On functors between module categories for associative algebras and for ℕ-graded vertex algebras

Yi-Zhi Huang and Jinwei Yang

Abstract

We prove that the weak associativity for modules for vertex algebras are equivalent to a residue formula for iterates of vertex operators, obtained using the weak associativity and the lower truncation property of vertex operators, together with a known formula expressing products of components of vertex operators as linear combinations of iterates of components of vertex operators. By requiring that these two formulas instead of the commutator formula hold, we construct a functor S from the category of modules for Zhu's algebra of a vertex operator algebra V to the category of ℕ-gradable weak V-modules. We prove that S has a universal property and the functor T of taking top levels of ℕ-gradable weak V-modules is a left inverse of S. In particular, S is equal to a functor implicitly given by Zhu and explicitly constructed by Dong, Li and Mason and we obtain a new construction without using relations corresponding to the commutator formula. The hard part of this new construction is a technical theorem stating roughly that in a module for Zhu's algebra, the relation corresponding to the residue formula mentioned above can in fact be obtained from the relations corresponding to the action of Zhu's algebra.

1 Introduction

In the study of vertex operator algebras and their modules, commutator formulas for vertex operators often play an important role. Because of the commutator formula, the components of the vertex operators of a vertex operator algebra form a Lie algebra. The vertex operator algebra itself and its modules become modules for this Lie algebra. This fact allows one to apply many techniques in the representation theory of Lie algebras to the study of vertex operator algebras and their modules. For example, the proof by Zhu [Z] of the modular invariance of the space spanned by the q-traces of products of vertex operators depends heavily on the commutator formula, especially in the case of q-traces of products of at least two vertex operators.

On the other hand, the main interesting mathematical objects in the representation theory of vertex operator algebras are intertwining operators, not just vertex operators for vertex operator algebras and for modules. For intertwining operators, in general there is no commutator formula and therefore the Lie-algebra-theoretic techniques cannot be applied. Thus it is necessary to develop different, non-Lie-algebra-theoretic method to study intertwining operators. One example is the modular invariance of the space spanned by the q-traces of products of at least two intertwining operators. This modular invariance was conjectured first by physicists and was used explicitly by Moore and Seiberg in [MS] to derive their polynomial equations. But its proof was given by the first author in the preprint of [H2] 13 years after Zhu's proof of his modular invariance result in [Z]. The difficulty lies mainly in the development of new method without using the commutator formula on which Zhu's proof depends heavily. Instead of any commutator formula (which in fact does not exist in general), the first author used the associativity for intertwining operators proved in [H1].

Because of the limitation of the commutator formula discussed above, it is desirable to prove results in the representation theory of vertex operator algebras, even only for modules, without using the commutator formula so that it will be relatively easy to see whether such results can be generalized to intertwining operators.

Note that for modules for vertex operator algebras, the weak associativity can be taken to be the main axiom. It is natural to expect that all the results for modules can be proved using the weak associativity without the commutator formula. However, in many problems in the representation theory of vertex operator algebras and its applications, one has to work with components of vertex operators instead of vertex operators directly. Thus we need formulas expressed using residues of formal series obtained from products and iterates of vertex operators that are equivalent to the weak associativity.

It is known from [DLM] and [L] (see also [LL]) that for a vertex algebra, the weak associativity for vertex operators gives a formula that expresses products of components of vertex operators as linear combinations of iterates of components of vertex operators. Unfortunately, this formula is not equivalent to the weak associativity. In the present paper, we give a formula for the residues of certain formal series involving iterates of vertex operators obtained using the weak associativity and the lower truncation property of vertex operators. Then we prove that the weak associativity is equivalent to this formula together with the formula given in [DLM] and [L]. Thus, in principle, all the results for modules for vertex operator algebras can be proved using these two formulas without the help of the commutator formula.

We apply this result to give a new construction of $\mathbb{N}$-gradable weak modules for an $\mathbb{N}$-graded vertex algebra V from modules for Zhu's algebra $A(V)$. We construct a functor S from the category of $A(V)$-modules to the category of $\mathbb{N}$-gradable weak V-modules. We prove that S has a universal property and the functor T of taking top levels of $\mathbb{N}$-gradable weak V-modules is a left inverse of S. In particular, S is equal to the functor constructed by Dong, Li and Mason in [DLM] in the case $n = 0$ and we achieve our goal of finding a new construction without using relations corresponding to the commutator formula. The hard part of this new construction is a technical theorem stating roughly that in a module for

Zhu's algebra, the relation corresponding to the residue formula given in the present paper (see above) can in fact be obtained from the relations corresponding to the action of Zhu's algebra.

The importance of our construction is that it allows us to use the method in the present paper to give constructions and prove results in the cases that there is no commutator formula. The main example that motivated the present paper is a construction of generalized twisted modules associated to an infinite order automorphism of a vertex operator algebra in the sense of the first author in [H3]. For these generalized twisted modules, the twisted vertex operators involve the logarithm of the variable and thus do not have a commutator formula. We shall discuss this construction in a future publication.

This paper is organized as follows: We recall and derive some residue formulas for vertex operators from the weak associativity in Section 2. We prove that these residue formulas are equivalent to the weak associativity in Section 3. In Section 4, we construct our functor S. In Section 5, we prove a university property of S and prove that the functor T mentioned above is a left inverse of S.

2   Residue formulas from the weak associativity

In this section, we recall and derive some consequences of the weak associativity.

Let $(V, Y, \mathbf{1})$ be a vertex algebra and (W, Y_W) a V-module. (Note that since V as a vertex algebra might not have a grading, a V-module also might not have a grading.) The weak associativity for W states that for $u, v \in V$, $w \in W$ and l a nonnegative integer such that $u_n w = 0$ for $n \geq l$, we have

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w.$$

We recall the following result from [DLM] and [L] (Proposition 4.5.7 in [LL]):

Proposition 2.1 *Let W be a V-module and let $u, v \in V$, $p, q \in \mathbb{Z}$ and $w \in W$. Let l be a nonnegative integer such that*

$$u_n w = 0 \quad \text{for } n \geq l$$

and let k be a nonnegative integer such that $k - q$ is positive and

$$v_n w = 0 \quad \text{for } n \geq k.$$

Then

$$u_p v_q w = \sum_{i=0}^{k-q-1} \sum_{j=0}^{l} \binom{p-l}{i} \binom{l}{j} (u_{p-l-i+j}v)_{q+l+i-j}w. \tag{2.1}$$

3

Remark 2.2 Let

$$p(x_0, x_2) = \sum_{i=0}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i} x_2^i.$$

Then the formula (2.1), the formula

$$\mathrm{Res}_{x_1}\mathrm{Res}_{x_2} x_1^p x_2^q Y(u, x_1) Y(v, x_2) w$$
$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} p(x_0, x_2) x_2^q (x_0 + x_2)^l Y(Y(u, x_0)v, x_2) w \qquad (2.2)$$

and the formula

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2} (x_0 + x_2)^p x_2^q Y(u, x_0 + x_2) Y(v, x_2) w$$
$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} p(x_0, x_2) x_2^q (x_0 + x_2)^l Y(Y(u, x_0)v, x_2) w \qquad (2.3)$$

are equivalent to each other. See the proof of Proposition 4.5.7 in [LL].

