Vertex operator algebras, fractional quantum Hall states and topological orders

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Lie Group / Quamtum Math Seminar

Outline

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1. Fractional quantum Hall states
   - Quantum Hall effect
   - Fractional quantum Hall states and vertex operator algebras
2. Anyons and representation theory
   - Anyons
   - Modules and intertwining operators for vertex operator superalgebras
3. Vertex operator (super)algebras and topological orders
   - Topological orders
   - Some mathematical problems

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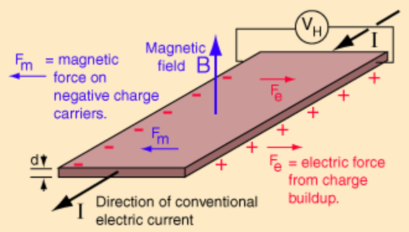
- Hall effect: When a magnetic field is applied perpendicular to the direction of a current flowing through a conductor, a measurable voltage is developed in the third perpendicular direction.



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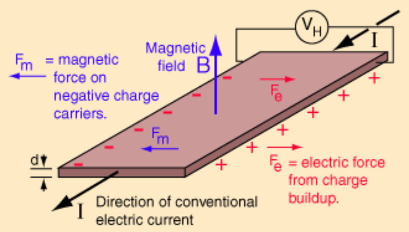

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Quantum Hall effect

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  $$R_H = \frac{h}{\nu e^2},$$

  where h is the Planck constant, e is the electric charge of a proton and ν is an integer (integer quantum Hall effect) or a rational number (fractional quantum Hall effect) called filling factor.
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Fractional quantum Hall states

- A quantum state in $2+1$ dimension is given by wave functions on the complex plane. When there are N electrons, we have a wave function $\Psi(z_1, \ldots, z_N)$ in the complex variaables $z_1, \ldots, z_N$. Since electrons are fermions, $\Psi(z_1, \ldots, z_N)$ is antisymmetric in the variables $z_1, \ldots, z_N$.
- The wave functions for electrons are products of polynomials and Gaussian factors of the form $e^{-\sum_k \frac{|z_k|^2}{4\ell_0}}$, where $\ell_0 = \sqrt{\frac{\hbar}{eB}}$ is the magnetic length.

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Fractional quantum Hall states

- Correlation functions of vertex operators for a vertex operator algebra are in general rational functions of $z_1, \ldots, z_N$. But for some vertex operator algebras, there are indeed vertex operators such that their correlation functions are in fact polynomials.
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Laughlin states

- Laughlin states: Laughlin found in 1983 the $m = 3$ case of the wave function

$$\prod_{i>j}(z_i - z_j)^m e^{-\sum_k \frac{|z_k|^2}{4\ell_0}}.$$

- The wave function in this $m = 3$ case explains theoretically the fractional quantum Hall effect discovered by Tsui, Strörmer and Gossard. The filling factor in this case is $\frac{1}{3}$. Tsui, Strörmer and Laughlin shared the Nobel prize in physics in 1998.
- For general odd positive integer m, this is the wave function for a fractional quantum Hall state with the filling factor $\nu = \frac{1}{m}$.

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Laughlin states and vertex operator algebras

- The polynomial parts $\prod_{i<j}(z_i - z_j)^m$ of these wave functions can be constructed from vertex superoperator algebras.
- L: the rank-one lattice generated by an element α with the $\mathbb{Z}$-bilinear form given by $(\alpha, \alpha) = m$.
- $V = V_L$: The vertex operator superalgebra associated to L. There is an invariant positive definite hermitian form $(\cdot, \cdot)$ on V_L extending the $\mathbb{Z}$-bilinear form on L.
- The analytic extension of

$$(\iota((e^\alpha)^N), Y(\iota(e^\alpha), z_1) \cdots Y(\iota(e^\alpha), z_N)\mathbf{1})$$

is equal to $\prod_{i<j}(z_i - z_j)^m$. Thus this correlation function can be interpreted as the wave function of N electrons. The conformal field corresponding to an electron is $Y(\iota(e^\alpha), z)$.

