Cofiniteness and $P(z)$-tensor product bifunctors in orbifold theories associated to abelian but not-necessarily-finite groups

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Abstract

Let V be a Möbius vertex algebra and G an abelian group of automorphisms of V. We construct $P(z)$-tensor product bifunctors for the category of C_n-cofinite grading-restricted generalized g-twisted V-modules (without g-actions) for $g \in G$ and the category of C_n-cofinite grading-restricted generalized g-twisted V-modules with G-actions for $g \in G$. In this paper, an automorphism g of V can be of infinite order and does not have to act semisimply on V, and the group G can be an infinite abelian group containing nonsemisimple automorphisms of V.

1 Introduction

Two-dimensional orbifold conformal field theories play an important role in the construction, classification and applications of two-dimensional conformal field theories. The moonshine module $V^{\natural}$ constructed in mathematics by Frenkel, Lepowsky and Meurman [FLM1] [FLM2] [FLM3] is the first example of two-dimensional orbifold conformal field theories. The systematic study of two-dimensional orbifold conformal field theories in physics was started by Dixon, Harvey, Vafa, and Witten [DHVW1] [DHVW2]. In [H4], the author introduced g-twisted modules for a vertex operator algebra V for an automorphism of V of infinite order. In [H7], the author initiated a program to construct general two-dimensional orbifold conformal field theories starting from a vertex operator algebra V and a group G of automorphisms of V. Here the group G is not necessarily finite abelian. A number of conjectures and open problems, including those on the convergence, associativity, commutativity, modular invariance of twisted intertwining operators, and on the corresponding G-crossed braided tensor category structures were formulated and discussed in [H7].

Recently, progress in this program have been made by Du and the author [DH] on twisted intertwining operators and $P(z)$-tensor product bifunctors for categories of suitable twisted modules for a vertex operator algebra V under certain assumptions (see below for more discussions on these assumptions, by Tan on the cofiniteness of suitable twisted V-modules [Ta2] and on differential equations satisfied by products of twisted intertwining operators in the mostly untwisted case [Ta1] and in the finite abelian case [Ta3], and by Du on the associativity of twisted intertwining operators under suitable convergence and extension assumptions [D]. The present paper is also one step in this program.

We want to emphasize the importance of $P(z)$-tensor product bifunctors. In the untwisted case, the braided and even modular tensor category structures has been constructed on suitable categories of modules for a vertex operator algebra satisfying certain conditions (see [HL2]–[HL4], [H1], [H2], [H3], [HLZ1]–[HLZ8], [H8]). However, from these braided or even modular tensor category structures, we cannot reconstruct the vertex operator algebra. For example, the modular tensor category for the moonshine module $V^{\natural}$ is equivalent to the tensor category of finite-dimensional vector spaces over $\mathbb{C}$. It is impossible to reconstruct the moonshine module from this trivial tensor category. On the other hand, the constructions in [HL2]–[HL4], [H1], [H2], [H3], [HLZ1]–[HLZ8], and [H8] give vertex tensor categories (see [HL1] for a definition of vertex tensor category). It is a conjecture that one can reconstruct vertex operator algebras from vertex tensor categories. In the twisted case, it is conjectured in [H7] that the category of grading-restricted generalized g-twisted modules for a C_2-cofinite vertex operator algebra of positive energy and for g in a finite group G of automorphisms has a natural G-crossed braided tensor category structure in the sense of of [Tu]. To prove this conjecture, we have to construct first a G-crossed vertex tensor category structure and then use the same procedure as in the untwisted case to obtain the G-crossed braided tensor category structure. $P(z)$-tensor product bifunctors are one set of the most important data of a G-crossed vertex tensor category.

In [DH], $P(z)$-tensor product bifunctors for suitable categories of grading-restricted generalized V-modules are constructed based on suitable assumptions (see Assumption 4.4 in [DH]). The results in [DH] are twisted generalizations of the results in the untwisted case in [HL2], [HL4], [HLZ3] and [HLZ4]. Theorem 4.8 and Corollary 4.9 in [DH] give some strong conditions on categories of twisted V-modules such that Assumption 4.4 in [DH] holds for such categories. But these conditions are very strong and are not easy to verify. We would like to find conditions on a vertex operator algebra V and categories of suitable twisted modules that are easy to verify or have been verified in the literature such that $P(z)$-tensor product bifunctors exist in the category.

In this paper, we construct $P(z)$-tensor product bifunctors on categories of suitable twisted modules satisfying some cofiniteness conditions in the case that the automorphisms involved commute but might be of infinite orders. Instead of trying to generalize the conditions in Theorem 4.8 and Corollary 4.9 in [DH], in this paper, we prove generalizations and analogues of some results in [H8] for lower-bounded generalized twisted V-modules and twisted intertwining operators among such g-twisted V-modules for g in an abelian group G of automorphisms of V. In [H8], a precisely formulated generalization of an inequality of Nahm [N] is proved and $P(z)$-tensor rpoduct bifunctors are constructed using some results in [HLZ3] and [HLZ4] and this inequality. In fact, in [N], Nahm introduced a notion of quasi-rational module for a $\mathcal{W}$-algebra. In mathematics, $\mathcal{W}$-algebras are suitable vertex operator algebras and quasi-rational modules means C_1-cofinite modules. For a suitable module W for a vertex operator aLgebra V with the vertex operator map Y_W, the C_1-subspace $C_1(W)$ of W is the subspace of V spanned by $\mathrm{Res}_x x^{-1} Y_W(v, x)w$ for $v \in V_+ = \coprod_{n\in\mathbb{Z}_+} V_{(n)}$ and

$w \in W$. The V-module W is C_1-cofinite if $\dim W/C_1(W) < \infty$. Assuming that a suitable fusion product W_{12} of two C_1-cofinite V-modules W_1 and W_2 exists, Nahm derived in [N] an inequality

$$\dim(W_{12}/C_1(W_{12})) \leq \dim(W_1/C_1(W_1)) \dim(W_2/C_1(W_2)). \tag{1.1}$$

Nahm did not give a construction of a fusion product in [N]. Instead, he derived the inequality (1.1) from some basic physical assumptions on conformal field theories. A generalization of the inequality (1.1) to surjective intertwining operators is formulated precisely and proved mathematically in [H8]. In particular, if W_{12} is taken to be the $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$ constructed in [H8], the inequality (1.1) holds. In [YZ], Yang and Zhu proved a twisted Nahm inequality for quotients of lower-bounded generalized twisted V-modules by their C_2-subspaces[1] (see below for the definition) in the case that the automorphisms involved commute and are of finite order. Using this inequality, it is proved in [YZ] that the category of C_2-cofinite lower-bounded generalized twisted V-modules twisted by commuting automorphisms of finite orders is closed under a suitable fusion product.

In this paper, we construct $P(z)$-tensor product bifunctors for categories of C_n-cofinite grading-restricted generalized g-twisted V-modules for g in an abelain group G of automorphisms of V. The group G does not have to be finite and can contain automorphisms that do not act semisimply on V. Our method is the same as the one in [H8].

Here we give more detailed descriptions of our results. Let V be a grading-restricted vertex algebra. For an automorphism g of V, we have $V = \coprod_{\alpha \in P_V^g} V^{[\alpha]}$, where $P_V^g \subset [0,1) + \mathbf{i}\mathbb{R} \subset \mathbb{C}$ and $V^{[\alpha]}$ for $\alpha \in P_V^g$ is the generalized eigenspace of g with the eigenvalue $e^{2\pi \mathbf{i}\alpha}$. For $n \in 2 + \mathbb{N}$ and a generalized g-twisted V-module W with the twisted vertex operator map Y_W, let the C_n-subspace $C_n(W)$ of W be the subspace of W spanned by elements of the form $\mathrm{Res}_x x^{\alpha - n} Y_W(v, x)w$ for $v \in V^{[\alpha]}$, $\alpha \in P_V^g$, and $w \in W$. Let g_1 and g_2 be commuting automorphisms of V, W_1 and W_2 lower-bounded generalized g_1- and g_2-twisted V-modules, respectively, and W_3 a generalized g_1g_2-twisted V-module. Assuming that there exists a surjective twisted intertwining operator of type $\binom{W_3}{W_1 W_2}$, we prove

$$\dim(W_3/C_{\min(p,q)}(W_3)) \leq \dim(W_1/C_p(W_1)) \dim(W_2/C_q(W_2)) \tag{1.2}$$

for $p, q \in 2 + \mathbb{N}$. Here g_1 and g_2 can be of infinite order and do not have to act semisimply on V. This inequality is proved using the same method as in [H8], but with the Jacobi identity for (untwisted) intertwining operators in [H8] replaced by a Jacobi identity for twisted intertwining operators in the case that the automorphisms of V involved commute with each other.