From the lower truncation property for vertex operators, we have the following:

Proposition 2.3 *For W, u, v, w, l, k as in Proposition 2.1, we have*

$$\mathrm{Res}_{x_2} x_2^q (x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w \;=\; 0, \qquad (2.4)$$
$$\mathrm{Res}_{x_2} x_2^q (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2) w \;=\; 0 \qquad (2.5)$$

for $q \geq k$.

Proof. Note that $x_2^q (x_0 + x_2)^l Y_W(u, x_0 + x_2)$ contains only terms with the powers of x_2 larger than or equal to $q \geq k$. Hence the left hand side of (2.4) are linear combinations of $u_m v_n w$ for $m \in \mathbb{Z}$ and $n \geq k$ with coefficients in $\mathbb{C}((x_0))$. Since $v_n w = 0$ for $n \geq k$, (2.4) holds.

By the weak associativity, the left-hand side of (2.5) is equal to the left-hand side of (2.4). Thus (2.5) also holds. ∎

Remark 2.4 Note that the proof of (2.4) uses only the property of Y_W that $Y_W(v, x)w \in W((x))$ for $v \in V$ and $w \in W$. Thus for a vector space W and a linear map Y_W from $V \otimes W$ to $W((x))$, (2.4) still holds. On the other hand, (2.5) does not hold in general for such a vector space W and such a map Y_W.

From (2.5), we obtain immediately:

Corollary 2.5 *For W, u, v, p, q, w, l, k as in Proposition 2.1 except that $k - q$ does not have to be positive, we have*

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2} x_0^{p-l-i} x_2^{q+i} (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2) w = 0 \qquad (2.6)$$

for $i \geq k - q$. ∎

4

Remark 2.6 Assume that V is a $\mathbb{Z}$-graded vertex algebra and W is an $\mathbb{N}$-gradable weak V-module. Here by a $\mathbb{Z}$-graded vertex algebra we mean a vertex algebra $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ such that for $u \in V$,

$$[d, Y(u, x)] = x \frac{d}{dx} Y(u, x) + Y(du, x),$$

where $d : V \to V$ is defined by $du = nu$ for $u \in V_{(n)}$. For $u \in V_{(n)}$, we use wt u to denote n. By an $\mathbb{N}$-gradable weak V-module, we mean a module $W = \coprod_{n \in \mathbb{N}} W_{(n)}$ for V when V is viewed as a vertex algebra such that for $u \in V_{(m)}$ $u_n = \mathrm{Res}_x x^n Y_W(u, x)$ maps $W_{(k)}$ to $W_{(k+m-n-1)}$. For $w \in W_{(n)}$, we use $\deg w$ to denote n. When u, v and w are homogeneous, we take $k = \mathrm{wt}\, v + \deg w$ and $l = \mathrm{wt}\, u + \deg w$ and set $K = p + q + 2 - \mathrm{wt}\, u - \mathrm{wt}\, v$ and $m = p - l - i$. Then (2.6) becomes

$$\mathrm{Res}_{x_0} \mathrm{Res}_{x_2} x_0^m x_2^{K + \mathrm{wt}\ v - m - \deg w - 2} (x_0 + x_2)^{\mathrm{wt}\ u + \deg w} Y_W(Y(u, x_0)v, x_2)w = 0 \qquad (2.7)$$

for $m \leq M - 2\deg w - 2$. The formula (2.7) can be further written without using the weights of u and v as

$$\mathrm{Res}_{x_0} \mathrm{Res}_{x_2} x_0^m x_2^{K - m - \deg w - 2} (x_0 + x_2)^{\deg w} Y_W(Y((x_0 + x_2)^{L(0)} u, x_0) x_2^{L(0)} v, x_2)w = 0, \quad (2.8)$$

which is in fact holds for general u and v, not necessarily homogeneous. The component form of (2.7) is

$$\sum_{j=0}^{\mathrm{wt}\ u + \deg w} \binom{\mathrm{wt}\ u + \deg w}{j} (u_{j+m}v)_{K + \mathrm{wt}\ u + \mathrm{wt}\ v - j - m - 2} w = 0 \qquad (2.9)$$

for $m \leq K - 2\deg w - 2$.

3 The equivalence between the residue formulas and the weak associativity

In this section, we prove that (2.3) and (2.6) imply, and therefore are equivalent to, the weak associativity.

Theorem 3.1 *Let V be a vertex operator algebra, W a vector space and Y_W a linear map from $V \otimes W$ to $W((x))$. For $v \in V$ and $w \in W$, as for a V-module, we denote the image of $u \otimes w$ under Y_W by $Y_W(u, x)w$ and $\mathrm{Res}_x x^n Y_W(u, x)w$ by u_n. Let $u, v \in V$ and $w \in W$ and let $k, l \in \mathbb{Z}$ such that*

$$v_n w = 0 \quad \text{for } n \geq k \qquad (3.1)$$

and

$$u_n w = 0 \quad \text{for } n \geq l. \qquad (3.2)$$

Assume that (i) the formulas (2.3) (or equivalently, (2.2) or (2.1)) holds for $p, q \in \mathbb{Z}$ such that $q < k$ and (ii) (2.6) holds for for $p, q \in \mathbb{Z}$ such that $i \geq k - q$. Then

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2)w = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w. \qquad (3.3)$$

Proof. The Laurent polynomial $p(x_0, x_2)$ in Remark 2.2 is in fact the first $k - q - 1$ terms of the formal series $(x_0 + x_2)^{p-l}$. But from (2.6), we obtain

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}((x_0 + x_2)^{p-l} - p(x_0, x_2))x_2^q((x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2))w = 0. \qquad (3.4)$$

Combining (2.3) and (3.4), we obtain

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}(x_0 + x_2)^{p-l}x_2^q((x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2))w$$
$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}(x_0 + x_2)^{p-l}x_2^q((x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2))w,$$

or equivalently,

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}(x_0 + x_2)^{p-l}x_2^q((x_0 + x_2)^l(Y(u, x_0 + x_2)Y(v, x_2) - Y_W(Y(u, x_0)v, x_2)))w = 0, \qquad (3.5)$$

On the other hand, by Remark 2.4, (2.4) holds for $m \geq k$. In particular, we have

