# Associative algebras and intertwining operators

Yi-Zhi Huang

**Abstract**

Let $V$ be a vertex operator algebra and $A^{\infty}(V)$ and $A^{N}(V)$ for $N \in \mathbb{N}$ the associative algebras introduced by the author in [H5]. For a lower-bounded generalized $V$-module $W$, we give $W$ a structure of graded $A^{\infty}(V)$-module and we introduce an $A^{\infty}(V)$-bimodule $A^{\infty}(W)$ and an $A^{N}(V)$-bimodule $A^{N}(W)$. We prove that the space of (logarithmic) intertwining operators of type $\binom{W_3}{W_1 W_2}$ for lower-bounded generalized $V$-modules $W_1$, $W_2$ and $W_3$ is isomorphic to the space $\mathrm{Hom}_{A^{\infty}(V)}(A^{\infty}(W_1) \otimes_{A^{\infty}(V)} W_2, W_3)$. Assuming that $W_2$ and $W_3'$ are equivalent to certain universal lower-bounded generalized $V$-modules generated by their $A^{N}(V)$-submodules consisting of elements of levels less than or equal to $N \in \mathbb{N}$, we also prove that the space of (logarithmic) intertwining operators of type $\binom{W_3}{W_1 W_2}$ is isomorphic to the space of $\mathrm{Hom}_{A^{N}(V)}(A^{N}(W_1) \otimes_{A^{N}(V)} \Omega_{N}^{0}(W_2), \Omega_{N}^{0}(W_3))$.

## 1 Introduction

In [H5], for a grading-restricted vertex algebra $V$, the author introduced associative algebras $A^{\infty}(V)$ and $A^{N}(V)$ for $N \in \mathbb{N}$ and proved basic properties of these algebras and their modules in connection with $V$ and lower-bounded generalized $V$-modules. In the present paper, we study the connections between (logarithmic) intertwining operators among lower-bounded generalized $V$-modules and module maps between suitable modules for these associative algebras in the case that $V$ is a vertex operator algebra so that there is a conformal vector $\omega \in V$. The results of the present paper will be used in the last step of a proof of the modular invariance of (logarithmic) intertwining operators in the case that $V$ satisfies the positive energy condition (or CFT type) and $C_2$-cofiniteness condition in a paper in preparation. For simplicity, we shall omit "(logarithmic)" in "(logarithmic) intertwining operator" in this paper so that by an intertwining operator, we always mean an intertwining operator which might contains logarithm of the variable.

In the case $N = 0$, the algebra $A^{0}(V)$ was proved in [H5] to be isomorphic to the Zhu algebra $A(V)$ (see [Z]). In this case, there is a theorem of Frenkel and Zhu [FZ] stating that for irreducible $V$-modules $W_1$, $W_2$ and $W_3$, the space of intertwining operators of type $\binom{W_3}{W_1 W_2}$ is linearly isomorphic to $\mathrm{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} \Omega(W_2), \Omega(W_3))$, where $\Omega(W_2)$ and

$\Omega(W_3)$ are the lowest weight spaces of $W_2$ and $W_3$, respectively, and $A(W_1)$ is an $A(V)$-bimodule introduced in the paper [FZ]. A proof of this theorem was given by Li in [L] under the assumption that every lower-bounded generalized $V$-module (or $\mathbb{N}$-gradable weak $V$-module) is completely reducible. In the same paper [L], a counterexample was also given to show that without this assumption, the theorem is not true. Since this result needs this semisimplicity assumption, it cannot be used to study intertwining operators in the case that lower-bounded generalized $V$-modules might not be completely reducible. In [HY1] and [HY2], under strong assumptions on $W_2$ and $W_3'$ but without this semisimplicity assumption, Yang and the author proved that the space of intertwining operators of type $\binom{W_3}{W_1W_2}$ is linearly isomorphic to $\mathrm{Hom}_{A_N(V)}(A_N(W_1)\otimes_{A_N(V)}\Omega_N^0(W_2), \Omega_N^0(W_3))$, where $\Omega_N^0(W_2)$ and $\Omega_N^0(W_3)$ are suitable subspaces of $W_2$ and $W_3$, respectively, and $A_N(W_1)$ is a bimodule for the generalization $A_N(V)$ of $A(V)$ by Dong, Li and Mason [DLM].

In the present paper, for a lower-bounded generalized $V$-module $W$, we introduce an $A^\infty(V)$-bimodule $A^\infty(W)$ and an $A^N(V)$-bimodule $A^N(W)$ for each $N\in\mathbb{N}$. We prove that the space of (logarithmic) intertwining operators of type $\binom{W_3}{W_1W_2}$ for lower-bounded generalized $V$-modules $W_1$, $W_2$ and $W_3$ is isomorphic to the space $\mathrm{Hom}_{A^\infty(V)}(A^\infty(W)\otimes_{A^\infty(V)} W_2, W_3)$. Assuming that $W_2$ and $W_3'$ are certain universal lower-bounded generated by their $A^N(V)$-submodules consisting of elements of levels less than or equal to $N\in\mathbb{N}$, we also prove that the space of (logarithmic) intertwining operators of type $\binom{W_3}{W_1W_2}$ is isomorphic to the space of $\mathrm{Hom}_{A^N(V)}(A^N(W)\otimes_{A^N(V)} W_2, W_3)$.

Here we give more discussions on the main results. In [H5], the associative algebra $A^\infty(V)$ is defined using the associated graded spaces given by suitable ascending filtrations of lower-bounded generalized $V$-modules for a grading-restricted vertex algebra. These associated graded spaces are by definition nondegenerate graded $A^\infty(V)$-modules (see [H5]). But it is not easy to work with the associated graded space of a lower-bounded module, since in general it is very difficult to determine the associated graded space explicitly. In this paper we shall work with a canonical $\mathbb{N}$-grading $W=\coprod_{n\in\mathbb{N}} W_{[\![n]\!]}$ of a lower-bounded generalized $V$-module $W$ and show that $W$ itself also has a structure of graded $A^\infty(V)$-module. We construct the structure of graded $A^\infty(V)$-module on $W$ by showing that $W$ is in fact a quotient of the associated graded space of another lower-bounded generalized $V$-module. But in general this structure of a graded $A^\infty(V)$-module on $W$ might not be nondegenerate (see [H5]). This result holds for a grading-restricted vertex algebra $V$ which might not have a conformal vector.

Given a lower-bounded generalized $V$-module $W$, let $U^\infty(W)$ be the space of all column-finite matrices with entries in $W$ and doubly index by $\mathbb{N}$. We introduce left and right actions of $A^\infty(V)$ on $U^\infty(W)$. Let $W_2$ and $W_3$ be lower-bounded generalized $V$-modules and let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{WW_2}$. We introduce a linear map $\vartheta_{\mathcal{Y}}:U^\infty(W)\to \mathrm{Hom}(W_2,W_3)$. Note that $A^\infty(V)$ is a quotient of a nonassociative algebra $U^\infty(V)$ of column-finite matrices with entries in $V$ and doubly index by $\mathbb{N}$. Since $W_2$ and $W_3$ are left $A^\infty(V)$-modules, $\mathrm{Hom}(W_2,W_3)$ has a natural structure of $A^\infty(V)$-bimodule and in particular, a natural structure of $U^\infty(V)$-bimodule. We prove that $\vartheta_{\mathcal{Y}}$ commutes with the left and right actions of $A^\infty(V)$. Let $Q^\infty(W)$ be the intersection of $\ker\vartheta_{\mathcal{Y}}$ for all such lower-bounded generalized $V$-modules $W_2$ and $W_3$ and intertwining operators $\mathcal{Y}$ of type $\binom{W_3}{WW_2}$. Then $A^\infty(W) = U^\infty(W)/Q^\infty(W)$ is an $A^\infty(V)$-bimodule. The construction of the $A^\infty(V)$-bimodule $A^\infty(W)$ works for a grading-restricted vertex algebra $V$ which might not have a conformal vector.

Given lower-bounded generalized $V$-modules $W_1$, $W_2$, $W_3$ and an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1W_2}$, we prove that the map $\vartheta_{\mathcal{Y}}$ discussed above in the case $W = W_1$ gives an $A^\infty(V)$-module map $\rho(\mathcal{Y}) : A^\infty(W_1) \otimes_{A^\infty(V)} W_2 \to W_3$. In particular, we obtain a linear map $\rho : \mathcal{V}_{W_1W_2}^{W_3} \to \mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3)$, where $\mathcal{V}_{W_1W_2}^{W_3}$ is the space of intertwining operators of type $\binom{W_3}{W_1W_2}$. Our first main theorem states that $\rho$ is an isomorphism. In the proof of the first main theorem, we need to assume that $V$ is a vertex operator algebra which has a conformal vector. This first main theorem can be generalized to a grading-restricted vertex algebra. But in this more general case, since there is no conformal vector, one has to introduce additional structures so that left actions of $L(-1)$ on the $A^\infty(V)$-bimodules can be introduced.

Let $N \in \mathbb{N}$ and $W$ a lower-bounded generalized $V$-module. Let $\Omega_N^0(W) = \coprod_{n=0}^N W_{[\![ n ]\!]}$. Then $\Omega_N^0(W)$ is in fact an $A^N(V)$-module. We consider the space $U^N(W)$ of $(N+1)\times(N+1)$ matrices with entries in $W$. Then $U^N(W)$ can be viewed as a subspace of $U^\infty(W)$. In particular, the cosets in $A^\infty(W)$ containing elements of $U^N(W)$ form an $A^N(V)$-bimodule $A^N(W)$. The construction of the $A^N(V)$-bimodule $A^N(W)$ works for a grading-restricted vertex algebra $V$.

Given lower-bounded generalized $V$-modules $W_1$, $W_2$, $W_3$ and an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1W_2}$, the $A^\infty(V)$-module map $\rho(\mathcal{Y}) : A^\infty(W_1) \otimes_{A^\infty(V)} W_2 \to W_3$ induces an $A^N(V)$-module map $\rho^N(\mathcal{Y}) : A^N(W_1) \otimes_{A^N(V)} \Omega_N^0(W_2) \to \Omega_N^0(W_3)$. Then we obtain a linear map $\rho^N : \mathcal{V}_{W_1W_2}^{W_3} \to \mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3))$. We prove that $\rho^N$ is injective. Our second main theorem states that $\rho^N$ is an isomorphism when $W_2$ and $W_3'$ are certain universal lower-bounded generalized $V$-mdoules generated by the $A^N(V)$-modules $\Omega_N^0(W_2)$ and $\Omega_N^0(W_3')$, respectively. This second main theorem is proved using the first main theorem above. In particular, $V$ is also assumed to be a vertex operator algebra. Again, this theorem can be generalized to the case that $V$ is a grading-restricted vertex algebra.

The main motivation of this paper is the modular invariance of intertwining operators in the nonsemisimple (or logarithmic) case. In [H1], the author proved the conjecture of Moore and Seiberg on modular invariance of intertwining operators for rational conformal field theories (see [MS]). Mathematically this modular invariance is for intertwining operators among modules for a vertex operator algebra $V$ satisfying the conditions that $V$ is of positive energy (or CFT-type), $V$ is $C_2$-cofinite and every lower-bounded generalized $V$-module (or every $\mathbb{N}$-gradable weak $V$-module) is completely reducible. After the convergence and analytic extensions of shifted $q$-traces of products of intertwining operators and the genus-one associativity is proved, the proof of the modular invariance is reduced to the proof that a genus-one one-point correlation function can always be written as the analytic extension of a shifted $q$-trace of an intertwining operator. It is in this last step of the proof that the theorem of Frenkel and Zhu mentioned above is used.

In [F1] and [F2], Fiordalisi proved the convergence and analytic extension property of shifted pseudo-$q$-traces of products of intertwining operators and the genus-one associativity under the condition that $V$ is of positive energy (or CFT-type) and $V$ is $C_2$-cofinite, but without assuming the condition that every lower-bounded generalized $V$-module (or every $\mathbb{N}$-gradable weak $V$-module) is completely reducible. The proof of the modular invariance in this nonsemisimple (or logarithmic) case is then also reduced to the proof that a genus-one one-point correlation function can always be written as a shifted pseudo-$q$-trace of an intertwining operator. But in this case, the theorem of Frenkel and Zhu mentioned above cannot be used. In a paper in preparation, we shall give this last step using the results obtained in the present paper.

This paper is organized as follows: We recall in Section 2 the associative algebras $A^\infty(V)$ and $A^N(V)$ and graded $A^\infty(V)$- and $A^N(V)$-modules introduced in [H5]. In Section 3, we give an $A^\infty(V)$-module structure to a lower-bounded generalized $V$-module $W$ . In Section 4, we construct the $A^\infty(V)$-bimodule $A^\infty(W)$ from $W$. Our first main theorem stating that $\rho$ is an isomorphism is formulated and proved in Section 5. Our second main theorem stating that $\rho^N$ is an isomorphism when when $W_2$ and $W_3'$ are certain universal lower-bounded generalized $V$-modules generated by the $A^N(V)$-modules $\Omega_N^0(W_2)$ and $\Omega_N^0(W_3')$, respectively, is formulated and proved in Section 6.

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# 2 The associative algebras $A^\infty(V)$ and $A^N(V)$ and graded $A^\infty(V)$- and $A^N(V)$-modules

In this section, we recall the associative algebras $A^\infty(V)$ and $A^N(V)$ for $N \in \mathbb{N}$ for a grading-restricted vertex algebra $V$ and $A^\infty(V)$- and $A^N(V)$-graded modules introduced in [H5].

Let $V$ be a grading-restricted vertex algebra. Let $U^\infty(V)$ be the space of column-finite infinite matrices with entries in $V$, but doubly indexed by $\mathbb{N}$ instead of $\mathbb{Z}_+$. Elements of $U^\infty(V)$ are of the form $\mathfrak{v} = [v_{kl}]$ for $v_{kl} \in V$, $k, l \in \mathbb{N}$ such that for each fixed $l \in \mathbb{N}$, there are only finitely many nonzero $v_{kl}$. For $k, l \in \mathbb{N}$ and $\in V$, let $[v]_{kl}$ be the element of $U^\infty(V)$ with the entry in the $k$-th row and $l$-th column equal to $v$ and all the other entries equal to 0.

We define a product $\diamond$ on $U^\infty(V)$ by

$$\mathfrak{u} \diamond \mathfrak{v} = [(\mathfrak{u} \diamond \mathfrak{v})_{kl}]$$

for $\mathfrak{u} = [u_{kl}], \mathfrak{v} = [v_{kl}] \in U^\infty(V)$, where

$$\begin{aligned}(\mathfrak{u} \diamond \mathfrak{v})_{kl} &= \sum_{n=k}^{l} \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l Y_V((1+x)^{L_V(0)} u_{kn}, x) v_{nl}\\ &= \sum_{n=k}^{n} \sum_{m=0}^{l} \binom{-k+n-l-1}{m} \mathrm{Res}_x x^{-k+n-l-m-1}(1+x)^l Y_V((1+x)^{L_V(0)} u_{kn}, x) v_{nl}\end{aligned} \tag{2.1}$$

for $k, l \in \mathbb{N}$, where

$$T_{k+l+1}((x+1)^{-k+n-l-1}) = \sum_{m=0}^{n} \binom{-k+n-l-1}{m} x^{-k+n-l-m-1}$$

is the Taylor polynomial in $x^{-1}$ of order $k+l+1$ of $(x+1)^{-k+n-l-1}$. Then $U^\infty(V)$ equipped with $\diamond$ is an algebra but in general is not even associative. By definition, for $u, v \in V$ and $k, m, n, l \in \mathbb{N}$, by definition,

$$[u]_{km} \diamond [v]_{nl} = 0$$

when $m \neq n$ and

$$\begin{aligned} [u]_{kn} \diamond [v]_{nl} &= \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_V((1+x)^{L_V(0)}u, x)v\right]_{kl} \\ &= \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_x x^{-k+n-l-m-1}(1+x)^l \left[Y_V((1+x)^{L_V(0)}u, x)v\right]_{kl}. \end{aligned} \tag{2.2}$$

Since $[u]_{km} \diamond [v]_{nl} = 0$ when $m \neq n$, we need only consider $[u]_{kn} \diamond [v]_{nl}$ for $u, v \in V$ and $k, n, l \in \mathbb{N}$.

Let $\mathbf{1}^\infty$ be the element of $U^\infty(V)$ with diagonal entries being $\mathbf{1} \in V$ and all the other entries being 0.

Let $W$ be a lower-bounded generalized $V$-module. For $n \in \mathbb{N}$, let

$$\Omega_n(W) = \{w \in W \mid (Y_W)_k(v)w = 0 \text{ for homogeneous } v \in V, \mathrm{wt}\, v - k - 1 < -n\}.$$

Then

$$\Omega_{n_1}(W) \subset \Omega_{n_2}(W)$$

for $n_1 \leq n_2$ and

$$W = \bigcup_{n \in \mathbb{N}} \Omega_n(W).$$

So $\{\Omega_n(w)\}_{n \in \mathbb{N}}$ is an ascending filtration of $W$. Let

$$Gr(W) = \sum_{n \in \mathbb{N}} Gr_n(W)$$

be the associated graded space, where

$$Gr_n(W) = \Omega_n(W)/\Omega_{n-1}(W).$$

Sometimes we shall use $[w]_n$ to denote the element $w + \Omega_{n-1}(W)$ of $Gr_n(W)$, where $w \in \Omega_n(W)$.

For $\mathfrak{v} = [v_{kl}] \in U^\infty(V)$, where $v_{kl} \in V$ and $k, l \in \mathbb{N}$, we have an operator $\vartheta_{Gr(W)}(\mathfrak{v})$ on $Gr(W)$ defined by

$$\vartheta_{Gr(W)}(\mathfrak{v})\mathfrak{w} = \sum_{k,l \in \mathbb{N}} \mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)} v_{kl}, x) \pi_{Gr_l(W)} \mathfrak{w},$$

for $\mathfrak{w} \in Gr(W)$, where $\pi_{Gr_l(W)}$ is the projection from $Gr(W)$ to $Gr_l(W)$. Then we have a linear map

$$\begin{aligned} \vartheta_{Gr(W)} : U^\infty(V) &\to \text{End } Gr(W) \\ \mathfrak{v} &\mapsto \vartheta_{Gr(W)}(\mathfrak{v}). \end{aligned}$$

Let $Q^\infty(V)$ be the intersection of $\ker \vartheta_{Gr(W)}$ for all lower-bounded generalized $V$-modules $W$ and $A^\infty(V) = U^\infty(V)/Q^\infty(V)$.

The following result gives the associative algebra $A^\infty(V)$:

**Theorem 2.1 ([H5])** *The product $\diamond$ on $U^\infty(V)$ induces a product, denoted still by $\diamond$, on $A^\infty(V) = U^\infty(V)/Q^\infty(V)$ such that $A^\infty(V)$ equipped with $\diamond$ is an associative algebra with $\mathbf{1}^\infty + Q^\infty(V)$ as identity. Moreover, the associated graded space $Gr(W)$ of the ascendant filtration $\{\Omega_n(W)\}_{n\in\mathbb{N}}$ of a lower-bounded generalized $V$-module $W$ is an $A^\infty(V)$-module.*

We also need the following notion introduced in [H5]:

**Definition 2.2** Let $G$ be an $A^\infty(V)$-module with the $A^\infty(V)$-module structure on $G$ given by a homomorphism $\vartheta_G : A^\infty(V) \to \text{End } G$ of associative algebras. We say that $G$ is a *graded $A^\infty(V)$-module* if the following conditions are satisfied:

1. $G$ is graded by $\mathbb{N}$, that is, $G = \coprod_{n\in\mathbb{N}} G_n$, and for $v \in V$, $k, l \in \mathbb{N}$, $\vartheta_G([v]_{kl} + Q^\infty(V))$ maps $G_n$ to 0 when $n \neq l$ and to $G_k$ when $n = l$.

2. $G$ is a direct sum of generalized eigenspaces of an operator $L_G(0)$ on $G$, $G_n$ for $n \in \mathbb{N}$ are invariant under $L_G(0)$ and the real parts of the eigenvalues of $L_G(0)$ have a lower bound.

3. There is an operator $L_G(-1)$ on $G$ mapping $G_n$ to $G_{n+1}$ for $n \in \mathbb{N}$.

4. The commutator relations

$$\begin{aligned} [L_G(0), L_G(-1)] &= L_G(-1), \\ [L_G(0), \vartheta_G([v]_{kl} + Q^\infty(V))] &= (k-l)\vartheta_G([v]_{kl} + Q^\infty(V)), \\ [L_G(-1), \vartheta_G([v]_{kl} + Q^\infty(V))] &= \vartheta_G([L_V(-1)v]_{(k+1)l} + Q^\infty(V)) \end{aligned}$$

   hold for $v \in V$ and $k, l \in \mathbb{N}$

A graded $A^\infty(V)$-algebra $G$ is said to be *nondegenerate* if it satisfies in addition the following condition: For $g \in G_l$, if $\vartheta_G([v]_{0l} + Q^\infty(V))g = 0$ for all $v \in V$, then $g = 0$. Let $G_1$ and $G_2$ be graded $A^\infty(V)$-modules. A *graded $A_\infty(V)$-module map* from $G_1$ to $G_2$ is an $A^N(V)$-module map $f : G_1 \to G_2$ such that $f((G_1)_n) \subset (G_2)_n$, $f \circ L_{G_1}(0) = L_{G_2}(0) \circ f$ and $f \circ L_{G_1}(-1) = L_{G_2}(-1) \circ f$. A *graded $A^\infty(V)$-submodule* of a graded $A^\infty(V)$-module $G$ is an $A^\infty(V)$-submodule of $G$ that is also an $\mathbb{N}$-graded subspace of $G$ and invariant under the operators $L_G(0)$ and $L_G(-1)$. A graded $A^\infty(V)$-module $G$ is said to be *generated by a subset*

$S$ if $G$ is equal to the smallest graded $A^\infty(V)$-submodule containing $S$, or equivalently, $G$ is spanned by homogeneous elements with respect to the $\mathbb{N}$-grading and the grading given by $L_G(0)$ obtained by applying elements of $A^\infty(V)$, $L_G(0)$ and $L_G(-1)$ to homogeneous summands of elements of $S$. A graded $A^\infty(V)$-module is said to be *irreducible* if it has no nonzero proper graded $A^\infty(V)$-submodules. A graded $A^\infty(V)$-module is said to be *completely reducible* if it is a direct sum of irreducible graded $A^\infty(V)$-modules.

