text{Res}_x x^{-k-l-2} (1 + x)^l Y((1 + x)^L(0) u^i, x)v^i]_{kl} \in O^\infty(V).$$

Thus

$$\text{Res}_x x^{-k-l-2} (1 + x)^l Y((1 + x)^L(0) u^i, x)v^i \in O^\infty(V).$$

Thus we have $v = v_1 + v_2$, where $v_1 \in X_{k+l}$ and $v_2 \in O_{kl}^\infty(V)$. By induction principle, we have $V = X_{k+l} + O_{kl}^\infty(V)$.

We now have proved $U^N(X) + (O^\infty(V) \cap U^N(V)) = U^N(V)$. Since $O^\infty(V) \cap U^N(V) \subset Q^\infty(V) \cap U^N(V)$, we also have $U^N(X) + (Q^\infty(V) \cap U^N(V)) = U^N(V)$. Since $U^N(X)$ is finite dimensional, $A^N(V)$ is finite dimensional.

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### 5 Lower-bounded generalized $V$-modules and graded $A^N(V)$-modules

By Theorem 2.8, the associated graded space $Gr(W)$ of a filtration of a lower-bounded generalized $V$-module $W$ is a graded $A^\infty(V)$-module. In this section, for $N \in \mathbb{N}$, we give an $A^N(V)$-module structure to a subspace of $Gr(W)$ and use it to study $W$.

Let $N \in \mathbb{N}$. Let $W$ be a lower-bounded generalized $V$-module. Since $A^N(V)$ is a subalgebra of $A^\infty(V)$, $Gr(W)$ as an $A^\infty(V)$-module is also an $A^N(V)$-module. Let

$$Gr^N(W) = \bigcap_{n=0}^N Gr_n(W) \subset Gr(W).$$

By the definition of $\vartheta_{Gr(W)}$, we see that for $v \in A^N(V)$ and $[w]_n \in Gr^N(W)$, $\vartheta_{Gr(W)}(v)[w]_n \in Gr^N(W)$. Thus $Gr^N(W)$ is an $A^N(W)$-submodule of $Gr(W)$. But $Gr^N(W)$ has some additional structures and properties and we are only interested in those $A^N(W)$-modules having
these additional structures and properties. Similar to Definition 3.1, we have the following notion:

**Definition 5.1** Let $M$ be an $A^N(V)$-module $M$ with the $A^N(V)$-module structure on $M$ given by $\partial_M : A^N(V) \to \text{End} M$. We say that $M$ is a graded $A^N(V)$-module if the following conditions are satisfied:

1. $M = \bigsqcup_{n=0}^{N} G_n(M)$ such that for $v \in V$ and $k, l = 0, \ldots, N$, $\partial_M([v]_{kl} + Q^\infty(V))$ maps $G_n(M)$ for $0 \leq n \leq N$ to 0 when $n \neq l$ and to $G_k(M)$ when $n = l$.
2. For $w \in G_l(M)$, if $\partial_M([v]_{0l} + Q^\infty(V))w = 0$ for all $v \in V$, then $w = 0$.
3. $M$ is a direct sum of generalized eigenspaces of of an operator $L_M(0)$ on $M$ and the real parts of the eigenvalues of $L_M(0)$ has a lower bound.
4. There is a linear map $L_M(-1) : \bigsqcup_{n=0}^{N-1} G_{n-1}(M) \to \bigsqcup_{n=1}^{N} G_n(M)$ mapping $G_n(M)$ to $G_{n+1}(M)$ for $n = 0, \ldots, N - 1$.
5. The commutator relations

$$[L_M(0), L_M(-1)] = L_M(-1),$$

$$[L_M(0), \partial_M([v]_{kl} + Q^\infty(V))] = (k - l)\partial_M([v]_{kl} + Q^\infty(V)),$$

$$[L_M(-1), \partial_M([v]_{pl} + Q^\infty(V))] = \partial_M([L_V(-1)v]_{(p+1)l} + Q^\infty(V))$$

hold for $v \in V$, $k, l = 0, \ldots, N$ and $p = 0, \ldots, N - 1$.

Let $M_1$ and $M_2$ be graded $A^N(V)$-modules. An graded $A_N(V)$-module map from $M_2$ to $M_2$ is an $A^N(V)$-module map $f : M_1 \to M_2$ such that $f(G_n(M_1)) \subset G_n(M_2)$ for $n = 0, \ldots, N$, $f \circ L_{M_1}(0) = L_{M_2}(0) \circ f$ and $f \circ L_{M_1}(-1) = L_{M_2}(-1) \circ f$. A graded $A^N(V)$-submodule of a graded $A^N(V)$-module $M$ is an $A^N(V)$-submodule $M_0$ of $M$ such that with the $A^N(V)$-module structure, the $N$-grading induced from $M$ and the operators $L_M(0)|_{M_0}$ and $L_M(-1)|_{M_0}$, $M_0$ is a graded $A^N(V)$-module. A graded $A^N(V)$-module $M$ is said to be generated by a subset $S$ if $M$ is equal to the smallest graded $A^N(V)$-submodule containing $S$, or equivalently, $M$ is spanned by homogeneous elements obtained by applying elements of $A^N(V)$, $L_M(0)$ and $L_M(-1)$ to homogeneous summands of elements of $S$. A graded $A^N(V)$-module is said to be irreducible if it has no nonzero proper graded $A^N(V)$-modules. A graded $A^N(V)$-module is said to be completely reducible if it is a direct sum of irreducible graded $A^N(V)$-modules.

From the discussion above and the property of $Gr^N(W)$, we obtain immediately:

**Proposition 5.2** For a lower-bounded generalized $V$-module $W$, $Gr^N(W)$ is a graded $A^N(V)$-module. Let $W_1$ and $W_2$ be lower-bounded generalized $V$-modules and $f : W_1 \to W_2$ a $V$-module map. Then $f$ induces a graded $A^N(V)$-module map $Gr^N(f) : Gr^N(W_1) \to Gr^N(W_2)$. 
We have the following results on irreducible and completely reducible lower-bounded generalized \( V \)-modules without additional conditions:

**Proposition 5.3** Let \( W \) be a lower-bounded generalized \( V \)-module. If \( W \) is irreducible or completely reducible, then \( \text{Gr}^N(W) \) is equivalent to \( T_N(W) \) as an \( A^N(V) \)-module and is also irreducible or completely reducible, respectively.

