Meromorphic open-string vertex algebras and Riemannian manifolds

Yi-Zhi Huang

Rutgers University

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Outline

The vertex operator algebra approach to two-dimensional conformal field theory

Nonlinear sigma models

Meromorphic open-string vertex algebras and representations

A sheaf $\mathcal{V}$ of "universal" meromorphic open-string vertex algebras

A sheaf $\mathcal{W}$ of left modules for the sheaf $\mathcal{V}$

A sheaf $\mathcal{V}^B$ of meromorphic open-string vertex algebra associated to $\mathcal{W}$

Lapacians and Ricci-flat manifolds

Further studies and the main challenge

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The vertex operator algebra approach (Belavin-Polyakov-Zamolodchikov, Borcherds, Frenkel-Lepowsky-Meurman) is in fact the most successful mathematical approach to conformal field theories. The program of constructing conformal field theories in the sense Kontsevich-Segal from repsentations of vertex operator algebras was developed by the speaker and collaboartors.

A vertex operator algebra is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ equipped with a vertex operator map

$$\begin{aligned} Y_V : V \otimes V &\to V[[z, z^{-1}]], \\ u \otimes v &\mapsto Y(u, z)v, \end{aligned}$$

a vacuum $\mathbf{1} \in V_{(0)}$ and a conformal vector $\omega \in V_{(2)}$, satisfying suitable axioms that will be briefly discussed later.

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Modules and intertwining operators

- Let V be a vertex operator algebra. A V-module is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ equipped with a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all those axioms for V which still make sense.
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- Let W_1, W_2 and W_3 be V-modules. An intertwining operator of type $\binom{W_3}{W_1\,W_2}$ is a linear map $\mathcal{Y} : W_1 \otimes W_2 \to W_3\{z\}$, where $W_3\{z\}$ is the space of all series in complex powers of z with coefficients in W_3, satisfying all those axioms for V which still make sense.

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Examples

- Free bosons: Representations of infinite-dimensional Heisenberg algebras. We will briefly discuss these theories later.
- Free bosons on tori: Vertex operator algebras, modules and classical vertex operators (intertwining operators) associated to lattices. We will also briefly discuss these theories later.
- Wess-Zumino-Novikov-Witten models: Representations of affine Lie algebras.
- Minimal models: Representations of the Virasoro algebra.
- Fermion theories: Representations of infinite-dimensional Clifford algebras, affine Lie superalgebras and superconformal algebras.
- Orbifolds, cosets and $\mathcal{W}$-algebras, including in particular the moonshine module.

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Rational conformal field theories

Theorem (H. 2002–2006)

Let V be a simple vertex operator algebra satisfying the following conditions:

1. $V_{(n)} = 0$ *for* $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ *and* V' *is isomorphic to* V *as a* V*-module.*
2. *Every lower-bounded generalized* V*-module is a direct sum of irreducible* V*-modules.*
3. V *is* C_2*-cofinite, that is,* $\dim V/C_2(V) < \infty$ *where* $C_2(V)$ *is the subspace of* V *spanned by elements of the form* $\mathrm{Res}_z z^{-2} Y(u, z)v$ *for* $u, v \in V$.

Then we have the following conclusion:

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Then we have the following conclusion:

1. *The Moore-Seiberg polynomial equations hold. In particular, the Verlinde conjecture holds.*
2. *The category of V-modules has a natural structure of a modular tensor category. In partciular, we have a modular functor (in all genera) and a 3-dimensional topological field theory.*
3. *All chiral and full correlation functions in genus-zero and genus-one can be constructed from intertwining operators (the part on full correlation functions being done jointly with Kong).*
4. *There exist locally convex topological completions of the spaces involved such that Segal's axioms in genus-zero and genus-one are satisfied.*

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Higher-genus and non-rational theories

- For higher-genus correlation functions, there is one problem still to be solved: Convergence in higher-genus case. Need results on meromorphic functions on the moduli space of Riemann surfaces with parametrized boundaries (generalizations of the q-expansion of the Weierstrass $\wp$-function).
- Many results for rational theories can be generalized to non-rational theories, including unitary non-rational theories and logarithmic theories. But there are still conjectures and open problems to be solved. The construction and study of non-rational theories is important for the study of the moduli space of conformal field theories. Nonlinear sigma models with Calabi-Yau manifolds as targets are important examples to be constructed and studied.