For a generalized twisted V-module, we say that W is C_n-cofinite if $\dim W/C_n(W) < \infty$. We now assume that V has in addition a Möbius vertex algebra or quasi-vertex operator

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[1] In [YZ], for a lower-bounded generalized g-twisted V-module for finite-order g, a subspace $C_1(W)$ of W is introduced. In fact, $C_1(W)$ in [YZ] is $C_2(W)$ in [Ta1] and the present paper, and C_1-cofiniteness in [YZ] is called C_2-cofiniteness in [Ta1] and the present paper. There is also a notion of C_1-cofiniteness in [Ta1]. Note that by the main result proved by Tan in [Ta1], finitely generated lower-bounded generalized twisted V-modules are C_2-cofinite in the sense of [Ta1] and the present paper if V is C_2-cofinite and of positive energy. Because of this result and the examples of twisted modules for the Heisenberg vertex operator algebras, in the author's opinion, $C_2(W)$ and C_2-cofinite are the correct notation and term to use.

algebra structure (see [FHL] and [HLZ1]). In particular, we have an operator $L_V(1)$ on V and for a lower-bounded generalized twisted V-mdoule W, we have a contragredient W' (see [H5]). Let G be an abelian group of automorphisms of V. Using some results in [DH], the inequality above in the case $p = q = n \in 2 + \mathbb{N}$, and other results proved in this paper on C_n-cofinite lower-bounded generalized twisted V-modules and twisted intertwining operators, we construct $P(z)$-tensor product bifunctors for the category of C_n-cofinite grading-restricted generalized g-twisted V-modules (without g-actions) for $g \in G$ and the category of C_n-cofinite grading-restricted generalized g-twisted V-modules with G-actions for $g \in G$. Though the second category is in fact a subcategory of the first category, the $P(z)$-tensor product bifunctor for the second category is not the restriction of the $P(z)$-tensor product bifunctor for the first category to the second tensor category. We also emphasize again that in this paper, an automorphism g of V can be of infinite order and does not have to act semisimply on V, and the group G can be an infinite abelian group containing nonsemisimple automorphisms of V.

However, even in the abelian case discussed in this paper, the results and the method in [H8] cannot be generalized directly to obtain results on C_1-cofinite lower-bounded generalized twisted V-modules (see, for example, [Ta1] for a definition of C_1-cofinite twisted modules) and the corresponding twisted intertwining operators. So the C_1-cofinite case is still an open problem.

Since Tan proved that for a C_2-cofinite vertex operator algebra V of positive energy, finitely-generated weak twisted V-modules are C_n-cofinite for $n \geq 2$, the construction of the $P(z)$-tensor product bifunctors in this paper can be applied to module categories for many vertex operator algebras that have been proved to be C_2-cofinite. One important class of examples is the simple affine vertex operator algebras at positive integral levels. They are C_2-cofinite and the Lie groups corresponding to the finite-dimensional simple Lie algebras are groups of automorphisms of these affine vertex operator algebras. The results of the present paper can be applied to such a vertex operator algebra and an infinite abelian subgroup of the corresponding Lie group. Another class of examples is related to $\mathcal{W}$-algebras obtained as the kernels of suitable screening operators. In fact, exponentials of screening operators are automorphisms of infinite orders of suitable vertex operator algebras and the kernels of these screening operators are exactly the fixed point subalgebras of the vertex operator algebras that one starts with (see [H4]). The results of the present paper can be used to initiate a study of such $\mathcal{W}$-algebras using the orbifold theory associated to infinite abelian groups.

The present paper is organized as follows: In the next section (Section 2), we give the definitions of various notions of twisted modules and twisted intertwining operators among twisted modules twisted by commuting automorphisms. We define twisted intertwining operators using a Jacobi identity, which is equivalent to the duality property in [DH] in this abelian case (but we will prove this equivalence in another paper). In Section 3, we prove some basic properties of lower-bounded generalized twisted modules. The inequality (1.2) is proved in Section 4. The constructions of the $P(z)$-tensor product bifunctors are given in Section 5: In Subsection 5.1, we give the construction of the $P(z)$-tensor product bifunctors for the category of C_n-cofinite grading-restricted generalized g-twisted V-modules (without

g-actions) for $g \in G$. In Subsection 5.2, we give the construction of the $P(z)$-tensor product bifunctors for the category of C_n-cofinite grading-restricted generalized g-twisted V-modules with G-actions for $g \in G$.

In this paper, we fix a grading-restricted vertex algebra V. In Section 5, we will assume in addition that V is a Möbius vertex algebra or quasi-vertex operator algebra (see [FHL] and [HLZ1]). Since we sometimes use i as an index, we will denote the imaginary number $\sqrt{-1}$ by $\mathbf{i}$.

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2 Twisted modules and twisted intertwining operators

We give the definitions of various notions of twisted module, module maps between these twisted modules, and twisted intertwining operators among various types of twisted modules twisted by commuting automorphisms of V in this section. We define twisted intertwining operators in this case using a Jacobi identity. We then derive a suitable component form of the Jacobi identity that will be needed in later sections.

For an automorphism g of V, V can be decomposed as a direct sum of generalized eigenspaces of g. An eigenvalue of g on V is of the form $e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{C}$. Since $e^{2\pi i(\alpha+m)} = e^{2\pi i\alpha}$ for $m \in \mathbb{Z}$, we can always find such an $\alpha \in \mathbb{C}$ satisfying $\Re(\alpha) \in [0,1)$.

For $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha), \Re(\beta) \in [0,1)$, either $\Re(\alpha+\beta) \in [0,1)$ or $\Re(\alpha+\beta-1) \in [0,1)$. Let

$$\epsilon(\alpha,\beta) = \begin{cases} 0 & \Re(\alpha+\beta) \in [0,1) \\ 1 & \Re(\alpha+\beta-1) \in [0,1). \end{cases}$$

Then for $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha), \Re(\beta) \in [0,1)$, we have $\Re(\alpha+\beta-\epsilon(\alpha,\beta)) \in [0,1)$. In general, for $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ such that $\Re(\alpha_1), \dots, \Re(\alpha_k) \in [0,1)$, there exists a unique $\epsilon(\alpha_1, \dots, \alpha_k) \in \mathbb{N}$ satisfying $\epsilon(\alpha_1, \dots, \alpha_k) \leq k-1$ such that $\Re(\alpha_1 + \cdots + \alpha_k - \epsilon(\alpha_1, \dots, \alpha_k)) \in [0,1)$. For simplicity, we also denote $\alpha_1 + \cdots + \alpha_k - \epsilon(\alpha_1, \dots, \alpha_k)$ by $\sigma(\alpha_1, \dots, \alpha_k)$.

Let

$$P_V^g = \{\alpha \in \mathbb{C} \mid \Re(\alpha) \in [0,1), e^{2\pi \mathbf{i}\alpha} \text{ is an eigenvalues of } g\}.$$

For $\alpha \in P_V^g$, we use $V^{[\alpha]}$ to denote the generalized eigenspace of g with eigenvalue $e^{2\pi \mathbf{i}\alpha}$. Then

$$V = \coprod_{\alpha \in P_V^g} V^{[\alpha]} = \coprod_{n \in \mathbb{Z}} \coprod_{\alpha \in P_V^g} V_{(n)}^{[\alpha]},$$

where $V_{(n)}^{[\alpha]} = V_{(n)} \cap V^{[\alpha]}$ for $n \in \mathbb{Z}$ and $\alpha \in P_V^g$.

From Section 2 of [HY] and Lemma 3.4 in [H6], we know that there exist a semisimple operator $\mathcal{S}_g$ and a local nilpotent operator $\mathcal{N}_g$ on V such that $g = e^{2\pi \mathbf{i}(\mathcal{S}_g + \mathcal{N}_g)} = e^{2\pi \mathbf{i}\mathcal{L}_g}$, where as in [H6], $\mathcal{L}_g = \mathcal{S}_g + \mathcal{N}_g$, and $V^{[\alpha]}$ is the eigenspace of $\mathcal{S}_g$ with eigenvalue α for $\alpha \in P_V^g$. For $\alpha, \beta \in P_V^g$, $u \in V^{[\alpha]}$, and $v \in V^{[\beta]}$, since $e^{2\pi \mathbf{i}\mathcal{S}_g}$ is also an automorphism of V by Proposition 3.5 in [H6], we have $e^{2\pi \mathbf{i}\mathcal{S}_g} Y_V(u,x)v = Y_V(e^{2\pi \mathbf{i}\mathcal{S}_g}u, x)e^{2\pi \mathbf{i}\mathcal{S}_g}v = e^{2\pi \mathbf{i}(\alpha+\beta)} Y_V(u,x)v$.

If there exist $u \in V^{[\alpha]}$, $v \in V^{[\beta]}$ such that $Y_V(u, x)v \neq 0$, then there exists $n \in \mathbb{Z}$ such that $u_n v = \mathrm{Res}_x x^n Y_V(u, x)v \neq 0$ is a generalized eigenvector of g with eigenvalue $e^{2\pi \mathbf{i}(\alpha+\beta)}$. In this case, if $\Re(\alpha + \beta) \in [0, 1)$, then $\alpha + \beta \in P_V^g$ and if $\Re(\alpha + \beta - 1) \in [0, 1)$, $\alpha + \beta - 1 \in P_V^g$.