**Proof.** Let \( W \) be irreducible. By Proposition 3.3, \( \Omega_n(W) = T_n(W) \) for \( n = 0, \ldots, N \). Then \( T_N(W) \) is a graded \( A^N(V) \)-module equivalent to \( \text{Gr}^N(W) \). We need to prove that the graded \( A^N(V) \)-module \( T_N(W) \) is irreducible.

Let \( M \) be a nonzero graded \( A^N(V) \)-submodule of \( T_N(W) \). We use the construction in Section 5 of [H3] to construct a universal lower-bounded generalized \( V \)-module \( \widehat{M} \) from \( M \). We take the generating fields for the grading-restricted vertex algebra \( V \) to be \( Y_{\nu}(v, x) \) for \( v \in V \). By definition, \( M \) is a direct sum of generalized eigenspaces of \( L_M(0) \) and the real parts of the eigenvalues of \( L_M(0) \) have a lower bound \( B \in \mathbb{R} \). We take \( M \) and \( B \) in Section 5 of [H3] to be the given graded \( A^N(V) \)-module \( M \) and the lower bound \( B \) above. Using the construction in Section 5 of [H3], we obtain a universal lower-bounded generalized \( V \)-module \( \widehat{M}_{\nu} \). For simplicity, we shall denote it simply by \( \widehat{M} \). By the universal property of \( \widehat{M} \) (Theorem 5.2 in [H3]), for the embedding map \( e_M : M \to T_N(W) \), there is a unique \( V \)-module map \( \widehat{e}_M : \widehat{M} \to W \) such that \( \widehat{e}_M|_M = e \). Then \( \widehat{e}_M(\widehat{M}) \) is a generalized \( V \)-submodule of \( W \) generated by \( M \). It is nonzero since \( M \subset \widehat{e}_M(\widehat{M}) \). Since \( W \) is irreducible, it must be \( W \). Then \( W \) is generated by \( M \). In particular, \( T_N(W) \) is obtained by applying the components of the vertex operators on \( W \), \( L_W(0) \) and \( L_W(-1) \) to elements of \( M \). Since the components of the vertex operators on \( W \) is by definition the actions of elements of \( A^N(V) \), \( L_W(0) \) and \( L_M(-1) \) preserving \( T_N(W) \), we see that as a graded \( A^N(V) \)-module, \( T_N(W) \) is generated by \( M \). But \( M \) itself is an \( A_N(V) \)-submodule of \( T_N(W) \). So we have \( M = T_N(W) \). Thus \( T_N(W) \) as a graded \( A^N \)-module is irreducible.

If \( W \) is completely reducible, by Proposition 3.3 again, \( \Omega_n(W) = T_n(W) \) for \( n = 0, \ldots, N \). Then \( T_N(W) \) is a graded \( A^N(V) \)-module equivalent to \( \text{Gr}^N(W) \). Since \( W \) is completely reducible, \( W = \bigoplus_{\mu \in \mathcal{M}} W^\mu \), where \( W^\mu \) for \( \mu \in \mathcal{M} \) are irreducible lower-bounded generalized \( V \)-modules. By the definition of \( T_N(W) \), we have \( T_N(W) = \bigoplus_{\mu \in \mathcal{M}} T_N(W^\mu) \). From what we have proved above, for \( \mu \in \mathcal{M} \), \( T_N(W^\mu) \) is an irreducible graded \( A^N(V) \)-module. Thus we see that \( T_N(W) \) is completely reducible.

Let \( M \) be a graded \( A_N(V) \)-module given by the map \( \vartheta_M : A^N(V) \to \text{End} M \) and operators \( L_M(0) \) and \( L_M(-1) \). We now construct a lower-bounded generalized \( V \)-module \( S^N(M) \) from \( M \). We use the construction in Section 5 of [H3]. We take the generating fields for the grading-restricted vertex algebra \( V \) to be \( Y_{\nu}(v, x) \) for \( v \in V \). By definition, \( M \) is a direct sum of generalized eigenspaces of \( L_M(0) \) and the real parts of the eigenvalues of \( L_M(0) \) has a lower bound \( B \in \mathbb{R} \). We take \( M \) and \( B \) in Section 5 of [H3] to be the given graded \( A_N(V) \)-module \( M \) and the lower bound \( B \) above. Using the construction in Section

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forms

Theorems 3.3 and 3.4 in [H4] state that elements of the form $M$ of elements of the form $(\hat{v})$ we shall denote it simply by $\hat{M}$. For simplicity, we shall denote it simply by $\hat{M}$.

By Theorem 3.4 in [H4] and the construction in Section 5 of [H3] and by identifying elements of the form $(\psi_{M})_{\hat{v},l}$ with basis elements $w^a \in M$ for $a \in A$ for a basis $\{w^a\}_{a \in A}$ of $M$, we see that $\hat{M}$ is generated by $M$ (in the sense of Definition 3.1 in [H4]). Moreover, Theorems 3.3 and 3.4 in [H4] state that elements of the form $L_{\hat{M}}(-1)^{p}w^a$ for $p \in \mathbb{N}$ and $a \in A$ are linearly independent and $\hat{M}$ is spanned by elements obtained by applying the components of the vertex operators to these elements. In particular, we identify $M$ as a subspace of $\hat{M}$. Let $J_{M}$ be the generalized $V$-submodule of $\hat{M}$ generated by elements of the forms

$$\text{Res}_{x}x^{l-k-1}Y_{\hat{M}}(x^{L_{V}(0)}v, x)w$$

for $l = 0, \ldots, N$, $k \in -\mathbb{Z}_{+}$ and $w \in G_{l}(M)$,

$$\text{Res}_{x}x^{l-k-1}Y_{\hat{M}}(x^{L_{V}(0)}v, x)w - \vartheta_{M}(\hat{v}_{kl})w$$

for $v \in V$, $k, l = 0, \ldots, N$ and $w \in G_{l}(M)$ and

$$L_{\hat{M}}(-1)w - L_{M}(-1)w$$

for $w \in \bigsqcup_{n=0}^{N-1} G_{n}(M)$.