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Open-closed conformal field theories

- Jointly with Kong, a notion of open-string vertex algebra is introduced. We also constructed open-string vertex algebras from modules and intertwining operators for a vertex operator algebra satisfying suitable conditions. These are the open parts of open-closed conformal field theories.
- Kong further studied the open-closed conformal field theories whose closed parts and open parts are constructed from modules and intertwining operators for vertex operator algebras, especially the important Cardy condition.
- In summary, genus-zero and genus-one open-closed conformal field theories can be constructed from vertex operator algebras, their modules and intertwining operators.

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Outline

Harmonic maps from a Riemann surface to a Riemnannian manifold

- Σ: 2-dimensional Riemannian manifold.
  M: n-dimensional Riemannian manifold.
  φ: smooth map from Σ to M.
- Action: $\int_{\Sigma} \| d\varphi \| dS$.
  Locally: $\| d\varphi \| = \eta^{ij} g_{\mu\nu} \frac{\partial \varphi^{\mu}}{\partial x^{i}} \frac{\partial \varphi^{\nu}}{\partial x^{j}}$.
- Classical nonlinear sigma models: Harmonic maps, that is, critical points of the action above.
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Free bosons: Quantization of linear sigma models

- Linear sigma models: M flat.
- $M = \mathbb{R}^n$ and $p \in \mathbb{R}^n$.
  $T_pM = T_p\mathbb{R}^n$: the tangent space of $M = \mathbb{R}^n$ at p.
- Heisenberg algebra: $\widehat{T_pM} = \widehat{T_pM}_- \oplus \widehat{T_pM}_0 \oplus \widehat{T_pM}_+$, where

$$\begin{aligned} \widehat{T_pM}_- &= T_pM \otimes t^{-1}\mathbb{C}[t^{-1}], \\ \widehat{T_pM}_0 &= T_pM \otimes \mathbb{C}t^0 \oplus \mathbb{C}\mathbf{I} \simeq T_pM \oplus \mathbb{C}\mathbf{I}, \\ \widehat{T_pM}_+ &= TM \otimes t\mathbb{C}[t]. \end{aligned}$$

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Conformal field theories associated to free bosons

- Let $\widehat{T_pM}_+$ act on $|\mathbf{k}\rangle$ as 0, $\mathbf{l}$ on $|\mathbf{k}\rangle$ as 1 and elements of T_pM on $|\mathbf{k}\rangle$ as i times the vector fields obtained by parallel transporting the elements of T_pM. Then $\mathbb{C}|\mathbf{k}\rangle$ becomes a module for $\widehat{T_pM}_0 \oplus \widehat{T_pM}_+$.
- Induced module: $W_{|\mathbf{k}\rangle} = U(\widehat{T_pM}) \otimes_{U(\widehat{T_pM}_0 \oplus \widehat{T_pM}_+)} \mathbb{C}|\mathbf{k}\rangle$ (Fock space generated by $|\mathbf{k}\rangle$).
- $W_{|\mathbf{0}\rangle}$ has an algebraic structure called vertex operator algebra and $W_{|\mathbf{k}\rangle}$ for any $\mathbf{k}$ is a module for this vertex operator algebra $W_{|\mathbf{0}\rangle}$.
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Sigma models on tori

- Given a torus, we still consider the vertex operator algebra $W_{|\mathbf{0}\rangle}$. But for modules we consider only $W_{|\mathbf{k}\rangle}$ for those $\mathbf{k}$ such that $e^{ik_\mu x^\mu}$ are well defined on the torus. Such $\mathbf{k}$ form a lattice.
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First difficulties in generalizing this construction to the curved case

- In general, it was conjectured by physicists that quantum nonlinear sigma models are not conformal field theories. In the case that the target manifold is a Calabi-Yau manifold, the quantum nonlinear sigma module is an $N = 2$ superconformal field theory.
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- If we choose a local coordinate patch near p and define the action using derivatives with respect to the coordinates, the results depend on the local coordinate patch.
- If we define the action using the covariant derivatives, then we cannot view T_pM as an abelian Lie algebra becuase the commutators of covariant derivatives involve the curvature tensor.
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Outline

Meromorphic open-string vertex algebras

A noncommutative generalization of a variant of the notion of vertex operator algebras.