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Moore-Read Pfaffian states

- Moore-Read Pfaffian states: In the case that $\nu = \frac{1}{m}$ for an even positive integer m, using conformal field theory, Gregory Moore and Nick Read found in 1991 the candidate of a wave function

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^m e^{-\sum_k \frac{|z_k|^2}{4\ell_0}},$$

where for even N,

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \frac{1}{2^{N/2}(N/2)!}\sum_{\sigma\in S_N}\mathrm{sgn}\,\sigma\prod_{k=1}^{N/2}\frac{1}{z_{\sigma(2k-1)} - z_{\sigma(2k)}}$$

is the Pfaffian for the skew-symmetric matrix whose off diagonal entries are $\frac{1}{z_i - z_j}$.

Moore-Read Pfaffian states and vertex operator algebras

- $L(\frac{1}{2}, 0)$: Vertex operator algebra for the minimal model of central charge $\frac{1}{2}$.
- $L(\frac{1}{2}, \frac{1}{2})$: Irreducible module of lowest weight $\frac{1}{2}$.
- $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$: A vertex operator superalgebra with an invariant positive definite hermitian form. This is constructed similarly to (but certainly easier than) the moonshine module.
- $V = V_L \otimes (L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}))$: A vertex operator superalgebra with an invariant positive definite hermitian form.

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for $i = 1, \ldots, N$ applied to the vacuum and the paired with a suitable element of V is equal to

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Read-Rezayi states

- For a positive integer k, Read and Rezayi proposed in 1999 the wave functions

$$S\left(\prod_{0\leq r<s<n} X_{r,s}(z_{kr+1},\ldots,z_{kr+k},z_{ks+1},\ldots z_{ks+k})\right)\cdot$$
$$\cdot\prod_{i<j}(z_i-z_j)^m e^{-\sum_p \frac{|z_p|^2}{4\ell_0}}$$

  where $N=kn$ (the number of variables), m is an odd positive integer, S means the symmetrization over all permutations, and

$$\begin{aligned} X_{r,s} &= (z_{kr+1}-z_{ks+1})(z_{kr+1}-z_{ks+2})(z_{kr+2}-z_{ks+2})\cdot \\ &\quad \cdots(z_{kr+2}-z_{ks+3})(z_{kr+k}-z_{ks+k})(z_{kr+k}-z_{ks+1}). \end{aligned}$$

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- $L(k,0)$: the vertex operator algebra associated to the affine Lie algebra $\mathfrak{sl}(2,\mathbb{C})\hat{}$ and level k.
- α: A positive root of $\mathfrak{sl}(2,\mathbb{C})$.
- $\Omega_{L(k,0)}^{\langle e_{\alpha_*}\rangle}$: The generalized vertex operator algebra of parafermions (Lepowsky-Wilson (1981), Lepowsky-Primc (1984), Fateev-Zamolodchikov(1985) and Dong-Lepowsky (1993)).
- V_L: The generalized vertex operator algebra associated to the rank-one lattice generated by α satisfying $(\alpha,\alpha) = m + \frac{2}{k}$.
- $V_L \otimes \Omega_{L(k,0)}^{A}$: A generalized vertex operator algebra.

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Abelian anyons

- The wave functions above are for electrons in pure material. If there are impurity in the material and so on, the points where impurity occurs behave like particles and are called quasi-particles. If these quasi-particles are indistinguishable, then we can exchange them using paths in the configuration space of n-tuples $(z_1, \ldots, z_n)$ satisfying $z_i \neq z_j$. These paths form the braid group B_n.
- If the wave function after the exchange of quasi-particles is equal to a complex number (not 1 or -1) of absolute value 1 multiplying the wave function before the exchange, these quasi-particles are called abelian anyons.
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- Wilczek in 1982 introduced the term anyons and proposed a model for abelian anyons in connection with fractional spin in two dimensions.