Let $S^{N}(M) = \hat{M}/J_{M}$. Then $S^{N}(M)$ is a lower-bounded generalized $V$-module. Let $\pi_{S^{N}(M)}$ be the projection from $\hat{M}$ to $S^{N}(M)$. Since $\hat{M}$ is generated by $M$ (in the sense of Definition 3.1 in [H4]), $S^{N}(M)$ is generated $\pi_{S^{N}(M)}(M)$ (in the same sense). In particular, $S^{N}(M)$ is spanned by elements of the form

$$\text{Res}_{x}x^{(l+p)-n-1}Y_{S^{N}(M)}(x^{L_{V}(0)}v, x)L_{S^{N}(M)}(-1)^{p}\pi_{S^{N}(M)}(w)$$

for $v \in V$, $l = 0, \ldots, N$, $n, p \in \mathbb{N}$ and $w \in G_{l}(M)$. For $n \in \mathbb{N}$, let $G_{n}(S^{N}(M))$ be the subspace of $S^{N}(M)$ spanned by elements of the form (5.4) for $v \in V$, $l = 0, \ldots, N$, $p \in \mathbb{N}$ and $w \in G_{l}(M)$.

Proposition 5.4 Let $M$ be a graded $A^{N}(V)$-module.

1. For $0 \leq n \leq N$, $G_{n}(S^{N}(M)) = \pi_{S^{N}(M)}(G_{n}(M))$ and for $0 \leq n_{1}, n_{2} \leq N$, $n_{1} \neq n_{2}$, $G_{n_{1}}(S^{N}(M)) \cap G_{n_{2}}(S^{N}(M)) = 0$. Moreover, $S^{N}(M) = \bigsqcup_{n=0}^{N} G_{n}(S^{N}(M))$ and

$$\pi_{S^{N}(M)}(M) = \bigsqcup_{n=0}^{N} G_{n}(S^{N}(M)).$$

2. For $n = 0, \ldots, N$,

$$\pi_{S^{N}(M)}\left(\prod_{j=0}^{n} G_{j}(M)\right) = \prod_{j=0}^{n} G_{j}(S^{N}(M)) \subset \Omega_{n}(S^{N}(M))$$

and

$$\pi_{S^{N}(M)}\left(\prod_{j=n}^{N} G_{j}(M)\right) \cap \Omega_{n}(S^{N}(M)) = \left(\prod_{j=n}^{N} G_{j}(S^{N}(M))\right) \cap \Omega_{n}(S^{N}(M)) = 0.$$
3. $M$ is equivalent to a graded $A^N(V)$-submodule of $Gr^N(S^N(M))$.

Proof. By definition, $G_n(S^N(M))$ for $0 \leq n \leq N$ is spanned by elements of the form (5.4) for $v \in V$, $l = 0, \ldots, N$, $p \in \mathbb{N}$ and $w \in G_l(M)$. Using the $L(-1)$-commutator formula for the vertex operator map $Y_{S^N(M)}$, we see that it is also spanned by elements of the form

$$\text{Res}_{x} x^{-k-1} L_{S^N(M)}(-1)^p Y_{S^N(M)}(x^{L_{V}(0)}v, x) \pi_{S^N(M)}(w)$$

(5.7)

for $v \in V$, $l, k = 0, \ldots, N$, $p = 0, \ldots, n - k$ and $w \in G_l(M)$. Since elements of the forms (5.2) and (5.3) are in $J_M$, we see that (5.7) is in fact equal to

$$\pi_{S^N(M)}(L_{M}(-1)^p \vartheta_{M}([v]_{kl} + Q^\infty(V))w) \in \pi_{S^N(M)}(G_n(M)).$$

(5.8)

Since $L_{M}(-1)^p \vartheta_{M}([v]_{kl+p} + Q^\infty(V))w$ for $v \in V$, $l, k = 0, \ldots, N$, $p = 0, \ldots, n - k$ and $w \in G_l(M)$ certainly span $G_n(M)$ (in fact, we need only $v = 1$, $k = l = n$, $p = 0$ and $w \in G_n(W)$) and elements of the form (5.7) for $v \in V$, $l, k = 0, \ldots, N$, $p = 0, \ldots, n - k$ and $w \in G_l(M)$ span $G_n(S(G))$ for $0 \leq n \leq N$, we see that elements of the form (5.8) for $v \in V$, $l, k = 0, \ldots, N$, $p = 0, \ldots, n - k$ and $w \in G_l(M)$ also span $G_n(S^N(M))$. Thus we obtain $G_n(S^N(M)) = \pi_{S^N(M)}(G_n(M))$ for $n = 0, \ldots, N$. When $n_1 \neq n_2$, we know $G_{n_1}(M) \cap G_{n_2}(M) = 0$. Then $G_{n_1}(S^N(M)) \cap G_{n_2}(S^N(M)) = \pi_{S(G)}(G_{n_1}(M) \cap G_{n_2}(M)) = 0$. Since $S^N(M)$ is spanned by elements of the form (5.4) for $v \in V$, $l = 0, \ldots, N$, $n, p \in \mathbb{N}$ and $w \in G_l(M)$, by the definition of $G_n(S^N(M))$, we have $S^N(M) = \bigcup_{n \in \mathbb{N}} G_n(S^N(M))$. Since $G_n(S^N(M)) = \pi_{S^N(M)}(G_n(M))$ for $n = 0, \ldots, N$, we have

$$\pi_{S^N(M)}(M) = \prod_{n=0}^{N} \pi_{S^N(M)}(G_n(M)) = \prod_{n=0}^{N} G_n(S^N(M)).$$

By definition, for $0 \leq j \leq n \leq N$, $G_j(S^N(M)) \subset \Omega_n(S^N(M))$. Then for $j = 0, \ldots, n$,

$$\pi_{S^N(M)}(G_j(M)) = G_j(S^N(M)) \subset \Omega_j(S^N(M)) \subset \Omega_n(S^N(M)).$$

So we obtain (5.5). By Condition 2 in Definition 5.1 of graded $A^N(V)$-module, nonzero elements of $G_j(M)$ for $N \geq j > n$ are not in $\Omega_n(M)$. From the construction of $\hat{M}$, nonzero elements of the form (5.1), (5.2) or (5.3) are not in $M \subset \hat{M}$. In particular, we see that the intersection of $J(M)$ with $M$ is 0. So $\pi_{S^N(M)}|_M$ is injective. Since $\pi_{S^N(M)}|_M$ is injective, we see that nonzero elements of $G_j(S^N(M)) = \pi_{S^N(M)}(G_j(M))$ for $N \geq j > n$ are not in $\Omega_n(S^N(M))$. Thus we obtain (5.6).