Let g_1 and g_2 be automorphisms of V. Then we have $g_1 = e^{2\pi \mathbf{i}(\mathcal{S}_{g_1}+\mathcal{N}_{g_1})} = e^{2\pi \mathbf{i}\mathcal{L}_{g_1}}$ and $g_1 = e^{2\pi \mathbf{i}(\mathcal{S}_{g_2}+\mathcal{N}_{g_2})} = e^{2\pi \mathbf{i}\mathcal{L}_{g_2}}$. Since g_1 and g_2 commute, the operators $\mathcal{S}_{g_1}$, $\mathcal{N}_{g_1}$, $\mathcal{S}_{g_2}$, and $\mathcal{N}_{g_2}$ also commute. In fact, since g_1 and g_2 commute, V can be decomposed as a direct sum of common generalized eigenspaces $V^{[\alpha_1,\alpha_2]}$ for $\alpha_1 \in P_V^{g_1}, \alpha_2 \in P_V^{g_2}$ of g_1 and g_2 with eigenvalues $e^{2\pi \mathbf{i}\alpha_1}$ and $e^{2\pi \mathbf{i}\alpha_2}$, respectively. Then $\mathcal{S}_{g_1}$ and $\mathcal{S}_{g_2}$ act on $V^{[\alpha_1,\alpha_2]}$ as the numbers α_1 and α_2, respectively. In particular, $\mathcal{S}_{g_1}$ and $\mathcal{S}_{g_2}$ commute. On the other hand, since $V^{[\alpha_1,\alpha_2]}$ is invariant under g_1 and g_2, $\mathcal{S}_{g_1}$ and $\mathcal{S}_{g_2}$ act on $g_1 v$ or $g_2 v$ for $v \in V^{[\alpha_1,\alpha_2]}$ are equal to $g_1 v$ or $g_2 v$ multiplied by α_1 and α_2, respectively. Thus we see that $\mathcal{S}_{g_1}$ and $\mathcal{S}_{g_2}$, g_1, and g_2 commute. and in particular, $\mathcal{S}_{g_1}$, $\mathcal{S}_{g_2}$, $\mathcal{L}_{g_1}$, and $\mathcal{L}_{g_2}$ commute. Then $\mathcal{S}_{g_1}$, $\mathcal{S}_{g_2}$, $\mathcal{N}_{g_1} = \mathcal{L}_{g_1} - \mathcal{S}_{g_1}$, and $\mathcal{N}_{g_2} = \mathcal{L}_{g_2} - \mathcal{S}_{g_2}$ commute.

We first give the definitions of various g-twisted V-modules for an automorphism g of V. The definitions below are generalizations or modifications of the definitions of suitable g-twisted V-modules given in [H4], [HY], and [H6]. It is important to note that g can be of infinite order and does not have to act semisimply on V.

Definition 2.1. A *weak g-twisted V-module without a g-action* is a vector space W equipped with operators $L_W(0)$ and $L_W(-1)$ on W, and a linear map

$$\begin{aligned} Y_W^g : V \otimes W &\to W\{x\}[\log x], \\ v \otimes w &\mapsto Y_W^g(v, x)w = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} v_{n,k} x^{-n-1} (\log x)^k \end{aligned}$$

called *twisted vertex operator map*, satisfying the following conditions:

1. The *lower-truncation condition*: For $v \in V$ and $w \in W$, $v_{n,k} w = 0$ for $\Re(n)$ sufficiently large.
2. The *equivariance property*: For $p \in \mathbb{Z}$, $z \in \mathbb{C}^\times$, $v \in V$ and $w \in W$,
$$(Y_W^g)^{p+1}(gv, z)w = (Y_W^g)^p(v, z)w,$$
where for $p \in \mathbb{Z}$, $(Y_W^g)^p(v, z)$ is the p-th analytic branch of $Y_W^g(v, x)$.
3. The *identity property*: For $w \in W$, $Y_W^g(\mathbf{1}, x)w = w$.
4. The *Jacobi identity*: For $u, v \in V$,
$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W^g(u, x_1)Y_W^g(v, x_2) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_W^g(v, x_2)Y_W^g(u, x_1) \\ &\quad = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) Y_W^g\left(Y_V\left(\left(\frac{x_2 + x_0}{x_1}\right)^{\mathcal{L}_g} u, x_0\right) v, x_2\right). \qquad (2.1)\end{aligned}$$

5. The $L(0)$-*commutator formula*:

$$[L_W(0), Y_W(v,x)] = z\frac{d}{dx}Y_W(v,x) + Y_W(L_V(0)v,x)$$

   for $v \in V$.

6. The $L(-1)$-*derivative property* and $L(-1)$-*commutator formula*: For $v \in V$,

$$\frac{d}{dx}Y_W(v,x) = Y_W(L_V(-1)v,x) = [L_W(-1), Y_W(v,x)].$$

A *weak g-twisted V-module with a g-action* is a weak g-twisted V-module without a g-action equipped with a $\langle g\rangle$-action (where $\langle g\rangle$ denote the cyclic group generated by g) satisfying the following *g-compatibility condition*: The g-action on W commutes with $L_W(0)$ and $L_W(-1)$ and $gY_W(u,x)w = Y_W(gu,x)gw$ for $u \in V$ and $w \in W$. Let G be a group of automorphisms of V such that $g \in G$. A *weak g-twisted V-module with a G-action* is a weak g-twisted V-module W with a g-action equipped with a G-module structure extending the g-action satisfying the following *G-compatibility condition*: The G-action on W commutes with $L_W(0)$, $L_W(-1)$ and $hY_W(u,x)w = Y_W(hu,x)hw$ for $h \in G$, $u \in V$ and $w \in W$. A *generalized g-twisted V-module without a g-action* is a weak g-twisted V-module without a g-action with a $\mathbb{C}$-grading

$$W = \coprod_{n\in\mathbb{C}} W_{[n]}$$

(graded by *weights*) satisfying the g-compatibility condition above and the following *$L(0)$-grading condition*: For $w \in W_{[n]} = \coprod_{\alpha\in P_W^g} W_{[n]}^{[\alpha]}$, $n \in \mathbb{C}$, there exists $K \in \mathbb{Z}_+$ such that $(L_W(0)-n)^K w = 0$. A *generalized g-twisted V-module with a g-action* is a generalized g-twisted V-module without a g-action with a $\mathbb{C}\times\mathbb{C}/\mathbb{Z}$-grading

$$W = \coprod_{n\in\mathbb{C}}\coprod_{\alpha\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$$

(graded by *weights* and *g-weights*) satisfying the following *g-grading condition*: For $\alpha \in \mathbb{C}/\mathbb{Z}$, $w \in W^{[\alpha]} = \coprod_{n\in\mathbb{C}} W_{[n]}^{[\alpha]}$, there exist $\Lambda \in \mathbb{Z}_+$ such that $(g - e^{2\pi \mathbf{i}\alpha})^\Lambda w = 0$. A *lower-bounded generalized g-twisted V-module with or without a g-action* is a generalized g-twisted V-module W with or without a g-action, respectively, such that $W_{[n]} = 0$ when $\Re(n) < N$ for some $N \in \mathbb{Z}$. A *grading-restricted generalized g-twisted V-module with or without a g-action* is a lower-bounded generalized g-twisted V-module W with or without a g-action, respectively, such that for each $n \in \mathbb{C}$, $\dim W_{[n]} < \infty$. A *quasi-finite-dimensional generalized g-twisted V-module with or without a g-action* is a generalized V-module W with or without a g-action, respectively, such that for $N \in \mathbb{Z}$, $\dim\left(\coprod_{\Re(n)\leq N}\coprod_{\alpha\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}\right) < \infty$. For a group G of automorphisms of V such that $g \in G$, a *generalized g-twisted V-module with a G-action*, a *lower-bounded generalized g-twisted V-module with a G-action*, a *grading-restricted generalized g-twisted V-module with a G-action*, and a *quasi-finite-dimensional generalized g-twisted V-module with a G-action* are defined by combining the definitions above accordingly.

We also need to define V-module maps between weak and generalized g-twisted modules without g-actions, with g-actions, or with G-actions for a fixed automorphism g of V.

Definition 2.2. For weak g-twisted V-modules W_1 and W_2 without g-actions, a *V-module map from W_1 to W_2* is a linear map$f : W_1 \to W_2$ such that $f \circ Y_{W_1}(v, x) = Y_{W_2}(v, x) \circ f$, $f \circ L_{W_1}(0) = L_{W_2}(0) \circ f$, and $f \circ L_{W_1}(-1) = L_{W_2}(-1) \circ f$. For weak g-twisted V-modules W_1 and W_2 with g-actions, a *V-module map from W_1 to W_2* is a V-module map $f : W_1 \to W_2$ when W_1 and W_2 are viewed as weak g-twisted V-modules without g-actions satisfying the additional condition $f \circ g = g \circ f$. Let G be a group of automorphisms of V containing g. For weak g-twisted V-modules W_1 and W_2 with G-actions, a *V-module map from W_1 to W_2* is a V-module map $f : W_1 \to W_2$ when W_1 and W_2 are viewed as weak g-twisted V-modules with g-actions such that f is also a G-module map. For generalized g-twisted V-modules W_1 and W_2 without g-actions, with g-actions or with G-actions, a *V-module map from W_1 to W_2* is a V-module map from W_1 to W_2 when W_1 and W_2 are viewed as weak g-twisted V-modules without g-actions, with g-actions or with G-actions, respectively.