A meromorphic vertex operator algebra consists the following data:

- A $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$.
- A vertex operator map

$$\begin{aligned} Y_V : V \otimes V &\to V[[z, z^{-1}]], \\ u \otimes v &\mapsto Y(u, z)v. \end{aligned}$$

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These data satisfy the following axioms:

- Grading-restriction property: $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $V_{(n)} = 0$ when n is sufficiently negative.
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- Rationality and associativity: For $u_1, u_2, v \in V$, $v' \in V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$, the series

$$\langle v', Y(u_1, z_1)Y(u_2, z_2)v\rangle$$

$$\langle v', Y(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

Vertex (operator) algebras

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- A $\mathbb{Z}$-graded vertex algebra V is a vertex operator algebra or conformal vertex algebra if there is an element $\omega \in V$ such that for $L(n) = \mathrm{Res}_z z^{n+1} Y(\omega, z)$, we have $[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+b,0}$ (the Virasoro relation), $\frac{d}{dz}Y(u, z) = Y(L(-1)u, z)$ ($L(-1)$-derivative property) for $u \in V$ and $L(0)u = nu$ for $u \in V_{(n)}$ ($L(0)$-grading property) conformal element

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Operator product expansion

In the region $|z_1| > |z_2| > |z_1 - z_2| > 0$, we have

$$Y(u_1, z_1)Y(u_2, z_2) = Y(Y(u_1, z_1 - z_2)u_2, z_2)$$

in a suitable sense. Write $Y(u_1, x)u_2 = \sum_{n\in\mathbb{Z}} Y_n(u_1)u_2x^{-n-1}$. Then we have

$$Y(u_1, z_1)Y(u_2, z_2) = \sum_{n\in\mathbb{Z}}(z_1 - z_2)^{-n-1}Y(Y_n(u_1)u_2, z_2),$$

where $Y(u_1, z_1)$ and $Y(u_2, z_2)$ are the values of the quantum fields $Y(u_1, z)$ and $Y(u_2, z)$ at z_1 and z_2, respectively, $Y(Y_n(u_1)u_2, z_2)$ for $n \in \mathbb{Z}$ are the values of the quantum fields $Y(Y_n(u_1)u_2, z)$ at z_2 and $(z_1 - z_2)^{-n-1}$ for $n \in \mathbb{Z}$ are analytic functions of $z_1 - z_2$.

Left modules for a meromorphic open-string vertex algebra

- Let V be a meromorphic open-string vertex algebra. A left V-module is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ equipped with a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all those axioms for V which still make sense.
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Meromorphic open-string vertex algebras as noncommutative generalizations of vertex algebras

- In general, a meromorphic open-string vertex algebra satisfies all the properties for a vertex algebra except for the commutativity. Thus meromorphic open-string vertex algebras should be viewed as noncommutative generalizations of vertex algebras.
- Many other properties, for example, Jacobi identity, locality, the commutator formula, skew-symmetry or even the associator formula, of vertex algebras are also not satisfied by a meromorphic open-string vertex algebra.
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- Many other properties, for example, Jacobi identity, locality, the commutator formula, skew-symmetry or even the associator formula, of vertex algebras are also not satisfied by a meromorphic open-string vertex algebra.
- But as we discussed above, the vertex operators for a meromorphic open-string vertex algebra and its left modules still have operator product expansion.

Examples of meromorphic open-string vertex algebras

Theorem (H. 2012)

Given a finite-dimensional inner product space $\mathfrak{h}$ over $\mathbb{R}$, let $\hat{\mathfrak{h}}_{-} = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$. Then the tensor algebra $T(\hat{\mathfrak{h}}_{-})$ of $\hat{\mathfrak{h}}_{-}$ has a natural structure of meromorphic open-string vertex algebra.

Example

Let M be a Riemannian manifold, $p \in M$ and T_pM the tangent space at p. Then $T(\widehat{T_pM}_{-})$ has a natural structure of meromorphic open-string vertex algebra.

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Examples of left modules for meromorphic open-string vertex algebras

As we discussed above, the vertex operators for a left module have operator product expansion.

Theorem (H. 2012)

Given a finite-dimensional inner product space $\mathfrak{h}$ over $\mathbb{R}$ and a module Λ for the tensor algebra $T(\mathfrak{h}^{\mathbb{C}})$ ($\mathfrak{h}^{\mathbb{C}}$ being the complexification of $\mathfrak{h}$), the space $T(\hat{\mathfrak{h}}_{-}) \otimes \Lambda$ has a natural structure of left module for the meromorphic open-string vertex algebra $T(\hat{\mathfrak{h}}_{-})$.

According to this theorem, to construct a left module for the meromorphic open-string vertex algebra $T(\hat{\mathfrak{h}}_{-})$, one needs only construct left modules for the tensor algebra $T(\mathfrak{h}^{\mathbb{C}})$.