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}x_0^{p-l-i}x_2^{q+i}(x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w = 0 \qquad (3.6)$$

for $i \geq k - q$. From (2.6) and (3.6), we obtain

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}x_0^{p-l-i}x_2^{q+i}(x_0 + x_2)^l(Y_W(u, x_0 + x_2)Y_W(v, x_2) - Y_W(Y(u, x_0)v, x_2))w = 0 \qquad (3.7)$$

for $i \geq k - q$. Combining (3.5) and (3.7), we obtain

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2} \sum_{i=0}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i}x_2^{q+i} \cdot$$
$$\cdot((x_0 + x_2)^l(Y(u, x_0 + x_2)Y(v, x_2) - Y_W(Y(u, x_0)v, x_2)))w$$
$$= 0. \qquad (3.8)$$

We now use induction on $k - q - 1$ to prove

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}x_0^{p-l}x_2^q((x_0 + x_2)^l(Y(u, x_0 + x_2)Y(v, x_2) - Y_W(Y(u, x_0)v, x_2)))w = 0 \qquad (3.9)$$

for $p \in \mathbb{Z}$ and $q < k$. When $k - q - 1 = 0$, (3.8) becomes

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}x_0^{p-l}x_2^q((x_0 + x_2)^l(Y(u, x_0 + x_2)Y(v, x_2) - Y_W(Y(u, x_0)v, x_2)))w = 0.$$

Assuming that (3.9) holds when $0 \leq k - q - 1 < n$. When $k - q - 1 = n$, $0 \leq k - q - i - 1 < n$ for $i = 1, \ldots, n = k - q - 1$. Since p is arbitrary, we can replace p by $p - i$ for any $i \in \mathbb{Z}$ in (3.9). Thus by the induction assumption,

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}x_0^{p-l-i}x_2^{q+i}((x_0 + x_2)^l(Y(u, x_0 + x_2)Y(v, x_2) - Y_W(Y(u, x_0)v, x_2)))w = 0 \qquad (3.10)$$

for $i = 1, \ldots, n = k - q - 1$. From (3.10) for $i = 1, \ldots, n = k - q - 1$ and (3.8), we obtain

$$\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}x_0^{p-l}x_2^q((x_0 + x_2)^l(Y(u, x_0 + x_2)Y(v, x_2) - Y_W(Y(u, x_0)v, x_2)))w$$
$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} \sum_{i=0}^{k-q-1} \binom{p-l}{i} x_0^{p-l-i}x_2^{q+i} \cdot$$
$$\cdot((x_0 + x_2)^l(Y(u, x_0 + x_2)Y(v, x_2) - Y_W(Y(u, x_0)v, x_2)))w$$
$$= 0,$$

6

proving (3.9) in this case.

Taking $i = 0$ in (2.6), we see that (3.9) also holds for $p \in \mathbb{Z}$ and $q \geq k$. Thus (2.6) holds for $p, q \in \mathbb{Z}$. But this means that

$$((x_0 + x_2)^l (Y(u, x_0 + x_2)Y(v, x_2) - Y_W(Y(u, x_0)v, x_2)))w = 0,$$

that is, (3.3) holds. ∎

The following result follows immediately from Theorem 3.1 above and Theorems 3.6.3 and 3.11.8 in [LL].

Theorem 3.2 *Let V be a $\mathbb{Z}$-graded vertex algebra, $W = \coprod_{n \in \mathbb{N}} W_{(n)}$ an $\mathbb{N}$-graded vector space and Y_W a linear map from $V \otimes W$ to $W((x))$. For $v \in V$ and $w \in W$, we denote the image of $u \otimes w$ under Y_W by $Y_W(u, x)w$ and $\mathrm{Res}_x x^n Y_W(u, x)w$ by u_n. Assume that u_n maps $W_{(k)}$ to $W_{(k+m-n-1)}$ for for $u \in V_{(m)}$ and $n \in \mathbb{Z}$ and $Y_W(\mathbf{1}, x) = 1_W$. Also assume that for $u, v \in V$, $w \in W$, there exist $k, l \in \mathbb{Z}$ such that (3.1) and (3.2) hold, and for $p, q \in \mathbb{Z}$, $u, v \in V$, $w \in W$, the formulas (2.2) (or equivalently, (2.1) or (2.3)) and (2.6) (or equivalently, (2.7)) hold. Then (W, Y_W) is an $\mathbb{N}$-gradable weak V-module.* ∎

4   A functor S

In this and the next section, $(V, Y, \mathbf{1})$ is a $\mathbb{Z}$-graded vertex algebra such that $V_{(n)} = 0$ for $n < 0$. We shall call such a $\mathbb{Z}$-graded vertex algebra an $\mathbb{N}$-graded vertex algebra. In this section, by dividing relations such that (2.3) and (2.6) hold, we construct a functor S from the category of modules for Zhu's algebra $A(V)$ to the category of $\mathbb{N}$-gradable weak V-modules.

We first recall the definition of Zhu's algebra in [Z]. Let $O(V)$ be the subspace of V spanned by elements of the form

$$\mathrm{Res}_x x^n Y((x + 1)^{L(0)} u, x)v$$

for $u, v \in V$ and $n \leq -2$. Zhu's algebra associated to V is the quotient space $A(V) = V/O(V)$ equipped with the multiplication

$$u * v = \mathrm{Res}_x x^{-1} Y((x + 1)^{L(0)} u, x)v$$

for $u, v \in V$. It was proved in [Z] that $A(V)$ equipped with the product $*$ is an associative algebra.

Let (W, Y_W) be an $\mathbb{N}$-gradable V-module. We shall use the same notations as in the preceding section. Let $T(W)$ be the subspace of elements w of W such that $u_n w = 0$ when wt $u - n - 1 < 0$. The subspace $T(W)$ is called the top level of W. For $u \in V$, let

$$o(u) = \mathrm{Res}_x x^{-1} Y_W(x^{L(0)} u, x).$$

Then for $w \in T(W)$, $o(u)w \in T(W)$. In fact, it was proved in [Z] that $u + O(V) \mapsto o(u)$ gives a $A(V)$-module structure on $T(W)$. Clearly, taking the top level of an $\mathbb{N}$-gradable V-module gives a functor T from the category of $A(V)$-modules to the category of $\mathbb{N}$-gradable V-modules.

Consider the affinization $V[t, t^{-1}] = V \otimes \mathbb{C}[t, t^{-1}]$ of V and the tensor algebra $T(V[t, t^{-1}])$ generated by $V[t, t^{-1}]$. (Note that although we use the same notation, $T(V[t, t^{-1}])$ has nothing to do with the top level of an $\mathbb{N}$-gradable weak V-module.) For simplicity, we shall denote $u \otimes t^m$ for $u \in V$ and $m \in \mathbb{Z}$ by $u(m)$ and we shall omit the tensor product sign $\otimes$ when we write an element of $T(V[t, t^{-1}])$. Thus $T(V[t, t^{-1}])$ is spanned by elements of the form $u_1(m_1) \cdots u_k(m_k)$ for $u_i \in V$ and $m_i \in \mathbb{Z}$, $i = 1, \ldots, k$.