In this paper, we mostly study lower-bounded, grading-restricted, quasi-finite-dimensional g-twisted V-modules with and without g-actions, and also with G-actions for a group of automorphisms of V. Since by definition, generalized g-twisted V-modules with g-actions and with G-actions are also generalized g-twisted V-modules without g-actions, the results obtained in this paper for suitable generalized g-twisted V-modules without g-actions hold also for the corresponding generalized g-twisted V-modules with g-actions and with G-actions when G is abelian. Since a large part of the results in the paper is about suitable g-twisted V-modules without g-actions, for simplicity, we will omit "without a g-action" in the term "a weak g-twisted V-module without a g-action" to call such a twisted V-module simply a weak g-twisted V-module. Similarly, we will omit "without a g-action" in the terms "a generalized g-twisted V-module without a g-action," "a lower-bounded generalized g-twisted V-module without a g-action," "a grading-restricted generalized g-twisted V-module without a g-action," "and a quasi-finite-dimensional generalized g-twisted V-module." If a result is specifically about suitable g-twisted V-modules with g- or G-actions, we will always explicitly say so by using the words "with g-actions" or "with G-actions."

For a weak g-twisted V-module W, let

$$Y_{W,0}(v, x) = \sum_{n\in\mathbb{C}} v_{n,0}x^{-n-1}$$

for $v \in V$. Then by Lemma 2.3 in [HY], we have

$$Y_W(v, x) = Y_{W,0}(x^{-\mathcal{N}_g}v, x) \tag{2.2}$$

for $v \in V$. For simplicity, we shall also denote $v_{n,0}$ sometimes simply by v_n for $v \in V$ and $n \in \mathbb{C}$.

For a generalized g-twisted V-module

$$W = \coprod_{n\in\mathbb{C}} \coprod_{\alpha\in\mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$$

with a g-action, let $W^{[\alpha]}=\coprod_{n\in\mathbb{C}}W^{[\alpha]}_{[n]}$ for $\alpha\in\mathbb{C}/\mathbb{Z}$. If $W^{[\alpha]}\neq 0$ for $\alpha\in\mathbb{C}/\mathbb{Z}$, then $W^{[\alpha]}$ is the generalized eigenspace of the action of g on W with eigenvalue $e^{2\pi\mathbf{i}\alpha}$. For $\alpha\in\mathbb{C}/\mathbb{Z}$, we can always find a unique complex number in α such that the real part of this complex number is in $[0,1)$. By abusing the notation, we denote this number by α and will always use $\alpha,\beta,\gamma,\ldots$ to denote such unique representatives of elements of $\mathbb{C}/\mathbb{Z}$. For $\alpha\in\mathbb{C}$ satisfying $\Re(\alpha)\in[0,1)$, We use $W^{[\alpha]}$ to denote the space $W^{[\alpha+\mathbb{Z}]}$. Let

$$P^g_W=\{\alpha\in\mathbb{C}\mid\Re(\alpha)\in[0,1),\ W^{[\alpha]}\neq 0\}.$$

Then we have

$$W=\coprod_{n\in\mathbb{C}}\coprod_{\alpha\in P^g_W}W^{[\alpha]}_{[n]}$$

General twisted intertwining operators, especially those among twisted modules twisted by noncommuting automorphisms, were introduced and studied in [H5] and [DH]. In the noncommuting case studied in these works, one must use the complex analytic approach. On the other hand, in the case of twisted intertwining operators among twisted modules twisted by commuting automorphisms, twisted intertwining operators can be defined and studied using a Jacobi identity formulated using the formal variable approach. In this paper, we study only the case of commuting automorphisms and we need the Jacobi identity. So here we give a formal variable definition of twisted intertwining operators. This formal variable definition can be derived from the analytic definition in [H5] and [DH]. We will discuss the connection between the complex analytic approach and the formal variable approach in another paper.

Definition 2.3. Let V be a grading-restricted vertex algebra, g_1 and g_2 commuting automorphisms of V, and W_1, W_2, W_3 weak g_1-, g_2, g_1g_2-twisted V-modules, respectively. A *twisted intertwining operator of type* $\binom{W_3}{W_1W_2}$ is a linear map

$$\begin{aligned}\mathcal{Y}:W_1\otimes W_2&\to W_3\{x\}[\log x]\\ w_1\otimes w_2&\mapsto\mathcal{Y}(w_1,x)w_2=\sum_{k=0}^{K}\sum_{n\in\mathbb{C}}\mathcal{Y}_{n,k}(w_1)w_2x^{-n-1}(\log x)^k\end{aligned}$$

satisfying the following axioms:

1. The *lower-truncation property*: For $w_1\in W_1$, $w_2\in W_2$, and $k=0,\ldots,K$, there exists $N\in\mathbb{N}$ such that $\mathcal{Y}_{n,k}(w_1)w_2=0$ when $\Re(n)>N$.

2. The *Jacobi identity*: For $v\in V$, $w_1\in W_1$, and $w_2\in W_2$,

$$\begin{aligned}&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{W_3}\left(\left(\frac{x_1-x_2}{x_0}\right)^{\mathcal{L}_{g_1}}v,x_1\right)\mathcal{Y}(w_1,x_2)\\ &\quad-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}(w_1,x_2)Y_{W_2}\left(e^{\pi\mathbf{i}\mathcal{L}_{g_1}}\left(\frac{x_2-x_1}{x_0}\right)^{\mathcal{L}_{g_1}}v,x_1\right)\end{aligned}$$

$$= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\mathcal{Y}\left(Y_{W_1}\left(\left(\frac{x_2+x_0}{x_1}\right)^{\mathcal{L}_{g_2}}v, x_0\right)w_1, x_2\right). \tag{2.3}$$

3. The $L(0)$-*commutator formula*: For $w_1 \in W_1$,

$$L_{W_3}(0)\mathcal{Y}(w_1, x) - \mathcal{Y}(w_1, x)L_{W_2}(0) = x\frac{d}{dx}\mathcal{Y}(w_1, x) + \mathcal{Y}(L_{W_1}(0)w_1, x).$$

4. The $L(-1)$-*derivative property*: For $w_1 \in W_1$,

$$\frac{d}{dx}\mathcal{Y}(w_1, x) = \mathcal{Y}(L_{W_1}(-1)w_1, x) = L_{W_3}(-1)\mathcal{Y}(w_1, x) - \mathcal{Y}(w_1, x)L_{W_2}(-1).$$

In the case that W_1, W_2, W_3 are weak g_1-, g_2, g_1g_2-twisted V-modules with G-actions when G is abelian and contains g_1 and g_2, a *twisted intertwining operator of type* $\binom{W_3}{W_1W_2}$ is a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$ when W_1, W_2, W_3 are viewed as weak g_1-, g_2, g_1g_2-twisted V-modules (without g_1-, g_2, g_1g_2-actions) satisfying the additional condition

$$h\mathcal{Y}(w_1, x)w_2 = \mathcal{Y}(hw_1, x)hw_2$$

for $h \in G$, $w_1 \in W_1$ and $w_2 \in W_2$. For generalized g_1-, g_2, g_1g_2-twisted V-modules W_1, W_2, and W_3, respectively, a *twisted intertwining operator of type* $\binom{W_3}{W_1W_2}$ is a twisted intertwining operator of the same type when W_1, W_2, and W_3 are viewed as weak g_1-, g_2, g_1g_2-twisted V-modules, respectively. For generalized g_1-, g_2, g_1g_2-twisted V-modules W_1, W_2, and W_3 with G-actions, respectively, a *twisted intertwining operator of type* $\binom{W_3}{W_1W_2}$ is a twisted intertwining operator of the same type when W_1, W_2, and W_3 are viewed as weak g_1-, g_2, g_1g_2-twisted V-modules with G-actions, respectively.

Remark 2.4. In the second term in the Jacobi identity (2.3), we choose the value of $(-1)^{\mathcal{L}_{g_1}}$ to be $e^{\pi \mathbf{i}\mathcal{L}_{g_1}}$. We can also choose the value of $(-1)^{\mathcal{L}_{g_1}}$ to be $e^{-\pi \mathbf{i}\mathcal{L}_{g_1}}$. Different choices of the values corresponding to different choices of single-valued branches of the multivalued analytic function to which the products and iterates of twisted vertex operators and the intertwining operator converge. These choices are related to the crossed braiding of the tensor category to be constructed. See [Ta3] for a discussion of both choices in the case that g_1 and g_2 are of finite orders.