We now recall the subalgebras $A^N(V)$ of $A^\infty(V)$ also introduced in [H5]. For $N \in \mathbb{N}$, let $U^N(V)$ be the space of all $(N+1)\times(N+1)$ matrices with entries in $V$. It is clear that $U^N(V)$ can be canonically embedded into $U_0^\infty(V)$ as a subspace. We view $U^N(V)$ as a subspace of $U_0^\infty(V)$. As a subspace of $U_0^\infty(V)$, $U^N(V)$ consists of infinite matrices in $U^\infty(V)$ whose $(k,l)$-th entries for $k > N$ or $l > N$ are all 0 and is spanned by elements of the form $[v]_{kl}$ for $v \in V$, $k, l = 0, \dots, N$.

By (4.1), for $u, v \in V$ and $k, n, l = 0, \dots, N$,

$$[u]_{kn} \diamond [v]_{nl} = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_V((1+x)^{L(0)}u, x)v\right]_{kl} \in U^N(V). \quad (2.3)$$

So $U^N(V)$ is closed under the product $\diamond$. Let

$$A^N(V) = \{\mathfrak{v} + Q^\infty(V) \mid \mathfrak{v} \in U^N(V)\} = \pi_{A^\infty(V)}(U^N(V)),$$

where $\pi_{A^\infty(V)}$ is the projection from $U^\infty(V)$ to $A^\infty(V)$. Then $A^N(V)$ is spanned by elements of the form $[v]_{kl} + Q^\infty(V)$ for $v \in V$ and $k, l = 0, \dots, N$. Let $\mathbf{1}^N = \sum_{k=0}^N [\mathbf{1}]_{kk}$, that is, $\mathbf{1}^N$ is the element of $U^N(V)$ with the only nonzero entries to be equal to $\mathbf{1}$ at the diagonal $(k,k)$-th entries for $k = 0, \dots, N$.

**Proposition 2.3** *The subspace $A^N(V)$ is closed under $\diamond$ and is thus a subalgebra of $A^\infty(V)$ with the identity $\mathbf{1}^N + Q^\infty(V)$.*

We also have the following notion introduced in [H5]:

**Definition 2.4** Let $M$ be an $A^N(V)$-module $M$ with the $A^N(V)$-module structure on $M$ given by $\vartheta_M : A^N(V) \to \mathrm{End}\, M$. We say that $M$ is a *graded $A^N(V)$-module* if the following conditions are satisfied:

1. $M = \coprod_{n=0}^N G_n(M)$ such that for $v \in V$ and $k, l = 0, \dots, N$, $\vartheta_M([v]_{kl} + Q^\infty(V))$ maps $G_n(M)$ for $0 \le n \le N$ to 0 when $n \ne l$ and to $G_k(M)$ when $n = l$.

2. $M$ is a direct sum of generalized eigenspaces of of an operator $L_M(0)$ on $M$. $G_n(M)$ for $n \in \mathbb{N}$ are invariant under $L_M(0)$ and the real parts of the eigenvalues of $L_M(0)$ has a lower bound.

3. There is a linear map $L_M(-1) : \coprod_{n=0}^{N-1} G_{n-1}(M) \to \coprod_{n=1}^N G_n(M)$ mapping $G_n(M)$ to $G_{n+1}(M)$ for $n = 0, \dots, N-1$.

4. The commutator relations

$$
\begin{aligned}
[L_M(0), L_M(-1)] &= L_M(-1),\\
[L_M(0), \vartheta_M([v]_{kl} + Q^\infty(V))] &= (k-l)\vartheta_M([v]_{kl} + Q^\infty(V)),\\
[L_M(-1), \vartheta_M([v]_{pl} + Q^\infty(V))] &= \vartheta_M([L_V(-1)v]_{(p+1)l} + Q^\infty(V))
\end{aligned}
$$

hold for $v \in V$, $k, l = 0, \dots, N$ and $p = 0, \dots, N-1$.

A graded $A^N(V)$-module $M$ is said to be *nondegenerate* if the following additional condition holds: For $w \in G_l(M)$, if $\vartheta_M([v]_{0l} + Q^\infty(V))w = 0$ for all $v \in V$, then $w = 0$. Let $M_1$ and $M_2$ be graded $A^N(V)$-modules. An *graded $A_N(V)$-module map* from $M_2$ to $M_2$ is an $A^N(V)$-module map $f : M_1 \to M_2$ such that $f(G_n(M_1)) \subset G_n(M_2)$ for $n = 0 \dots, N$, $f \circ L_{M_1}(0) = L_{M_2}(0) \circ f$ and $f \circ L_{M_1}(-1) = L_{M_2}(-1) \circ f$. A *graded $A^N(V)$-submodule* of a graded $A^N(V)$-module $M$ is an $A^N(V)$-submodule $M_0$ of $M$ such that with the $A^N(V)$-module structure, the $\mathbb{N}$-grading induced from $M$ and the operators $L_M(0)\Big|_{M_0}$ and $L_M(-1)\Big|_{M_0}$, $M_0$ is a graded $A^N(V)$-module. A graded $A^\infty(V)$-module $M$ is said to be *generated by a subset $S$* if $M$ is equal to the smallest graded $A^N(V)$-submodule containing $S$, or equivalently, $M$ is spanned by homogeneous elements obtained by applying elements of $A^N(V)$, $L_M(0)$ and $L_M(-1)$ to homogeneous summands of elements of $S$. A graded $A^N(V)$-module is said to be *irreducible* if it has no nonzero proper graded $A^N(V)$-modules. A graded $A^N(V)$-module is said to be *completely reducible* if it is a direct sum of irreducible graded $A^N(V)$-modules.

# 3 Lower-bounded generalized $V$-modules and $A^\infty(V)$-modules

In this section, $V$ is a grading-restricted vertex algebra. In particular, we do not assume that $V$ has a conformal vector. We give an $A^\infty(V)$-module structure to a lower-bounded generalized $V$-module $W$ in this section. For the associative algebra $A^\infty(V)$, its subalgebras, its modules and related structures and results, see Section 2 and [H5].

Let $W$ be a lower-bounded generalized $V$-module. Let $W^\mu$ for $\mu \in \mathbb{C}/\mathbb{Z}$ be the generalized $V$-submodule of $W$ spanned by homogeneous elements of weights in $\mu$. Let

$$\Gamma(W) = \{\mu \in \mathbb{C}/\mathbb{Z} \mid W^\mu \neq 0\}.$$

We call $\Gamma(W)$ the *sets of congruence classes of weights of $W$*.

For $\mu \in \Gamma(W)$, there exists $h^\mu \in \mathbb{C}$ such that

$$W^\mu = \coprod_{n \in \mathbb{N}} W_{[h^\mu + n]}$$

and $W_{[h^\mu]} \neq 0$. Then

$$W = \coprod_{\mu \in \Gamma(W)} W^\mu = \coprod_{\mu \in \Gamma(W)} \coprod_{n \in \mathbb{N}} W_{[h^\mu + n]}.$$

For $n \in \mathbb{N}$, let

$$W_{\llbracket n \rrbracket} = \coprod_{\mu \in \Gamma(W)} W_{[h^\mu + n]}.$$

Then $W$ has a canonical $\mathbb{N}$-grading

$$W = \coprod_{n \in \mathbb{N}} W_{\llbracket n \rrbracket}.$$

For $n \in \mathbb{N}$, we call the space $W_{\llbracket n \rrbracket}$ the *$n$-the level of $W$* and for an element $w \in W_{\llbracket n \rrbracket}$, we call $n$ the *level of $w$*. From the definition, we have $W_{\llbracket n \rrbracket} \subset \Omega_n(W)$.

We define a linear map $\vartheta_W : U^\infty(V) \to \mathrm{End}\, W$ by

$$\vartheta_W([v]_{kl})w = \delta_{ln} \mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w$$

for $k, l \in \mathbb{N}$, $v \in V$ and $w \in W_{\llbracket n \rrbracket}$.

We now want to show that $W$ is in fact a graded $A^\infty(V)$-module. For simplicity, we discuss only the case that $W = W^\mu = \coprod_{n \in \mathbb{N}} W_{[h^\mu + n]}$ for some $\mu \in \Gamma(W)$. The general case follows immediately from the decomposition $W = \coprod_{\mu \in \Gamma(W)} W^\mu$.

We need to use the construction in Section 5 of [H2]. Take the generating fields for the grading-restricted vertex algebra $V$ to be $Y_V(v, x)$ for $v \in V$. By definition, $W$ is a direct sum of generalized eigenspaces of $L_W(0)$ and the real parts of the eigenvalues of $L_W(0)$ has a lower bound $\Re(h^\mu) \in \mathbb{R}$. We take $M$ and $B$ in Section 5 of [H2] to be $W$ and $\Re(h^\mu)$, respectively. Using the construction in Section 5 of [H2], we obtain a universal lower-bounded generalized $V$-module $\widehat{W}^{[1_V]}_{\Re(h^\mu)}$. For simplicity, we shall denote it simply by $\widehat{W}$.

By Theorem 3.3 in [H3] and the construction in Section 5 of [H2] and by identifying elements of the form $(\psi^a_{\widehat{W}})_{-1,0}\mathbf{1}$ with basis elements $w^a \in W$ for $a \in A$ for a basis $\{w^a\}_{a \in A}$ of $W$, we see that $\widehat{W}$ is generated by $W$ (in the sense of Definition 3.1 in [H3]). Moreover, after identifying $(\psi^a_{\widehat{W}})_{-1,0}\mathbf{1}$ with basis elements $w^a \in G$ for $a \in A$, Theorems 3.3 and 3.4 in [H3] say that elements of the form $L_{\widehat{W}}(-1)^p w^a$ for $p \in \mathbb{N}$ and $a \in A$ are linearly independent and $\widehat{W}$ is spanned by elements obtained by applying the components of the vertex operators to these elements. In particular, $W$ can be embedded into $\widehat{W}$ as a subspace. So we shall view $W$ as a subspace of $\widehat{W}$. But note that $W$ is not a $V$-submodule of $\widehat{W}$ since $W$ is not invariant under the action of vertex operators on $\widehat{W}$.

Since $W = \coprod_{n \in \mathbb{N}} W_{[h^\mu + n]}$, from the construction of $\widehat{W}$ in Section 5 of [H2], we also have $\widehat{W} = \coprod_{n \in \mathbb{N}} \widehat{W}_{[h^\mu + n]}$. For $n \in \mathbb{N}$, $W_{[h^\mu + n]}$ is a subspace of $\widehat{W}_{[h^\mu + n]}$.

**Lemma 3.1** *For $n \in \mathbb{N}$, $W_{[h^\mu + n]} \cap \Omega_{n-1}(\widehat{W}) = 0$. In particular, we can view $W_{[h^\mu + n]}$ as a subspace of $Gr_n(\widehat{W})$.*

*Proof.* Let $w \in W_{[h^\mu + n]} \cap \Omega_{n-1}(\widehat{W})$. Then for homogeneous $v \in V$,

$$\mathrm{Res}_x x^{n-1} Y_{\widehat{W}}(x^{L_V(0)}v, x)w = (Y_{\widehat{W}})_{\mathrm{wt}\, + n-1}(v)w = 0.$$

When $w$ is a basis element $w^a$ of $W$ as in [H2], the element $(Y_{\widehat{W}})_{\mathrm{wt}\,+n-1}(v)w^a$ can be written as $(Y_{\widehat{W}})_{\mathrm{wt}\,v+n-1}(v)(\psi^a_{\widehat{W}})_{-1,0}\mathbf{1}$. Since $w^a \neq 0$ as a basis element,

$$(Y_{\widehat{W}})_{\mathrm{wt}\,+n-1}(v)(\psi^a_{\widehat{W}})_{-1,0}\mathbf{1} = 0 \tag{3.1}$$

for $v \in V$ and $a \in A$ is a relation in $\widehat{W}$ with the left-hand side being of weight $h^\mu$. But relations in $\widehat{W}$ of weight $h^\mu$ (or of level 0) and involving the operator $(\psi^a_{\widehat{W}})_{-1,0}$ must have elements of $V$ not equal to $\mathbf{1}$ to the right of $(\psi^a_{\widehat{W}})_{-1,0}$ (see Section 5 of [H2]). So (3.1) is not a relation in $\widehat{W}$. Thus we must have $w^a = 0$ for $a \in A$, that is, $W_{[h^\mu+n]} \cap \Omega_{n-1}(\widehat{W}) = 0$.

By the construction of $\widehat{W}$, $W_{[h^\mu+n]} \subset \Omega_n(\widehat{W})$. We define a linear map from $W_{[h^\mu+n]}$ to $Gr_n(\widehat{W})$ by $w \mapsto w + \Omega_{n-1}(\widehat{W})$. Since $W_{[h^\mu+n]} \cap \Omega_{n-1}(\widehat{W}) = 0$, this map is injective. So we can view $W_{[h^\mu+n]}$ as a subspace of $Gr_n(\widehat{W})$. ∎

Let $J_W$ be the generalized $V$-submodule of $\widehat{W}$ generated by elements of the forms

$$\mathrm{Res}_x x^{l-k-1} Y_{\widehat{W}}(x^{L_V(0)}v, x)w - \mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w \tag{3.2}$$

for $v \in V$, $k, l \in \mathbb{N}$, $w \in W_{[h^\mu+l]}$ and

$$L_{\widehat{W}}(-1)w - L_W(-1)w \tag{3.3}$$

for $w \in W$. Let

$$G_n(J_W) = \{w + \Omega_{n-1}(\widehat{W}) \mid w \in \Omega_n(J_W)\} \subset Gr_n(\widehat{W})$$

for $n \in \mathbb{N}$ and let

$$G(J_W) = \coprod_{n\in\mathbb{N}} G_n(J_W).$$

Since $J_W$ is a generalized $V$-submodule of $\widehat{W}$ and $Gr_n(\widehat{W})$ is an $A^\infty(V)$-module, $G(J_W)$ is an $A^\infty(V)$-submodule of $Gr(\widehat{W})$.

**Proposition 3.2** *The $\mathbb{N}$-graded space $W$ with the action of $U^\infty(V)$ given by $\vartheta_W$ induces a graded $A^\infty(V)$-module structure on $W$ canonically equivalent to the quotient $A^\infty(V)$-module $Gr(\widehat{W})/G(J_W)$.*

*Proof.* As we have done above, we prove only the case that $W = W^\mu$ for some $\mu \in \Gamma(W)$. The general case follows immediately.

Since $\widehat{W}$ is generated by $W$, $\widehat{W}$ is spanned by elements of the form

$$\begin{aligned}
&\mathrm{Res}_x x^{l-k-1} Y_{\widehat{W}}(x^{L_V(0)}v, x)w \\
&\quad = (\mathrm{Res}_x x^{l-k-1} Y_{\widehat{W}}(x^{L_V(0)}v, x)w - \mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w) \\
&\qquad + \mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w \\
&\quad \in J_W + W
\end{aligned}$$

for $v \in V$, $k, l \in \mathbb{N}$, $w \in W_{[h^\mu+l]}$ and

$$L_{\widehat{W}}(-1)w = (L_{\widehat{W}}(-1)w - L_W(-1)w) + L_W(-1)w \in J_W + W$$

for $w \in W$. From the definition of $J_W$, we have $J_W \cap W = 0$. Hence $\widehat{W} = J_W \oplus W$. From this decomposition and Lemma 3.1, we have $Gr_n(\widehat{W}) = G_n(J_W) \oplus W_{[h^\mu+n]}$ and $Gr(\widehat{W}) = G(J_W) \oplus W$. Thus the $\mathbb{N}$-graded space $W$ is canonically isomorphic to $Gr(\widehat{W})/G(J_W)$. Since $Gr(\widehat{W})$ is an $A^\infty(V)$-module and $G(J_W)$ is an $A^\infty(V)$-submodule of $Gr(\widehat{W})$, we see that $Gr(\widehat{W})/G(J_W)$ as a quotient of $A^\infty(V)$ is also an $A^\infty(V)$-module.

Let $f$ be the canonical isomorphism from $W$ to $Gr(\widehat{W})/G(J_W)$. Then

$$f(w) = (w + \Omega_{l-1}(\widehat{W})) + G_l(J_W)$$

for $w \in W_{[h^\mu+l]}$. We use $\vartheta_{Gr(\widehat{W})/G(J_W)}([v]_{kn})\tilde{w}$ to denote the action of $[v]_{kn}$ on $\tilde{w} \in Gr(\widehat{W})/G(J_W)$. Then

$$\begin{aligned}
&\vartheta_{Gr(\widehat{W})/G(J_W)}([v]_{kn})f(w)\\
&\quad = \vartheta_{Gr(\widehat{W})/G(J_W)}([v]_{kn})((w + \Omega_{l-1}(\widehat{W})) + G_l(J_W))\\
&\quad = \delta_{nl}(\mathrm{Res}_x x^{l-k-1} Y_{\widehat{W}}(x^{L_V(0)}v, x)w + \Omega_{l-1}(\widehat{W})) + G_l(J_W)\\
&\quad = \delta_{nl}(\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w + \Omega_{l-1}(\widehat{W})) + G_l(J_W)\\
&\quad = (\vartheta_W([v]_{kn})w + \Omega_{l-1}(\widehat{W})) + G_l(J_W)\\
&\quad = f(\vartheta_W([v]_{kn})w)
\end{aligned}$$

for $v \in V$, $k, l, n \in \mathbb{N}$ and $w \in W_{[h^\mu+l]}$. So the isomorphism $f$ from $W$ to $Gr(\widehat{W})/G(J_W)$ commutes with the actions of $A^\infty(V)$ on $W$ and $Gr(\widehat{W})/G(J_W)$. Thus $f$ is an $A^\infty(V)$-module map. Since $f$ is also a linear isomorphism, it is an equivalence of $A^\infty(V)$-modules.

The four conditions for $W$ to be a graded $A^\infty(V)$-module in Definition 2.2 are in fact the properties of the $\mathbb{N}$-grading of $W$, the operators $L_W(0)$ and $L_W(-1)$ and $\vartheta_W$. ∎

**Remark 3.3** Note that in general, $W$ is not nondegenerate as a graded $A^\infty(V)$-module (see Definition 2.2).

**Remark 3.4** It is easy to see that $\ker \vartheta_W \subset \ker \vartheta_{Gr(W)}$. In fact, let $\mathfrak{v} \in \ker \vartheta_W$. Then for $k, l \in \mathbb{N}$, $w \in W_{\llbracket l \rrbracket}$, $\pi_{W_{\llbracket k \rrbracket}} \vartheta_W(\mathfrak{v})w = 0$, where $\pi_{W_{\llbracket k \rrbracket}}$ is the projection from $W$ to $W_{\llbracket k \rrbracket}$. By the definition of $\vartheta_W$, we know that $\pi_{W_{\llbracket k \rrbracket}} \vartheta_W(\mathfrak{v})w$ must be of the form $\vartheta_W([v]_{kl})w$ for some $v \in V$. So we have $\vartheta_W([v]_{kl})w = 0$, or explicitly, $\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w = 0$. Then we also have

$$\pi_{Gr_k(W)} \vartheta_{Gr(W)}(\mathfrak{v})[w]_l = \vartheta_{Gr(W)}([v]_{kl})[w]_l = [\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w]_k = 0$$

in $Gr_k(W)$. Since $k, l$ and $w$ are arbitrary, we obtain $\vartheta_{Gr(W)}(\mathfrak{v}) = 0$, that is, $\mathfrak{v} \in \ker \vartheta_{Gr(W)}$. So $\ker \vartheta_W \subset \ker \vartheta_{Gr(W)}$. This fact together with Proposition 3.2 means that $Q^\infty(V)$ is also equal to the intersection of $\ker \vartheta_W$ for all lower-bounded generalized $V$-modules $W$. This fact gives new definitions of $Q^\infty(V)$ and $A^\infty(V)$. In this paper, we shall use these new definitions.

## 4 Intertwining operators and $A^\infty(V)$-bimodules

In this section, $V$ is still a grading-restricted vertex algebra which does not have to have a conformal vector. We construct an $A^\infty(V)$-bimodule $A^\infty(W)$ from a lower-bounded generalized $V$-module using all intertwining operators of type $\binom{W_3}{WW_2}$ for all lower-bounded generalized $V$-modules $W_2$ and $W_3$.

Let $U^\infty(W)$ be the space of all column-finite infinite matrices with entries in $W$ and doubly index by $\mathbb{N}$. For $w \in W$ and $k, l \in \mathbb{N}$, we use $[w]_{kl}$ to denote the infinite matrix in $U^\infty(W)$ with the $(k,l)$-th entry equal to $w$ and all the other entries equal to 0. Then elements of $U^\infty(W)$ are infinite linear combinations of elements of the form $[w]_{kl}$ for $w \in W$ and $k, l \in \mathbb{N}$ with only finitely many elements of such a form for each fixed $k, l \in \mathbb{N}$. We can use elements of the form $[w]_{kl}$ for $w \in W$ and $k, l \in \mathbb{N}$ to study $U^\infty(W)$.