For $0 \leq n \leq N$ and $w \in G_n(M)$, we define $f_M(w) = \pi_{S^N(M)}(w) + \Omega_{n-1}(S^N(M))$. Since $\pi_{S^N(M)}(w) \in \Omega_n(S^N(M))$, $f_M(w) \in Gr_n(S^N(M))$. Therefore we obtain linear map $f_M : M \rightarrow Gr^N(S^N(M))$. It is clear from the definition that $f_M$ is in fact a graded $A^N(V)$-module map. If for some $0 \leq n \leq N$ and $w \in G_n(M)$, $f_M(w) = 0$, then $\pi_{S^N(M)}(w) \in \Omega_{n-1}(S^N(M))$. But we have proved above that nonzero elements of $\pi_{S^N(M)}(G_n(M))$ are not in $\Omega_{n-1}(S^N(M))$. So $\pi_{S^N(M)}(w) = 0$. Since $\pi_{S^N(M)}|_M$ is injective, we obtain $w = 0$. So $f_M$ is injective. Thus $M$ is equivalent to the graded $A^N(V)$-submodule $f_M(M)$ of $Gr^N(S^N(M))$. 

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Remark 5.5 As in the case of $S(G)$ in Section 3, our construction of the lower-bounded generalized $V$-module $\widehat{M}$ depends on the lower bound $B$ of the real parts of the eigenvalues of $L_M(0)$. But by Proposition 5.4, $S^N(M)$ depends only on $M$, not on $B$.

Theorem 5.6 For $N \in \mathbb{N}$, the set of the equivalence classes of irreducible lower-bounded generalized $V$-modules is in bijection with the set of the equivalence classes of irreducible graded $A^N(V)$-modules.

Proof. Recall the set $[\mathfrak{M}]_{\text{irr}}$ of the equivalence classes of irreducible lower-bounded generalized $V$-modules in the proof of Theorem 3.7. Let $[\mathfrak{M}^N]$ be the set of the equivalence classes of irreducible graded $A^N(V)$-modules. Given an irreducible lower-bounded generalized $V$-module $W$, by Theorem 5.3, $Gr^N(W) = T_N(W)$ is an irreducible graded $A^N(V)$-module. Thus we obtain a map $f : [\mathfrak{M}]_{\text{irr}} \to [\mathfrak{M}^N]$ given by $f([W]) = [T_N(W)]$, where $[W] \in [\mathfrak{M}]_{\text{irr}}$ is the equivalence class containing the irreducible lower-bounded generalized $V$-module $W$ and $[T_N(W)] \in [\mathfrak{M}^N]$ is the equivalence class containing the irreducible graded $A^N(V)$-module $T_N(W)$.

Given an irreducible graded $A^N(V)$-module $M$, we have the lower-bounded generalized $V$-module $S^N(M)$ generated by $\pi_{S^N(M)}(M)$. The main difference of the proof here and the proof of Theorem 3.7 is that we do not know whether $S^N(M)$ is irreducible. So we need to take a quotient of $S^N(M)$. Since $M$ is an irreducible graded $A^N(V)$-module, it is generated by any nonzero element. Since $S^N(M)$ is generated by $\pi_{S^N(M)}(M)$, it is also generated by any element $w_0 \in \pi_{S^N(M)}(M)$. Then by Theorem 4.7 in [H4], there is a maximal generalized $V$-submodule $J_{\pi_{S^N(M)},w_0}$ of $S^N(M)$ such that $J_{\pi_{S^N(M)},w_0}$ does not contain $w_0$ and $S^N(M)/J_{\pi_{S^N(M)},w_0}$ is irreducible. The maximal generalized $V$-submodule $J_{\pi_{S^N(M)},w_0}$ is in fact independent of $w_0 \in \pi_{S^N(M)}(M)$. We prove this fact by proving that no nonzero element of $\pi_{S^N(M)}(M)$ is in $J_{\pi_{S^N(M)},w_0}$. In fact, if a nonzero $w \in \pi_{S^N(M)}(M)$ is also in $J_{\pi_{S^N(M)},w_0}$, since the actions of components of vertex operators on $w$ are equal to the actions of elements of $A^N(V)$ and $M$ is generated also by $w$, we see that $w_0$ must also be in $J_{\pi_{S^N(M)},w_0}$. Contradiction. Thus $J_{\pi_{S^N(M)},w_0}$ is in fact the maximal generalized $V$-submodule of $S^N(M)$ such that it does not contain nonzero elements of $M$. We denote it by $\widetilde{J}_M$, which depends only on $\pi_{S^N(M)}(M)$, or equivalently, $M$. Thus we obtain a map $g : [\mathfrak{M}^N]_{\text{irr}} \to [\mathfrak{M}]_{\text{irr}}$ given by $g([M]) = [S^N(M)/\widetilde{J}_M]$.