The definition of twisted intertwining operator above is given using formal variables. We will need the evaluation of the formal series $\mathcal{Y}(w_1, x)$ for $w_1 \in W_1$ at $z \in \mathbb{C}^\times$ in later sections. Since $\mathcal{Y}(w_1, x)$ contains nonintegral powers and the logarithm of x, the evaluation depends on a choice of the values of the logarithm of z. Let $l_p(z) = \log|z| + \mathbf{i}\arg z + 2\pi\mathbf{i}p$, where $0 \leq \arg z < 2\pi$. We also use the notation $\log z = l_0(z)$. For $w_1 \in W_1$ and $w_2 \in W_2$, we have

$$\mathcal{Y}(w_1, x)w_2 = \sum_{k=0}^{K}\sum_{n\in\mathbb{C}}\mathcal{Y}_{n,k}(w_1)w_2x^{-n-1}(\log x)^k.$$

Then for $p \in \mathbb{Z}$, we define the p-th value $\mathcal{Y}^p(w_1, z)w_2$ of $\mathcal{Y}(w_1, x)w_2$ at $x = z$ to be

$$\mathcal{Y}^p(w_1, z)w_2 = \sum_{k=0}^{K} \sum_{n \in \mathbb{C}} \mathcal{Y}_{n,k}(w_1)w_2 e^{(-n-1)l_p(z)} l_p(z)^k \in \overline{W}_3 = \prod_{n \in \mathbb{C}} (W_3)_{[n]}.$$

In the case $p = 0$, we denote $\mathcal{Y}^0(w_1, w_1)w_2$ simply by $\mathcal{Y}(w_1, z)w_2$.

We now derive a version of the Jacobi identity expressed in terms of the components of the twisted vertex operators. We need this version in later sections. Recall that we use v_n to denote $v_{n,0}$ for $v \in V$ and $n \in \mathbb{C}$.

Proposition 2.5. *Let V be a grading-restricted vertex algebra, g_1 and g_2 commuting automorphisms of V, W_1, W_2, W_3 weak g_1-, g_2-, g_1g_2-twisted V-modules and $\mathcal{Y}$ a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. Then for $\alpha_1 \in P_V^{g_1}$, $\alpha_2 \in P_V^{g_2}$, $v \in V^{[\alpha_1,\alpha_2]}$, $w_1 \in W_1$,*

$$\begin{aligned}
&\sum_{k \in \mathbb{N}} \sum_{l \in \mathbb{N}} \binom{\alpha_1 - n_1}{k} (-x)^{k+l} \left(\binom{\mathcal{N}_{g_1}}{l} v \right)_{\alpha_1+\alpha_2-n_1-n_2-k-l} \mathcal{Y}(w_1, x)\\
&\quad - \sum_{k \in \mathbb{N}} \sum_{l \in \mathbb{N}} \binom{\alpha_1 - n_1}{k} e^{\pi \mathbf{i}(\alpha_1-n_1-k-l)} x^{\alpha_1-n_1-k-l} \mathcal{Y}(w_1, x) \left(e^{\pi \mathbf{i} \mathcal{N}_{g_1}} \binom{\mathcal{N}_{g_1}}{l} x^{\mathcal{N}_{g_1}} v \right)_{\alpha_2-n_2+k+l}\\
&= \sum_{k \in \mathbb{N}} \sum_{l \in \mathbb{N}} \binom{\alpha_2 - n_2}{k} x^{\alpha_2-n_2-k-l} \mathcal{Y} \left(\left(\binom{\mathcal{N}_{g_2}}{l} x^{\mathcal{N}_{g_2}} v \right)_{\alpha_1-n_1+k+l} w_1, x \right). \qquad (2.4)
\end{aligned}$$

Proof. For $\alpha_1 \in P_V^{g_1}$ and $\alpha_2 \in P_V^{g_2}$, let $V^{[\alpha_1,\alpha_2]} = V^{]\alpha_1]} \cap V^{[\alpha_2]}$. Then $V^{[\alpha_1,\alpha_2]}$ the space of common eigenvectors of g_1 and g_2 with eigenvalues $e^{2\pi \mathbf{i}\alpha_1}$ and $e^{2\pi \mathbf{i}\alpha_2}$, respectively, and we have $V = \coprod_{\alpha_1 \in P_V^{g_1}, \alpha_2 \in P_V^{g_2}} V^{[\alpha_1,\alpha_2]}$. For $v \in V_{g_1,g_2}^{[\alpha_1,\alpha_2]}$ and $w_1 \in W_1$, we have $\mathcal{S}_{g_1}v = \alpha_1 v$ and $\mathcal{S}_{g_2}v = \alpha_2 v$. Then

$$\begin{aligned}
\left(\frac{x_1 - x_2}{x_0} \right)^{\mathcal{L}_{g_1}} v &= \left(\frac{x_1 - x_2}{x_0} \right)^{\alpha_1} \left(\frac{x_1 - x_2}{x_0} \right)^{\mathcal{N}_{g_1}} v,\\
e^{\pi \mathbf{i} \mathcal{L}_{g_1}} \left(\frac{x_2 - x_1}{x_0} \right)^{\mathcal{L}_{g_1}} v &= e^{\pi \mathbf{i}\alpha_1} \left(\frac{x_2 - x_1}{x_0} \right)^{\alpha_1} e^{\pi \mathbf{i} \mathcal{N}_{g_1}} \left(\frac{x_2 - x_1}{x_0} \right)^{\mathcal{N}_{g_1}} v,\\
\left(\frac{x_2 + x_0}{x_1} \right)^{\mathcal{L}_{g_2}} v &= \left(\frac{x_2 + x_0}{x_1} \right)^{\alpha_2} \left(\frac{x_2 + x_0}{x_1} \right)^{\mathcal{N}_{g_2}} v.
\end{aligned}$$

Using these formulas, we see that for $v \in V_{g_1,g_2}^{[\alpha_1,\alpha_2]}$ and $w_1 \in W_1$, the Jacobi identity (2.3) becomes

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0} \right) \left(\frac{x_1 - x_2}{x_0} \right)^{\alpha_1} Y_{W_3} \left(\left(\frac{x_1 - x_2}{x_0} \right)^{\mathcal{N}_{g_1}} v, x_1 \right) \mathcal{Y}(w_1, x_2)\\
&\quad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0} \right) e^{\pi \mathbf{i}\alpha_1} \left(\frac{x_2 - x_1}{x_0} \right)^{\alpha_1} \mathcal{Y}(w_1, x_2) Y_{W_2} \left(e^{\pi \mathbf{i} \mathcal{N}_{g_1}} \left(\frac{x_2 - x_1}{x_0} \right)^{\mathcal{N}_{g_1}} v, x_1 \right)
\end{aligned}$$

$$= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^{\alpha_2}\mathcal{Y}\left(Y_{W_1}\left(\left(\frac{x_2+x_0}{x_1}\right)^{\mathcal{N}_{g_2}}v, x_0\right)w_1, x_2\right). \tag{2.5}$$

By (2.2), we have

$$\begin{aligned}
Y_{W_3}\left(\left(\frac{x_1-x_2}{x_0}\right)^{\mathcal{N}_{g_1}}v, x_1\right) &= Y_{W_3;0}((1-x_1^{-1}x_2)^{\mathcal{N}_{g_1}}x_0^{-\mathcal{N}_{g_1}}x_1^{-\mathcal{N}_{g_2}}v, x_1),\\
Y_{W_2}\left(e^{\pi\mathbf{i}\mathcal{N}_{g_1}}\left(\frac{x_2-x_1}{x_0}\right)^{\mathcal{N}_{g_1}}v, x_1\right) &= Y_{W_2,0}(e^{\pi\mathbf{i}\mathcal{N}_{g_1}}(1-x_1x_2^{-1})^{\mathcal{N}_{g_1}}x_0^{-\mathcal{N}_{g_1}}x_1^{-\mathcal{N}_{g_2}}x_2^{\mathcal{N}_{g_1}}v, x_1),\\
Y_{W_1}\left(\left(\frac{x_2+x_0}{x_1}\right)^{\mathcal{N}_{g_2}}v, x_0\right) &= Y_{W_1,0}((1+x_0x_2^{-1})^{\mathcal{N}_{g_2}}x_0^{-\mathcal{N}_{g_1}}x_1^{-\mathcal{N}_{g_2}}x_2^{\mathcal{N}_{g_2}}v, x_0)
\end{aligned}$$

Using these formulas, and replacing v by $x_0^{\mathcal{N}_{g_1}}x_1^{\mathcal{N}_{g_2}}v$, we see that (2.5) becomes

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\left(\frac{x_1-x_2}{x_0}\right)^{\alpha_1}Y_{W_3;0}((1-x_1^{-1}x_2)^{\mathcal{N}_{g_1}}v, x_1)\mathcal{Y}(w_1, x_2)\\
&\quad - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)e^{\pi\mathbf{i}\alpha_1}\left(\frac{x_2-x_1}{x_0}\right)^{\alpha_1}\mathcal{Y}(w_1, x_2)Y_{W_2,0}(e^{\pi\mathbf{i}\mathcal{N}_{g_1}}(1-x_1x_2^{-1})^{\mathcal{N}_{g_1}}x_2^{\mathcal{N}_{g_1}}v, x_1)\\
&= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^{\alpha_2}\mathcal{Y}(Y_{W_1,0}((1+x_0x_2^{-1})^{\mathcal{N}_{g_2}}x_2^{\mathcal{N}_{g_2}}v, x_0)w_1, x_2).
\end{aligned}\tag{2.6}$$