Let M be an $A(V)$-module and $\rho : A(V) \to \operatorname{End} M$ the corresponding representation of $A(V)$. Consider $T(V[t, t^{-1}]) \otimes M$. Again for simplicity we shall omit the tensor product sign. So $T(V[t, t^{-1}]) \otimes M$ is spanned by elements of the form $u_1(m_1) \cdots u_k(m_k)w$ for $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \ldots, k$ and $w \in M$ and for any $u \in V$, $m \in Z$, $u(m)$ acts from the left on $T(V[t, t^{-1}]) \otimes M$. For homogeneous $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \ldots, k$ and $w \in M$, we define the degree of $u_1(m_1) \cdots u_k(m_k)w$ to be $(\operatorname{wt} u_1 - m_1 - 1) + \cdots (\operatorname{wt} u_k - m_k - 1)$. In particular, the degrees of elements of M are 0.

For $u \in V$, let

$$Y_t(u, x) : T(V[t, t^{-1}]) \otimes M \longrightarrow T(V[t, t^{-1}]) \otimes M[[x, x^{-1}]]$$

be defined by

$$Y_t(u, x) = \sum_{m \in \mathbb{Z}} u(m) x^{-m-1}.$$

For a homogeneous element $u \in V$, let

$$o_t(u) = u(\operatorname{wt} u - 1).$$

Using linearity, we extend $o_t(u)$ to non-homogeneous u.

Let $\mathcal{I}$ be the $\mathbb{Z}$-graded $T(V[t, t^{-1}])$-submodule of $T(V[t, t^{-1}]) \otimes W$ generated by elements of the forms

$$u(m)w$$

for homogeneous $u \in V$, $w \in T(V[t, t^{-1}]) \otimes M$, $\operatorname{wt} u - m - 1 + \deg w < 0$,

$$o_t(u)w - \rho(u + O(V))w$$

for $u \in V$, $w \in M$ and

$$\operatorname{Res}_{x_1} \operatorname{Res}_{x_2} x_1^p x_2^q Y_t(u, x_1) Y_t(v, x_2) w$$
$$- \operatorname{Res}_{x_0} \operatorname{Res}_{x_2} p(x_0, x_2) x_2^q (x_0 + x_2)^{\operatorname{wt} u + \deg w} Y_t(Y(u, x_0)v, x_2)w, \qquad (4.1)$$

where

$$p(x_0, x_2) = \sum_{i=0}^{\operatorname{wt} v + \deg w - q - 1} \binom{p - \operatorname{wt} u - \deg w}{i} x_0^{p - \operatorname{wt} u - \deg w - i} x_2^i,$$

or equivalently,

$$u(p)v(q)w \ - \sum_{i=0}^{\operatorname{wt} v + \deg w - q - 1} \sum_{j=0}^{\operatorname{wt} u + \deg w} \binom{p - \operatorname{wt} u - \deg w}{i} \binom{\operatorname{wt} u + \deg w}{j}.$$
$$\cdot (u_{p - \operatorname{wt} u - \deg w - i + j} v)(q + \operatorname{wt} u + \deg w + i - j)w \qquad (4.2)$$

8

for homogeneous $u, v \in V$, $w \in T(V[t, t^{-1}]) \otimes M$, $p, q \in \mathbb{Z}$ such that wt $v + \deg w > q$ and wt $u - p - 1 + $ wt $v - q - 1 + \deg w \geq 0$.

Let $S_1(M) = (T(V[t, t^{-1}]) \otimes M)/\mathcal{I}$. Then $S_1(M)$ is also a $\mathbb{Z}$-graded $T(V[t, t^{-1}])$-module. In fact, by definition of $\mathcal{I}$, we see that $S_1(M)$ is spanned by elements of the form $u(m)w + \mathcal{I}$ for homogeneous $u \in V$, $m \in \mathbb{Z}$ such that $m <$ wt $u - 1$ and $w \in M$. In particular, we see that $S_1(M)$ has an $\mathbb{N}$-grading. Note that since $\mathcal{I} \cap M = \{0\}$, M can be embedded into $S_1(M)$ and the homogeneous subspace $(S_1(M))_{(0)}$ of degree 0 is the image of M under this embedding. We shall identify M with $(S_1(M))_{(0)}$. In particular, M is now viewed as a subspace of $S_1(M)$. The vertex operator map Y_t induces a vertex operator map $Y_{S_1(M)} :$ $V \otimes S_1(M) \to S_1(M)[[x, x^{-1}]]$ by $Y_{S_1(M)}(u, x)w = Y_t(u, x)w$.

Let $\mathcal{J}$ be the $\mathbb{N}$-graded $T(V[t, t^{-1}])$-submodule of $S_1(M)$ generated by

$$\text{Res}_{x_0}\text{Res}_{x_2} x_0^{p - \text{wt } u - \deg w - i} x_2^{q+i}(x_0 + x_2)^{\text{wt } u + \deg w} Y_{S_1(M)}(Y(u, x_0)v, x_2)w. \tag{4.3}$$

for homogeneous $u, v \in V$, $w \in S_1(M)$, $p, q \in \mathbb{Z}$ and $i \geq$ wt $v + \deg w - q$. By Remark 2.6, $\mathcal{J}$ is generated by

$$\text{Res}_{x_0}\text{Res}_{x_2} x_0^m x_2^{K - m - \deg w - 2}(x_0 + x_2)^{\deg w} Y_{S_1(M)}(Y((x_0 + x_2)^{L(0)}u, x_0)x_2^{L(0)}v, x_2)w$$

for homogeneous $u, v \in V$, $w \in S_1(M)$, $K, m \in \mathbb{Z}$ satisfying $m \leq K - 2 \deg w - 2$, or equivalently, by

$$\sum_{j=0}^{\text{wt } u + \deg w} \binom{\text{wt } u + \deg w}{j} (u_{j+m}v)(K + \text{wt } u + \text{wt } v - j - m - 2)w, \tag{4.4}$$

for homogeneous $u, v \in V$ and the same w, K and m.