For $v \in V$, $w \in W$ and $k, m, n, l \in \mathbb{N}$, we define

$$[v]_{km} \diamond [w]_{nl} = 0$$

when $m \neq n$ and

$$\begin{aligned}
[v]_{kn} \diamond [w]_{nl} &= \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_W((1+x)^{L_V(0)}v, x)w\right]_{kl} \\
&= \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_x x^{-k+n-l-m-1}(1+x)^l \left[Y_W((1+x)^{L_V(0)}v, x)w\right]_{kl}.
\end{aligned} \tag{4.1}$$

This is a left action of $U^\infty(V)$ on $U^\infty(W)$, that is, a linear map from $U^\infty(V) \otimes U^\infty(W)$ to $U^\infty(W)$.

For $v \in V$, $w \in W$ and $k, m, n, l \in \mathbb{N}$, we define

$$[w]_{km} \diamond [v]_{nl} = 0$$

when $m \neq n$ and

$$\begin{aligned}
[w]_{kn} \diamond [v]_{nl} &= \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^k \left[Y_W((1+x)^{-L_V(0)}v, -x(1+x)^{-1})w\right]_{kl} \\
&= \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_x x^{-k+n-l-m-1}(1+x)^k \cdot \\
&\quad\quad \cdot \left[Y_W((1+x)^{-L_V(0)}v, -x(1+x)^{-1})w\right]_{kl}.
\end{aligned} \tag{4.2}$$

We then obtain a right action of $U^\infty(V)$ on $U^\infty(W)$, that is, a linear map from $U^\infty(W) \otimes U^\infty(V)$ to $U^\infty(W)$. With the left and right actions of $U^\infty(V)$, $U^\infty(W)$ becomes a $U^\infty(V)$-bimodule. The definition (4.2) can also be rewritten using the $L(0)$-conjugation formula as

$$\begin{aligned}
&[w]_{kn} \diamond [v]_{nl} \\
&\quad = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^k \left[(1+x)^{-L_W(0)} Y_W(v, -x)(1+x)^{L_W(0)}w\right]_{kl}
\end{aligned}$$

$$= \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_x x^{-k+n-l-m-1}(1+x)^k \cdot$$
$$\cdot \left[(1+x)^{-L_W(0)} Y_W(v,-x)(1+x)^{L_W(0)} w\right]_{kl}. \quad (4.3)$$

The definition (4.2) can also be rewritten using the right vertex operator map $Y_{WV}^V : W \otimes V \to W[[x, x^{-1}]]$ defined by

$$Y_{WV}^V(w,x)v = e^{xL_W(-1)} Y_W(v,-x)w \quad (4.4)$$

for $v \in V$ and $w \in W$ (see [FHL]). To do this, we need a formula involving $L_W(-1)$ and $L_W(0)$. It is straightforward to show

$$\frac{d}{dx} e^{xyL_W(-1)}(1+x)^{L_W(0)}(1+x)^{-(yL_W(-1)+L_W(0))} = 0$$

Then $e^{xyL_W(-1)}(1+x)^{L_W(0)}(1+x)^{-(yL_W(-1)+L_W(0))}$ must be independent of $x$. Setting $x=0$, we obtain

$$e^{xyL_W(-1)}(1+x)^{L_W(0)}(1+x)^{-(yL_W(-1)+L_W(0))} = 1_W,$$

which is equivalent to

$$e^{xyL_W(-1)}(1+x)^{L_W(0)} = (1+x)^{yL_W(-1)+L_W(0)}.$$

Let $y=1$. We obtain

$$e^{xL_W(-1)}(1+x)^{L_W(0)} = (1+x)^{L_W(-1)+L_W(0)}. \quad (4.5)$$

Using (4.5) and (4.4), we see that (4.3) can also be rewritten as

$$\begin{aligned}
[w]_{kn} \diamond [v]_{nl} &= \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^k \cdot \\
&\qquad \cdot \left[(1+x)^{-(L_W(-1)+L_W(0))} Y_{WV}^W((1+x)^{L_W(0)}w, x)v\right]_{kl} \\
&= \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_x x^{-k+n-l-m-1}(1+x)^k \cdot \\
&\qquad \cdot \left[(1+x)^{-(L_W(-1)+L_W(0))} Y_{WV}^W((1+x)^{L_W(0)}w, x)v\right]_{kl}. \qquad (4.6)
\end{aligned}$$

The definition (4.6) is more conceptual since it says that the right action of $U^\infty(V)$ on $U^\infty(W)$ is defined using the right action (the right vertex operator map) of $V$ on $W$.

Now let $W_2$ and $W_3$ be lower-bounded generalized $V$-modules and $\mathcal{Y}$ an intertwining operator of type $\binom{W_3}{WW_2}$. As we have mentioned in the introduction, for simplicity, by an intertwining operator, we always mean a logarithmic intertwining operator defined in Definition 3.10 in [HLZ], except that in the case that $V$ is a grading-restricted vertex algebra so that there are no $L_V(1)$ on $V$ and no $L_W(1)$ on a lower-bounded generalized $V$-module $W$, we do not require that the $L(1)$-commutator formula for intertwining operator hold. Such an intertwining operator might contain the logarithm of the variable and might even be an infinite power series in the logarithm of the variable. See [HLZ] for more details. But in addition, we are interested in only intertwining operators $\mathcal{Y}$ such that the powers of the formal variable $x$ in $\mathcal{Y}(w, x)w_2$ for $w \in W$ and $w_2 \in W_2$ belong to only finitely many congruence classes in $\mathbb{C}/Z$. For such an intertwining operator, its image is in a finite direct sum of $W_3^\nu$ for $\nu \in \mathbb{C}/\mathbb{Z}$. We say that such an intertwining operator has *locally-finite sets of congruence classes of powers* or is an *intertwining operator with locally-finite sets of congruence classes of powers*. If $\Gamma(W_3)$ is a finite set, then every intertwining operator of type $\binom{W_3}{WW_2}$ in the sense of Definition 3.10 in [HLZ] has locally-finite sets of congruence classes of powers. In this paper, by an intertwining operator, we always mean an intertwining operator with locally-finite sets of congruence classes of powers. In particular, the space of intertwining operators also means the space of intertwining operators with locally-finite sets of congruence classes of powers.

As is discussed in the preceding section, there exist $h_2^\mu, h_3^\nu \in \mathbb{C}$ for $\mu \in \Gamma(W_2)$, $\nu \in \Gamma(W_3)$ such that

$$W_2 = \coprod_{n\in\mathbb{N}} (W_2)_{\llbracket n \rrbracket} = \coprod_{n\in\mathbb{N}} \coprod_{\mu\in\Gamma(W_2)} (W_2)_{[h_2^\mu+n]},\ W_3 = \coprod_{n\in\mathbb{N}} (W_3)_{\llbracket n \rrbracket} = \coprod_{n\in\mathbb{N}} \coprod_{\nu\in\Gamma(W_3)} (W_3)_{[h_3^\nu+n]}.$$

For $w \in W$, we have

$$\mathcal{Y}(w, x) = \sum_{k\in\mathbb{N}} \sum_{m\in\mathbb{C}} \mathcal{Y}_{m,k}(w)x^{-m-1}(\log x)^k,$$

where for $m \in \mathbb{C}$, $k \in \mathbb{N}$ and homogeneous $w$, the map $\mathcal{Y}_{m,k}(w) : W_2 \to W_3$ is homogeneous of weight $\mathrm{wt}\, w - m - 1$. For $w \in W$, let

$$\mathcal{Y}^k(w, x) = \sum_{m\in\mathbb{C}} \mathcal{Y}_{m,k}(w)x^{-m-1}.$$

Note that $\mathcal{Y}^k(w, x)$ for $k \in \mathbb{N}$ satisfy the same Jacobi identity as the one for intertwining operators but do not satisfy the $L(-1)$-derivative property for intertwining operators.

For a vector space $U$ and $k \in \mathbb{N}$, let $\mathrm{Coeff}^k_{\log x} : U\{x\}[[\log x]] \to U\{x\}$ be the linear map given by taking the coefficients of $(\log x)^k$. Then we have

$$\mathrm{Coeff}^k_{\log x}\mathcal{Y}(w, x) = \mathcal{Y}^k(w, x)$$

for $w \in W$ and $k \in \mathbb{N}$.

For $k, l \in \mathbb{N}$, $\mu \in \Gamma(W_2)$, $\nu \in \Gamma(W_3)$, $w \in W$ and $w_2 \in (W_2)_{[h_2^\mu+l]} \subset \Omega_l^0(W_2)$,

$$\begin{aligned}
&\mathrm{Coeff}^0_{\log x}\mathrm{Res}_x x^{h_2^\mu-h_3^\nu+l-k-1}\mathcal{Y}(x^{L_W(0)}w, x)w_2\\
&\quad = \mathrm{Res}_x x^{h_2^\mu-h_3^\nu+l-k-1}\mathcal{Y}^0(x^{L_W(0)_S}w, x)w_2\\
&\quad \in (W_3)_{[h_3^\nu+k]} \subset W_{\llbracket k \rrbracket}.
\end{aligned}$$

For $k, l, n \in \mathbb{N}$, $\mu \in \Gamma(W_2)$, $w \in W$ and $w_2 \in (W_2)_{[h_2^\mu+l]} \subset (W_2)_{\llbracket l \rrbracket}$, we define

$$\vartheta_{\mathcal{Y}}([w]_{kl})w_2 = \sum_{\nu\in\Gamma(W_3)} \mathrm{Coeff}^0_{\log x}\mathrm{Res}_x x^{h_2^\mu-h_3^\nu+l-k-1}\mathcal{Y}(x^{L_W(0)}w, x)w_2$$

$$= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_x x^{h_2^\mu - h_3^\nu + l - k - 1}\mathcal{Y}^0(x^{L_W(0)_S}w, x)w_2$$
$$\in W_{\llbracket k\rrbracket}.$$

In this definition, the sum is finite since there are only finitely many congruence classes of powers of $x$ in $\mathcal{Y}^0(x^{L_W(0)_S}w, x)w_2$. We now have a linear map

$$\vartheta_{\mathcal{Y}}([w]_{kl}) : W_2 \to W_3,$$

or equivalently, we obtain an element

$$\vartheta_{\mathcal{Y}}([w]_{kl}) \in \mathrm{Hom}(W_2, W_3).$$

The maps $\vartheta_{\mathcal{Y}}([w]_{kl})$ for $w \in W$ and $k, l \in \mathbb{N}$give a linear map

$$\vartheta_{\mathcal{Y}} : U^\infty(W) \to \mathrm{Hom}(W_2, W_3).$$

Since $W_2$ and $W_3$ are both left $A^\infty(V)$-modules, $\mathrm{Hom}(W_2, W_3)$ is an $A^\infty(V)$-bimodule. In particular, we have left and right actions of $U^\infty(V)$ on $\mathrm{Hom}(W_2, W_3)$ such that both the left and right actions of $Q^\infty(V)$ on $\mathrm{Hom}(W_2, W_3)$ are 0.

**Proposition 4.1** *The linear map $\vartheta_{\mathcal{Y}}$ commutes with the left and right actions of $U^\infty(V)$. In particular, $U^\infty(W)/\ker\vartheta_{\mathcal{Y}}$ is an $A^\infty(V)$-bimodule.*

*Proof.* We first prove that $\vartheta_{\mathcal{Y}}$ commutes with the left action of $U^\infty(V)$. Let $k, m, n, l, p \in \mathbb{N}$, $\mu \in \Gamma(W_2)$, $v \in V$, $w \in W$, $w_2 \in (W_2)_{[h_2^\mu + p]} \subset (W_2)_{\llbracket p\rrbracket}$. In the case $m \neq n$ or $p \neq l$ by definition, we have

$$\vartheta_{\mathcal{Y}}([v]_{km} \diamond [w]_{nl})w_2 = 0 = \vartheta_{W_3}([v]_{km})\vartheta_{\mathcal{Y}}([w]_{nl})w_2.$$

In the case $m = n$ and $p = l$,

$$\begin{aligned}
&\vartheta_{\mathcal{Y}}([v]_{kn} \diamond [w]_{nl})w_2\\
&\quad= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0} T_{k+l+1}((x_0+1)^{-k+n-l-1})(1+x_0)^l \vartheta_{\mathcal{Y}}([Y_W((1+x_0)^{L_V(0)}v, x_0)w]_{kl})w_2\\
&\quad= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} T_{k+l+1}((x_0+1)^{-k+n-l-1})(1+x_0)^l x_2^{h_2^\mu - h_3^\nu + l - k - 1}\cdot\\
&\qquad\quad \cdot \mathcal{Y}^0(x_2^{L_W(0)_S} Y_W((1+x_0)^{L_V(0)}v, x_0)w, x_2)w_2\\
&\quad= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} T_{k+l+1}((x_0+1)^{-k+n-l-1})(1+x_0)^l x_2^{h_2^\mu - h_3^\nu + l - k - 1}\cdot\\
&\qquad\quad \cdot \mathcal{Y}^0(Y_W((x_2+x_0x_2)^{L_V(0)}v, x_0x_2) x_2^{L_W(0)_S} w, x_2)w_2\\
&\quad= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_1^{-1}\delta\left(\frac{x_2+x_0x_2}{x_1}\right) T_{k+l+1}((x_0+1)^{-k+n-l-1})\cdot
\end{aligned}$$

$$
\begin{aligned}
&\cdot x_1^l x_2^{h_2^\mu - h_3^\nu - k-1}\mathcal{Y}^0(Y_W(x_1^{L_V(0)}v, x_0x_2)x_2^{L_W(0)_S}w, x_2)w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_0^{-1}x_2^{-1}\delta\left(\frac{x_1-x_2}{x_0x_2}\right) T_{k+l+1}((x_0+1)^{-k+n-l-1})\cdot\\
&\quad\cdot x_1^l x_2^{h_2^\mu - h_3^\nu - k-1} Y_{W_3}(x_1^{L_V(0)}v, x_1)\mathcal{Y}^0(x_2^{L_W(0)_S}w, x_2)w_2\\
&\quad - \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_0^{-1}x_2^{-1}\delta\left(\frac{x_2-x_1}{x_0x_2}\right) T_{k+l+1}((x_0+1)^{-k+n-l-1})\cdot\\
&\quad\cdot x_1^l x_2^{h_2^\mu - h_3^\nu - k-1}\mathcal{Y}^0(x_2^{L_W(0)_S}w, x_2) Y_{W_2}(x_1^{L_V(0)}v, x_1)w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_2}\mathrm{Res}_{x_1} T_{k+l+1}((x_0+1)^{-k+n-l-1})\Big|_{x_0=(x_1-x_2)x_2^{-1}}\cdot\\
&\quad\cdot x_1^l x_2^{h_2^\mu - h_3^\nu - k-2} Y_{W_3}(x_1^{L_V(0)}v, x_1)\mathcal{Y}^0(x_2^{L_W(0)_S}w, x_2)w_2\\
&\quad - \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_2}\mathrm{Res}_{x_1} T_{k+l+1}((x_0+1)^{-k+n-l-1})\Big|_{x_0=(-x_2+x_1)x_2^{-1}}\cdot\\
&\quad\cdot x_1^l x_2^{h_2^\mu - h_3^\nu - k-2}\mathcal{Y}^0(x_2^{L_W(0)_S}w, x_2) Y_{W_2}(x_1^{L_V(0)}v, x_1)w_2.
\end{aligned}
\tag{4.7}
$$

The second term in the right hand side of (4.7) is 0 since $w_2 \in (W_2)_{[h_2^\mu + l]}$. Expanding $T_{k+l+1}((x_0+1)^{-k+n-l-1})$ explicitly, we see that the first term in the right-hand side of (4.7) is equal to

$$
\begin{aligned}
&\sum_{\nu\in\Gamma(W_3)}\sum_{m=0}^{n}\binom{-k+n-l-1}{m}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1}(x_1-x_2)^{-k+n-l-m-1}x_2^{k-n+l+m+1}\cdot\\
&\quad\cdot x_1^l x_2^{h_2^\mu - h_3^\nu - k-2} Y_{W_3}(x_1^{L_V(0)}v, x_1)\mathcal{Y}^0(x_2^{L_W(0)_S}w, x_2)w_2\\
&= \sum_{\nu\in\Gamma(W_3)}\sum_{m=0}^{n}\sum_{j\in\mathbb{N}}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{j}(-1)^j\cdot\\
&\quad\cdot \mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_1^{-k+n-m-1-j}x_2^{h_2^\mu - h_3^\nu - n+l+m-1+j} Y_{W_3}(x_1^{L_V(0)}v, x_1)\mathcal{Y}^0(x_2^{L_W(0)_S}w, x_2)w_2.
\end{aligned}
\tag{4.8}
$$

Since $w_2 \in (W_2)_{[h_2^\mu + l]}$, we know that $\mathrm{Res}_x x^{h_2^\mu - h_3^\nu + q-1}\mathcal{Y}^0(x^{L_{W_1}(0)_S}w, x)w_2 \in (W_3)_{[h_3^\nu + l-q]}$ and thus is equal to 0 if $q > l$. In the case $j > n-m$, we have $-n+l+m+j > l$ and hence

$$
\mathrm{Res}_{x_2} x_2^{h_2^\mu - h_3^\nu - n+l+m-1+j}\mathcal{Y}^0(x_2^{L_W(0)_S}w, x_2)w_2 = 0.
$$

In particular, those terms in the right-hand side of (4.8) with $j > n-m$ is 0. So the right-hand side of (4.8) is equal to

$$
\sum_{\nu\in\Gamma(W_3)}\sum_{m=0}^{n}\sum_{j=0}^{n-m}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{j}(-1)^j\cdot
$$

$$\cdot \operatorname{Res}_{x_2}\operatorname{Res}_{x_1} x_1^{-k+n-m-1-j} x_2^{h_2^\mu - h_3^\nu - n+l+m-1+j} Y_{W_3}(x_1^{L_V(0)}v, x_1)\mathcal{Y}^0(x_2^{L_W(0)_S}w, x_2)w_2$$

$$= \sum_{\nu\in\Gamma(W_3)} \sum_{m=0}^{n}\sum_{q=m}^{n} \binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{q-m}(-1)^{q-m}\cdot$$

$$\cdot \operatorname{Res}_{x_2}\operatorname{Res}_{x_1} x_1^{-k+n-1-q} x_2^{h_2^\mu - h_3^\nu - n+l-1+q} Y_{W_3}(x_1^{L_V(0)}v, x_1)\mathcal{Y}^0(x_2^{L_W(0)}v, x_2)w$$

$$= \sum_{\nu\in\Gamma(W_3)} \sum_{q=0}^{n} \left(\sum_{m=0}^{q} \binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{q-m}(-1)^{q-m}\right)\cdot$$

$$\cdot \operatorname{Res}_{x_2}\operatorname{Res}_{x_1} x_1^{-k+n-1-q} x_2^{h_2^\mu - h_3^\nu - n+l-1+q} Y_{W_3}(x_1^{L_V(0)}v, x_1)\mathcal{Y}^0(x_2^{L_W(0)_S}v, x_2)w. \quad (4.9)$$

Using (2.18) in [H5], that is,

$$\sum_{m=0}^{q} \binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{q-m}(-1)^{q-m} = \binom{-k+n-l-1}{q}\delta_{q,0} \qquad (4.10)$$

for $q = 0, \dots, n$, we see that the right-hand side of (4.9) is equal to

$$\sum_{\nu\in\Gamma(W_3)} \operatorname{Res}_{x_2}\operatorname{Res}_{x_1} x_1^{-k+n-1} x_2^{h_2^\mu - h_3^\nu - n+l-1} Y_{W_3}(x_1^{L_V(0)}v, x_1)\mathcal{Y}^0(x_2^{L_W(0)_S}w, x_2)w_2$$

$$= \vartheta_{W_3}([v]_{kn}) \sum_{\nu\in\Gamma(W_3)} \operatorname{Res}_{x_2} x_2^{h_2^\mu - h_3^\nu + l-n-1}\mathcal{Y}^0(x_2^{L_W(0)_S}w, x_2)w_2$$

$$= \vartheta_{W_3}([v]_{kn})\vartheta_{\mathcal{Y}}([w]_{nl})w_2. \qquad (4.11)$$

From (4.7)–(4.11), we obtain

$$\vartheta_{\mathcal{Y}}([v]_{kn} \diamond [w]_{nl})w_2 = \vartheta_{W_3}([v]_{kn})\vartheta_{\mathcal{Y}}([w]_{nl})w_2.$$

We have now proved

$$\vartheta_{\mathcal{Y}}([v]_{km} \diamond [w]_{nl})w_2 = \vartheta_{W_3}([v]_{km})\vartheta_{\mathcal{Y}}([w]_{nl})w_2$$

for $k, l, m, n \in \mathbb{N}$, $v \in V$, $w \in W$ and $w_2 \in W_2$. This shows that $\vartheta_{\mathcal{Y}}$ commutes with the left actions of $U^\infty(V)$.