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- Moore and Read in 1991 and Wen in 1991 pointed out explicitly that nonabelian anyons can be realized in fractional quantum Hall effect.
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Outline

1. Fractional quantum Hall states
   - Quantum Hall effect
   - Fractional quantum Hall states and vertex operator algebras
2. Anyons and representation theory
   - Anyons
   - Modules and intertwining operators for vertex operator superalgebras
3. Vertex operator (super)algebras and topological orders
   - Topological orders
   - Some mathematical problems

Modules

- When the material has an impurity point, say at 0, electrons still go through the material but in general might not pass the impurity point. We then have wave functions with 0 as a possible singular point.
- These wave functions still have the same properties as the wave functions without impurity points.
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Intertwining perators

- When the material has several impurity points, the wave functions locally are analytic functions of the positions of the impurity points. These wave functions can be obtained as the correlation functions of intertwining operators.
- Each impurity point corresponds to a quasi-particle. The algebra of intertwining operators (intertwining operator algebra) is the algebra underlying anyons.
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Tensor categories

- When a vertex operator algebra satisfying suitable conditions, a suitable category of modules has a natural structure of a braided tensor category or even a modular tensor category.
- The braided tensor category structure or the modular tensor category structure describes the topological properties of nonabelian anyons.
- If the algebraic structure of modular tensor categories can be realized in a physical system such as a fractional quantum Hall system, then one can use such a physical system to do quantum computation. Since modular tensor category is essentially topological, this is called topological quantum computation.

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Orders in states of matter

- Matter has many different states, such as gas, liquid and solid. Different states are distinguished by some of their internal structures. These internal structures are called orders.
- Classically, in Landau's symmetry-breaking theory, different orders correspond to different symmetries (or more precisely, the breaking of symmetries). But there exist different fractional quantum Hall states that have the same symmetry. This suggests that there is a new type of orders that cannot be described by symmetry or symmetry breaking.
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Topological orders in $2 + 1$ dimension

- Macroscopically, a topological order in $2 + 1$ dimension can be studied using tensor categories satisfying suitable properties.
- But there is still no satisfactory microscopic theory of topological orders. Ideally, one would want to define a topological order to be an equivalence class of suitable quantum states (such as fractional quantum Hall states) under a suitable equivalence relation such that the quantum states in the equivalence class can be deformed to each other smoothly and give equivalent tensor categories.
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Vertex operator (super)algebras as "symmetries"

- Vertex operator (super)algebras are analogous to Lie (super)algebras. The Jacobi identity for vertex operator (super)algebras is analogous to the Jacobi identity for Lie super)algebras. There is also a skew-symmetry for vertex operator (super)algebras.
- In 1990, I introduced a notion of vertex group that should be viewed as an analogue of the notion of group. The relation between vertex operator algebras and vertex groups is analogous to the relation between between Lie algebras and Lie groups.

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- It is clear that the representation theory of vertex operator (super)algebras must play an important role in the still-to-be-established microscopic theory of topological orders in $2+1$ dimension.
- Based on the role played by vertex operator (super)algebras in the study of fractional quantum Hall states, I propose that, topological orders also correspond to "symmetries," just as in Landau's theory. But the "symmetries" are described by vertex operator (super)algebras or vertex (super)groups, not by groups.

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- The vertex operator (super)algebra or the "(infinitesimal) symmetry" is $V = V_L \otimes (L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}))$.
- The wave function for N electrons without quasi-particles is given by the correlation function obtained from the product of the vertex operators $Y_V\left(\iota(e^\alpha) \otimes v_{\frac{1}{2},\frac{1}{2}}, z_i\right)$ for $i = 1, \ldots, N$.
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- The category of V-modules has a natural stucture of braided tensor categories. Probably it is a modular tensor category.
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A fundamental conjecture

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Classification of wave functions for electrons

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- Is it possible to classify such vertex operator superalgebras and thus to classify the possible wave functions for electrons?
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