Let $S(M) = S_1(M)/\mathcal{J}$. Then $S(M)$ is also an $\mathbb{N}$-graded $T(V[t, t^{-1}])$-module. We can still use elements of $T(V[t, t^{-1}]) \otimes M$ to represent elements of $S(M)$. But note that these elements now satisfy more relations. We equip $S(M)$ with the vertex operator map

$$Y_{S(M)} : V \otimes S(M) \longrightarrow S(M)[[x, x^{-1}]]$$

given by

$$u \otimes w \to Y_{S(M)}(u, x)w = Y_{S_1(M)}(u, x)w = Y_t(u, x)w.$$

Theorem 4.1 *The pair $(S(M), Y_{S(M)})$ is an $\mathbb{N}$-gradable weak V-module.*

Proof. As in $S_1(M)$, for $u \in V$ and $w \in S(M)$, we have $u(m)w = 0$ when $m >$ wt $u + $ wt $w - 1$. Clearly,

$$Y(\mathbf{1}, x) = 1_{S(M)},$$

where $1_{S(M)}$ is the identity operator on $S(M)$.

By Theorems 3.1 and 3.2, $S(M)$ is an $\mathbb{N}$-gradable weak V-module. $\blacksquare$

Let M_1 and M_2 be $\mathbb{N}$-gradable weak V-modules and $f : M_1 \to M_2$ a module map. Then F induces a linear map from $T(V[t, t^{-1}]) \otimes M_1$ to $T(V[t, t^{-1}]) \otimes M_2$. By definition, this

induced linear map in turn induces a linear map $S(f)$ from $S(M_1)$ to $S(M_2)$. Since $Y_{S(M_1)}$ and $Y_{S(M_2)}$ are induced by Y_t on $T(V[t, t^{-1}]) \otimes M_1$ and $T(V[t, t^{-1}]) \otimes M_2$, respectively, we have

$$S(f)(Y_{S(M_1)}(u, x)w_1) = Y_{S(M_2)}(u, x)S(f)(w_1)$$

for $u \in V$ and $w_1 \in S(M_1)$. Thus $S(f)$ is a module map. The following result is now clear:

Corollary 4.2 *Let V be a $\mathbb{N}$-graded vertex algebra. Then the correspondence sending an $A(V)$-module M to an $\mathbb{N}$-gradable weak V-module $(S(M), Y_{S(M)})$ and an $A(V)$-module map $M_1 \to M_2$ to a V-module map $S(f) : S(M_1) \to S(M_2)$ is a functor from the category of $A(V)$-modules to the category of $\mathbb{N}$-gradable weak V-modules.*

5 A universal property and T as a left inverse of S

In this section, we prove that S satisfies a natural universal property and thus is the same as the functor constructed in [DLM]. In particular, we achieve our goal of constructing $\mathbb{N}$-gradable weak V-modules from $A(V)$-modules without dividing relations corresponding to the commutator formula for weak modules. We also prove that the functor T of sending an $\mathbb{N}$-gradable weak V-module W to its top level $T(W)$ is a left inverse of S.

As in the preceding section, we let V be a $\mathbb{N}$-graded vertex algebra.

We shall need the following lemma:

Lemma 5.1 *In the setting of Theorem 3.1, if we assume only (2.3) (or equivalently, (2.1)), then for $u, v \in V$, homogeneous $w \in W$ and $p', q' \in \mathbb{Z}$, $q' < \deg w$, we have*

$$\mathrm{Res}_{x_1} \mathrm{Res}_{x_2} x_1^{p'} x_2^{q'} Y_W(x_1^{L(0)} u, x_1) Y_W(x_2^{L(0)} v, x_2) w$$
$$= \mathrm{Res}_{x_0} \mathrm{Res}_{x_2} q(x_0, x_2) x_2^{q'} (x_0 + x_2)^{\deg w} Y_W(Y((x_0 + x_2)^{L(0)} u, x_0) x_2^{L(0)} v, x_2) w, \quad (5.5)$$

where $q(x_0, x_2) \in C[x_0, x_0^{-1}, x_2]$ such that when w is homogeneous,

$$q(x_0, x_2) = \sum_{i=0}^{\deg w - q' - 1} \binom{p' - \deg w}{i} x_0^{p' - \deg w - i} x_2^i.$$

Proof. In the case that u, v are also homogeneous, the lemma follows from Remark 2.2 by taking $p = p' + \mathrm{wt}\ u$, $q = q' + \mathrm{wt}\ v$, $k = \mathrm{wt}\ v + \deg w$ and $l = \mathrm{wt}\ u + \deg w$. ∎

For an $A(V)$-module M, let $\mathcal{I}$, $S_1(M)$ and $\mathcal{J}$ be the same as in the preceding section. The following theorem is the main technical result in this section and its proof is the hardest in the present paper:

Theorem 5.2 *Let M be a an $A(V)$-module. Then in $S_1(M)$,*

$$\mathcal{J} \cap M = 0.$$

Proof. Note that elements of the form (4.2) are in $\mathcal{I}$ and that M is identified with $(S_1(M))_{(0)}$. Using the definitions of $\mathcal{I}$ and $\mathcal{J}$ and noticing that the degree of elements of M is 0, we see that elements of $\mathcal{J}$ are linear combinations of elements of the form

$$a(\text{wt } a - K + N - 1)\text{Res}_{x_0}\text{Res}_{x_2}x_0^m x_2^{K-m-N-2} \cdot$$
$$\cdot(x_0 + x_2)^N Y_{S_1(M)}(Y((x_0+x_2)^{L(0)}u, x_0)x_2^{L(0)}v, x_2)b(\text{wt } b - N - 1)w + \mathcal{I}$$
$$= a(\text{wt } a - K + N - 1)\sum_{j=0}^{\text{wt } u+N}\binom{\text{wt } u + N}{j} \cdot$$
$$\cdot(u_{j+m}v)(K + \text{wt } u + \text{wt } v - j - m - 2)b(\text{wt } b - N - 1)w + \mathcal{I}$$

$$(5.6)$$

for homogeneous $a, b, u, v \in V$, $w \in M$, $K, N, m \in \mathbb{Z}$ satisfying $m \leq K - 2N - 2$. To prove this theorem, we need only prove that (5.6) is 0 in $S_1(M)$.

When $N < 0$, $\deg b(\text{wt } b - N - 1)w = N < 0$. So $b(\text{wt } b - N - 1)w \in \mathcal{I}$ and hence (5.6) is 0 in $S_1(M)$.

When $N \geq 0$ but $K > N$,

$$\deg(u_{j+m}v)(K + \text{wt } u + \text{wt } v - j - m - 2)b(\text{wt } b - N - 1)w = N - K < 0.$$

So $(u_{j+m}v)(K + \text{wt } u + \text{wt } v - j - m - 2)b(\text{wt } b - N - 1)w \in \mathcal{I}$ and hence (5.6) is 0 in $S_1(M)$.