Next we prove that $\vartheta_{\mathcal{Y}}$ commutes with the right actions of $U^\infty(V)$. Let $k, m, n, l, p \in \mathbb{N}$, $\mu \in \Gamma(W_2)$, $v \in V$, $w \in W$, $w_2 \in (W_2)_{[h_2^\mu+p]} \subset (W_2)_{\llbracket p \rrbracket}$. In the case $m \neq n$ or $p \neq l$, by definition, we have

$$\vartheta_{\mathcal{Y}}([w]_{km} \diamond [v]_{nl})w_2 = 0 = \vartheta_{\mathcal{Y}}([w]_{km})\vartheta_{W_2}([v]_{nl})w_2.$$

In the case $m = n$ and $p = l$, using (4.3), we have

$$\vartheta_{\mathcal{Y}}([w]_{kn} \diamond [v]_{nl})w_2$$
$$= \operatorname{Res}_{x_0} T_{k+l+1}((x_0+1)^{-k+n-l-1})(1+x_0)^k \cdot$$

$$
\begin{aligned}
&\quad \cdot \vartheta_{\mathcal{Y}}([(1+x_0)^{-L_W(0)}Y_W(v,-x_0)(1+x_0)^{L_W(0)}w]_{kl})w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}T_{k+l+1}((x_0+1)^{-k+n-l-1})(1+x_0)^k x_2^{h_2^\mu-h_3^\nu+l-k-1}\cdot\\
&\quad \cdot \mathcal{Y}(x_2^{L_W(0)}(1+x_0)^{-L_W(0)}Y_W(v,-x_0)(1+x_0)^{L_W(0)}w,x_2)w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}T_{k+l+1}((x_0+1)^{-k+n-l-1})(1+x_0)^k x_2^{h_2^\mu-h_3^\nu+l-k-1}\cdot\\
&\quad \cdot (1+x_0)^{-L_{W_3}(0)}\mathcal{Y}(x_2^{L_W(0)}Y_W(v,-x_0)(1+x_0)^{L_W(0)}w,x_2(1+x_0))(1+x_0)^{L_{W_2}(0)}w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}T_{k+l+1}((x_0+1)^{-k+n-l-1})\cdot\\
&\quad \cdot (1+x_0)^{h_2^\mu-h_3^\nu+l}x_2^{h_2^\mu-h_3^\nu+l-k-1}(1+x_0)^{-L_{W_3}(0)}\cdot\\
&\quad \cdot \mathcal{Y}(Y_W(x_2^{L_W(0)}v,-x_0x_2)(x_2+x_0x_2)^{L_W(0)}w,x_2+x_0x_2)(1+x_0)^{L_{W_2}(0)}w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1}x_1^{-1}\delta\left(\frac{x_2+x_0x_2}{x_1}\right)T_{k+l+1}((x_0+1)^{-k+n-l-1})\cdot\\
&\quad \cdot (x_2+x_0x_2)^{h_2^\mu-h_3^\nu+l}x_2^{-k-1}(1+x_0)^{-L_{W_3}(0)_N}\cdot\\
&\quad \cdot \mathcal{Y}(Y_W(x_2^{L_V(0)}v,-x_0x_2)(x_2+x_0x_2)^{L_W(0)}w,x_2+x_0x_2)(1+x_0)^{L_{W_2}(0)_N}w_2. \qquad (4.12)
\end{aligned}
$$

From the $L(0)$-conjugation property for intertwining operators,

$$(1+x_0)^{-L_{W_3}(0)}\mathcal{Y}((1+x_0)^{L_W(0)}w,x_2+x_0x_2)(1+x_0)^{L_{W_2}(0)}=\mathcal{Y}(w,x_2),$$

or equivalently,

$$
\begin{aligned}
&\sum_{k=0}^{K}(1+x_0)^{-L_{W_3}(0)}\mathcal{Y}^k((1+x_0)^{L_W(0)}w,x_2+x_0x_2)(1+x_0)^{L_{W_2}(0)}(\log(x_2+x_0x_2))^k\\
&\quad =\sum_{k=0}^{K}\mathcal{Y}^k(w,x_2)(\log x_2)^k. \qquad (4.13)
\end{aligned}
$$

Taking $\mathrm{Coeff}^0_{\log x_2}$ on both sides of (4.13), we obtain

$$
\begin{aligned}
&\sum_{k=0}^{K}(1+x_0)^{-L_{W_3}(0)}\mathcal{Y}^k((1+x_0)^{L_W(0)}w,x_2+x_0x_2)(1+x_0)^{L_{W_2}(0)}(\log(1+x_0))^k\\
&\quad =\mathcal{Y}^0(w,x_2).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\mathrm{Coeff}^0_{\log x_2}(1+x_0)^{-L_{W_3}(0)_N}\mathcal{Y}((1+x_0)^{L_W(0)_N}w,x_2+x_0x_2)(1+x_0)^{L_{W_2}(0)_N}\\
&\quad =\sum_{k=0}^{K}(1+x_0)^{-L_{W_3}(0)_N}\mathcal{Y}^k((1+x_0)^{L_W(0)_N}w,x_2+x_0x_2)(1+x_0)^{L_{W_2}(0)_N}(\log(1+x_0))^k
\end{aligned}
$$

$$= (1+x_0)^{L_{W_3}(0)_S}\mathcal{Y}^0((1+x_0)^{-L_W(0)_S}w, x_2)(1+x_0)^{-L_{W_2}(0)_S}$$
$$= \mathcal{Y}^0(w, x_2+x_0x_2).$$

In particular, we have

$$\begin{aligned}
&\mathrm{Coeff}^0_{\log x_2}(1+x_0)^{-L_{W_3}(0)_N}\cdot\\
&\qquad\cdot\mathcal{Y}(Y_W(x_2^{L_V(0)}v, -x_0x_2)(x_2+x_0x_2)^{L_W(0)}w, x_2+x_0x_2)(1+x_0)^{L_{W_2}(0)_N}\\
&= \mathrm{Coeff}^0_{\log x_2}(1+x_0)^{-L_{W_3}(0)_N}\cdot\\
&\qquad\cdot\mathcal{Y}((1+x_0)^{L_W(0)_N}Y_W(x_2^{L_V(0)}v, -x_0x_2)(x_2+x_0x_2)^{L_W(0)_S}x_2^{L_W(0)_N}w, x_2+x_0x_2)\cdot\\
&\qquad\cdot(1+x_0)^{L_{W_2}(0)_N}\\
&= \mathcal{Y}^0(Y_W(x_2^{L_V(0)}v, -x_0x_2)(x_2+x_0x_2)^{L_W(0)_S}w, x_2+x_0x_2).
\end{aligned}\tag{4.14}$$

Using (4.14), we see that the right-hand side of (4.12) is equal to

$$\begin{aligned}
&\sum_{\nu\in\Gamma(W_3)}\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1}x_1^{-1}\delta\left(\frac{x_2+x_0x_2}{x_1}\right)T_{k+l+1}((x_0+1)^{-k+n-l-1})\cdot\\
&\quad\cdot(x_2+x_0x_2)^{h_2^\mu-h_3^\nu+l}x_2^{-k-1}\mathcal{Y}^0(Y_W(x_2^{L_V(0)}v, -x_0x_2)(x_2+x_0x_2)^{L_W(0)_S}w, x_2+x_0x_2)w_2.
\end{aligned}\tag{4.15}$$

On the other hand,

$$(x_2+x_0x_2)^{h_2^\mu-h_3^\nu+l}\mathcal{Y}^0(Y_W(x_2^{L_V(0)}v, -x_0x_2)(x_2+x_0x_2)^{L_W(0)_S}w, x_2+x_0x_2)w_2$$

contains only integral powers of $x_2+x_0x_2$ and does not contain $\log(x_2+x_0x_2)$. Then (4.15) is equal to

$$\begin{aligned}
&\sum_{\nu\in\Gamma(W_3)}\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1}x_2^{-1}\delta\left(\frac{x_1-x_0x_2}{x_2}\right)T_{k+l+1}((x_0+1)^{-k+n-l-1})\cdot\\
&\qquad\cdot x_1^{h_2^\mu-h_3^\nu+l}x_2^{-k-1}\mathcal{Y}^0(Y_W(x_2^{L_V(0)}v, -x_0x_2)x_1^{L_W(0)_S}w, x_1)w_2.
\end{aligned}\tag{4.16}$$

The Jacobi identity for $\mathcal{Y}^0$ gives

$$\begin{aligned}
&\mathrm{Res}_{x_1}x_2^{-1}\delta\left(\frac{x_1-x_0x_2}{x_2}\right)x_1^{h_2^\mu-h_3^\nu+l}\mathcal{Y}^0(Y_W(x_2^{L_V(0)}v, -x_0x_2)x_1^{L_W(0)_S}w, x_1)w_2\\
&\quad= -\mathrm{Res}_{x_1}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0x_2}\right)x_1^{h_2^\mu-h_3^\nu+l}Y_{W_3}(x_2^{L_V(0)}v, x_2)\mathcal{Y}^0(x_1^{L_W(0)_S}w, x_1)w_2\\
&\qquad+\mathrm{Res}_{x_1}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0x_2}\right)x_1^{h_2^\mu-h_3^\nu+l}\mathcal{Y}^0(x_1^{L_W(0)_S}w, x_1)Y_{W_2}(x_2^{L_V(0)}v, x_2)w_2.
\end{aligned}\tag{4.17}$$

Since $w_2\in (W_2)_{[h_2^\mu+l]}$, we have

$$\mathrm{Res}_{x_1}x_1^{h_2^\mu-h_3^\nu+l+j}\mathcal{Y}^0(x_1^{L_W(0)_S}w, x_1)w_2\in (W_2)_{[h_3^\nu-1-j]}=0\tag{4.18}$$

for $j \in \mathbb{N}$. Using (4.17) and (4.18), we see that (4.16) is equal to

$$\begin{aligned}
&\sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0x_2}\right) T_{k+l+1}((x_0+1)^{-k+n-l-1})\cdot\\
&\qquad \cdot x_1^{h_2^\mu-h_3^\nu+l} x_2^{-k-2}\mathcal{Y}^0(x_1^{L_W(0)_S}w, x_1)Y_{W_2}(x_2^{L_V(0)}v, x_2)w_2\\
&\quad = \sum_{\nu\in\Gamma(W_3)}\sum_{m=0}^{n}\binom{-k+n-l-1}{m}\mathrm{Res}_{x_2}\mathrm{Res}_{x_1}\left(\frac{x_1-x_2}{x_2}\right)^{-k+n-l-m-1}\cdot\\
&\qquad \cdot x_1^{h_2^\mu-h_3^\nu+l} x_2^{-k-2}\mathcal{Y}^0(x_1^{L_W(0)_S}w, x_1)Y_{W_2}(x_2^{L_V(0)}v, x_2)w_2\\
&\quad = \sum_{\nu\in\Gamma(W_3)}\sum_{m=0}^{n}\sum_{r=0}^{N}\sum_{j\in\mathbb{N}}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{j}(-1)^j\cdot\\
&\qquad \cdot \mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_1^{h_2^\mu-h_3^\nu-k+n-m-1-j} x_2^{-n+l+m-1+j}\mathcal{Y}^0(x_1^{L_W(0)_S}w, x_1)Y_{W_2}(x_2^{L_V(0)}v, x_2)w_2.
\end{aligned} \tag{4.19}$$

Since $w_2 \in (W_2)_{[h_2^\mu+l]}$, $\mathrm{Res}_x x^{q-1}Y_{W_2}(x^{L_V(0)}v, x) \in (W_2)_{[h_2^\mu+l-q]}$ and thus is equal to 0 if $q > l$. In the case $j > n-m$, we have $-n+l+m+j > l$ and hence

$$\mathrm{Res}_{x_2} x_2^{-n+l+m-1+j}Y_{W_2}(x_2^{L_V(0)}v, x_2)w_2 = 0.$$

In particular, those terms in the right-hand side of (4.19) with $j > n-m+r$ is 0. So the right-hand side of (4.19) is equal to

$$\begin{aligned}
&\sum_{\nu\in\Gamma(W_3)}\sum_{m=0}^{n}\sum_{j=0}^{n-m}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{j}(-1)^j\cdot\\
&\qquad \cdot \mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_1^{h_2^\mu-h_3^\nu-k+n-m-1-j} x_2^{-n+l+m-1+j}\mathcal{Y}^0(x_1^{L_W(0)_S}w, x_1)Y_{W_2}(x_2^{L_V(0)}v, x_2)w_2\\
&\quad = \sum_{\nu\in\Gamma(W_3)}\sum_{m=0}^{n}\sum_{q=m}^{n}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{q-m}(-1)^{q-m}\cdot\\
&\qquad \cdot \mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_1^{h_2^\mu-h_3^\nu-k+n-1-q} x_2^{-n+l-1+q}\mathcal{Y}^0(x_1^{L_W(0)_S}w, x_1)Y_{W_2}(x_2^{L_V(0)}v, x_2)w_2\\
&\quad = \sum_{\nu\in\Gamma(W_3)}\sum_{q=0}^{n}\left(\sum_{m=0}^{q}\binom{-k+n-l-1}{m}\binom{-k+n-l-m-1}{q-m}(-1)^{q-m}\right)\cdot\\
&\qquad \cdot \mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_1^{h_2^\mu-h_3^\nu-k+n-1-q} x_2^{-n+l-1+q}\mathcal{Y}^0(x_1^{L_W(0)_S}w, x_1)Y_{W_2}(x_2^{L_V(0)}v, x_2)w_2.
\end{aligned} \tag{4.20}$$

Using (4.10), we see that the right-hand side of (4.20) is equal to

$$\begin{aligned}
&\sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_2}\mathrm{Res}_{x_1} x_1^{h_2^\mu-h_3^\nu-k+n-1} x_2^{-n+l-1}\mathcal{Y}^0(x_1^{L_W(0)_S}w, x_1)Y_{W_2}(x_2^{L_V(0)}v, x_2)w_2\\
&\qquad = \vartheta_{\mathcal{Y}}([w]_{kn})\mathrm{Res}_{x_2} x_2^{-n+l-1}Y_{W_2}(x_2^{L_V(0)}v, x_2)w_2
\end{aligned}$$

$$= \vartheta_{\mathcal{Y}}([w]_{kn})\vartheta_{W_2}([v]_{nl})w_2. \tag{4.21}$$

The calculations from (4.12) to (4.21) gives

$$\vartheta_{\mathcal{Y}}([w]_{kn} \diamond [v]_{nl})w_2 = \vartheta_{\mathcal{Y}}([w]_{kn})\vartheta_{W_2}([v]_{nl})w_2.$$

Thus we have proved

$$\vartheta_{\mathcal{Y}}([w]_{km} \diamond [v]_{nl})w_2 = \vartheta_{\mathcal{Y}}([w]_{km})\vartheta_{W_2}([v]_{nl})w_2$$

for $k, m, n, l \in \mathbb{N}$, $v \in V$, $w \in W$ and $w_2 \in W_2$. This shows that $\vartheta_{\mathcal{Y}}$ commutes with the right actions of $U^\infty(V)$.

We know that $U^\infty(W)/\ker \vartheta_{\mathcal{Y}}$ is linearly isomorphic to the image $\vartheta_{\mathcal{Y}}(U^\infty(W))$ of $U^\infty(W)$ in $\mathrm{Hom}(W_2, W_3)$ under $\vartheta_{\mathcal{Y}}$. Since $\vartheta_{\mathcal{Y}}$ commutes with both the left and right actions of $U^\infty(V)$, $U^\infty(W)/\ker \vartheta_{\mathcal{Y}}$ is in fact equivalent to $\vartheta_{\mathcal{Y}}(U^\infty(W))$ as $U^\infty(V)$-bimodules. But $\mathrm{Hom}(W_2, W_3)$ is an $A^\infty(V)$-bimodule and $\vartheta_{\mathcal{Y}}(U^\infty(W))$ is an $A^\infty(V)$-subbimodule of $\mathrm{Hom}(W_2, W_3)$. Thus $U^\infty(W)/\ker \vartheta_{\mathcal{Y}}$ is an $A^\infty(V)$-bimodule equivalent to the $A^\infty(V)$-bimodule $\vartheta_{\mathcal{Y}}(U^\infty(W))$. ∎

Let $Q^\infty(W)$ be the intersection of $\ker \vartheta_{\mathcal{Y}}$ for all lower-bounded generalized $V$-modules $W_2$ and $W_3$ and all intertwining operators $\mathcal{Y}$ of type $\binom{W_3}{WW_2}$.

**Theorem 4.2** *The quotient $A^\infty(W) = U^\infty(W)/Q^\infty(W)$ is an $A^\infty(V)$-bimodule.*

*Proof.* For every pair of lower-bounded generalized $V$-modules $W_2$ and $W_3$ and every intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{WW_2}$, $U^\infty(W)/\ker \vartheta_{\mathcal{Y}}$ is an $A^\infty(V)$-bimodule. Hence $U^\infty(W)$ is an $A^\infty(V)$-bimodule modulo elements of $\ker \vartheta_{\mathcal{Y}}$ for all lower-bounded generalized $V$-modules $W_2$ and $W_3$ and all intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{WW_2}$. Thus $U^\infty(W)$ is an $A^\infty(V)$-bimodule modulo elements of the intersection $Q^\infty(W)$ of all such $\ker \vartheta_{\mathcal{Y}}$. This shows that $A^\infty(W) = U^\infty(W)/Q^\infty(W)$ is an $A^\infty(V)$-bimodule. ∎

**Proposition 4.3** *In the case $W = V$, the $A^\infty(V)$-bimodule $A^\infty(W)$ is equal to the associative algebra $A^\infty(V)$ introduced in [H5] (see also Section 2).*

*Proof.* We need only prove that $Q^\infty(W)$ for $W = V$ as a lower-bounded generalized $V$-module is equal to $\mathbb{Q}^\infty(V)$ in [H5] (see also Section 2). By Remark 3.4, $Q^\infty(V)$ in [H5] is equal to the intersection of $\ker \vartheta_W$ for all lower-bounded generalized $V$-modules $W$. Note that $\ker \vartheta_{W_2} = \ker \vartheta_{Y_{W_2}}$. If for every pair of lower-bounded generalized $V$-modules $W_2$ and $W_3$ and every intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{VW_2}$, $\ker \vartheta_{W_2} \subset \ker \vartheta_{\mathcal{Y}}$, then the intersection of $\ker \vartheta_{W_2}$ for all lower-bounded generalized $V$-module $W_2$ is equal to the intersection of all $\ker \vartheta_{\mathcal{Y}}$ for all lower-bounded generalized $V$-modules $W_2$ and $W_3$ and all intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{VW_2}$, that is, the proposition is true.

We now prove this fact. From the $L(-1)$-derivative property and $L_V(-1)\mathbf{1} = 0$, we see that $\mathcal{Y}(\mathbf{1}, x)$ must be independent of $x$. We denote it as $f$ and then $f$ is a linear map from $W_2$ to $W_3$. It is easy to verify that $f$ is in fact a $V$-module map.