Multiplying $x_0^{\alpha_1}x_1^{\alpha_2}$ to both sides of (2.5), we obtain the following version of the Jacobi identity:

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(x_1-x_2)^{\alpha_1}x_1^{\alpha_2}Y_{W_3;0}((1-x_1^{-1}x_2)^{\mathcal{N}_{g_1}}v, x_1)\mathcal{Y}(w_1, x_2)\\
&\quad - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)e^{\pi\mathbf{i}\alpha_1}(x_2-x_1)^{\alpha_1}x_1^{\alpha_2}\mathcal{Y}(w_1, x_2)Y_{W_2,0}(e^{\pi\mathbf{i}\mathcal{N}_{g_1}}(1-x_1x_2^{-1})^{\mathcal{N}_{g_1}}x_2^{\mathcal{N}_{g_1}}v, x_1)\\
&= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)x_0^{\alpha_1}(x_2+x_0)^{\alpha_2}\mathcal{Y}(Y_{W_1,0}((1+x_0x_2^{-1})^{\mathcal{N}_{g_2}}x_2^{\mathcal{N}_{g_2}}v, x_0)w_1, x_2).
\end{aligned}\tag{2.7}$$

Taking $\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}x_0^{-n_1}x_1^{-n_2}$ of the first term in the left-hand side of the Jacobi identity (2.7), we obtain

$$\begin{aligned}
&\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}x_0^{-n_1}x_1^{-n_2}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(x_1-x_2)^{\alpha_1}x_1^{\alpha_2}Y_{W_3;0}((1-x_1^{-1}x_2)^{\mathcal{N}_{g_1}}v, x_1)\mathcal{Y}(w_1, x_2)\\
&\quad = \mathrm{Res}_{x_1}x_1^{-n_2}(x_1-x_2)^{\alpha_1-n_1}x_1^{\alpha_2}Y_{W_3;0}((1-x_1^{-1}x_2)^{\mathcal{N}_{g_1}}v, x_1)\mathcal{Y}(w_1, x_2)\\
&\quad = \sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\binom{\alpha_1-n_1}{k}\mathrm{Res}_{x_1}x_1^{\alpha_1+\alpha_2-n_1-n_2-k-l}(-x_2)^{k+l}Y_{W_3;0}\left(\binom{\mathcal{N}_{g_1}}{l}v, x_1\right)\mathcal{Y}(w_1, x_2)\\
&\quad = \sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\binom{\alpha_1-n_1}{k}(-x_2)^{k+l}\left(\binom{\mathcal{N}_{g_1}}{l}v\right)_{\alpha_1+\alpha_2-n_1-n_2-k-l}\mathcal{Y}(w_1, x_2).
\end{aligned}$$

Taking $\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}x_0^{-n_1}x_1^{-n_2}$ of the second term in the left-hand side of the Jacobi identity (2.7), we obtain

$$\begin{aligned}
&-\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}x_0^{-n_1}x_1^{-n_2}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)e^{\pi\mathbf{i}\alpha_1}(x_2-x_1)^{\alpha_1}x_1^{\alpha_2}\cdot\\
&\qquad\qquad\cdot\mathcal{Y}(w_1,x_2)Y_{W_2,0}(e^{\pi\mathbf{i}\mathcal{N}_{g_1}}(1-x_1x_2^{-1})^{\mathcal{N}_{g_1}}x_2^{\mathcal{N}_{g_1}}v,x_1)\\
&\quad=-\mathrm{Res}_{x_1}x_1^{-n_2}e^{\pi\mathbf{i}(\alpha_1-n_1)}(x_2-x_1)^{\alpha_1-n_1}x_1^{\alpha_2}\cdot\\
&\qquad\qquad\cdot\mathcal{Y}(w_1,x_2)Y_{W_2,0}(e^{\pi\mathbf{i}\mathcal{N}_{g_1}}(1-x_1x_2^{-1})^{\mathcal{N}_{g_1}}x_2^{\mathcal{N}_{g_1}}v,x_1)\\
&\quad=-\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\binom{\alpha_1-n_1}{k}\mathrm{Res}_{x_1}e^{\pi\mathbf{i}(\alpha_1-n_1)}x_2^{\alpha_1-n_1-k-l}(-1)^{k+l}x_1^{\alpha_2-n_2+k+l}\cdot\\
&\qquad\qquad\cdot\mathcal{Y}(w_1,x_2)Y_{W_2,0}\left(e^{\pi\mathbf{i}\mathcal{N}_{g_1}}\binom{\mathcal{N}_{g_1}}{l}x_2^{\mathcal{N}_{g_1}}v,x_1\right)\\
&\quad=-\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\binom{\alpha_1-n_1}{k}e^{\pi\mathbf{i}(\alpha_1-n_1-k-l)}x_2^{\alpha_1-n_1-k-l}\cdot\\
&\qquad\qquad\cdot\mathcal{Y}(w_1,x_2)\left(e^{\pi\mathbf{i}\mathcal{N}_{g_1}}\binom{\mathcal{N}_{g_1}}{l}x_2^{\mathcal{N}_{g_1}}v\right)_{\alpha_2-n_2+k+l}.
\end{aligned}$$

Taking $\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}x_0^{-n_1}x_1^{-n_2}$ of the right-hand side of the Jacobi identity (2.7), we obtain

$$\begin{aligned}
&\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}x_0^{-n_1}x_1^{-n_2}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)x_0^{\alpha_1}(x_2+x_0)^{\alpha_2}\cdot\\
&\qquad\qquad\cdot\mathcal{Y}(Y_{W_1,0}((1+x_0x_2^{-1})^{\mathcal{N}_{g_2}}x_2^{\mathcal{N}_{g_2}}v,x_0)w_1,x_2)\\
&\quad=\mathrm{Res}_{x_0}x_0^{\alpha_1-n_1}(x_2+x_0)^{\alpha_2-n_2}\mathcal{Y}(Y_{W_1,0}((1+x_0x_2^{-1})^{\mathcal{N}_{g_2}}x_2^{\mathcal{N}_{g_2}}v,x_0)w_1,x_2)\\
&\quad=\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\binom{\alpha_2-n_2}{k}\mathrm{Res}_{x_0}x_0^{\alpha_1-n_1+k+l}x_2^{\alpha_2-n_2-k-l}\mathcal{Y}\left(Y_{W_1,0}\left(\binom{\mathcal{N}_{g_2}}{l}x_2^{\mathcal{N}_{g_2}}v,x_0\right)w_1,x_2\right)\\
&\quad=\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\binom{\alpha_2-n_2}{k}x_2^{\alpha_2-n_2-k-l}\mathcal{Y}\left(\left(\binom{\mathcal{N}_{g_2}}{l}x_2^{\mathcal{N}_{g_2}}v\right)_{\alpha_1-n_1+k+l}w_1,x_2\right).
\end{aligned}$$

Using these calculations and (2.7), and changing x_2 to x, we obtain (2.4). □

Remark 2.6. In the special case that g_1 and g_2 acts semisimply on V, (2.4) becomes

$$\begin{aligned}
&\sum_{k\in\mathbb{N}}\binom{\alpha_1-n_1}{k}(-x)^k v_{\alpha_1+\alpha_2-n_1-n_2-k}\mathcal{Y}(w_1,x)\\
&\qquad-\sum_{k\in\mathbb{N}}\binom{\alpha_1-n_1}{k}e^{\pi\mathbf{i}(\alpha_1-n_1-k)}x^{\alpha_1-n_1-k}\mathcal{Y}(w_1,x)v_{\alpha_2-n_2+k}\\
&\quad=\sum_{k\in\mathbb{N}}\binom{\alpha_2-n_2}{k}x^{\alpha_2-n_2-k}\mathcal{Y}(v_{\alpha_1-n_1+k}w_1,x).
\end{aligned}$$

3 Cofiniteness of twisted modules

In this section, we prove some basic properties of generalized twisted V-modules.

Definition 3.1. Let W be a weak g-twisted V-module. For each $n \in 2+\mathbb{N}$, we define $C_n(W)$ to be the subspace of W spanned by the elements of the form $u_{\alpha-n}w$ for $u \in V_+^{[\alpha]}$, $\alpha \in P_V^g$, $w \in W$. We say that W is *C_n-cofinite* if $\dim W/C_n(W) < \infty$.