We still need to show that the two maps above are inverses of each other. Let $M$ be an irreducible graded $A^N(V)$-module. Since $S^N(M)/\widetilde{J}_M$ is irreducible, by Proposition 3.3, $Gr_N(S^N(M)/\widetilde{J}_M)$ is an irreducible graded $A^N(V)$-module. By Proposition 5.4, $M$ is equivalent to a graded $A^N(V)$-submodule of $Gr_N(S^N(M))$. As in the proof of Proposition 5.4, we denote this equivalence by $f_M$. Let $\pi_{\widetilde{J}_M} : S^N(M) \to S^N(M)/\widetilde{J}_M$ be the projection map. Since $\widetilde{J}_M \cap \pi_{S^N(M)}(M) = 0$, $\pi_{\widetilde{J}_M}\big|_{\pi_{S^N(M)}(M)}$ is injective and in particular, is not 0. The $V$-module map $\pi_{\widetilde{J}_M}$ induces a graded $A^N(V)$-module map $Gr^N(\pi_{\widetilde{J}_M}) : Gr_N(S^N(M)) \to Gr_N(S^N(M)/\widetilde{J}_M)$. Since $\pi_{\widetilde{J}_M}\big|_{\pi_{S^N(M)}(M)}$ is not 0, the restriction $Gr^N(\pi_{\widetilde{J}_M})\big|_{\pi_{S^N(M)}(M)}$ to the image of $M$ under $f_M$ is also not 0. Consider the $A^N(V)$-module map $Gr^N(\pi_{\widetilde{J}_M}) \circ f_M$:
\[ M \to Gr_N(S^N(M)/\tilde{J}_M). \] Since \( f_M \) is injective and \( Gr_N(\pi_{\tilde{J}_M})|_{f_M(M)} \neq 0 \), \( Gr_N(\pi_{\tilde{J}_M}) \circ f_M \) is not 0. But both \( M \) and \( Gr_N(S^N(M)/\tilde{J}_M) \) are irreducible. So \( Gr_N(\pi_{\tilde{J}_M}) \circ f_M \) must be an equivalence of graded \( A^N(V) \)-modules. Moreover, by Proposition 5.3, \( Gr_N(S^N(M)/\tilde{J}_M) \) is equivalent to \( T_N(S^N(M)/\tilde{J}_M) \). So \( M \) is equivalent to \( T_N(S^N(M)/\tilde{J}_M) \). Thus \([M] = [T_N(M/\tilde{J}_M)]\). This means \( f(g([M])) = [M] \). So we obtain \( f \circ g = 1_{[\mathfrak{M}]_{\text{irr}}} \).

Let \( W \) be an irreducible lower-bounded generalized \( V \)-module. By Theorem 5.3, \( T_N(W) \) is an irreducible \( A^N(V) \)-module. We then have a lower-bounded generalized \( V \)-module \( S^N(T_N(W)) \). By the universal property of \( T_N(W) \), there is a unique \( V \)-module map \( \tilde{1}_{T_N(W)} : T_N(W) \to W \) such that \( \tilde{1}_{T_N(W)}|_{T_N(W)} = 1_{T_N(W)} \), where \( 1_{T_N(W)} \) is the identity operator on \( T_N(W) \). Since \( W \) is irreducible, the image of \( T_N(W) \) under \( \tilde{1}_{T_N(W)} \) is either 0 or \( W \). Since \( \tilde{1}_{T_N(W)}|_{T_N(W)} = 1_{T_N(W)} \), the image of \( T_N(W) \) under \( \tilde{1}_{T_N(W)} \) cannot be 0 and thus must be \( W \). In particular, \( \tilde{1}_{T_N(W)} \) is surjective. Moreover, since \( J_{T_N(W)} \) is generated by \( (5.1), (5.2) \) and \( (5.3) \) with \( M = T_N(W) \), the image of \( J_{T_N(W)} \) under \( \tilde{1}_{T_N(W)} \) is 0, that is, \( J_{T_N(W)} \in \ker\tilde{1}_{T_N(W)} \). In particular, \( \tilde{1}_{T_N(W)} \) induces a surjective \( V \)-module map \( f_{T_N(W)} : S^N(T_N(W)) = T_N(W)/J_{T_N(W)} \to W \). Since \( J_{T_N(W)} \cap T_N(W) = 0 \), \( f_{T_N(W)}(T_N(W)) = T_N(W) \). We have a maximal generalizer \( \tilde{J}_{T_N(W)} \) of \( S^N(T_N(W)) \) as in the construction above such that \( T_N(W) \cap \tilde{J}_{T_N(W)} = 0 \) and \( S^N(T_N(W))/\tilde{J}_{T_N(W)} \) is irreducible. Since \( f_{T_N(W)}(T_N(W)) = T_N(W), \ker f_{T_N(W)} \) is a generalizer \( V \)-module of \( S^N(T_N(W)) \) that does not contain nonzero elements of \( M \). Hence \( \ker f_{T_N(W)} \subset \tilde{J}_{T_N(W)} \). Thus we obtain a surjective \( V \)-module map from \( S^N(T_N(W))/\tilde{J}_{T_N(W)} \) to \( W \). Since both \( S^N(T_N(W))/\tilde{J}_{T_N(W)} \) and \( W \) are irreducible, this surjective \( V \)-module map must be an equivalence. So we obtain \([S^N(T_N(W))/\tilde{J}_{T_N(W)}] = [W]\), that is, \( g(f([W])) = [W] \). So we obtain \( g \circ f = 1_{[\mathfrak{M}]_{\text{irr}}} \). This finishes the proof that \([\mathfrak{M}]_{\text{irr}} \) is in bijection with \([\mathfrak{M}^N]_{\text{irr}}\).

**Corollary 5.7** For \( N_1, N_2 \in \mathbb{N} \) or equal to \( \infty \), the set of the equivalence classes of irreducible graded \( A^{N_1}(V) \)-modules is in bijection with the set of the equivalence classes of irreducible graded \( A^{N_2}(V) \)-modules.

We now assume that \( V \) is a Möbius vertex algebra, that is, a grading-restricted vertex algebra equipped with an operator \( L_V(1) \) such that \( L_V(1), L_V(0) \) and \( L_V(-1) \) satisfying the usually commutator relations for the standard basis of \( \mathfrak{sl}_2 \) and the usual commutator formula between \( L_V(1) \) and vertex operators for a vertex operator algebra. See, for example, Definition 7.1 in [H4] for the precise definition. In this case, a lower-bounded generalized \( V \)-module should also have an operator \( L_W(1) \) satisfying the same relations as \( L_V(1) \). We assume that \( V \) is a grading-restricted Möbius vertex algebra in the remaining part of the paper because in this case, a lowest weight of a lower-bounded generalized \( V \)-module is well-defined. See Remark 7.3 in [H4].
Proposition 5.8 Let $V$ be a Möbius vertex algebra. Assume that $A_N(V)$ for all $N \in \mathbb{N}$ are finite dimensional (for example, when $V$ is $C_2$-cofinite and of positive energy by Theorem 4.6). Then every irreducible lower-bounded generalized $V$-module is an ordinary $V$-module and every lower-bounded generalized $V$-module of finite length is grading restricted.