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A sheaf $\mathcal{V}$ of "universal" meromorphic open-string vertex algebras

- Given a Riemannian manifold M, we need to put the the meromorphic open-string vertex algebras constructed from the tangent spaces together to obtain a meromorphic open-string vertex algebra containing the global information of M.
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A sheaf $\mathcal{V}$ of "universal" meromorphic open-string vertex algebras

Let U be an open subset of M, $TM^{\mathbb{C}}$ the complexification of the tangent bundle of M, $\widehat{TM}_{-}$ the vector bundle whose fiber at $p \in M$ is $\widehat{T_pM}_{-}$, $T(\widehat{TM}_{-})$ the vector bundle whose fiber at $p \in M$ is $T(\widehat{T_pM}_{-})$ and $\Pi_U(T(\widehat{TM}_{-}))$ the space of parallel sections on U of the vector bundle $T(\widehat{TM}_{-})$.

Theorem (H. 2012)

The space $\Pi_U(T(\widehat{TM}_{-}))$ has a natural structure of meromorphic open-string vertex algebra. The assignment $U \to \Pi_U(T(\widehat{TM}_{-}))$ gives a sheaf of meromorphic open-string vertex algebras.

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Constructing left modules for the sheaf $\mathcal{V}$ from smooth functions

- The interesting and difficult part of our construction is the construction left modules for the sheaf $\mathcal{V}$ from smooth functions.
- This construction is based on an action of the associative algebra $\Pi_U(T(TM^{\mathbb{C}}))$ on the space of smooth functions. Such an action is equivalent to a homomorphism from $\Pi_U(T(TM^{\mathbb{C}}))$ to the associative algebra of operators on the space of smooth functions.

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A map ϕ_U from $\Pi_U(T(TM^{\mathbb{C}}))$ to $L(C^\infty(U))$

- Let $T^m(TM^{\mathbb{C}})$ be the vector bundle whose fibers are the m-th tensor powers of the complexifications of the tangent spaces of M, $T(TM^{\mathbb{C}})$ the vector bundle whose fibers are the tensor algebras of the complexifications of the tangent spaces of M and $\Pi_U(T(TM^{\mathbb{C}}))$ the space of parallel sections on U of the vector bundle $T(TM^{\mathbb{C}})$.
- Let $C^\infty(U)$ be the space of smooth complex functions on U and $L(C^\infty(U))$ the algebra of linear operators on the space $C^\infty(U)$ of complex-valued smooth function on U.
- We define $\phi_U^m : \Pi_U(T^m(TM^{\mathbb{C}})) \to L(C^\infty(U))$ to be the map $\left(\sqrt{-1}\right)^m \nabla^m$, where the m-th order covariant derivative ∇^m is viewed as a map $\Pi_U(T^m(TM^{\mathbb{C}}))$ to $L(C^\infty(U))$. Puting ϕ_U^m together, we obtain $\phi_U : \Pi_U(T(TM^{\mathbb{C}})) \to L(C^\infty(U))$.

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A homomorphism ϕ_U of associative algebras

Example: Let g^{-1} be the element of $\Gamma_U(T^2(TM^{\mathbb{C}}))$ corresponding to the metric on the cotagent bundle. Then g^{-1} is parallel, that is, $g^{-1} \in \Pi_U(T^2(TM^{\mathbb{C}}))$ and $\phi_U(g^{-1}) = -\Delta$.

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ϕ_U is a homomorphism of associative algebras.

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A presheaf

- By the theorem above, $C^\infty(U)$ is a $\Pi_U(T(TM^{\mathbb{C}}))$-module. For any $p \in M$, $\Pi_U(T(TM^{\mathbb{C}}))$ is isomorphic to the fixed point subspace $(T(T_pM^{\mathbb{C}}))^{H_p(U)}$ of $T(T_pM^{\mathbb{C}})$ under the holonormy group $H_p(U)$. Let

$$C_p(U) = T(T_pM^{\mathbb{C}}) \otimes_{(T(T_pM^{\mathbb{C}}))^{H_p(U)}} C^\infty(U).$$

- We know that $T(\widehat{T_pM}\_)$ has a natural structure of meromorphic open-string vertex algebra and $T(\widehat{T_pM}\_) \otimes C_p(U)$ has a natural structure of left $T(\widehat{T_pM}\_)$-module. Then $(T(\widehat{T_pM}\_))^{H_p(U)}$ is a meromorphic open-string vertex subalgebra of $T(\widehat{T_pM}\_)$ and it is isomorphic to $\Pi_U(T(\widehat{TM}_-))$. In particular, $T(\widehat{T_pM}\_) \otimes C_p(U)$ is also a left $\Pi_U(T(\widehat{TM}_-))$-module.