As in the case of (untwisted) weak modules, we also have the following useful fact:

Proposition 3.2. *Let W be a weak g-twisted V-module. If W is C_n-cofinite for some $n \in 2+\mathbb{N}$, then it is also C_m-cofinite for $m = 2, \dots, n$.*

Proof. As in the untwisted case, this result follows from the $L(-1)$-derivative property. □

A C_n-cofinite lower-bounded generalized g-twisted V-module has the following properties:

Proposition 3.3. *Let $W = \coprod_{m\in\mathbb{C}} W_{[n]}$ be a C_n-cofinite lower-bounded generalized g-twisted V-module. Then W has the following properties:*

1. *W is quasi-finite-dimensional.*
2. *There exists a finite-dimensional subspace M of W such that W is spanned by elements of the form*
$$v^{(1)}_{\alpha_1-n} \cdots v^{(i)}_{\alpha_i-n} w \tag{3.1}$$
for $i \in \mathbb{N}$, $v^{(1)} \in V_+^{[\alpha_1]}, \dots, v^{(i)} \in V_+^{[\alpha_i]}$ and $w \in M$. In particular, W is finitely generated.
3. *If W has an action of another automorphism h of V such that the h-action commutes with $L_W(0)$, then W can be decomposed as a direct sum of generalized eigenspaces of the action of h.*

Proof. To prove Property 1, we need to prove that

$$\dim\left(\coprod_{\Re(m)\leq N} W_{[m]}\right) < \infty$$

for $N \in \mathbb{Z}$. Since W is lower bounded, there exists $N_0 \in \mathbb{Z}$ such that $W_{[m]} = 0$ when $\Re(m) < N_0$. We use induction on $N - N_0$. In the case $N - N_0 = 0$, for $v \in V_{(l)}^{[\alpha]}$ and $w \in W_{[m]}$, where $l \in \mathbb{Z}_+$ and $m \in \mathbb{C}$ satisfying $\Re(m) \geq N_0$, $v_{\alpha-n}w \in W_{[l-\alpha+n-1+m]}$. Since $l \in \mathbb{Z}_+$, $n \in 2+\mathbb{N}$, and $\Re(m) \geq N_0$, $\Re(l-\alpha+n-1+m) > N_0$. So nonzero elements of $W_{[l-\alpha+n-1+m]}$ such as $v_{\alpha-n}w$ cannot be in $\coprod_{\Re(m)=N_0} W_{[m]}$. Therefore $\dim \coprod_{\Re(m)=N_0} W_{[m]} \leq \dim W/C_n(W) < \infty$ since W is C_n-cofinite. This proves Property 1 in the case $N - N_0 = 0$.

Assume that Property 1 is true for $0 \leq N - N_0 \leq k$. Now let $N = N_0 + k + 1$. By the induction assumption, we need only prove

$$\dim \left(\coprod_{k < \Re(m) - N_0 \leq k+1} W_{[m]} \right) \tag{3.2}$$

For $m \in \mathbb{C}$ satisfying $k < \Re(m) - N_0 \leq k + 1$, let $(C_n(W) \cap W_{[m]})_{\alpha,l}$ be the subspace of $C_n(W) \cap W_{[m]}$ spanned by elements of the form $v_{\alpha-n}w$ for $v \in V^{[\alpha]}_{(m+\alpha-n+1-l)}$ and $w \in W_{[l]}$ for $l \in \mathbb{C}$ satisfying $0 \leq \Re(l) - N_0 \leq k$ and $m + \alpha - n + 1 - l \in \mathbb{Z}_+$. For $m, l \in \mathbb{C}$, $\alpha \in P^g_V$ such that $m+\alpha-n+1-l \in \mathbb{Z}_+$, or equivalently, $m+\alpha-n-l \in \mathbb{N}$, we have $\Im(l) = \Im(m)+\Im(\alpha)$ and $\Re(l) \in \Re(m) + \Re(\alpha) - n + 1 - \mathbb{Z}_+ = \Re(m) + \Re(\alpha) - n - \mathbb{N}$. Let $p = m + \alpha - n - l \in \mathbb{N}$. If m and l further satisfy $0 \leq \Re(l) - N_0 \leq k < \Re(m) - N_0 \leq k + 1$, then we have

$$p = m + \alpha - n - l = \Re(m) + \Re(\alpha) - n - \Re(l) \leq N_0 + k + 1 + \Re(\alpha) - n - N_0 < k + 1 - n.$$

So we obtain $0 \leq p \leq k - n$. In particular, in the case $k < n$, there does not exist such p, or equivalently, $(C_n(W) \cap W_{[m]})_{\alpha,l} = 0$. In the case $k \geq n$, for $p \in \mathbb{N}$ satisfying $0 \leq p \leq k - n$ and $l = m + \alpha - n - p$, we have

$$\begin{aligned}
k \geq \Re(m) - N_0 - 1 &= \Re(l) - \Re(\alpha) + n + p - N_0 - 1\\
&> \Re(l) - N_0\\
&= \Re(m) + \Re(\alpha) - n - p - N_0\\
&\geq \Re(m) - N_0 + \Re(\alpha) - k\\
&> k.
\end{aligned}$$

So summing over $l \in m+\alpha-n+1-\mathbb{Z}_+$ satisfying $0 \leq \Re(l)-N_0 \leq k$ is the same as summing over $p = 0, \ldots, k - n$. Thus in the case $k \geq n$, , we have

$$\begin{aligned}
&\dim \left(C_n(W) \cap \left(\coprod_{k<\Re(m)-N_0\leq k+1} W_{[m]} \right) \right)\\
&= \dim \left(\coprod_{k<\Re(m)-N_0\leq k+1} C_n(W) \cap W_{[m]} \right)\\
&= \dim \left(\coprod_{k<\Re(m)-N_0\leq k+1} \coprod_{\alpha\in P^g_V} \coprod_{\substack{l\in m+\alpha-n+1-\mathbb{Z}_+\\ 0\leq\Re(l)-N_0\leq k}} (C_n(W) \cap W_{[l]})_{\alpha,l} \right)\\
&= \sum_{k<\Re(m)-N_0\leq k+1} \sum_{\alpha\in P^g_V} \sum_{\substack{l\in m+\alpha-n+1-\mathbb{Z}_+\\ 0\leq\Re(l)-N_0\leq k}} \dim(C_n(W) \cap W_{[m]})_{\alpha,l}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k<\Re(m)-N_0\leq k+1}\ \sum_{\alpha\in P_V^g}\ \sum_{\substack{l\in m+\alpha-n+1-\mathbb{Z}_+\\ 0\leq \Re(l)-N_0\leq k}} \dim V^{[\alpha]}_{(m+\alpha-n+1-l)} \dim W_{[l]}\\
&\leq \sum_{k<\Re(m)-N_0\leq k+1}\ \sum_{\alpha\in P_V^g}\ \sum_{p=0}^{k-n} \dim V^{[\alpha]}_{(p)} \dim W_{[m+\alpha-n-p]}\\
&\leq \sum_{\alpha\in P_V^g}\ \sum_{p=0}^{k-n} \dim V^{[\alpha]}_{(p)} \left(\sum_{k<\Re(m)-N_0\leq k+1} \dim W_{[m+\alpha-n-p]}\right)\\
&= \sum_{p=0}^{k-n} \left(\sum_{\alpha\in P_V^g} \dim V^{[\alpha]}_{(p)}\right) \dim \left(\coprod_{k<\Re(m)-N_0\leq k+1} W_{[m+\alpha-n-p]}\right)\\
&\leq \sum_{p=0}^{k-n} \left(\sum_{\alpha\in P_V^g} \dim V^{[\alpha]}_{(p)}\right) \dim \left(\coprod_{0\leq\Re(s)-N_0\leq k} W_{[s]}\right)\\
&= \sum_{p=0}^{k-n} \dim V_{(p)} \dim \left(\coprod_{0\leq\Re(s)-N_0\leq k} W_{[s]}\right)\\
&< \infty, \qquad\qquad (3.3)
\end{aligned}
$$

where the last step follows from the grading-restriction property of V and the induction assumption.

On the other hand, the quotient

$$
\left(\coprod_{k<\Re(m)-N_0\leq k+1} W_{[m]}\right) \Bigg/ \left(C_n(W)\cap \left(\coprod_{k<\Re(m)-N_0\leq k+1} W_{[m]}\right)\right)
$$

as a subspace of the finite-dimensional space $W/C_n(W)$ is also finite-dimensional. Thus by (3.3) and the finite-dimensionality of this quotient, we obtain

$$
\begin{aligned}
\dim &\left(\coprod_{k<\Re(m)-N_0\leq k+1} W_{[m]}\right)\\
&\leq \dim \left(C_n(W)\cap \left(\coprod_{k<\Re(m)-N_0\leq k+1} W_{[m]}\right)\right)\\
&\quad + \dim \left(\coprod_{k<\Re(m)-N_0\leq k+1} W_{[m]}\right) \Bigg/ \left(C_n(W)\cap \left(\coprod_{k<\Re(m)-N_0\leq k+1} W_{[m]}\right)\right)\\
&< \infty,
\end{aligned}
$$

proving (3.2).