Using the associativity between the intertwining operator $\mathcal{Y}$ and the vertex operator maps $Y_V$ and $Y_{W_2}$, we have

$$\begin{aligned}\langle w_3', \mathcal{Y}(v, z_2)w_2\rangle &= \langle w_3', \mathcal{Y}(Y_V(\mathbf{1}, z_1 - z_2)v, z_2)w_2\rangle \\ &= \langle w_3', \mathcal{Y}(\mathbf{1}, z_1)Y_{W_2}(v, z_2)w_2\rangle \\ &= \langle w_3', f(Y_{W_2}(v, z_2)w_2)\rangle \end{aligned} \tag{4.22}$$

for $v \in V$, $w_2 \in W_2$ and $w_3' \in W_3'$. Since every term in (4.22) is defined for all $z_1$ and $z_2 \neq 0$, (4.22) holds for all such $z_1$ and $z_2$. In particular, we obtain

$$\mathcal{Y}(v, x)w_2 = f(Y_{W_2}(v, x)w_2)$$

for $v \in V$ and $w_2 \in W_2$. By the definition of $\vartheta_{W_2}$ and $\vartheta_{\mathcal{Y}}$ we have

$$\begin{aligned}\vartheta_{\mathcal{Y}}([v]_{kl})w_2 &= \mathrm{Res}_x x^{l-k-1}\mathcal{Y}(x^{L_V(0)}v, x)w_2 \\ &= \mathrm{Res}_x x^{l-k-1} f(Y_{W_2}(x^{L_V(0)}v, x)w_2) \\ &= f(\vartheta_{W_2}([v]_{kl})w_2)\end{aligned}$$

for $k, l \in \mathbb{N}$, $v \in V$ and $w_2 \in (W_2)_{[\![l]\!]}$. Then we have

$$\vartheta_{\mathcal{Y}}(\mathfrak{v}) = f \circ \vartheta_{W_2}(\mathfrak{v})$$

for $\mathfrak{v} \in U^\infty(V)$. Now $\ker \vartheta_{W_2} \subset \ker \vartheta_{\mathcal{Y}}$ follows immediately. ∎

Using the Jacobi identity of intertwining operators, we have the following result giving some particular elements of $Q^\infty(W)$:

**Proposition 4.4** *For $k, l, n \in \mathbb{N}$, $p \in \mathbb{Z}$ such that $l + p \in \mathbb{N}$, homogeneous $v \in V$ and $w \in W$, the element*

$$\begin{aligned}&\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}} (-1)^j \binom{p}{j} [v]_{k,n+p-j} \diamond [w]_{n+p-j,l+p} \\ &- \sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}} (-1)^{p-j} \binom{p}{j} [w]_{k,l-n+k+p-j} \diamond [v]_{l-n+k+p-j,l+p} \\ &- \sum_{j\in\mathbb{N}} \binom{\mathrm{wt}\, v + n - k - 1}{j} [(Y_W)_{p+j}(v)w]_{k,l+p}\end{aligned} \tag{4.23}$$

*of $U^\infty(W)$ is in fact in $Q^\infty(W)$.*

*Proof.* Let $W_2$ and $W_3$ be lower-bounded generalized $V$-modules and $\mathcal{Y}$ an intertwining operator of type $\binom{W_3}{W W_2}$. For $k, l, n \in \mathbb{N}$, $p \in \mathbb{Z}$ such that $l + p \in \mathbb{N}$, $\mu \in \Gamma(W_2)$, homogeneous $v \in V$, $w \in W$ and $w_2 \in (W_2)_{[h_2^\mu + l + p]}$,

$$\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}} (-1)^j \binom{p}{j} \vartheta_{\mathcal{Y}}([v]_{k,n+p-j} \diamond [w_1]_{n+p-j,l+p})w_2$$

$$
\begin{aligned}
&= \sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}} (-1)^j \binom{p}{j} \vartheta_{W_3}([v]_{k,n+p-j})\vartheta_{\mathcal{Y}}([w]_{n+p-j,l+p})w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}} (-1)^j \binom{p}{j} \vartheta_{W_3}([v]_{k,n+p-j})\cdot\\
&\qquad\qquad \cdot \mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2} x_2^{h_2^\mu-h_3^\nu+(l+p)-(n+p-j)-1}\mathcal{Y}(x_2^{L_W(0)}w,x_2)w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}} \binom{p}{j}(-1)^j \mathrm{Res}_{x_1}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2} x_1^{(n+p-j)-k-1}\cdot\\
&\qquad\qquad \cdot x_2^{h_2^\mu-h_3^\nu+(l+p)-(n+p-j)-1} Y_{W_3}(x_1^{L_V(0)}v,x_1)\mathcal{Y}(x_2^{L_W(0)}w,x_2)w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_1}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}(x_1-x_2)^p\cdot\\
&\qquad\qquad \cdot x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1} Y_{W_3}(x_1^{L_V(0)}v,x_1)\mathcal{Y}(x_2^{L_W(0)}w,x_2)w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_1}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0} x_0^p x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\cdot\\
&\qquad\qquad \cdot x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1} Y_{W_3}(x_1^{L_V(0)}v,x_1)\mathcal{Y}(x_2^{L_W(0)}w,x_2)w_2.
\end{aligned}\tag{4.24}
$$

Using the Jacobi identity for the intertwining operator $\mathcal{Y}$, we see that the right-hand side of (4.24) is equal to

$$
\begin{aligned}
&\sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_1}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0} x_0^p x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\cdot\\
&\qquad\qquad \cdot x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1}\mathcal{Y}(x_2^{L_W(0)}w,x_2)Y_{W_2}(x_1^{L_V(0)}v,x_1)w_2\\
&\quad + \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_1}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0} x_0^p x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\cdot\\
&\qquad\qquad \cdot x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1}\mathcal{Y}(Y_{W_1}(x_1^{L_V(0)}v,x_0)x_2^{L_W(0)}w,x_2)w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_1}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}(-x_2+x_1)^p\cdot\\
&\qquad\qquad \cdot x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1}\mathcal{Y}(x_2^{L_W(0)}w,x_2)Y_{W_2}(x_1^{L_V(0)}v,x_1)w_2\\
&\quad + \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_1}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0} x_0^p x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\cdot\\
&\qquad\qquad \cdot x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1}\mathcal{Y}(x_2^{L_W(0)}Y_{W_1}(x_1^{L_V(0)}x_2^{-L_V(0)}v,x_0x_2^{-1})w,x_2)w_2\\
&= \sum_{\nu\in\Gamma(W_3)} \sum_{j\in\mathbb{N}}(-1)^{p-j}\binom{p}{j}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2} x_2^{h_2^\mu-h_3^\nu+(l-n+k+p-j)-k-1}\cdot\\
&\qquad\qquad \cdot \mathcal{Y}(x_2^{L_W(0)}w,x_2)\mathrm{Res}_{x_1}x_1^{(l+p)-(l-n+k+p-j)-1}Y_{W_2}(x_1^{L_V(0)}v,x_1)w_2
\end{aligned}
$$

$$+\sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_{x_1}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0} x_0^p x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\cdot$$
$$\cdot\, x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1}\mathcal{Y}(x_2^{L_W(0)}Y_{W_1}(x_1^{L_V(0)}x_2^{-L_V(0)}v, x_0x_2^{-1})w, x_2)w_2. \tag{4.25}$$

Using the definitions of $\vartheta_{\mathcal{Y}}$ and $\vartheta_{W_2}$, the property of the formal $\delta$-function, noting that $v$ is homogeneous and $\mathrm{Res}_{x_1}x_1^{(l+p)-(l-n+k+p-j)-1}Y_{W_2}(x_1^{L_V(0)}v, x_1)w_2 = 0$ when $l-n+k+p-j<0$, and changing the variable $x_0$ to $y = x_0x_2^{-1}$ in the second term, we see that the right-hand side of (4.25) is equal to

$$\sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}} (-1)^{p-j}\binom{p}{j}\vartheta_{\mathcal{Y}}([w]_{k,l-n+k+p-j})\cdot$$
$$\cdot\,\mathrm{Res}_{x_1}x_1^{(l+p)-(l-n+k+p-j)-1}Y_{W_2}(x_1^{L_V(0)}v, x_1)w_2$$
$$+\sum_{\nu\in\Gamma(W_3)}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0}x_0^p(x_2+x_0)^{n-k-1}\cdot$$
$$\cdot\, x_2^{h_2^\mu-h_3^\nu+l-n-1}\mathcal{Y}(x_2^{L_W(0)}Y_{W_1}((1+x_0x_2^{-1})^{L_V(0)}v, x_0x_2^{-1})w, x_2)w_2$$
$$=\sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}} (-1)^{p-j}\binom{p}{j}\vartheta_{\mathcal{Y}}([w]_{k,l-n+k+p-j})\vartheta_{W_2}([v]_{l-n+k+p-j,l+p})w_2$$
$$+\sum_{\nu\in\Gamma(W_3)}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0}x_0^p(1+x_0x_2^{-1})^{\mathrm{wt}\, v+n-k-1}\cdot$$
$$\cdot\, x_2^{h_2^\mu-h_3^\nu+l-k-2}\mathcal{Y}(x_2^{L_W(0)}Y_{W_1}(v, x_0x_2^{-1})w, x_2)w_2$$
$$=\sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}} (-1)^{p-j}\binom{p}{j}\vartheta_{\mathcal{Y}}([w]_{k,l-n+k+p-j}\diamond[v]_{l-n+k+p-j,l+p})w_2$$
$$+\sum_{\nu\in\Gamma(W_3)}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}\mathrm{Res}_{y}y^p(1+y)^{\mathrm{wt}\, v+n-k-1}\cdot$$
$$\cdot\, x_2^{h_2^\mu-h_3^\nu+(l+p)-k-1}\mathcal{Y}(x_2^{L_W(0)}Y_{W_1}(v, y)w, x_2)w_2. \tag{4.26}$$

Expanding $(1+y)^{\mathrm{wt}\, v+n-k-1}$ and $Y_{W_1}(v, y)$ and evaluating $\mathrm{Res}_y$, we see that the second term in the right-hand side of (4.26) is equal to

$$\sum_{\nu\in\Gamma(W_3)}\sum_{j\in\mathbb{N}}\binom{\mathrm{wt}\, v+n-k-1}{j}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}x_2^{h_2^\mu-h_3^\nu+(l+p)-k-1}\mathcal{Y}(x_2^{L_W(0)}(Y_{W_1})_{p+j}(v)w, x_2)w_2$$
$$=\sum_{j\in\mathbb{N}}\binom{\mathrm{wt}\, v+n-k-1}{j}\vartheta_{\mathcal{Y}}([(Y_{W_1})_{p+j}(v)w]_{k,l+p})w_2. \tag{4.27}$$

From (4.24)–(4.27), we obtain

$$\begin{aligned}
\vartheta_{\mathcal{Y}}\Bigg(&\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}}(-1)^{j}\binom{p}{j}[v]_{k,n+p-j}\diamond[w_1]_{n+p-j,l+p}\\
&-\sum_{\substack{j\in\mathbb{N}\\ l-n+k+j\geq 0}}(-1)^{p-j}\binom{p}{j}[w_1]_{k,l-n+k+j}\diamond[v]_{l-n+k+j,l+p}\\
&-\sum_{j\in\mathbb{N}}\binom{\mathrm{wt}\, v+n-k-1}{j}[(Y_{W_1})_j(v)w_1]_{k,l+p}\Bigg)w_2\\
=\;&0. \qquad (4.28)
\end{aligned}$$

Since $w_2$, $l$ and $p$ are arbitrary, we see from (4.28) that (4.23) is in $\ker\vartheta_{\mathcal{Y}}$. Since $W_2, W_3$ and $\mathcal{Y}$ are arbitrary, (4.23) is in the intersection $Q^{\infty}(W)$ of all $\ker\vartheta_{\mathcal{Y}}$. ∎

**Remark 4.5** In the case that $V$ is a vertex operator algebra, that is, we have a conformal vector $\omega\in V$. Let $\omega^{\infty}(0)$ and $\omega^{\infty}(-1)$ be the elements of $U^{\infty}(V)$ with diagonal entries being $\omega\in V$ and all the other entries being 0 and with the $(k+1,k)$-entries being $\omega$ for $k\in\mathbb{N}$ and all the other entries being 0, respectively. Then

$$\omega^{\infty}(0)=\sum_{k\in\mathbb{N}}[\omega]_{kk},\ \omega^{\infty}(-1)=\sum_{k\in\mathbb{N}}[\omega]_{k+1,k}.$$

From the definitions of the left and right actions of $U^{\infty}(V)$ on $U^{\infty}(W)$, we have $[\omega]_{kk}\diamond[w]_{kl}=\omega^{\infty}(0)\diamond[w]_{kl}$, $[w]_{kl}\diamond[\omega]_{ll}=[w]_{kl}\diamond\omega^{\infty}(0)$, $[\omega]_{k+1,k}\diamond[w]_{kl}=\omega^{\infty}(-1)\diamond[w]_{kl}$ and $[w]_{kl}\diamond[\omega]_{l+1,l}=[w]_{kl}\diamond\omega^{\infty}(-1)$ for $k,l\in\mathbb{N}$ and $w\in W$. Taking $n=k$, $p=0$ and $v=\omega$ in (4.23), we see that elements of the form

$$\begin{aligned}
&\omega^{\infty}(0)\diamond[w]_{nl}-[w]_{nl}\diamond\omega^{\infty}(0)-[(L_W(-1)+L_W(0))w]_{nl}\\
&\quad=[\omega]_{nn}\diamond[w]_{nl}-[w]_{nl}\diamond[\omega]_{ll}-[(L_W(-1)+L_W(0))w]_{nl}
\end{aligned}$$

for $n,l\in\mathbb{N}$ and $w\in W$ are in $Q^{\infty}(W)$. Taking $k=n+1$, $p=0$ and $v=\omega$ in (4.23), we see that elements of the form

$$\begin{aligned}
&\omega^{\infty}(-1)\diamond[w]_{nl}-[w]_{n+1,l+1}\diamond\omega^{\infty}(-1)-[L_W(-1)w]_{n+1,l}\\
&\quad=[\omega]_{n+1,n}\diamond[w]_{nl}-[w]_{n+1,l+1}\diamond[\omega]_{l+1,l}-[L_W(-1)w]_{n+1,l}
\end{aligned}$$

for $n,l\in\mathbb{N}$ and $w\in W$ are in $Q^{\infty}(W)$. In particular, we have

$$\begin{aligned}
&(\omega^{\infty}(0)+Q^{\infty}(W))\diamond([w]_{nl}+Q^{\infty}(W))-([w]_{nl}+Q^{\infty}(W))\diamond(\omega^{\infty}(0)+Q^{\infty}(W))\\
&\quad=[(L_W(-1)+L_W(0))w]_{nl}+Q^{\infty}(W),\\
&(\omega^{\infty}(-1)+Q^{\infty}(W))\diamond([w]_{nl}+Q^{\infty}(W))-([w]_{n+1,l+1}+Q^{\infty}(W))\diamond(\omega^{\infty}(-1)+Q^{\infty}(W))\\
&\quad=[L_W(-1)w]_{n+1,l}+Q^{\infty}(W)
\end{aligned}$$

in $A^{\infty}(W)$.

In the special case that $W=V$, we have:

**Corollary 4.6** *For $k,l,n\in\mathbb{N}$, $p\in\mathbb{Z}$ such that $l+p\in\mathbb{N}$, homogeneous $u\in V$ and $v\in V$, the elements*

$$\begin{aligned}&\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}}(-1)^j\binom{p}{j}[u]_{k,n+p-j}\diamond[v]_{n+p-j,l+p}\\ &\quad-\sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}}(-1)^{p-j}\binom{p}{j}[v]_{k,l-n+k+p-j}\diamond[u]_{l-n+k+p-j,l+p}\\ &\quad-\sum_{j\in\mathbb{N}}\binom{\mathrm{wt}\,u+n-k-1}{j}[(Y_V)_{p+j}(u)v]_{k,l+p}\end{aligned}\tag{4.29}$$

*is in $Q^\infty(V)$.*

*Proof.* This corollary follows immediately from Propositions 4.3 and 4.4. ∎

**Remark 4.7** We can also give a subspace $O^\infty(W)$ of $Q^\infty(W)$ analogous to $O^\infty(V)$ in [H5]. We conjecture that $Q^\infty(W)$ is spanned by $O^\infty(W)$ and the elements of the form (4.23) for $k,l,n,p\in\mathbb{N}$, homogeneous $v\in V$ and $w\in W$. In particular, we also conjecture that $Q^\infty(V)$ is spanned by $O^\infty(V)$ in [H5] and elements of the form (4.29) of $Q^\infty(V)$. We shall discuss these in a future paper.

# 5 Isomorphisms between spaces of intertwining operators and $A^\infty(V)$-module maps

We formulate and prove the first main theorem of the present paper in this section. For lower-bounded generalized $V$-modules $W_1$, $W_2$ and $W_3$, we define a linear map $\rho:\mathcal{V}_{W_1W_2}^{W_3}\to\mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1)\otimes_{A^\infty(V)}W_2,W_3)$, where $W_1$, $W_2$ and $W_3$ are lower-bounded generalized $V$-module and $\mathcal{V}_{W_1W_2}^{W_3}$ is the space of intertwining operators of type $\binom{W_3}{W_1W_2}$. Our first main theorem states that $\rho$ is an isomorphism.

Before we formulate and prove this theorem, we prove first that the category lower-bounded generalized $V$-modules and the category of graded $A^\infty(V)$-modules are isomorphic (not just equivalent since the underlying vector space are the same). In Section 2, we have obtained a functor from the category of lower-bounded generalized $V$-modules to the category of graded $A^\infty(V)$-modules. We now have:

**Theorem 5.1** *The functor from the category lower-bounded generalized $V$-modules to the category of graded $A^\infty(V)$-modules is in fact an isomorphism of categories.*

*Proof.* Given a graded $A^\infty(V)$-module $W$, we need to construct a lower-bounded generalized $V$-module structure on $W$. We define

$$(Y_W)_{\mathrm{wt}\, v+l-k-1}(v)w = \vartheta_W([v]_{kl})w$$

for $k, l \in \mathbb{N}$, homogeneous $v \in V$ and $w \in W_{\llbracket l \rrbracket}$ and

$$Y_W(v, x)w = \sum_{k\in\mathbb{N}} (Y_W)_{\mathrm{wt}\, v+l-k-1}(v)wx^{-\mathrm{wt}\, v-l+k}$$

for $l \in \mathbb{N}$, homogeneous $v \in V$ and $w \in W_{\llbracket l \rrbracket}$. It is clear that $Y_W(v, x)w$ is lower truncated. For any lower-bounded generalized $V$-module $W_0$, by definition,

$$\vartheta_{W_0}([\mathbf{1}]_{kl})w_0 = \mathrm{Res}_x x^{l-k-1}w_0 = \delta_{kl}[\mathbf{1}]_{ll}w_0$$

for $k, l \in \mathbb{N}$ and $w_0 \in W_0$. So we have $[\mathbf{1}]_{kl} - \delta_{kl}[\mathbf{1}]_{ll} \in \ker \vartheta_{W_0}$. Since $W_0$ is arbitrary, by Remark 3.4, we see that $[\mathbf{1}]_{kl} - \delta_{kl}[\mathbf{1}]_{ll} \in Q^\infty(V)$. In particular, $\vartheta_W([\mathbf{1}]_{kl})w = \delta_{kl}w$ for $k, l \in \mathbb{N}$ and $w \in W_{\llbracket l \rrbracket}$. Thus we have $Y_W(\mathbf{1}, x)w = w$. The $L(0)$- and $L(-1)$-commutator formulas and the $L(0)$-grading condition follow hold since $W$ is a graded $A^\infty(V)$-algebra. The only remaining axiom to be proved is the Jacobi identity.

We now prove the Jacobi identity using Corollary 4.6. From Corollary 4.6, we know that (4.29) with $w$ replaced by $w_1$ is an element of $Q^\infty(V)$. In particular,

$$\begin{aligned}
&\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}} (-1)^j \binom{p}{j} \vartheta_W([u]_{k,n+p-j} \diamond [v]_{n+p-j,l+p})w_2\\
&\quad - \sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}} (-1)^{p-j} \binom{p}{j} \vartheta_W([v]_{k,l-n+k+p-j} \diamond [u]_{l-n+k+p-j,l+p})w\\
&\quad - \sum_{j\in\mathbb{N}} \binom{\mathrm{wt}\, u+n-k-1}{j} \vartheta_W([(Y_V)_{p+j}(u)v]_{k,l+p})w\\
&= 0
\end{aligned} \tag{5.1}$$

for $k, l, n \in \mathbb{N}$, $p \in \mathbb{Z}$ such that $l + p \in \mathbb{N}$, homogeneous $u, v \in V$ and $w \in W_{\llbracket l+p \rrbracket}$. The left-hand side of (5.1) is equal to

$$\begin{aligned}
&\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}} (-1)^j \binom{p}{j} \vartheta_W([u]_{k,n+p-j})\vartheta_W([v]_{n+p-j,l+p})w\\
&- \sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}} (-1)^{p-j} \binom{p}{j} \vartheta_W([v]_{k,l-n+k+p-j})\vartheta_W([v]_{l-n+k+p-j,l+p})w_2\\
&- \sum_{j\in\mathbb{N}} \binom{\mathrm{wt}\, u+n-k-1}{j} \vartheta_W([(Y_V)_{p+j}(u)v]_{k,l+p})w.
\end{aligned} \tag{5.2}$$

From the definitions of $\vartheta_W$, we see that (5.2) is equal to

$$\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}}(-1)^j\binom{p}{j}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n+p-j-k-1}x_2^{l-n+j-1}Y_W(x_1^{L_V(0)}u,x_1)Y_W(x_2^{L_V(0)}v,x_2)w$$
$$-\sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}}(-1)^{p-j}\binom{p}{j}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k+j-1}\cdot$$
$$\cdot\, x_2^{l-n+p-j-1}Y_W(x_2^{L_V(0)}v,x_2)Y_{W_2}(x_1^{L_V(0)}u,x_1)w$$
$$-\sum_{j\in\mathbb{N}}\binom{\mathrm{wt}\,u+n-k-1}{j}\mathrm{Res}_{x_2}x_2^{l+p-k-1}Y_W(x_2^{L_V(0)}(Y_V)_{p+j}(u)v,x_2)w$$
$$=\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{l-n-1}(x_1-x_2)^pY_W(x_1^{L_V(0)}u,x_1)Y_W(x_2^{L_V(0)}v,x_2)w$$
$$-\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{l-n-1}(-x_2+x_1)^pY_W(x_2^{L_V(0)}v,x_2)Y_W(x_1^{L_V(0)}u,x_1)w$$
$$-\mathrm{Res}_{x_2}\mathrm{Res}_{y}y^p(1+y)^{\mathrm{wt}\,u+n-k-1}x_2^{l+p-k-1}Y_W(x_2^{L_V(0)}Y_V(u,y)v,x_2)w. \tag{5.3}$$

Since the left-hand side of (5.3) is equal to the left-hand side of (5.1), we see that the right-hand side of (5.3) is 0 for $k,l,n\in\mathbb{N}$, $p\in\mathbb{Z}$ such that $l+p\in\mathbb{N}$, homogeneous $u,v\in V$ and $w\in W_{\llbracket l+p\rrbracket}$. In fact, the right-hand side of (5.3) is also 0 for all $p\in\mathbb{Z}$ even if $l+p<0$ since $w=0$. Multiplying the right-hand side of (5.3) by $x_0^{-p-1}$ and then take sum over $p\in\mathbb{Z}$, we obtain

$$\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{l-n-1}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(x_1^{L_V(0)}u,x_1)Y_W(x_2^{L_V(0)}v,x_2)w$$
$$-\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{l-n-1}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_W(x_2^{L_V(0)}v,x_2)Y_W(x_1^{L_V(0)}u,x_1)w$$
$$-\mathrm{Res}_{x_2}\mathrm{Res}_{y}x_0^{-1}\delta\left(\frac{yx_2}{x_0}\right)(1+y)^{n-k-1}x_2^{l-k-1}Y_W(x_2^{L_V(0)}Y_V((1+y)^{L_V(0)}u,y)v,x_2)w$$
$$=0. \tag{5.4}$$

The third term in the left-hand side of (5.4) is equal to

$$-\mathrm{Res}_{x_2}\mathrm{Res}_{y}x_0^{-1}\delta\left(\frac{yx_2}{x_0}\right)(1+y)^{n-k-1}x_2^{l-k-1}Y_W(x_2^{L_V(0)}Y_V((1+y)^{L_V(0)}u,y)v,x_2)w$$
$$=-\mathrm{Res}_{x_2}\mathrm{Res}_{y}x_0^{-1}\delta\left(\frac{yx_2}{x_0}\right)(1+x_0x_2^{-1})^{n-k-1}x_2^{l-k-1}\cdot$$
$$\cdot\, Y_W(x_2^{L_V(0)}Y_V((1+x_0x_2^{-1})^{L_V(0)}u,x_0x_2^{-1})v,x_2)w$$
$$=-\mathrm{Res}_{x_2}(x_2+x_0)^{n-k-1}x_2^{l-n-1}Y_W(Y_V((x_2+x_0)^{L_V(0)}u,x_0)x_2^{L_V(0)}v,x_2)w$$
$$=-\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{l-n-1}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W(Y_V(x_1^{L_V(0)}u,x_0)x_2^{L_V(0)}v,x_2)w. \tag{5.5}$$

Using (5.5) and substituting $x_1^{-L_V(0)}v$ and $x_2^{-L_{W_1}(0)_S}w_1$ for $v$ and $w_1$, respectively, we see that (5.4) becomes

$$\begin{aligned}
&\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{l-n-1}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u,x_1)Y_W(v,x_2)w\\
&\quad -\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{l-n-1}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_W(v,x_2)Y_{W_2}(u,x_1)w\\
&\quad -\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{l-n-1}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W(Y_V(u,x_0)v,x_2)w\\
&=0.
\end{aligned}\tag{5.6}$$

Since $k,l,n\in\mathbb{N}$ and $w\in W$ are arbitrary, (5.18) gives the Jacobi identity for $Y_W$.