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A presheaf

- Let W_U^0 be the left $\Pi_U(T(\widehat{TM_-}))$-submodule of $T(\widehat{T_pM\_}) \otimes C_p(U)$ generated by elements of the form $1 \otimes (1 \otimes_{(T(T_pM^{\mathbb{C}}))^{H_p(U)}} f)$ for $f \in C^\infty(U)$, where $1 \otimes_{(T(T_pM^{\mathbb{C}}))^{H_p(U)}} f$ is the image of $1 \otimes f$ under the projection from $T(T_pM^{\mathbb{C}}) \otimes C^\infty(U)$ to $C_p(U)$.
- The assignment $U \to W^0(U)$ gives a presheaf $\mathcal{W}^0$.

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A sheaf $\mathcal{W}$ of left modules for the sheaf $\mathcal{V}$

Let $\mathcal{W}$ be the sheafification of $\mathcal{W}^0$. The section of $\mathcal{W}$ on U is denoted W_U.

Theorem (H. 2012)

The space W_U of sections of $\mathcal{W}$ on U is a left $\Pi_U(T(\widehat{TM_-}))$-module and $\mathcal{W}$ is a sheaf of left modules for the sheaf $\mathcal{V}$ of meromorphic open-string vertex algebras.

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A sheaf $\mathcal{V}^B$ of meromorphic open-string vertex algebra associated to $\mathcal{W}$

- For an open subset U of M, let $\tilde{V}_U$ be the subspace of V_U consisting of elements u such that $Y_{W_U}(u, x) = 0$. Then $\tilde{V}_U$ is a meromorphic open-string vertex subalgebra of V_U and $U \mapsto \tilde{V}_U$ for all U give a subsheaf $\tilde{\mathcal{V}}$ of $\mathcal{V}$ of meromorphic open-string vertex algebras. Let $\mathcal{V}^B$ be the sheafification of the quotient presheaf $\mathcal{V}/\tilde{\mathcal{V}}$.
- Let V_M^B be the space of global sections of $\mathcal{V}^B$. The correspondance $M \mapsto V_M^B$ gives a functor from the category of Riemannian manifolds to the category of meromorphic open-string vertex algebras. Moreover, when $M = \mathbb{R}^n$, $V_{\mathbb{R}^n}^B$ is isomorphic to the vertex operator algebra $W_{|\mathbf{0}\rangle}$. We believe that V_M^B is the canonical meromorphic open-string vertex algebra associated to the nonlinear sigma model with the target space M.

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Lapacian as a component of a vertex operator

Let $\{E_i\}_{i=1}^n$ be an orthonormal frame in an open subset U of M. Let

$$g_{\mathbb{C}}^{-1}(-1,-1) = \sum_{i=1}^{n} (E_i \otimes_{\mathbb{R}} t^{-1}) \otimes (E_i \otimes_{\mathbb{R}} t^{-1}) \in \Pi_U(T^2(\widehat{TM_-})).$$

Then $g_{\mathbb{C}}^{-1}(-1,-1)$ is in fact well defined on any open subset of M. In particular, it is well defined on M. Also, we can identify $C^\infty(M)$ with a subspace of W_M.

Theorem (H. 2012)

The coefficient of the x^{-2} term of $Y_{W_M}(-g_{\mathbb{C}}^{-1}(-1,-1), x)$ acting on $f \in C^\infty(M)$ is equal to Δf.

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The Laplacian of a Ricci-flat manifold

Proposition

Let M be a Riemannian manifold and X a parallel tangent field. Then

$$[\Delta, \phi(X)] = \mathrm{Ric}(X).$$

In particular, if M is Ricci flat, Δ and $\phi(X)$ is commutative.

Conjecture

Let (M, g) be Ricci flat. Then there exists a metric $\tilde{g}$ on M such that for any parallel tensor field $\mathcal{X}$ on $(M, \tilde{g})$, $\Delta_{\tilde{g}}$ and $\phi_{\tilde{g}}(\mathcal{X})$ is commutative, where $\Delta_{\tilde{g}}$ and $\phi_{\tilde{g}}$ are the Laplacian and the map defined before with the metric $\tilde{g}$.

If this conjecture is true, then for a Ricci-flat manifold M, we can find a metric $\tilde{g}$ on M such that eigenspaces of the Laplacian $\Delta_{\tilde{g}}$ generate modules for the meromorphic open-string vertex algebra of global sections of the sheaf $\mathcal{V}_{\tilde{g}}$, where $\mathcal{V}_{\tilde{g}}$ is the sheaf of meromorphic open-string vertex algebras constructed using the metric $\tilde{g}$.

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Thank you!