Proof. Since for $N \in \mathbb{N}$, $A^N(V)$ is finite dimensional, there are only finitely many irreducible $A^N(V)$-modules. By Theorem 5.6, there are also finitely many irreducible lower-bounded generalized $V$-modules. For an irreducible lower-bounded generalized $V$-module $W$ with lowest weight $h_W$ and $N \in \mathbb{N}$, $T_N(W)$ is an irreducible graded $A^N(V)$-module by Proposition 5.3. Since $A^N(V)$ is finite dimensional, $T_N(W)$ is also finite dimensional. Thus $G_N(W) = W[h_W + N] \subset T_N(W)$ is also finite dimensional. Since this is true for $N \in \mathbb{N}$, we see that $W$ is grading restricted. Since $W$ is irreducible, $L_W(0)$ must act semisimply on $W$. So $W$ is an irreducible ordinary $V$-module.

Since as a graded vector space, a lower-bounded generalized $V$ module $W$ of finite length is a finite sum of irreducible lower-bounded generalized $V$-modules, which are all ordinary $V$-modules from what we have proved above. Then $W$ must be grading restricted. 

Since $V$ is a Möbius vertex algebra, the associative algebras $A^\infty(V)$ and $A^N(V)$ for $N \in \mathbb{N}$ have an additional operator $L_V(1)$ induced from the operator $L_V(1)$ acting on $V$. For a lower-bounded generalized $V$-module $W$, there is also an operator $L_{Gr(W)}(1)$ on the $A^\infty(V)$-module $Gr(W)$ induced from $L_W(1)$ on $W$ such that $L_{Gr(W)}(1)$ maps $Gr_n(W)$ to $Gr_{n-1}(W)$. Restricting $L_{Gr(W)}(1)$ to $Gr^N(W)$, we obtain an operator $L_{Gr^N(W)}(1)$ on $Gr^N(W)$.

Definition 5.9 Let $V$ be a Möbius vertex algebra. A graded $A^N(V)$-module is a graded $A^N(V)$-module $M$ when $V$ is viewed as a grading-restricted vertex algebra together with an operator $L_M(1)$ satisfying the following conditions:

1. $L_M(1)$ maps $G_n(M)$ to $G_{n-1}(M)$ for $n = 0, \ldots, N$, where $G_{-1}(M) = 0$.

2. The operators $L_M(1)$ satisfies the commutator relations

\[
\begin{align*}
[L_M(0), L_M(1)] &= -L_M(1), \\
[L_M(1), L_M(-1)] &= 2L_M(0), \\
[L_M(1), \vartheta_M([v]_{kl} + Q^\infty(V))] &= \vartheta_M([(L_V(1) + 2L_V(0) + L_V(-1)v]_{(k-1)l} + Q^\infty(V)).
\end{align*}
\]

Let $M_1$ and $M_2$ be graded $A^N(V)$-modules. An graded $A_N(V)$-module map from $M_1$ to $M_2$ is an $A^N(V)$-module map $f : M_1 \to M_2$ such that $f(G_n(M_1)) \subset G_n(M_2)$ for $n = 0, \ldots, N$, $f \circ L_{M_1}(1) = L_{M_2}(1) \circ f$, $f \circ L_{M_1}(0) = L_{M_2}(0) \circ f$ and $f \circ L_{M_1}(-1) = L_{M_2}(-1) \circ f$. A graded $A^N(V)$-submodule of a graded $A^N(V)$-module $M$ is an $A^N(V)$-submodule $M_0$ of $M$ such that with the $A^N(V)$-module structure and the $\mathbb{N}$-grading induced from $M$ and the operators $L_M(1)|_{M_0}$, $L_M(0)|_{M_0}$ and $L_M(-1)|_{M_0}$, $M_0$ is a graded $A^N(V)$-module. A graded $A^\infty(V)$-module $M$ is said to be generated by a subset $S$ if $M$ is equal to the smallest graded $A^N(V)$-submodule containing $S$, or equivalently, $M$ is spanned by homogeneous elements obtained by applying elements of $A^N(V)$, $L_M(1)$ and $L_M(-1)$ to homogeneous summands.
of elements of $S$. Irreducible and completely reducible graded $A^N(V)$-module are defined in the same way as in the case that $V$ is a grading-restricted vertex algebra.

From Proposition 5.2 and the property of $L_W(1)$, we immediately obtain the following:

**Proposition 5.10** Let $V$ be a Möbius vertex algebra. For a lower-bounded generalized $V$-module $W$, $Gr^N(W)$ is a graded $A^N(V)$-module. Let $W_1$ and $W_2$ be lower-bounded generalized $V$-modules and $f: W_1 \to W_2$ a $V$-module map. Then $f$ induces a graded $A^N(V)$-module map $Gr^N(f): Gr^N(W_1) \to Gr^N(W_2)$.

As is mentioned above, in the remaining part of this paper, we assume that $V$ is a Möbius vertex algebra. We shall not repeat this assumption except in the statements of propositions, theorems, corollaries and so on. Lower-bounded generalized $V$-modules and graded $A^N(V)$-modules always mean those for $V$ as a Möbius vertex algebra, not as a grading-restricted vertex algebra. All the results that we have obtained above certainly still hold.

We recall the notion of lower-bounded generalized $V$-module of finite length. A lower-bounded generalized $V$-module $W$ is said to be of finite length if there is a composition series $W = W_0 \supset \cdots \supset W_{i+1} = 0$ of lower-bounded generalized $V$-modules such that $W_i/W_{i+1}$ for $i = 0, \ldots, l$ are irreducible lower-bounded generalized $V$-modules.

**Proposition 5.11** Let $V$ be a Möbius vertex algebra. Assume that the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized $V$-modules are all less than or equal to $N \in \mathbb{N}$. Then a lower-bounded generalized $V$-module $W$ of finite length is generated by

$$\bigoplus_{\Re(h_W) \leq \Re(n) \leq \Re(h_W) + N} W[n] \subset \Omega_N(W),$$

where $h_W$ is a lowest weight of $W$.

**Proof.** Let $W = W_0 \supset W_1 \supset \cdots \supset W_{i+1} = 0$ be a finite composition series such that $W_i/W_{i+1}$ for $i = 0, \ldots, l$ are irreducible lower-bounded generalized $V$-modules. As a graded vector space, $W$ is isomorphic to $\bigoplus_{i=0}^l W_i/W_{i+1}$. In particular, the lowest weight of one of the irreducible lower-bounded generalized $V$-modules $W_i/W_{i+1}$ for $i = 0, \ldots, l$ is a lowest weight $h_W$ of $W$.