This construction of a lower-bounded generalized $V$-module structure from a graded $A^\infty(V)$-module gives us a functor from the category of lower-bounded generalized $V$-modules to the category of graded $A^\infty(V)$-modules. It is clear that the graded $A^\infty(V)$-module structure on $W$ obtained from $Y_W$ is the same as the graded $A^\infty(V)$-module structure that we start with. So this functor is the inverse of the functor from the category of graded $A^\infty(V)$-modules to the category of lower-bounded generalized $V$-modules. The proposition is proved. ∎

Let $V$ be a grading-restricted vertex algebra and $W_1$ , $W_2$ and $W_3$ lower-bounded generalized $V$-modules. Let $A^\infty(W_1)\otimes_{A^\infty(V)}W_2$ be the tensor product over $A^\infty(V)$ of the $A^\infty(V)$-bimodule $A^\infty(W_1)$ and the left $A^\infty(V)$-module $W_2$.

We first prove some lemmas which we shall need later.

**Lemma 5.2** *The $A^\infty(V)$-module $A^\infty(W_1)\otimes_{A^\infty(V)}W_2$ is spanned by elements of the form*

$$([w_1]_{kl}+Q^\infty(W_1))\otimes_{A^\infty(V)}w_2$$

*for $k,l\in\mathbb{N}$, $w_1\in W_1$, $w_2\in(W_2)_{[\![l]\!]}$.*

*Proof.* We know that elements of the form

$$([w_1]_{kn}+Q^\infty(W_1))\otimes_{A^\infty(V)}w_2$$

for $k,l,n\in\mathbb{N}$, $w_1\in W_1$ span $A^\infty(W_1)\otimes_{A^\infty(V)}W_2$. If $l\neq n$, then by definition $[w_1]_{kn}\diamond[\mathbf{1}]_{ll}=0$. Also for $l\in\mathbb{N}$ and $w_2\in(W_2)_{[\![l]\!]}$, by definition, $\vartheta_{W_2}([\mathbf{1}]_{ll})w_2=w_2$. Hence for $k,l,n\in\mathbb{N}$, $w_1\in W_1$, $w_2\in(W_2)_{[\![l]\!]}$, if $n\neq l$, then

$$\begin{aligned}
&([w_1]_{kn}+Q^\infty(W_1))\otimes_{A^\infty(V)}w_2\\
&\quad=([w_1]_{kn}+Q^\infty(W_1))\otimes_{A^\infty(V)}\vartheta_{W_2}([\mathbf{1}]_{ll})w_2\\
&\quad=([w_1]_{kn}\diamond[\mathbf{1}]_{ll}+Q^\infty(W_1))\otimes_{A^\infty(V)}w_2\\
&\quad=0.
\end{aligned}$$

Hence the lemma is true. ∎

**Lemma 5.3** *Let*

$$f \in \mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3).$$

*For* $k, l \in \mathbb{N}$, $w_1 \in W_1$, $w_2 \in (W_2)_{\llbracket l \rrbracket}$,

$$f(([w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2) \in (W_3)_{\llbracket k \rrbracket}.$$

*Proof.* For $k, l \in \mathbb{N}$, $w_1 \in W_1$, $w_2 \in (W_2)_{\llbracket l \rrbracket}$, since $f$ is an $A^\infty(V)$-module map,

$$\begin{aligned}
&f(([w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)\\
&\quad = f(([\mathbf{1}]_{kk} \diamond [w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)\\
&\quad = \vartheta_{W_3}([\mathbf{1}]_{kk}) f(([w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2).
\end{aligned}$$

By the definition of $\vartheta_{W_3}$, $\vartheta_{W_3}([\mathbf{1}]_{kk})w_3 = 0$ for $w_3 \in (W_3)_{\llbracket n \rrbracket}$, $n \neq k$. Hence the lemma is true. ∎

Let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{W_1 W_2}$. We define a linear map

$$\rho(\mathcal{Y}) : A^\infty(W_1) \otimes W_2 \to W_3$$

by

$$(\rho(\mathcal{Y}))((\mathfrak{w}_1 + Q^\infty(W_1)) \otimes w_2) = \vartheta_{\mathcal{Y}}(\mathfrak{w}_1) w_2$$

for $\mathfrak{w}_1 \in U^\infty(W_1)$ and $w_2 \in W_2$. Since $Q^\infty(W_1) \subset \ker \vartheta_{\mathcal{Y}}$, $\rho(\mathcal{Y})$ is well defined.

Using Proposition 4.1, we have:

**Proposition 5.4** *The linear map* $\rho(\mathcal{Y})$ *is in fact an* $A^\infty(V)$*-module map from* $A^\infty(W_1) \otimes_{A^\infty(V)} W_2$ *to* $W_3$*, that is,*

$$\rho(\mathcal{Y}) \in \mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3).$$

*Proof.* By Proposition 4.1, for $\mathfrak{v} \in U^\infty(V)$, $\mathfrak{w}_1 \in U^\infty(W_1)$ and $w_2 \in W_2$, we have

$$\begin{aligned}
&(\rho(\mathcal{Y}))(((\mathfrak{v} + Q^\infty(V)) \diamond (\mathfrak{w}_1 + Q^\infty(W_1))) \otimes w_2)\\
&\quad = (\rho(\mathcal{Y}))((\mathfrak{v} \diamond \mathfrak{w}_1 + Q^\infty(W_1)) \otimes w_2)\\
&\quad = \vartheta_{\mathcal{Y}}(\mathfrak{v} \diamond \mathfrak{w}_1) w_2\\
&\quad = \vartheta_{W_3}(\mathfrak{v}) \vartheta_{\mathcal{Y}}(\mathfrak{w}_1) w_2\\
&\quad = \vartheta_{W_3}(\mathfrak{v})(\rho(\mathcal{Y}))(([w_1]_{nl} + Q^\infty(W_1)) \otimes w_2)
\end{aligned}$$

and

$$\begin{aligned}
&(\rho(\mathcal{Y}))(((\mathfrak{w}_1 + Q^\infty(W_1)) \diamond (\mathfrak{v} + Q^\infty(V))) \otimes w_2)\\
&\quad = (\rho(\mathcal{Y}))((\mathfrak{w}_1 \diamond \mathfrak{v} + Q^\infty(V)) \otimes w_2)\\
&\quad = \vartheta_{\mathcal{Y}}(\mathfrak{w}_1 \diamond \mathfrak{v}) w_2\\
&\quad = \vartheta_{\mathcal{Y}}(\mathfrak{w}_1) \vartheta_{W_2}(\mathfrak{v}) w_2
\end{aligned}$$

$$= (\rho(\mathcal{Y}))((\mathfrak{w}_1 + Q^\infty(W_1)) \otimes \vartheta_{W_2}(\mathfrak{v})w_2),$$

proving that $\rho(\mathcal{Y})$ is indeed an $A^\infty(V)$-module map from $A^\infty(W_1) \otimes_{A^\infty(V)} W_2$ to $W_3$. ∎

We now have a linear map

$$\begin{aligned}\rho : \mathcal{V}_{W_1 W_2}^{W_3} &\to \mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3)\\ \mathcal{Y} &\mapsto \rho(\mathcal{Y}).\end{aligned}$$

In the proof of Theorem 5.5 below (in fact only in the surjectivity part), we need a conformal vector of $V$ (that is, $V$ is a vertex operator algebra) so that we do not have to verify separately the $L(0)$- and $L(-1)$-commutator formulas when we construct an intertwining operator.

**Theorem 5.5** *Let $V$ be a vertex operator algebra. Then the linear map $\rho$ is an isomorphism.*

*Proof.* Let $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$. Assume that $\rho(\mathcal{Y}) = 0$. Then for $k, l \in \mathbb{N}$, $\mu \in \Gamma(W_2)$, $w_1 \in W_1$, $w_2 \in (W_2)_{[h_2^\mu + l]}$, we have

$$\begin{aligned}&\sum_{\nu \in \Gamma(W_3)} \mathrm{Res}_x x^{h_2^\mu - h_3^\nu + l - k - 1} \mathcal{Y}^0(x^{L_{W_1}(0)_S} w_1, x) w_2\\ &\quad = \vartheta_{\mathcal{Y}}([w]_{kl}) w_2\\ &\quad = (\rho(\mathcal{Y}))(([w]_{kl} + Q^\infty) \otimes w_2)\\ &\quad = 0.\end{aligned}$$

So for $w_3' \in (W_3')_{[h_3^\nu + k]}$, we have

$$\langle w_3', \mathrm{Res}_x x^{h_2^\mu - h_3^\nu + l - k - 1} \mathcal{Y}^0(x^{L_{W_1}(0)_S} w_1, x) w_2 \rangle = 0. \tag{5.7}$$

On the other hand,

$$\begin{aligned}\mathcal{Y}^0(w, x) &= \mathrm{Coeff}_{\log x}^0 \mathcal{Y}(w, x)\\ &= \mathrm{Coeff}_{\log x}^0 x^{L_{W_3}(0)} \mathcal{Y}(x^{-L_W(0)} w, 1) x^{-L_{W_2}(0)}\\ &= x^{L_{W_3}(0)_S} \mathcal{Y}^0(x^{-L_W(0)_S} w, 1) x^{-L_{W_2}(0)_S}.\end{aligned} \tag{5.8}$$

By (5.8),

$$\begin{aligned}&\langle w_3', \mathrm{Res}_x x^{h_2^\mu - h_3^\nu + l - k - 1} \mathcal{Y}^0(x^{L_{W_1}(0)_S} w_1, x) w_2 \rangle\\ &\quad = \mathrm{Res}_x x^{-1} \langle x^{-L_{W_3'}(0)_S} w_3', \mathcal{Y}^0(x^{L_{W_1}(0)_S} w_1, x) x^{L_{W_2}(0)_S} w_2 \rangle\\ &\quad = \mathrm{Res}_x x^{-1} \langle w_3', \mathcal{Y}^0(w_1, 1) w_2 \rangle\\ &\quad = \langle w_3', \mathcal{Y}^0(w_1, 1) w_2 \rangle.\end{aligned} \tag{5.9}$$

From (5.7) and (5.9), we obtain

$$\langle w_3', \mathcal{Y}^0(w_1, 1)w_2\rangle = 0 \tag{5.10}$$

for $k, l \in \mathbb{N}$, $\mu \in \Gamma(W_2)$, $\nu \in \Gamma(W_3)$, $w_1 \in W_1$, $w_2 \in (W_2)_{[h_2^\mu+l]}$ and $w_3' \in (W_3')_{[h_3^\nu+k]}$. Since $\mu, \nu, k, l$ are also arbitrary, we see that (5.10) holds for $w_1 \in W_1$, $w_2 \in W_2$ and $w_3' \in W_3'$. Hence $\mathcal{Y}^0(w_1, 1) = 0$ for $w_1 \in W_1$. Then by the $L(0)$-conjugation property of intertwining operators, we have

$$\begin{aligned}\mathcal{Y}(w_1, x) &= x^{L_{W_3}(0)}\mathcal{Y}(x^{-L_{W_1}(0)}w_1, 1)x^{-L_{W_2}(0)}\\ &= x^{L_{W_3}(0)}\mathcal{Y}^0(x^{-L_{W_1}(0)}w_1, 1)x^{-L_{W_2}(0)}\\ &= 0\end{aligned}$$

for $w_1 \in W_1$. Since $w_1$ is arbitrary, we obtain $\mathcal{Y} = 0$, proving the injectivity of $\rho$.

We now prove the surjectivity of $\rho$. Let

$$f \in \mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3).$$

We want to construct an intertwining operator $\mathcal{Y}^f$ of type $\binom{W_3}{W_1 W_2}$ such that $\rho(Y^f) = f$. For simplicity, we construct $\mathcal{Y}^f$ and prove the surjectivity only in the case $W_2 = W_2^\mu \neq 0$ and $W_3 = W_3^\nu \neq 0$ for $\mu, \nu \in \mathbb{C}/\mathbb{Z}$. The surjectivity in the general case follows immediately.

We define

$$\mathcal{Y}^f_{h_2^\mu - h_3^\nu + l - k + \mathrm{wt}\, w_1 - 1, 0}(w_1)w_2 = f(([w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)$$

for $k, l \in \mathbb{N}$, homogeneous $w_1 \in W_1$ and $w_2 \in (W_2)_{[h_2^\mu+l]}$. Then by Lemma 5.3,

$$\mathcal{Y}^f_{h_2^\mu - h_3^\nu + l - k + \mathrm{wt}\, w_1 - 1, 0}(w_1)w_2 \in (W_3)_{\llbracket k \rrbracket} = (W_3)_{[h_3^\nu + k]}.$$

We define a map

$$\begin{aligned}(\mathcal{Y}^f)^0 : W_1 \otimes W_2 &\to W_3\{x\}\\ w_1 \otimes w_2 &\mapsto (\mathcal{Y}^f)^0(w_1, x)w_2\end{aligned}$$

by

$$(\mathcal{Y}^f)^0(w_1, x)w_2 = \sum_{k\in\mathbb{N}} \mathcal{Y}^f_{h_2^\mu - h_3^\nu + l - k + \mathrm{wt}\, w_1 - 1, 0}(w_1)x^{-h_2^\mu + h_3^\nu - l + k - \mathrm{wt}\, w_1}$$

for $w_1 \in W_1$, $w_2 \in (W_2)_{[h_2^\mu+l]}$ and $l \in \mathbb{N}$. We then define

$$\begin{aligned}\mathcal{Y}^f(w_1, x)w_2 &= x^{L_{W_3}(0)}(\mathcal{Y}^f)^0(x^{-L_{W_1}(0)}w_1, 1)x^{-L_{W_2}(0)}w_2\\ &= x^{L_{W_3}(0)_N}(\mathcal{Y}^f)^0(x^{-L_{W_1}(0)_N}w_1, x)x^{-L_{W_2}(0)_N}w_2\end{aligned} \tag{5.11}$$

for $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$. Since $L_{W_1}(0)_N$ and $L_{W_2}(0)_N$ are locally nilpotent operators, $x^{-L_{W_1}(0)_N}w_1 \in W_1[\log x]$ and $x^{-L_{W_2}(0)_N}w_2 \in W_2[\log x]$. Then the coefficient of $x$ to a power in

$$(\mathcal{Y}^f)^0(x^{-L_{W_1}(0)_N}w_1, x)x^{-L_{W_2}(0)_N}w_2$$

is an element of $W_3[\log x]$. Since $L_{W_3}(0)_N$ is also locally nilpotent, the coefficient of $x$ to a power in the right-hand side of (5.11) is also in $W_3[\log x]$. This shows that $\mathcal{Y}^f(w_1, x)w_2$ is in fact in $W_3[\log x]\{x\}$. In particular, we obtain a linear map

$$\begin{aligned}\mathcal{Y}^f : W_1 \otimes W_2 &\to W_3[\log x]\{x\}\\ w_1 \otimes w_2 &\mapsto \mathcal{Y}^f(w_1, x)w_2.\end{aligned}$$

We now prove that $\mathcal{Y}^f$ is indeed an intertwining operator of type $\binom{W_3}{W_1 W_2}$. From the definition of $\mathcal{Y}^f$, we see that it is lower truncated. Also by definition, for $w_1 \in W_1$, $w_2 \in (W_2)_{[h_2^\mu + l]}$, $l \in \mathbb{N}$,

$$\begin{aligned}(\mathcal{Y}^f)^0(w_1, 1)w_2 &= \sum_{k\in\mathbb{N}} \mathcal{Y}^f_{h_2^\mu - h_3^\nu + l - k + \mathrm{wt}\, w_1 - 1, 0}(w_1)w_2\\ &= \sum_{k\in\mathbb{N}} f(([w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2).\end{aligned}$$

Then we have

$$\mathcal{Y}^f(w_1, x)w_2 = \sum_{k\in\mathbb{N}} x^{L_{W_3}(0)} f(([x^{-L_{W_1}(0)} w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} x^{-L_{W_1}(0)} w_2)$$

and

$$\begin{aligned}&f(([w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)\\ &\quad = \mathrm{Res}_x x^{h_2^\mu - h_3^\nu + l - k - 1} (\mathcal{Y}^f)^0(x^{L_{W_1}(0)_S} w_1, x)w_2\\ &\quad = \mathrm{Coeff}^0_{\log x} \mathrm{Res}_x x^{h_2^\mu - h_3^\nu + l - k - 1} \mathcal{Y}^f(x^{L_{W_1}(0)} w_1, x)w_2. \end{aligned} \tag{5.12}$$

From Proposition 4.4, we know that (4.23) with $w$ replaced by $w_1$ is an element of $Q^\infty(W_1)$. In particular,

$$\begin{aligned}&\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}} (-1)^j \binom{p}{j} f(([v]_{k,n+p-j} \diamond [w_1]_{n+p-j,l+p} + Q^\infty(W_1)) \otimes_{A^\infty(W_1)} w_2)\\ &\quad - \sum_{\substack{i\in\mathbb{N}\\ l-n+k+p-j\geq 0}} (-1)^{p-j} \binom{p}{j}\cdot\\ &\qquad \cdot f(([w_1]_{k,l-n+k+p-j} \diamond [v]_{l-n+k+p-j,l+p} + Q^\infty(W_1)) \otimes_{A^\infty(W_1)} w_2)\\ &\quad - \sum_{j\in\mathbb{N}} \binom{\mathrm{wt}\, v + n - k - 1}{j} f(([(Y_{W_1})_{p+j}(v)w_1]_{k,l+p} + Q^\infty(W_1)) \otimes_{A^\infty(W_1)} w_2)\\ &= 0\end{aligned} \tag{5.13}$$

for $k, l, n \in \mathbb{N}$, $p \in \mathbb{Z}$ such that $l + p \in \mathbb{N}$, $v \in V$, $w_1 \in W_1$ and $w_2 \in (W_2)_{[h_2^\mu + l + p]}$. Since

$$f \in \mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3),$$

the left-hand side of (5.13) is equal to

$$
\begin{aligned}
&\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}}(-1)^{j}\binom{p}{j}\vartheta_{W_3}([v]_{k,n+p-j})f(([w_1]_{n+p-j,l+p}+Q^{\infty}(W_1))\otimes_{A^{\infty}(W_1)}w_2)\\
&\quad-\sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}}(-1)^{p-j}\binom{p}{j}\cdot\\
&\qquad\cdot f(([w_1]_{k,l-n+k+p-j}+Q^{\infty}(W_1))\otimes_{A^{\infty}(W_1)}\vartheta_{W_2}([v]_{l-n+k+p-j,l+p})w_2)\\
&\quad-\sum_{j\in\mathbb{N}}\binom{\mathrm{wt}\,v+n-k-1}{j}f(([(Y_{W_1})_{p+j}(v)w_1]_{k,l+p}+Q^{\infty}(W_1))\otimes_{A^{\infty}(W_1)}w_2). \qquad (5.14)
\end{aligned}
$$

From the definitions of $\vartheta_{W_3}$, $\vartheta_{W_2}$ and (5.12), we see that (5.14) is equal to

$$
\begin{aligned}
&\sum_{\substack{j\in\mathbb{N}\\ n+p-j\geq 0}}(-1)^{j}\binom{p}{j}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n+p-j-k-1}\cdot\\
&\qquad\cdot x_2^{h_2^{\mu}-h_3^{\nu}+l-n+j-1}Y_{W_3}(x_1^{L_V(0)}v,x_1)(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}w_1,x_2)w_2\\
&\quad-\sum_{\substack{j\in\mathbb{N}\\ l-n+k+p-j\geq 0}}(-1)^{p-j}\binom{p}{j}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k+j-1}\cdot\\
&\qquad\cdot x_2^{h_2^{\mu}-h_3^{\nu}+l-n+p-j-1}(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}w_1,x_2)Y_{W_2}(x_1^{L_V(0)}v,x_1)w_2\\
&\quad-\sum_{j\in\mathbb{N}}\binom{\mathrm{wt}\,v+n-k-1}{j}\mathrm{Res}_{x_2}x_2^{h_2^{\mu}-h_3^{\nu}+l+p-k-1}(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}(Y_{W_1})_{p+j}(v)w_1,x_2)w_2\\
&=\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{h_2^{\mu}-h_3^{\nu}+l-n-1}(x_1-x_2)^pY_{W_3}(x_1^{L_V(0)}v,x_1)(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}w_1,x_2)w_2\\
&\quad-\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{h_2^{\mu}-h_3^{\nu}+l-n-1}(-x_2+x_1)^p(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}w_1,x_2)Y_{W_2}(x_1^{L_V(0)}v,x_1)w_2\\
&\quad-\mathrm{Res}_{x_2}\mathrm{Res}_{y}y^p(1+y)^{\mathrm{wt}\,v+n-k-1}x_2^{h_2^{\mu}-h_3^{\nu}+l+p-k-1}(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}Y_{W_1}(v,y)w_1,x_2)w_2. \quad (5.15)
\end{aligned}
$$