We now prove that (5.6) is 0 in $S_1(M)$ in the case $N \geq 0$ and $N \geq K$. We rewrite (5.6) as

$$\text{Res}_{y_1}\text{Res}_{y_2}\text{Res}_{x_0}\text{Res}_{x_2}y_1^{-(K-N)-1}y_2^{-N-1}x_0^m x_2^{K-m-N-2}(x_0 + x_2)^N \cdot$$
$$\cdot Y_{S_1(M)}(y_1^{L(0)}a, y_1)Y_{S_1(M)}(Y((x_0+x_2)^{L(0)}u, x_0)x_2^{L(0)}v, x_2)Y_{S_1(M)}(y_2^{L(0)}b, y_2)w$$

$$(5.7)$$

where $a, b, u, v \in V$, $w \in M$, $K, N, m \in \mathbb{Z}$ satisfying $m \leq K - 2N - 2$.

Note that in $S_1(M)$, $Y_{S_1(M)}$ satisfies (5.5) with Y_W replaced by $Y_{S_1(M)}$. We shall use (5.5) and other formulas that still hold to show that (5.7) is in fact a linear combination of an element of $O(V)$ acting on $w \in M$ and hence equals 0 in $S_1(M)$.

Let

$$\tilde{w} = \text{Res}_{y_2}y_2^{-N-1}Y_{S_1(W)}(y_2^{L(0)}b, y_2)w.$$

Using the properties of vertex operators, changing variables and using (5.5), we see that

(5.7) is equal to

$$\operatorname{Res}_{y_1}\operatorname{Res}_{x_0}\operatorname{Res}_{x_2}y_1^{-(K-N)-1}x_0^m x_2^{K-m-N-2}(x_0+x_2)^N\cdot$$
$$\cdot Y_{S_1(M)}(y_1^{L(0)}a,y_1)Y_{S_1(M)}(Y((x_0+x_2)^{L(0)}u,x_0)x_2^{L(0)}v,x_2)\tilde{w}$$
$$=\operatorname{Res}_{y_1}\operatorname{Res}_{x_0}\operatorname{Res}_{x_2}y_1^{-(K-N)-1}x_0^m x_2^{K-m-2}(x_0x_2^{-1}+1)^N\cdot$$
$$\cdot Y_{S_1(M)}(y_1^{L(0)}a,y_1)Y_{S_1(M)}(x_2^{L(0)}Y((x_0x_2^{-1}+1)^{L(0)}u,x_0x_2^{-1})v,x_2)\tilde{w}$$
$$=\operatorname{Res}_{x_0}x_0^m(x_0+1)^N\operatorname{Res}_{y_1}\operatorname{Res}_{x_2}y_1^{-(K-N)-1}x_2^{K-1}\cdot$$
$$\cdot Y_{S_1(M)}(y_1^{L(0)}a,y_1)Y_{S_1(M)}(x_2^{L(0)}Y((x_0+1)^{L(0)}u,x_0)v,x_2)\tilde{w}$$
$$=\operatorname{Res}_{x_0}x_0^m(x_0+1)^N\operatorname{Res}_{y_0}\operatorname{Res}_{x_2}q_1(y_0,x_2)x_2^{K-1}(y_0+x_2)^N\cdot$$
$$\cdot Y_{S_1(M)}(Y((y_0+x_2)^{L(0)}a,y_0)x_2^{L(0)}Y((x_0+1)^{L(0)}u,x_0)v,x_2)\tilde{w},\qquad(5.8)$$

where

$$\begin{aligned}q_1(y_0,x_2)&=\sum_{i=0}^{N-K}\binom{-K-1}{i}y_0^{-K-1-i}x_2^i\\
&=x_2^{-K-1}\sum_{i=0}^{N-K}\binom{-K-1}{i}(y_0x_2^{-1})^{-K-1-i}\\
&=x_2^{-K-1}q_1(y_0x_2^{-1},1).\end{aligned}$$

The right-hand side of (5.8) can now be written as

$$\operatorname{Res}_{x_0}x_0^m(x_0+1)^N\operatorname{Res}_{y_0}\operatorname{Res}_{x_2}q_1(y_0x_2^{-1},1)x_2^{N-2}(y_0x_2^{-1}+1)^N\cdot$$
$$\cdot Y_{S_1(M)}(x_2^{L(0)}Y((y_0x_2^{-1}+1)^{L(0)}a,y_0x_2^{-1})Y((x_0+1)^{L(0)}u,x_0)v,x_2)\tilde{w}$$
$$=\operatorname{Res}_{x_0}x_0^m(x_0+1)^N\operatorname{Res}_{y_0}\operatorname{Res}_{x_2}q_1(y_0,1)x_2^{N-1}(y_0+1)^N\cdot$$
$$\cdot Y_{S_1(M)}(x_2^{L(0)}Y((y_0+1)^{L(0)}a,y_0)Y((x_0+1)^{L(0)}u,x_0)v,x_2)\tilde{w}.\qquad(5.9)$$

Using the commutator formula for V, the right-hand side of (5.9) is equal to

$$\operatorname{Res}_{y_0}q_1(y_0,1)(y_0+1)^N\operatorname{Res}_{x_2}\operatorname{Res}_{x_0}x_2^{N-1}x_0^m(x_0+1)^N\cdot$$
$$\cdot Y_{S_1(M)}(x_2^{L(0)}Y((x_0+1)^{L(0)}u,x_0)Y((y_0+1)^{L(0)}a,y_0)v,x_2)\tilde{w}$$
$$+\operatorname{Res}_{y_0}q_1(y_0,1)(y_0+1)^N\operatorname{Res}_{x_2}\operatorname{Res}_{x_0}\operatorname{Res}_x x_2^{N-1}x_0^m(x_0+1)^N\cdot$$
$$\cdot x_0^{-1}\left(\frac{y_0-x}{x_0}\right)Y_{S_1(M)}(x_2^{L(0)}Y(Y((y_0+1)^{L(0)}a,x)(x_0+1)^{L(0)}u,x_0)v,x_2)\tilde{w}.$$
$$(5.10)$$

The first term of (5.10) is a linear combination of the elements of the form

$$\operatorname{Res}_{x_2}\operatorname{Res}_{x_0}x_2^{N-1}x_0^n(x_0+1)^N Y_{S_1(M)}(x_2^{L(0)}Y((x_0+1)^{L(0)}\tilde{u},x_0)\tilde{v},x_2)\tilde{w}\qquad(5.11)$$