Let $w_i \in W_i$ be homogeneous for $i = 0, \ldots, l$ such that $w_i + W_{i+1}$ is a lowest weight vector of $W_i/W_{i+1}$. Then by assumption, the differences between the real parts of the lowest weights of $W_i/W_{i+1}$ for $i = 0, \ldots, l$ are less than or equal to $N$. Since one of these lowest weights is a lowest weight $h_W$ of $W$, we see that the the differences between the real parts of the lowest weights of $W_i/W_{i+1}$ for $i = 0, \ldots, l$ and $\Re(h_W)$ are less than or equal to $N$. In particular $w_i \in \bigoplus_{\Re(h_W) \leq \Re(n) \leq \Re(h_W) + N} W[n]$. Since for each $i$, $W_i/W_{i+1}$ is generated by $w_i + W_{i+1}$, $W_i$ is generated by $w_i$ and $W_{i+1}$. Thus $W$ is generated by $w_i$ for $i = 0, \ldots, l$.

Since $w_i \in \bigoplus_{\Re(h_W) \leq \Re(n) \leq \Re(h_W) + N} W[n]$, $W$ is generated by $\bigoplus_{\Re(h_W) \leq \Re(n) \leq \Re(h_W) + N} W[n]$. It is clear that $\bigoplus_{\Re(h_W) \leq \Re(n) \leq \Re(h_W) + N} W[n]$ is a subspace of $\Omega_N(W)$.  

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Corollary 5.12 Let $V$ be a Möbius vertex algebra. Assume that $A_{N'}(V)$ for all $N' \in \mathbb{N}$ are finite dimensional (for example, when $V$ is $C_2$-cofinite and of positive energy by Theorem 4.6). Let $N \in \mathbb{N}$ such that the differences between the real parts of the lowest weights of the finitely many (inequivalent) irreducible ordinary $V$-modules are less than or equal to $N$. Then a lower-bounded generalized $V$-module $W$ of finite length or a grading-restricted generalized $V$-module $W$ is generated by

$$
\prod_{\Re(h_W) \leq \Re(n) \leq \Re(h_W) + N} W_{[n]} \subset \Omega_N(W).
$$

Proof. Since by Proposition 5.8, the finitely many (inequivalent) irreducible lower-bounded generalized $V$-modules are all ordinary $V$-modules, the condition in Proposition 5.11 is satisfied. Also, by Corollary 3.16 in [H1], every grading-restricted generalized $V$-module is of finite length. Thus $W$ is generated by $\prod_{\Re(h_W) \leq \Re(n) \leq \Re(h_W) + N} W_{[n]}$.

Theorem 5.13 Let $V$ be a Möbius vertex algebra. Assume that the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized $V$-modules are all less than or equal to $N \in \mathbb{N}$. Then a lower-bounded generalized $V$-module $W$ of finite length is irreducible or completely reducible if and only if the graded $A^N(V)$-module $Gr^N(W)$ is irreducible or completely reducible, respectively.

Proof. By Proposition 5.3, we already know that if $W$ is irreducible, $Gr^N(W) = T_N(W)$ is irreducible. Conversely, assume that the graded $A^N(V)$-module $Gr^N(W)$ is irreducible. Let $W_0$ be a nonzero generalized $V$-submodule of $W$. Let $e_{W_0} : W_0 \to W$ be the embedding map. Then we have a graded $A^N(V)$-module map $Gr(e_{W_0}) : Gr^N(W_0) \to Gr^N(W)$ given by $(Gr(e_{W_0}))(w_0 + \Omega_n(W_0)) = w_0 + \Omega_n(W)$ for $n = 0, \ldots, N$ and $w_0 \in \Omega_n(W_0)$. Since $e_{W}$ is injective, $Gr(e_{W_0})$ is also injective. So $(Gr(e_{W_0}))(Gr^N(W_0))$ is a graded $A^N(V)$-submodule of $Gr^N(W)$. Since $W_0$ is nonzero, $Gr^N(W_0)$ is nonzero. Since $Gr^N(W)$ is irreducible and $Gr(e_{W_0})$ is injective, $(Gr(e_{W_0}))(Gr^N(W_0))$ is equal to $Gr^N(W)$. We now prove $W_0 = W$. In fact, for $n = 0, \ldots, N$, $Gr^N(W_0) = \{w_0 + \Omega_n(W_0) \mid w_0 \in \Omega_n(W_0)\}$. So $Gr_n(W) = \{w_0 + \Omega_n(W_0) \mid w_0 \in \Omega_n(W_0)\}$. For $n = 0$, we obtain $\Omega_0(W) = Gr_0(W) = Gr(W) = \Omega(W)$. Assume that $\Omega_n(W) = \Omega_{n-1}(W_0)$ for $n < N$. Given $w \in \Omega_n(W)$, $w + \Omega_{n-1}(W) \in Gr_n(W)$. By $Gr_n(W) = \{w_0 + \Omega_{n-1}(W) \mid w_0 \in \Omega_n(W_0)\}$, there exists $w_0 \in \Omega_n(W_0)$ such that $w + \Omega_{n-1}(W) = w_0 + \Omega_{n-1}(W)$, or equivalently, $w - w_0 \in \Omega_n(W_0)$. Thus $w \in \Omega_n(W)$. This shows $\Omega_n(W) = \Omega_n(W_0)$ for $n = 0, \ldots, N$. In particular, $\Omega_N(W) = \Omega_N(W_0)$. But by Proposition 5.11, $W_0$ and $W$ are generated by $\Omega_n(W_0)$ and $\Omega_N(W)$, respectively. Since $\Omega_N(W) = \Omega_N(W_0)$, we must have $W = W_0$. So $W$ is irreducible.