Since the left-hand side of (5.15) is equal to the left-hand side of (5.13), we see that the right-hand side of (5.15) is 0 for $k,l,n\in\mathbb{N}$, $p\in\mathbb{Z}$ such that $l+p\in\mathbb{N}$, $v\in V$, $w_1\in W_1$ and $w_2\in(W_2)_{[h_2^{\mu}+l+p]}$. The right-hand side of (5.15) is also 0 when $l+p<0$ since in this case $w_2=0$. Multiplying the right-hand side of (5.15) by $x_0^{-p-1}$ and then take sum over $p\in\mathbb{Z}$, we obtain

$$
\begin{aligned}
&\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{h_2^{\mu}-h_3^{\nu}+l-n-1}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{W_3}(x_1^{L_V(0)}v,x_1)(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}w_1,x_2)w_2\\
&\quad-\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{h_2^{\mu}-h_3^{\nu}+l-n-1}\cdot\\
&\qquad\cdot x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}w_1,x_2)Y_{W_2}(x_1^{L_V(0)}v,x_1)w_2
\end{aligned}
$$

$$
\begin{aligned}
&- \mathrm{Res}_{x_2}\mathrm{Res}_y x_0^{-1}\delta\left(\frac{yx_2}{x_0}\right)(1+y)^{n-k-1}\cdot\\
&\qquad\qquad\cdot x_2^{h_2^\mu-h_3^\nu+l-k-1}(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}Y_W((1+y)^{L_V(0)}v,y)w_1,x_2)w_2\\
&= 0. \qquad (5.16)
\end{aligned}
$$

The third term in the left-hand side of (5.16) is equal to

$$
\begin{aligned}
&- \mathrm{Res}_{x_2}\mathrm{Res}_y x_0^{-1}\delta\left(\frac{yx_2}{x_0}\right)(1+y)^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-k-1}\cdot\\
&\qquad\qquad\cdot(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}Y_W((1+y)^{L_V(0)}v,y)w_1,x_2)w_2\\
&= -\mathrm{Res}_{x_2}\mathrm{Res}_y x_0^{-1}\delta\left(\frac{yx_2}{x_0}\right)(1+x_0x_2^{-1})^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-k-1}\cdot\\
&\qquad\qquad\cdot(\mathcal{Y}^f)^0(x_2^{L_{W_1}(0)_S}Y_W((1+x_0x_2^{-1})^{L_V(0)}v,x_0x_2^{-1})w_1,x_2)w_2\\
&= -\mathrm{Res}_{x_2}(x_2+x_0)^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1}\cdot\\
&\qquad\qquad\cdot(\mathcal{Y}^f)^0(Y_W((x_2+x_0)^{L_V(0)}v,x_0)x_2^{L_{W_1}(0)_S}w_1,x_2)w_2\\
&= -\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\cdot\\
&\qquad\qquad\cdot(\mathcal{Y}^f)^0(Y_W(x_1^{L_V(0)}v,x_0)x_2^{L_{W_1}(0)_S}w_1,x_2)w_2. \qquad (5.17)
\end{aligned}
$$

Using (5.17) and substituting $x_1^{-L_V(0)}v$ and $x_2^{-L_{W_1}(0)_S}w_1$ for $v$ and $w_1$, respectively, we see that (5.16) becomes

$$
\begin{aligned}
&\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{W_3}(v,x_1)(\mathcal{Y}^f)^0(w_1,x_2)w_2\\
&\quad- \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)(\mathcal{Y}^f)^0(w_1,x_2)Y_{W_2}(v,x_1)w_2\\
&\quad- \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{n-k-1}x_2^{h_2^\mu-h_3^\nu+l-n-1}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(\mathcal{Y}^f)^0(Y_W(v,x_0)w_1,x_2)w_2\\
&= 0. \qquad (5.18)
\end{aligned}
$$

Since $k,l,n\in\mathbb{N}$ are arbitrary, (5.18) gives

$$
\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{W_3}(v,x_1)(\mathcal{Y}^f)^0(w_1,x_2)w_2 - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)(\mathcal{Y}^f)^0(w_1,x_2)Y_{W_2}(v,x_1)w_2\\
&\quad= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(\mathcal{Y}^f)^0(Y_W(v,x_0)w_1,x_2)w_2. \qquad (5.19)
\end{aligned}
$$

This is the Jacobi identity for $(\mathcal{Y}^f)^0$. To obtain the Jacobi identity for $\mathcal{Y}^f$, we replace $x_0$, $x_1$, $x_2$, $v$, $w_1$ and $w_2$ by $x_0x_2^{-1}$, $x_1x_2^{-1}$, 1, $x_2^{-L_V(0)}v$, $x_2^{-L_{W_1}(0)}w_1$ and $x^{-L_{W_2}(0)}w_2$, respectively in (5.19) and then multiply $x^{L_{W_3}(0)}$ from the left of both sides of (5.19). Then we obtain

$$\begin{aligned}
&x_0^{-1}x_2\delta\left(\frac{x_1x_2^{-1}-1}{x_0x_2^{-1}}\right)x_2^{L_{W_1}(0)}Y_{W_3}(x_2^{-L_V(0)}v,x_1x_2^{-1})(\mathcal{Y}^f)^0(x_2^{-L_{W_1}(0)}w_1,1)x_2^{-L_{W_1}(0)}w_2\\
&\quad-x_0^{-1}x_2\delta\left(\frac{1-x_1x_2^{-1}}{-x_0x_2^{-1}}\right)x_2^{L_{W_1}(0)}(\mathcal{Y}^f)^0(x_2^{-L_{W_1}(0)}w_1,1)Y_{W_2}(x_2^{-L_V(0)}v,x_1x_2^{-1})x_2^{-L_{W_1}(0)}w_2\\
&\quad=x_1^{-1}x_2\delta\left(\frac{1+x_0x_2^{-1}}{x_1x_2^{-1}}\right)x_2^{L_{W_1}(0)}(\mathcal{Y}^f)^0(Y_W(x_2^{-L_V(0)}v,x_0x_2^{-1})x_2^{-L_{W_1}(0)}w_1,1)w_2.
\end{aligned}\tag{5.20}$$

Using the $L(0)$-conjugation formulas for vertex operators and the definition

$$\mathcal{Y}^f(w_1,x_2)=x_2^{L_{W_1}(0)}(\mathcal{Y}^f)^0(x_2^{-L_{W_1}(0)}w_1,1)x_2^{-L_{W_1}(0)}$$

of $\mathcal{Y}^f$, we see that (5.20) is equivalent to

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{W_3}(v,x_1)\mathcal{Y}^f(w_1,x_2)w_2-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}^f(w_1,x_2)Y_{W_2}(v,x_1)w_2\\
&\quad=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\mathcal{Y}^f(Y_W(v,x_0)w_1,x_2)w_2,
\end{aligned}$$

the Jacobi identity for $\mathcal{Y}$.

We also need to show that $\mathcal{Y}^f$ satisfies the $L(-1)$-derivative property. This follows from (5.11) and the $L(0)$-commutator formula for $(\mathcal{Y}^f)^0$ which is a special case of the Jacobi identity for $(\mathcal{Y}^f)^0$. In fact, applying $\frac{d}{dx}$ to both sides of (5.11), we obtain

$$\begin{aligned}
\frac{d}{dx}\mathcal{Y}^f(w_1,x)w_2&=x^{-1}x^{L_{W_3}(0)}L_{W_3}(0)(\mathcal{Y}^f)^0(x^{-L_{W_1}(0)}w_1,1)x^{-L_{W_2}(0)}w_2\\
&\quad-x^{-1}x^{L_{W_3}(0)}(\mathcal{Y}^f)^0(L_{W_1}(0)x^{-L_{W_1}(0)}w_1,1)x^{-L_{W_2}(0)}w_2\\
&\quad-x^{-1}x^{L_{W_3}(0)}(\mathcal{Y}^f)^0(x^{-L_{W_1}(0)}w_1,1)L_{W_2}(0)x^{-L_{W_2}(0)}w_2
\end{aligned}\tag{5.21}$$

for $w_1\in W_1$ and $w_2\in W_2$. Using the $L(0)$-commutator formula

$$\begin{aligned}
&L_{W_3}(0)(\mathcal{Y}^f)^0(x^{-L_{W_1}(0)}w_1,1)-(\mathcal{Y}^f)^0(x^{-L_{W_1}(0)}w_1,1)L_{W_2}(0)\\
&\quad=(\mathcal{Y}^f)^0((L_{W_1}(-1)+L_{W_1}(0))x^{-L_{W_1}(0)}w_1,1)
\end{aligned}$$

and (5.11) with $w_1$ replaced by $L_{W_1}(-1)w_1$, the right-hand side of (5.21) is equal to

$$\begin{aligned}
&x^{-1}x^{L_{W_3}(0)}(\mathcal{Y}^f)^0(L_{W_1}(-1)x^{-L_{W_1}(0)}w_1,1)x^{-L_{W_2}(0)}w_2\\
&\quad=x^{L_{W_3}(0)}(\mathcal{Y}^f)^0(x^{-L_{W_1}(0)}L_{W_1}(-1)w_1,1)x^{-L_{W_2}(0)}w_2\\
&\quad=\mathcal{Y}^f(L_{W_1}(-1)w_1,x)w_2.
\end{aligned}\tag{5.22}$$

From (5.21) and (5.22) , we obtain the $L(-1)$-derivative property for $\mathcal{Y}^f$.

Since $V$ is a vertex operator algebra and $\mathcal{Y}^f$ satisfies the lower-truncation property, the Jacobi identity and the $L(-1)$-derivative property, $\mathcal{Y}^f$ is an intertwining operator of type

$\binom{W_3}{W_1 W_2}$ (see Definition 3.10 in [HLZ] for the definition of intertwining operator in the case that $V$ is a vertex operator algebra).

Finally we prove $\rho(Y^f) = f$. By definition,

$$\begin{aligned}
(\rho(\mathcal{Y}^f))(([w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)& \\
&= \vartheta_{\mathcal{Y}^f}([w_1]_{kl}) w_2 \\
&= \mathrm{Res}_x x^{h_2^\mu - h_3^\nu + l - k - 1} (\mathcal{Y}^f)^0 (x^{L_{W_1}(0)} w_1, x) w_2 \\
&= f(([w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)
\end{aligned} \tag{5.23}$$

for $k, l \in \mathbb{N}$, $w_1 \in W_1$ and $w_2 \in (W_2)_{[h_2^\mu + l]}$. From Lemma 5.2 and (5.23), we obtain $\rho(Y^f) = f$. This finishes the proof that $\rho$ is surjective. ∎

# 6 $A^N(V)$-bimodules and intertwining operators

In this section, we formulate and prove the second main theorem of the present paper. For $N \in \mathbb{N}$, we use the results obtained in the preceding two sections to give an $A^N(V)$-bimodule $A^N(W)$ for a lower-bounded generalized $V$-module $W$. For lower bounded generalized $V$-modules $W_1$, $W_2$ and $W_3$, we obtain a linear map $\rho^N : \mathcal{V}_{W_1 W_2}^{W_3} \to \mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3))$ induced from the linear isomorphism $\rho$ in the preceding section (see below for the definition of $\Omega_n^0(W)$ for a lower-bounded generalized $V$-module). We prove that $\rho^N$ is surjective. Our second main theorem states that $\rho^N$ is an isomorphism when $W_2$ and $W_3'$ are certain universal lower-bounded generalized $V$-modules generated by $\Omega_N^0(W_2)$ and $\Omega_N^0(W_3')$, respectively.

Let $V$ be a grading-restricted vertex algebra and $W$ a lower-bounded generalized $V$-module. For $n \in \mathbb{N}$, let

$$\Omega_n^0(W) = \coprod_{m=0}^{n} W_{[\![ m ]\!]} = \coprod_{m=0}^{n} \coprod_{\mu \in \Gamma(W)} W_{[h^\mu + m]}.$$

For homogeneous $v \in V$, $k, l, n \in \mathbb{N}$, $\mu \in \Gamma(W)$ and $w \in W_{[h^\mu + l]}$, we have

$$\vartheta_W([v]_{kn}) w = \delta_{nl} \mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)} v, x) w \in W_{[h^\mu + k]} \subset W_{[\![ k ]\!]}.$$

We fix $N \in \mathbb{N}$ in the rest of this section. Recall the associative algebra $A^N(V)$ (see Section 2 and Subsection 4.2 in [H5]). Then the discussion above in particular shows that when restricted to $A^N(V)$, $\vartheta_W$ gives $\Omega_N^0(W)$ a graded $A^N(V)$-module structure. But in general, $\Omega_N^0(W)$ is not nondegenerate as a graded $A^N(V)$-module (see Definition 2.4 for the definition of nondegenerate graded $A^N(V)$-module).

Let $U^N(W)$ be the space of all $(N+1) \times (N+1)$ matrices with entries in $W$. Then $U^N(W)$ can be viewed as subspaces of $U^\infty(W)$. In this paper, we shall always view $U^N(W)$ as subspaces of $U^\infty(W)$ and we shall use the notations for elements of $U^\infty(W)$ to denote elements of $U^N(W)$. Using these notations, we see that $U^N(W)$ are spanned by elements of the forms $[w]_{kl}$ for $w \in W$ and $k, l = 0, \dots, N$.

Let $A^N(W)$ be the subspace of $A^\infty(W)$ consisting of elements of the form $\mathfrak{w} + Q^\infty(W)$ for $\mathfrak{w} \in U^N(W)$. Since $A^N(V)$ is a subalgebra of $A^\infty(V)$, $A^\infty(W)$ as an $A^\infty(V)$-bimodule is also an $A^N(V)$-bimodule. For $v \in V$, $w \in W$ and $k, m, n, l = 0, \dots, N$, by definition, $[v]_{km} \diamond [w]_{nl}$ and $[w]_{km} \diamond [v]_{nl}$ are still in $U^N(W)$. Thus $A^N(W)$ is in fact an $A^N(V)$-subbimodule of $A^\infty(W)$.

Let $W_1$, $W_2$ and $W_3$ be lower-bounded generalized $V$-modules and $\mathcal{Y}$ an intertwining operator of type $\binom{W_3}{W_1 W_2}$. Let

$$\eta^N : A^N(W_1) \otimes_{A^N(V)} \Omega_N^0(W_2) \to A^\infty(W_1) \otimes_{A^\infty(V)} W_2$$

be the $A^N(V)$-module map defined by

$$\eta^N((\mathfrak{w}_1 + Q^\infty(V)) \otimes_{A^N(V)} w_2) = (\mathfrak{w}_1 + Q^\infty(V)) \otimes_{A^\infty(V)} w_2$$

for $\mathfrak{w}_1 \in U^N(W_1)$ and $w_2 \in \Omega_N^0(W_2)$. Given

$$f \in \mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3),$$

we have a map $f^N = f \circ \eta^N$. By Lemma 5.3, the image of $A^N(W_1) \otimes_{A^N(V)} \Omega_N^0(W_2)$ under $f^N$ is in fact in $\Omega_N^0(W_3)$. So $f^N$ is an element of

$$\mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3)).$$

Hence $f \mapsto f^N$ gives a linear map from

$$\mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3)$$

to

$$\mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3)).$$

In particular, the image of

$$\rho(\mathcal{Y}) \in \mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3)$$

under this map is an element

$$\rho^N(\mathcal{Y}) = \rho(\mathcal{Y}) \circ \eta^N \in \mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3))$$

More explicitly, $(\rho^N(\mathcal{Y}))$ is given by

$$\begin{aligned}
&(\rho^N(\mathcal{Y}))([w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^N(V)} w_2) \\
&\quad = (\rho(\mathcal{Y}))([w_1]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2) \\
&\quad = \vartheta_{\mathcal{Y}}([w_1]_{kl}) w_2
\end{aligned}$$

$$= \sum_{\nu\in\Gamma(W_3)} \mathrm{Res}_x x^{h_2^\mu - h_3^\nu + l - k - 1}\mathcal{Y}^0(x^{L_{W_1}(0)}w_1, x)w_2$$

for $k, l = 0, \dots, N$, $w_1 \in W_1$, $w_2 \in (W_2)_{[h_2^\mu + l]}$ and $\mu \in \Gamma(W_2)$.

We now have a linear map

$$\begin{aligned}\rho^N : \mathcal{V}_{W_1W_2}^{W_3} &\to \mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3))\\ \mathcal{Y} &\mapsto \rho^N(\mathcal{Y}).\end{aligned}$$

**Proposition 6.1** *Assume that $W_2$ and $W_3'$ are generated by $\Omega_N^0(W_2)$ and $\Omega_N^0(W_3')$. Then the linear map $\rho^N$ is injective.*

*Proof.* Assume that $\rho^N(\mathcal{Y}) = 0$. Then for $\mathfrak{w}_1 \in U^N(W_1)$, $w_2 \in \Omega_N^0(W_2)$,

$$(\rho(\mathcal{Y}))(\mathfrak{w}_1 + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2) = 0. \tag{6.1}$$

Since $W_2$ is generated by $\Omega_N^0(W_2)$, $W_2$ is spanned by elements of the form $(Y_{W_2})_{\mathrm{wt}\, v+l-k-1}(v)w_2$ for $k \in \mathbb{N}$, $0 \le l \le N$, homogeneous $v \in V$ and $w_2 \in (W_2)_{\llbracket l\rrbracket}$. For $k \in \mathbb{N}$, $0 \le l \le N$, $\mathfrak{w}_1 \in U^N(W_1)$ homogeneous $v \in V$ and $w_2 \in (W_2)_{\llbracket l\rrbracket} \subset \Omega_N^0(W_2)$, we have

$$\begin{aligned}&(\rho(\mathcal{Y}))(\mathfrak{w}_1 + Q^\infty(W_1)) \otimes_{A^\infty(V)} (Y_{W_2})_{\mathrm{wt}\, v+l-k-1}(v)w_2)\\ &\quad = (\rho(\mathcal{Y}))((\mathfrak{w}_1 + Q^\infty(W_1 : X)) \otimes_{A^\infty(V)} \vartheta_{W_2}([v]_{kl})w_2)\\ &\quad = (\rho(\mathcal{Y}))((\mathfrak{w}_1 \diamond [v]_{kl} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)\\ &\quad = 0.\end{aligned}$$

So (6.1) holds for $\mathfrak{w}_1 \in U^N(W_1)$ and $w_2 \in W_2$.

Since $W_3'$ is generated by $\Omega_N^0(W_3')$, every element of $W_3'$ is a linear combination of elements of the form $(Y_{W_3})_{\mathrm{wt}\, v+k-n-1}(v)'w_3'$ for $n \in \mathbb{N}$, $0 \le k \le N$, homogeneous $v \in V$, $w_3' \in (W_3')_{\llbracket n\rrbracket}$, where $(Y_{W_3})_{\mathrm{wt}\, v+k-n-1}(v)'$ is the adjoint of $(Y_{W_3})_{\mathrm{wt}\, v+k-n-1}(v)$. Then for $n \in \mathbb{N}$, $0 \le k, l \le N$, $\mathfrak{w}_1 \in U^\infty(W_1)$ homogeneous $v \in V$, $w_2 \in W_2$ and $w_3' \in (W_3')_{\llbracket k\rrbracket}$, we have

$$\begin{aligned}&\langle (Y_{W_3})_{\mathrm{wt}\, v+k-n-1}(v)'w_3', (\rho(\mathcal{Y}))(\mathfrak{w}_1 + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)\rangle\\ &\quad = \langle w_3', (Y_{W_3})_{\mathrm{wt}\, v+k-n-1}(v)(\rho(\mathcal{Y}))(\mathfrak{w}_1 + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)\rangle\\ &\quad = \langle w_3', \vartheta_{W_3}([v]_{nk})(\rho(\mathcal{Y}))(\mathfrak{w}_1 + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)\rangle\\ &\quad = \langle w_3', (\rho(\mathcal{Y}))([v]_{nk} \diamond \mathfrak{w}_1 + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2)\rangle\\ &\quad = 0.\end{aligned}$$

Then we see that (6.1) holds for $\mathfrak{w}_1 \in U^\infty(W_1)$ and $w_2 \in W_2$. Thus we obtain $\rho(\mathcal{Y}) = 0$. By Theorem 5.5, $\rho$ is injective. So $\mathcal{Y} = 0$, proving the injectivity of $\rho^N$. ∎

In general, $\rho^N$ is not surjective. But we shall prove that in the case that $W_2$ and $W_3'$ are equivalent to certain universal lower-bounded generalized $V$-modules, it is also surjective. Such a universal lower-bounded generalized $V$-module $S^N(M)$ has been constructed from an $A^N(V)$-module $M$ in Section 5 of [H5] using the construction in Section 5 of [H2] for a grading-restricted vertex algebra.

Since in our result below on the surjectivity, we assume for simplicity that $V$ is a vertex operator algebra, we shall instead use the modified construction for vertex operator algebra in subsection 4.2 of [H4].