12

for $n = m \leq -N - 2$, $\tilde{u} = u, \tilde{v} \in V$. The second term of (5.10) is equal to

$$\mathrm{Res}_{x_2}\mathrm{Res}_{x_0}\mathrm{Res}_x\mathrm{Res}_{y_0}q_1(x_0 + x, 1)(x_0 + x + 1)^N x_2^{N-1} x_0^m (x_0 + 1)^N x_0^{-1}\left(\frac{y_0 - x}{x_0}\right)\cdot$$

$$\cdot Y_{S_1(M)}(x_2^{L(0)}Y((x_0 + 1)^{L(0)}Y((1 + x(x_0 + 1)^{-1})^{L(0)}a, x(x_0 + 1)^{-1})u, x_0)v, x_2)\tilde{w}$$

$$= \mathrm{Res}_{x_2}\mathrm{Res}_{x_0}\mathrm{Res}_x q_1(x_0 + x, 1)(x_0 + x + 1)^N x_2^{N-1} x_0^m (x_0 + 1)^N$$

$$\cdot Y_{S_1(M)}(x_2^{L(0)}Y((x_0 + 1)^{L(0)}Y((1 + x(x_0 + 1)^{-1})^{L(0)}a, x(x_0 + 1)^{-1})u, x_0)v, x_2)\tilde{w}$$

$$= \mathrm{Res}_{x_2}\mathrm{Res}_{x_0}\mathrm{Res}_y q_1(x_0 + y(x_0 + 1), 1)(x_0 + 1)^N (1 + y)^N x_2^{N-1} x_0^m (x_0 + 1)^{N+1}$$

$$\cdot Y_{S_1(M)}(x_2^{L(0)}Y((x_0 + 1)^{L(0)}Y((1 + y)^{L(0)}a, y)u, x_0)v, x_2)\tilde{w}. \tag{5.12}$$

By the definition of $q_1(y_0, x_2)$ above, we see that $q_1(x_0 + y(x_0 + 1), 1)$ is a Laurent series in x_0 and y such that the largest power of x_0 in $q_1(x_0 + y(x_0 + 1), 1)$ is $-K - 1$. Then the powers of x_0 in

$$q_1(x_0 + y(x_0 + 1), 1)x_0^m (x_0 + 1)^{N+1}$$

are less than or equal to $-N - 2$. Thus the right-hand side of (5.12) is a linear combination of the elements of the form (5.11) for $n \leq -N - 2$ and $\tilde{u}, \tilde{v} = v \in V$.

It remains to prove that (5.11) for $n \leq -N - 2$ and $\tilde{u}, \tilde{v} \in V$ is 0 in $S_1(M)$. We prove that elements of the form (5.11) for $n \leq -N - 2$ and $\tilde{u}, \tilde{v} \in V$ are linear combinations of elements obtained from the action of $O(V)$ on M. Then by the definition of $S_1(M)$, (5.11) for $n \leq -N - 2$ and $\tilde{u}, \tilde{v} \in V$ is 0.

Writing the element $\tilde{w}$ explicitly and using (5.5), we see that (5.11) is equal to

$$\mathrm{Res}_{x_0} x_0^n (x_0 + 1)^N \mathrm{Res}_{x_2}\mathrm{Res}_{y_2} x_2^{N-1} y_2^{-N-1} Y_{S_1(M)}(x_2^{L(0)}Y((x_0 + 1)^{L(0)}\tilde{u}, x_0)\tilde{v}, x_2)Y_{S_1(M)}(y_2^{L(0)}b, y_2)w$$

$$= \mathrm{Res}_{x_0} x_0^n (x_0 + 1)^N \mathrm{Res}_{y_0}\mathrm{Res}_{y_2} q_2(y_0, y_2)y_2^{-N-1} \cdot$$

$$\cdot Y_{S_1(M)}(Y((y_0 + y_2)^{L(0)}Y((x_0 + 1)^{L(0)}\tilde{u}, x_0)\tilde{v}, y_0)y_2^{L(0)}b, y_2)w$$

$$= \mathrm{Res}_{x_0} x_0^n (x_0 + 1)^N \mathrm{Res}_{y_0}\mathrm{Res}_{y_2} q_2(y_0, y_2)y_2^{-N-1} \cdot$$

$$\cdot Y_{S_1(M)}(y_2^{L(0)}Y((y_0 y_2^{-1} + 1)^{L(0)}Y((x_0 + 1)^{L(0)}\tilde{u}, x_0)\tilde{v}, y_0 y_2^{-1})b, y_2)w$$

$$= \mathrm{Res}_{x_0} x_0^n (x_0 + 1)^N \mathrm{Res}_y\mathrm{Res}_{y_2} q_2(yy_2, y_2)y_2^{-N} \cdot$$

$$\cdot Y_{S_1(M)}(y_2^{L(0)}Y((y + 1)^{L(0)}Y((x_0 + 1)^{L(0)}\tilde{u}, x_0)\tilde{v}, y)b, y_2)w$$

$$= \mathrm{Res}_{x_0} x_0^n (x_0 + 1)^N \mathrm{Res}_y\mathrm{Res}_{y_2} q_2(yy_2, y_2)y_2^{-N} \cdot$$

$$\cdot Y_{S_1(M)}(y_2^{L(0)}Y(Y((x_0 + 1)^{L(0)}(y + 1)^{L(0)}\tilde{u}, x_0(y + 1))(y + 1)^{L(0)}\tilde{v}, y)b, y_2)w$$

$$= \mathrm{Res}_{y_2}\mathrm{Res}_y\mathrm{Res}_x x^n (x + y + 1)^N q_2(yy_2, y_2)y_2^{-N}(y + 1)^{-N-1-n} \cdot$$

$$\cdot Y_{S_1(M)}(y_2^{L(0)}Y(Y((x + y + 1)^{L(0)}\tilde{u}, x)(y + 1)^{L(0)}\tilde{v}, y)b, y_2)w$$

$$= \mathrm{Res}_{y_2}\mathrm{Res}_y\mathrm{Res}_{x_1} (x_1 - y)^n (x_1 + 1)^N q_2(yy_2, y_2)y_2^{-N}(y + 1)^{-N-1-n} \cdot$$

$$\cdot Y_{S_1(M)}(y_2^{L(0)}Y((x_1 + 1)^{L(0)}\tilde{u}, x_1)Y((y + 1)^{L(0)}\tilde{v}, y)b, y_2)w$$

$$- \mathrm{Res}_{y_2}\mathrm{Res}_y\mathrm{Res}_{x_1} (-y + x_1)^n (x_1 + 1)^N q_2(yy_2, y_2)y_2^{-N}(y + 1)^{-N-1-n} \cdot$$

$$\cdot Y_{S_1(M)}(y_2^{L(0)}Y((y + 1)^{L(0)}\tilde{v}, y)Y((x_1 + 1)^{L(0)}\tilde{u}, x_1)b, y_2)w$$

$$\tag{5.13}$$

Where
$$q_2(y_0, y_2) = \sum_{i=0}^{N} \binom{N-1}{i} y_0^{N-1-i} y_2^i.$$

From the expression of $q_2(y_0, y_2)$, we have
$$q_2(yy_2, y_2) = y_2^{N-1} q_2(y, 1).$$