If $W$ is completely reducible, then by Proposition 5.3, $Gr^N(W) = T_N(W)$ is completely reducible. Conversely, assume that the graded $A^N(V)$-module $Gr^N(W)$ is completely reducible. Then $Gr^N(W) = \prod_{\mu \in \mathcal{M}} M^\mu$, where $M^\mu$ for $\mu \in \mathcal{M}$ are irreducible graded $A^N(V)$-submodules of $Gr^N(W)$. For $\mu \in \mathcal{M}$, since $M^\mu$ is a graded $A^N(V)$-module of $Gr^N(W)$, we have $M^\mu \subset Gr_n(W) = \Omega_n(W)/\Omega_{n-1}(W)$ for $n = 0, \ldots, N$. Let $W^\mu$ be the generalized $V$-submodule of $W$ generated by the set of elements of the form $w^\mu \in \Omega_n(W)$ such that

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\[ w^\mu + \Omega_{n-1}(W) \in M_n^\mu \] for for \( n = 0, \ldots, N \). Since \( W^\mu \) is a generalized \( V \)-submodule of \( W \), for \( v \in V, k, l \in \mathbb{N} \) and \( w^\mu \in \Omega_l(W) \) such that \( w^\mu + \Omega_{l-1}(W) \in M_l^\mu \),

\[
\text{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x) w^\mu + \Omega_{k-1}(W) \in Gr_k(W^\mu).
\]

By the definition of \( W^\mu \), we see that \( \text{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x) w^\mu \in W^\mu \). Since \( w^\mu \in \Omega_l(W) \), \( \text{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x) w^\mu = 0 \) for \( k \in -\mathbb{Z}_+ \). Therefore \( \text{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x) w^\mu \in W^\mu \) for \( k \in \mathbb{N} \) are all the nonzero coefficients of \( Y_W(v, x) w^\mu \). So \( W^\mu \) is closed under the action of the vertex operators on \( W \). Since \( M^\mu \) is invariant under the actions of \( L_{Gr(W)}(0) \) and \( L_{Gr(W)}(-1) \) and is a direct sum of generalized eigenspaces of \( L_{Gr(W)}(0) \), \( W^\mu \) is invariant under the actions of \( L_W(0) \) and \( L_W(-1) \) and is a direct sum of generalized eigenspaces of \( L_W(0) \). Thus \( W^\mu \) is a generalized \( V \)-submodule of \( W \).

Let \( w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu) \), where \( 0 \leq n \leq N \) and \( w^\mu \in \Omega_n(W^\mu) \subset \Omega_n(W) \). By the definition of \( W^\mu \), we see that since \( w^\mu \) is an element of \( W^\mu \), \( w^\mu + \Omega_{n-1}(W^\mu) \in G_n(M^\mu) \). So we obtain a linear map from \( Gr_n(W^\mu) \) to \( G_n(M^\mu) \) given by \( w^\mu + \Omega_{n-1}(W^\mu) \mapsto w^\mu + \Omega_{n-1}(W) \) for \( w^\mu + \Omega_{n-1}(W^\mu) \in Gr_n(W^\mu) \). These maps for \( n = 0, \ldots, N \) give a map from \( Gr^N(W^\mu) \) to \( M^\mu \). It is clear that this map is a graded \( A^N(V) \)-module map. If for \( 0 \leq n \leq N \), the image \( w^\mu + \Omega_{n-1}(W) \) of \( w^\mu + \Omega_{n-1}(W^\mu) \) in \( Gr_n(W^\mu) \) under this map is 0 in \( M^\mu \), then \( w^\mu \in \Omega_{n-1}(W) \). But \( w^\mu \in \Omega_n(W^\mu) \subset W^\mu \). So \( w^\mu \in \Omega_{n-1}(W^\mu) \) and \( w^\mu + \Omega_{n-1}(W^\mu) = 0 \) in \( Gr^N(W^\mu) \). This means that this graded \( A^N(V) \)-module map is injective. In particular, the image of \( Gr^N(W^\mu) \) under this map is a nonzero graded \( A^N(V) \)-submodule of \( M^\mu \). But \( M^\mu \) is irreducible. So \( Gr^N(W^\mu) \) must be equivalent to \( M^\mu \) and is therefore also irreducible. From what we have proved above, since \( Gr^N(W^\mu) \) is irreducible, \( W^\mu \) is irreducible. Thus \( W \) is complete reducible.

From Corollary 5.12 and Theorem 5.13, we obtain the following result:

**Corollary 5.14** Let \( V \) be a M"obius vertex algebra. Assume that \( A_{N'}(V) \) for all \( N' \in \mathbb{N} \) are finite dimensional (for example, when \( V \) is \( C_2 \)-cofinite and of positive energy by Theorem 4.6). Let \( N \in \mathbb{N} \) such that the differences between the real parts of the lowest weights of the finitely many (inequivalent) irreducible ordinary \( V \)-modules are less than or equal to \( N \). Then a lower-bounded generalized \( V \)-module \( W \) of finite length or a grading-restricted generalized \( V \)-module \( W \) is a direct sum of irreducible ordinary \( V \)-modules if and only if the graded \( A^N(V) \)-module \( Gr^N(W) \) is completely reducible.

**Proof.** Since \( A^N(V) \) is finite dimensional, there are only finitely many (inequivalent) irreducible graded \( A^N(V) \)-modules. By Theorem 5.6, there are finitely many irreducible lower-bounded generalized \( V \)-modules. By Corollary 5.12, these finitely many irreducible lower-bounded generalized \( V \)-modules are all irreducible ordinary \( V \)-modules. There exists \( N \in \mathbb{N} \) such that the differences between the real parts of the lowest weights of the finitely many irreducible ordinary \( V \)-modules are less than or equal to \( N \). For such \( N \), the condition in Theorem 5.13 holds. So by Theorem 5.13, a lower-bounded generalized \( V \)-module \( W \) of finite length is a direct sum of irreducible ordinary \( V \)-modules if and only if \( Gr^N(W) \) is completely reducible as a graded \( A^N(V) \)-module.
By Corollary 3.16 in [H1], every grading-restricted generalized $V$-module is of finite length. Thus the conclusion holds also for a grading-restricted generalized $V$-module $W$.

**Remark 5.15** Note that the assumption or condition on the lowest weights of irreducible $V$-modules in Proposition 5.11, Corollary 5.12, Theorem 5.13 and Corollary 5.14 can be weakened to the assumption or condition that the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized $V$-modules appearing as a quotient in a composition series of $W$ are all less than or equal to $N \in \mathbb{N}$. This is because the proofs used only this weaker assumption or condition. For the study of some particular lower-bounded generalized $V$-modules of finite length or some grading-restricted generalized $V$-modules, this weaker assumption or condition is certainly easier to verify than the more general ones in the statements of these results.

**References**


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