In the remaining part of this section, $V$ is a vertex operator algebra. Let $M = \coprod_{n=0}^{N} G_n(M)$ be a graded $A^N(V)$-module given by a linear map $\vartheta_M : A^N(V) \to \mathrm{End}\, M$ and operators $L_M(0)$ and $L_M(-1)$ (see Definition 2.4). Take $g = 1_V$ and $B \in \mathbb{R}$ a lower bound of the real parts of the eigenvalues of $L_M(0)$. From Subsection 4.2 of [H4], we have a lower-bounded generalized $V$-module $\widehat{M}_B^{1_V}$ satisfying the universal property given by Theorem 4.7 in [H4]. For simplicity, we shall denote it simply by $\widehat{M}$.

Using the construction in Subsection 4.2 of [H4] and the results in [H2] and [H3], we see that $\widehat{M}$ is generated by $M$. In particular, we identify $M$ as a subspace of $\widehat{M}$. Let $J_M$ be the generalized $V$-submodule of $\widehat{M}$ generated by elements of the forms

$$\mathrm{Res}_x x^{l-k-1} Y_{\widehat{M}}(x^{L_V(0)}v, x)w \tag{6.2}$$

for $l = 0, \dots, N$, $k \in -\mathbb{Z}_+$ and $w \in G_l(M)$,

$$\mathrm{Res}_x x^{l-k-1} Y_{\widehat{M}}(x^{L_V(0)}v, x)w - \vartheta_M([v]_{kl})w \tag{6.3}$$

for $v \in V$, $k, l = 0, \dots, N$ and $w \in G_l(M)$.

Let $S^N_{\mathrm{voa}}(M) = \widehat{M}/J_M$. Then $S^N_{\mathrm{voa}}(M)$ is a lower-bounded generalized $V$-module. From (6.2) and (6.3), we see that $J_M \cap M = 0$. Then we can also identify $M$ as a subspace of $S^N_{\mathrm{voa}}(M)$. Since $\widehat{M}$ is generated by $M$, $S^N_{\mathrm{voa}}(M)$ is also generated by $M$ and hence is spanned by elements of the form

$$\mathrm{Res}_x x^{l-n-1} Y_{S^N_{\mathrm{voa}}(M)}(x^{L_V(0)}v, x)w \tag{6.4}$$

for $v \in V$, $l = 0, \dots, N$, $n \in \mathbb{N}$ and $w \in G_l(M)$. In Section 5 of [H5], the nondegenerate graded $A^\infty(V)$-module $Gr(S^N(M))$ is studied. What we are interested here is the graded $A^\infty(V)$-module structure on $S^N_{\mathrm{voa}}(M)$ and the graded $A^N(V)$-module structure on $\Omega^0_N(S^N_{\mathrm{voa}}(M))$.

**Proposition 6.2** *For a graded $A^N(V)$-module $M$, the graded $A^N(V)$-module $\Omega^0_N(S^N_{\mathrm{voa}}(M))$ is equal to $M$.*

*Proof.* We prove only the case that the eigenvalues of $L_M(0)$ are all congruent to each other modulo $\mathbb{Z}$. The general case can be obtained by taking direct sums. In this case, there exists $\mu \in \mathbb{C}/\mathbb{Z}$ and $h^\mu \in \mathbb{C}$ such that $M = \coprod_{n=0}^{N} M_{[h^\mu + l]}$, where for $l = 0, \dots, N$, $M_{[h^\mu+l]}$ is the generalized eigenspace of $L_M(0)$ with eigenvalue $h^\mu + l$.

Since $S^N_{\mathrm{voa}}(M)$ is spanned by elements of the form (6.4), $(S^N_{\mathrm{voa}}(M))_{\llbracket n \rrbracket}$ for $n \in \mathbb{N}$ is spanned by elements of the form (6.4) for $v \in V$, $l = 0, \dots, N$ and $w \in G_l(M)$. But from the definition of $J_M$, when $0 \le n \le N$, an element of this form is equal to $\vartheta_M([v]_{nl})w \in M_{[h^\mu+n]}$.

So $(S^N_{\mathrm{voa}}(M))_{\llbracket n\rrbracket} \subset M$ for $n = 0, \ldots, N$. In particular, we have $\Omega^0_N(S^N_{\mathrm{voa}}(M)) \subset M$. But by the construction, $M \subset \Omega^0_N(S^N_{\mathrm{voa}}(M))$. Thus we obtain $\Omega^0_N(S^N_{\mathrm{voa}}(M)) = M$. ∎

From the construction of $S^N_{\mathrm{voa}}(M)$, we have the following universal property:

**Proposition 6.3** *Let $W$ be a lower-bounded generalized $V$-module and $f : M \to \Omega^0_N(W)$ be a graded $A^N(V)$-module map. Then there exists a unique $V$-module map $f^\vee : S^N_{\mathrm{voa}}(M) \to W$ such that $f^\vee|_M = f$.*

*Proof.* By Theorem 4.7 in [H4], there is a unique $V$-module map $\widehat{f} : \widehat{M} \to W$ such that $\widehat{f}|_M = f$. Since $\widehat{f}$ is a $V$-module map, the image of $f$ is in $\Omega^0_N(W)$ and $f$ is a graded $A^N(V)$-module map,

$$\widehat{f}(\mathrm{Res}_x x^{l-n-1} Y_{\widehat{M}}(x^{L_V(0)}v, x)w) = \mathrm{Res}_x x^{l-n-1} Y_W(x^{L_V(0)}v, x) f(w) = 0$$

for $l = 0, \ldots, N$, $k \in -\mathbb{Z}_+$ and $w \in G_l(M)$ and

$$\begin{aligned}
&\widehat{f}(\mathrm{Res}_x x^{l-k-1} Y_{\widehat{M}}(x^{L_V(0)}v, x)w - \vartheta_M([v]_{kl})w)\\
&\quad = \mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x) f(w) - \vartheta_W([v]_{kl}) f(w)\\
&\quad = 0
\end{aligned}$$

for $v \in V$, $k, l = 0, \ldots, N$ and $w \in G_l(M)$. So we obtain $J_M \subset \ker \widehat{f}$. In particular, $\widehat{f}$ induces a $V$-module map $f^\vee : S^N_{\mathrm{voa}}(M) \to W$ such that $f^\vee|_M = f$. The uniqueness of $f^\vee$ follows from the uniqueness of $\widehat{f}$. ∎

Using Propositions 5.1 and 6.3, we have the following consequence:

**Corollary 6.4** *Let $W$ be an $A^\infty(V)$-module. Let $f : M \to W$ be an $A^N(V)$-module map with $W$ viewed as an $A^N(V)$-module. Then there exists an $A^\infty(V)$-module map $f^\vee : S^N_{\mathrm{voa}}(M) \to W$ such that $f^\vee|_M = f$.*

*Proof.* Since $V$ is a vertex operator algebra, we have a conformal vector $\omega$ such that $L_{S^N_{\mathrm{voa}}(M)}(0)$ and $L_{S^N_{\mathrm{voa}}(M)}(-1)$ are given by the actions of $\omega^\infty(0)$ and $\omega^\infty(-1)$ on $S^N_{\mathrm{voa}}(M)$ (see Remark 4.5). The actions of $\omega^\infty(0)$ and $\omega^\infty(-1)$ on $W$ also give operators $L_W(0)$ and $L_W(-1)$ on $W$. In particular, $L_W(0)$ and $L_W(-1)$ acts on $M$ and on $\coprod_{n=0}^{N-1} G_n(M)$, respectively (recall from [H5] that $G_n(M)$ is the homogeneous subspace of $M$ of level $n$). Since $f$ is an $A^N(V)$-module map, it in particular commutes with the actions of $\omega^\infty(0)$ and $\omega^\infty(-1)$. Then $f(M)$ is also a graded $A^\infty(V)$-module. Let $W_0$ be the $A^\infty(V)$-submodule of $W$ generated by $f(M)$. Then $W_0$ is a graded $A^\infty(V)$-module containing $f(M)$.

We now have a graded $A^\infty(V)$-module $W_0$ and an $A^N(V)$-module map from $M$ to $f(M) \subset W_0$. By Propositions 5.1 and 6.3, there exists a unique $A^\infty(V)$-module map $f^\vee : S^N_{\mathrm{voa}}(M) \to W_0$ such that $f^\vee|_M = f$. Since $W_0 \subset W$, we can also view $f^\vee$ as an $A^\infty(V)$-module map from $S^N_{\mathrm{voa}}(M)$ to $W$. ∎

We also need a right $A^{\infty}(V)$-module structure on $S^{N}_{\mathrm{voa}}(M)$ and a universal property of this right $A^{\infty}(V)$-module structure. These can be obtained using the following result and Corollary 6.4:

**Proposition 6.5** *The linear map $O: U^{\infty}(V) \to U^{\infty}(V)$ given by*

$$O([v]_{kl}) = [-e^{L_V(1)}(-1)^{-L_V(0)}v]_{kl}$$

*for $k, l \in \mathbb{N}$ and $v \in V$ induces an isomorphism from $A^{\infty}(V)$ to its opposite algebra.*

*Proof.* We need only prove

$$O(\mathfrak{u} \diamond \mathfrak{v}) - O(\mathfrak{v}) \diamond O(\mathfrak{u}) \in Q^{\infty}(V)$$

for $\mathfrak{u}, \mathfrak{v} \in U^{\infty}(V)$.

Let $W$ be a lower-bounded generalized $V$-module. Then

$$\begin{aligned}
&\langle \vartheta_{W'}([v]_{kl})w', w\rangle \\
&\quad = \mathrm{Res}_x x^{l-k-1}\langle Y_{W'}(x^{L_V(0)}v, x)w', w\rangle \\
&\quad = \mathrm{Res}_x x^{l-k-1}\langle w', Y_W(e^{xL_V(1)}(-x^{-2})^{L_V(0)}x^{L_V(0)}v, x^{-1})w\rangle \\
&\quad = \mathrm{Res}_x x^{l-k-1}\langle w', Y_W(e^{xL_V(1)}x^{-L_V(0)}(-1)^{-L_V(0)}v, x^{-1})w\rangle \\
&\quad = \mathrm{Res}_x x^{l-k-1}\langle w', Y_W(x^{-L_V(0)}e^{L_V(1)}(-1)^{-L_V(0)}v, x^{-1})w\rangle \\
&\quad = -\mathrm{Res}_y y^{k-l-1}\langle w, Y_W(y^{L_V(0)}e^{L_V(1)}(-1)^{-L_V(0)}v, y)w\rangle \\
&\quad = \langle w', \vartheta_W([-e^{L_V(1)}(-1)^{-L_V(0)}v]_{lk})w\rangle \\
&\quad = \langle w', \vartheta_W(O([v]_{kl}))w\rangle
\end{aligned} \tag{6.5}$$

for $k, l \in \mathbb{N}$, $v \in V$, $w \in W$ and $w' \in W'$. From (6.5), we obtain

$$\langle w', \vartheta_W(O(\mathfrak{v}))w\rangle = \langle \vartheta_{W'}(\mathfrak{v})w', w\rangle \tag{6.6}$$

for $\mathfrak{v} \in U^{\infty}(V)$, $w \in W$ and $w' \in W'$.

Using (6.6), we have

$$\begin{aligned}
&\langle w', \vartheta_W(O(\mathfrak{u} \diamond \mathfrak{v}))w\rangle \\
&\quad = \langle \vartheta_{W'}(\mathfrak{u} \diamond \mathfrak{v})w', w\rangle \\
&\quad = \langle \vartheta_{W'}(\mathfrak{u})\vartheta_{W'}(\mathfrak{v})w', w\rangle \\
&\quad = \langle \vartheta_{W'}(\mathfrak{v})w', \vartheta_W(O(\mathfrak{u}))w\rangle \\
&\quad = \langle w', \vartheta_W(O(\mathfrak{v}))\vartheta_W(O(\mathfrak{u}))w\rangle \\
&\quad = \langle w', \vartheta_W(O(\mathfrak{v}) \diamond O(\mathfrak{u}))w\rangle
\end{aligned} \tag{6.7}$$

for $\mathfrak{u}, \mathfrak{v} \in U^{\infty}(V)$, $w \in W$ and $w' \in W'$. Since $w'$ and $w$ are arbitrary, we obtain from (6.7)

$$O(\mathfrak{u} \diamond \mathfrak{v}) - O(\mathfrak{v}) \diamond O(\mathfrak{u}) \in \ker \vartheta_W$$

for $\mathfrak{u}, \mathfrak{v} \in U^{\infty}(V)$. Since $W$ is arbitrary, we obtain

$$O(\mathfrak{u} \diamond \mathfrak{v}) - O(\mathfrak{v}) \diamond O(\mathfrak{u}) \in Q^{\infty}(V),$$

proving that $O$ indeed induces an isomorphism from $A^{\infty}(V)$ to its opposite algebra. ∎

Since $A^{\infty}(V)$ is isomorphic to its opposite algebra, a left $A^{\infty}(V)$-module also give a right $A^{\infty}(V)$-module and vice versa. In particular, $S^N_{\mathrm{voa}}(M)$ is also a right $A^{\infty}(V)$-module. Then from Corollary 6.4, we obtain the following consequence immediately:

**Corollary 6.6** *Let $W$ be a right $A^{\infty}(V)$-module. Let $f: M \to W$ be a right $A^N(V)$-module map when $W$ is viewed as a right $A^N(V)$-module. Then there exists a right $A^{\infty}(V)$-module map $f^{\vee}: S^N_{\mathrm{voa}}(M) \to W$ such that $f^{\vee}|_M = f$.*

We now prove our second main theorem.

**Theorem 6.7** *Let $V$ be a vertex operator algebra. Assume that $W_2$ and $W_3'$ are equivalent to $S^N_{\mathrm{voa}}(\Omega^0_N(W_2))$ and $S^N_{\mathrm{voa}}(\Omega^0_N(W_3'))$, respectively. Then $\rho^N$ is a linear isomorphism.*

*Proof.* By Theorem 5.5, $\rho$ is a linear isomorphism. Then the composition $\rho^N \circ \rho^{-1}$ is a linear map from

$$\mathrm{Hom}_{A^{\infty}(V)}(A^{\infty}(W_1) \otimes_{A^{\infty}(V)} W_2, W_3)$$

to

$$\mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} \Omega^0_N(W_2), \Omega^0_N(W_3)).$$

We need only prove that $\rho^N \circ \rho^{-1}$ is an isomorphism. Then from the definition of $\rho^N$ and Theorem 5.5, $\rho^N$ is also a linear isomorphism.

By Proposition 6.1, $\rho^N$ is injective. Since both $\rho^N$ and $\rho^{-1}$ are injective, $\rho^N \circ \rho^{-1}$ is also injective. We still need to prove that $\rho^N \circ \rho^{-1}$ is surjective. Let

$$f^N \in \mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} \Omega^0_N(W_2), \Omega^0_N(W_3)).$$

We want to find

$$f \in \mathrm{Hom}_{A^{\infty}(V)}(A^{\infty}(W_1) \otimes_{A^{\infty}(V)} W_2, W_3)$$

such that the $(\rho^N \circ \rho^{-1})(f)$ is $f^N$.

Let $A^{N,\infty}(W_1)$ be the subspace of $A^{\infty}(W_1)$ obtained by taking sums of elements of the form $[w_1]_{np} + Q^{\infty}(W_1)$ for $w_1 \in W_1$, $0 \leq n \leq N$ and $p \in \mathbb{N}$, including certain infinite sums as in the case of $U^{\infty}(W_1)$. Then $A^{N,\infty}(W_1)$ is an $A^N(V)$-$A^{\infty}(V)$-bimodule. Since $\Omega^0_N(W_3') \otimes_{A^N(V)} A^{N,\infty}(W_1)$ is a right $A^{\infty}(V)$-module, its dual space $(\Omega^0_N(W_3') \otimes_{A^N(V)} A^{N,\infty}(W_1))^*$ a left $A^{\infty}(V)$-module. The map $f^N$ gives an $A^N(V)$-module map from $\Omega^0_N(W_2)$ to $(\Omega^0_N(W_3') \otimes_{A^N(V)} A^{N,\infty}(W_1))^*$ as follows: For $w_2 \in \Omega^0_N(W_2)$, we define the image of $w_2$ in $(\Omega^0_N(W_3') \otimes_{A^N(V)} A^{N,\infty}(W_1))^*$ to be the linear functional given by

$$w_3' \otimes_{A^N(V)} ([w_1]_{np} + Q^{\infty}(W_1)) \mapsto \langle w_3', f^N(([w_1]_{np} + Q^{\infty}(W_1)) \otimes_{A^N(V)} w_2)\rangle$$

for $0 \leq n, p \leq N$, $w_1 \in W_1$ and $w_3' \in \Omega_N^0(W_3')$ and

$$w_3' \otimes_{A^N(V)} ([w_1]_{np} + Q^\infty(W_1)) \mapsto 0$$

for $0 \leq n \leq N$, $p \in N + 1 + \mathbb{N}$, $w_1 \in W_1$ and $w_3' \in \Omega_N^0(W_3')$. For $0 \leq k, l \leq N$, $v \in V$, we have

$$\begin{aligned}
&\langle w_3', f^N(([w_1]_{np} + Q^\infty(W_1)) \otimes_{A^N(V)} \vartheta_{W_2}([v]_{kl})w_2)\rangle \\
&\quad = \langle w_3', f^N(([w_1]_{np} + Q^\infty(W_1)) \diamond ([v]_{kl} + Q^\infty(V)) \otimes_{A^N(V)} w_2)\rangle.
\end{aligned}$$

This means that the linear map from $\Omega_N^0(W_2)$ to $(\Omega_N^0(W_3') \otimes_{A^N(V)} A^{N,\infty}(W_1))^*$ is an $A^N(V)$-module map.

Since $W_2$ is equivalent to $S_{\mathrm{voa}}^N(\Omega_N^0(W_2))$, by Corollary 6.4, there exists a unique $A^\infty(V)$-module map from $W_2$ to $(\Omega_N^0(W_3') \otimes_{A^N(V)} A^{N,\infty}(W_1))^*$ such that when restricted to $\Omega_N^0(W_2)$, it is equal to the $A^N(V)$-module map given above. But such an $A^\infty(V)$-module map is equivalent to an element

$$f^{N,\infty} \in (\Omega_N^0(W_3') \otimes_{A^N(V)} A^{N,\infty}(W_1) \otimes_{A^\infty(V)} W_2)^*.$$

Since $A^\infty(W_1) \otimes_{A^\infty(V)} W_2$ is a left $A^\infty(V)$-module, its dual space $(A^\infty(W_1) \otimes_{A^\infty(V)} W_2)^*$ is a right $A^\infty(V)$-module. The map $f^{N,\infty}$ gives a right $A^N(V)$-module map from $\Omega_N^0(W_3')$ to $(A^{N,\infty}(W_1) \otimes_{A^\infty(V)} W_2)^*$ as follows: For $w_3' \in \Omega_N^0(W_3')$, we define the image of $w_3'$ in $(A^{N,\infty}(W_1) \otimes_{A^\infty(V)} W_2)^*$ to be the linear functional given by

$$([w_1]_{np} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2 \mapsto f^{N,\infty}(w_3' \otimes_{A^N(V)} ([w_1]_{np} + Q^\infty(W_1)) \otimes_{A^N(V)} w_2)\rangle$$

for $0 \leq n \leq N$, $p \in \mathbb{N}$, $w_1 \in W_1$ and $w_2 \in W_2$ and

$$([w_1]_{np} + Q^\infty(W_1)) \otimes_{A^\infty(V)} w_2 \mapsto 0$$

for $n \in N + 1 + \mathbb{N}$, $p \in \mathbb{N}$, $w_1 \in W_1$ and $w_2 \in W_2$. We use $w_3'\vartheta_{W_3'}^r(\mathfrak{v})$ to denote the right action of $\mathfrak{v}$ on $W_3'$. Then for $0 \leq k, l \leq N$, $v \in V$, we have

$$\begin{aligned}
&f^{N,\infty}(w_3'\vartheta_{W_3'}^r([v]_{kl}) \otimes_{A^N(V)} ([w_1]_{np} + Q^\infty(W_1)) \otimes_{A^N(V)} w_2)\rangle \\
&\quad = f^{N,\infty}(w_3' \otimes_{A^N(V)} ([v]_{kl} \diamond [w_1]_{np} + Q^\infty(W_1)) \otimes_{A^N(V)} w_2)\rangle.
\end{aligned}$$

This means that the linear map from $\Omega_N^0(W_3')$ to $(A^{N,\infty}(W_1) \otimes_{A^\infty(V)} W_2)^*$ is a right $A^N(V)$-module map.

Since $W_3'$ is equivalent to $S_{\mathrm{voa}}^N(\Omega_N^0(W_3'))$, by Corollary 6.6, there exists a unique right $A^\infty(V)$-module map from $W_3'$ to $(A^\infty(W_1) \otimes_{A^\infty(V)} W_2)^*$ such that when restricted to $\Omega_N^0(W_3')$, it is equal to the right $A^N(V)$-module map given above. But such a right $A^\infty(V)$-module map is equivalent to a left $A^\infty(V)$-module map from $(A^\infty(W_1) \otimes_{A^\infty(V)} W_2)^{**}$ to $W_3$. Since $A^\infty(W_1) \otimes_{A^\infty(V)} W_2$ is a $A^\infty(V)$-submodule of $(A^\infty(W_1) \otimes_{A^\infty(V)} W_2)^{**}$, we obtain a unique left $A^\infty(V)$-module map from $A^\infty(W_1) \otimes_{A^\infty(V)} W_2$ to $W_3$, or equivalently, an element

$$f \in \mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3).$$

From our construction of $f$, it is clear that $(\rho^N \circ \rho^{-1})(f) = f^N$. This proves the surjectivity of $\rho^N \circ \rho^{-1}$. ∎

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Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019
*E-mail address*: yzhuang@math.rutgers.edu