Thus the first term in the right-hand side of (5.13) is a linear combination of terms of the form
$$\mathrm{Res}_{y_2} y_2^{-1} Y_{S_1(M)}(y_2^{L(0)} \mathrm{Res}_{x_1} x_1^{\tilde{n}} Y((x_1+1)^{L(0)} \tilde{u}, x_1) \tilde{\tilde{v}}, y_2) w$$
for $\tilde{n} \le -2$ and $\tilde{\tilde{v}} \in V$. By definition,
$$\tilde{\tilde{u}} = \mathrm{Res}_{x_1} x_1^{\tilde{n}} Y((x_1+1)^{L(0)} \tilde{u}, x_1) \tilde{\tilde{v}} \in O(V)$$
and, by definition,
$$\mathrm{Res}_{y_2} y_2^{-1} Y_{S_1(M)}(y_2^{L(0)} \tilde{\tilde{u}}, y_2) = o(\tilde{u}).$$

Since $o(\tilde{\tilde{u}})w = 0$ in $S_1(M)$, $\mathrm{Res}_{y_2} y_2^{-1} Y_{S_1(M)}(y_2^{L(0)} \tilde{\tilde{u}}, y_2) w$ is also 0. This proves that the first term in the right-hand side of (5.13) is 0.

The largest power of y in
$$(-y + x_1)^n q_2(y, 1)(y+1)^{-N-1-n}$$

is -2. Thus the second term in the right-hand side of (5.13) is a linear combination of terms of the form
$$\mathrm{Res}_{y_2} y_2^{-1} Y_{S_1(M)}(y_2^{L(0)} \mathrm{Res}_y y^{\tilde{n}} Y((y+1)^{L(0)} \tilde{v}, y) \tilde{\tilde{u}}, y_2) w$$
for $\tilde{n} \le -2$ and $\tilde{\tilde{u}} \in V$. Then the same argument as above shows that the second term in the right-hand side of (5.13) is 0. ∎

Remark 5.3 Note that though the proof of the theorem above does not need the commutator formula for $Y_{S_1(M)}$, the commutator formula for V is indeed used. It will be interesting to see whether there is a proof without using the commutator formula for V.

Corollary 5.4 *Let M be an $A(V)$-module. Then the embedding from M to $S_1(M)$ induces an isomorphism e_M of $A(V)$-modules from M to $T(S(M))$.*

Proof. Given an N-gradable weak V-module M, we have an embedding from M to $S_1(M)$. By Theorem 5.2, the induced map from M to $S(M) = S_1/\mathcal{J}$ is still an embedding. By definition, $T(S(M))$ is exactly the image of M in $S(M)$. Clearly, when we view the induced map from M to $S(M) = S_1/\mathcal{J}$ as a map from M to $T(S(M))$, it is an $A(V)$-module map and thus is a an isomorphism of $A(V)$-modules. ∎

Theorem 5.5 *The functor S has the following universal property: Let M be an $A(V)$-module. For any $\mathbb{N}$-gradable weak V-module W and any $A(V)$-module map $f : M \to T(W)$, there exists a unique V-module map $\tilde{f} : S(M) \to W$ such that $\tilde{f}|_{T(S(M))} = f \circ e_M^{-1}$.*

Proof. Elements of $S(M)$ are linear combinations of elements of the form $u(m)w$ for homogeneous $u \in V$, $w \in M$ and $m \in \mathbb{Z}$ satisfying wt $- m - 1 + \deg w \geq 0$. We define $\tilde{f}(u(m)w) = u_m f(w)$. By definition, relations among elements of the form $u(m)w$ for homogeneous $u \in V$, $w \in M$ and $m \in \mathbb{Z}$ satisfying wt $- m - 1 + \deg w \geq 0$ are also satisfied by elements of W of the form $u_m f(w)$. Thus the map $\tilde{f}$ is well defined. Using (2.2) (or equivalently, (2.1) or (2.3)), we see that $\tilde{f}$ is indeed a V-module map. By definition, $\tilde{f}|_M = f \circ e_M^{-1}$. If $g : S(M) \to W$ is another such V-module map, then $g(u(m)w) = u_m g(w) = u_m f(w) = \tilde{f}(u(m)w)$, proving the uniqueness of $\tilde{f}$. ∎

We also have the following:

Corollary 5.6 *The functor T is a left inverse of S, that is, $T \circ S$ is naturally isomorphic to is the identity functor on the category of $A(V)$-modules.*

Proof. For an $A(V)$-module M, by Corollary 5.4, we have an isomorphism e_M^{-1} from $(T \circ S)(M)$ to M. For an $A(V)$-module map $f : M_1 \to M_2$, by definition, the $A(V)$-module map $(T \circ S)(f) : (T \circ S)(M_1) \to (T \circ S)(M_2)$ is $e_{M_2} \circ f \circ e_{M_1}^{-1}$. Thus we obtain a natural transformation e^{-1} from $T \circ S$ to the identity functor on the category of $A(V)$-modules. This natural transformation e^{-1} is clearly a natural isomorphism since it has an inverse e which is given by the isomorphism e_M from an $A(V)$-module M to $T(S(M))$. ∎

In [Z], in his proof that the set of equivalence classes of irreducible $\mathbb{N}$-gradable weak V-modules is in bijection with the set of equivalence classes of irreducible modules for $A(V)$, Zhu in fact obtained implicitly a functor from the category of $A(V)$-modules to the category of $\mathbb{N}$-gradable weak V-modules. In [DLM], this functor, denoted by $\bar{M}_0$, was constructed explicitly and was proved to satisfy the same universal property above. The following result achieves our goal of constructing this functor without dividing relations corresponding to the commutator formula for weak modules:

Corollary 5.7 *The functor S is equal to the functor $\bar{M}_0$ constructed in [DLM].*

Proof. This result follows immediately from the universal property. ∎

REFERENCES

[DLM] C. Dong, H. Li and G. Mason, Vertex operator algebras and associative algebras, *J. Algebra* **206** (1998), 67-96.

[FHL]   I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. 104, Amer. Math. Soc., Providence, 1993, no. 494 (preprint, 1989).

[H1]   Y.-Z. Huang, Differential equations and intertwining operators, *Comm. Contemp. Math.* **7** (2005), 375–400.

[H2]   Y.-Z. Huang, Differential equations, duality and modular invariance, *Comm. Contemp. Math.* **7** (2005), 649-706.

[H3]   Y.-Z. Huang, Generalized twisted modules associated to general automorphisms of a vertex operator algebra, *Comm. Math. Phys.* **298** (2010), 265–292.

[LL]   J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Math., Vol. 227, Birkhäuser, Boston, 2003.

[L]   H. Li, The regular representation, Zhu's $A(V)$-theory and induced modules, *J. Alg.* **238** (2001), 159–193.

[MS]   G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177–254.

[Z]   Y.-C. Zhu, Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.* **9**(1996), 237-302.