We prove Property 2 now. Take a finite-dimensional subspace M of W such that $W = C_n(W) + M$. But every finite-dimensional subspace of W must be in $\coprod_{\Re(m)\leq N} W_{[m]}$ for some $N \in \mathbb{N}$. In particular, there exists $N_M \in \mathbb{N}$ such that $M \subset \coprod_{\Re(m)\leq N_M} W_{[m]}$ and thus $\coprod_{\Re(m)>N_M} W_{[m]} \subset C_n(W)$. Since $\coprod_{\Re(m)\leq N_M} W_{[m]}$ is also finite-dimensional, we take $M = \coprod_{\Re(m)\leq N_M} W_{[m]}$ from now on.

Denote the space spanned by elements of the form (3.1) by $\widetilde{W}$. What we want to prove is $W = \widetilde{W}$. We need only prove that $W_{[m]} \subset \widetilde{W}$ for every $m \in \mathbb{C}$. For each $m \in \mathbb{C}$ such that $W_{[m]} \neq 0$, there exists a unique $N_m \in \mathbb{Z}$ such that $N_m - 1 < \Re(m) \leq N_m$. We use induction on N_m. When $N_m \leq N_M$, $W_{[m]} \subset M \subset \widetilde{W}$. Assume that for $N_m \leq p \in N_M + \mathbb{N}$, $W_{[m]} \subset \widetilde{W}$. Then in the case $N_m = p+1$, since $\Re(m) > N_m - 1 = p \geq N_M$ and $\coprod_{\Re(m)>N_M} W_{[m]} \subset C_n(W)$, we obtain $W_{[m]} \subset C_n(W)$. Then $W_{[m]}$ is spanned by elements of the form $v_{\alpha-n}w$ for $v \in V^{[\alpha]}_{(m+\alpha-n+1-l)}$ and $w \in W_{[l]}$ for $l \in \mathbb{C}$ satisfying $\Re(l) \leq p$ and $m+\alpha-n+1-l \in \mathbb{Z}_+$. Since $\Re(l) \leq p$, we have $N_p \leq p$. By induction assumption, $w \in \widetilde{W}$. So w is a linear combination of elements of the form $v^{(1)}_{\alpha_1-n}\cdots v^{(i)}_{\alpha_i-n}\tilde{w}$ for homogeneous $v^{(1)} \in V_+^{[\alpha_1]}, \dots, v^{(i)} \in V_+^{[\alpha_i]}$ and $\tilde{w} \in M$. Then $W_{\llbracket m \rrbracket}$ is spanned by elements of the form $v_{\beta-n}v^{(1)}_{\alpha_1-n}\cdots v^{(i)}_{\alpha_i-n}\tilde{w}$ for $v \in V^{[\beta]}_{(l)}$, $v^{(1)} \in V_+^{[\alpha_1]}, \dots, v^{(i)} \in V_+^{[\alpha_i]}$, and $\tilde{w} \in M$. So $W_{[m]} \subset \widetilde{W}$, proving Property 2.

Finally, we prove Property 3. Since W is quasi-finite-dimensional, $W_{[m]}$ for $m \in \mathbb{C}$ is Since W is in particular finite-dimensional. If W has an action of another automorphism h of V such that the h-action commutes with $L_W(0)$, $W_{[m]}$ is invariant under the action of h. So $W_{[m]}$ can be decomposed as a direct sum of generalized eigenspaces of the action of h, and thus W can also be decomposed as a direct sum of generalized eigenspaces of the action of h. $\square$

Remark 3.4. From Property 1 in Proposition 3.3, we see that C_n-cofinite lower-bounded generalized g-twisted V-modules, C_n-cofinite grading-restricted generalized g-twisted V-modules, and C_n-cofinite quasi-finite-dimensional generalized g-twisted V-modules are the same. In this paper, we will call these g-twisted modules C_n-cofinite grading-restricted generalized g-twisted V-modules, but note that these g-twisted modules are quasi-finite-dimensional.

For the next result and the twisted Nahm inequality to be proved in the next section, we need the notion of surjectivity of a twisted intertwining operator. Let W_1, W_2, W_3 be weak g_1-, g_2-, g_3-twisted V-modules, respectively. A twisted intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1W_2}$ is said to be surjective if W_3 is spanned by the coefficients of $\mathcal{Y}(w_1, x)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$. If for a weak V-module W_3, there is an surjective twisted intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1W_2}$, we say that $(W_3, \mathcal{Y})$ is a *weak surjective product of* W_1 *and* W_2. For simplicity, we shall often call W_3 a weak surjective product of W_1 and W_2. If W_3 is generalized g_3-twisted V-module or grading-restricted generalized g_3-twisted V-module or other classes of g_3-twisted V-modules, we also use the terms *generalized surjective product of* W_1 *and* W_2, *grading-restricted generalized surjective product of* W_1 *and* W_2 and so on.

Proposition 3.5. *Let W_1 and W_2 be generalized g_1 and g_2-twisted V-modules, respectively, and W_3 a weak g_3-twisted V-module. If W_3 is a weak surjective product of W_1 and W_2, then W_3 is also a generalized g_3-twisted V-module.*

Proof. Let V^{g_1,g_2,g_3} be the fixed point subalgebra of V under the group generated by g_1, g_2 and g_3. Then W_1 and W_2 are also generalized V^{g_1,g_2,g_3}-modules, respectively, and W_3 also a weak V^{g_1,g_2,g_3}-module. Let $\mathcal{Y}$ be the surjective twisted intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1W_2}$ for the weak surjective product W_3 of W_1 and W_2. Then $\mathcal{Y}$ is an intertwining operator of the same type when W_1, W_2, and W_3 are viewed as generalized or weak V^{g_1,g_2,g_3}-modules. Moreover, since $\mathcal{Y}$ as a twisted intertwining operator among the generalized or weak twisted V-modules is surjective, it is also surjective as an intertwining operator among the generalized or weak V^{g_1,g_2,g_3}-modules. Then by Proposition 3.4 in [H8], W_3 is a generalized V^{g_1,g_2,g_3}-module. Thus W_3 is a generalized g_3-twisted V-module. □

4 A twisted Nahm inequality

In this section, by using the same method as in [H8], we prove the following twisted Nahm inequality in the case that the automorphisms of V involved commute, but in general can be of infinite orders and do not have to act semisimply on V:

Theorem 4.1. *Let g_1 and g_2 be commuting automorphisms of V and W_1, W_2 lower-bounded generalized g_1, g_2-twisted V-modules, respectively. Then for a generalized surjective product W_3 of W_1 and W_2 and $p, q \in 2 + \mathbb{N}$,*

$$\dim(W_3/C_{\min(p,q)}(W_3)) \leq \dim(W_1/C_p(W_1)) \dim(W_2/C_q(W_2)). \tag{4.1}$$

Proof. In the case that at least one of $\dim(W_1/C_p(W_1))$ and $\dim(W_2/C_q(W_2))$ is ∞, (4.1) holds. So we need only prove (4.1) in the case that $\dim W_1/C_p(W_1), \dim W_2/C_q(W_2) < \infty$, that is, in the case that W_1 and W_2 are C_p-cofinite and C_q-cofinite lower-bounded generalized twisted V-modules, respectively. In this case, the properties in Proposition 3.3 hold for W_1 and W_2. In particular, W_1 and W_2 are quasi-finite-dimensional generalized twisted V-modules.

Let M_1 and M_2 be finite-dimensional graded subspaces of W_1 and W_2, respectively, such that $W_1 = C_p(W_1) \oplus M_1$ and $W_2 = C_q(W_2) \oplus M_2$. Then $\dim(W_1/C_p(W_1)) = \dim M_1$ and $\dim(W_2/C_q(W_2)) = \dim M_2$.

Since W_3 is a generalized g_1g_2-twisted V-module, W_3 is $\mathbb{C}$-graded and we have the contragredient W_3' of W_3. Let M_3 be a graded subspace of W_3 such that $W_3 = C_{\min(p,q)}(W_3) \oplus M_3$. Then $W_3/C_{\min(p,q)}(W_3)$ is isomorphic to M_3. We need only prove $\dim M_3 \leq \dim M_1 \dim M_2$.

Let

$$C_{\min(p,q)}(W_3)^{\perp} = \{w_3' \in W_3' \mid \langle w_3', C_{\min(p,q)}(W_3)\rangle = 0\} \subset W_3'.$$

and let M_3' be the graded dual of M_3 with respect to the $\mathbb{C}$-grading induced from the one on W_3. We define a linear map $r : C_{\min(p,q)}(W_3)^{\perp} \to M_3'$ to be the restriction map sending elements of $C_{\min(p,q)}(W_3)^{\perp} \subset W_3'$ to their restrictions to M_3, that is, $(r(w_3'))(w_3) = \langle w_3', w_3\rangle$

for $w_3' \in C_{\min(p,q)}(W_3)^{\perp}$ and $w_3 \in M_3$. If $r(w_3') = 0$ for $w_3' \in C_{\min(p,q)}(W_3)^{\perp}$, then $\langle w_3', w_3\rangle = 0$ for all $w_3 \in M_3$. But by definition, $\langle w_3', C_{\min(p,q)}(W_3)\rangle = 0$. Since $W_3 = C_{\min(p,q)}(W_3) \oplus M_3$, we obtain $\langle w_3', w_3\rangle = 0$ for all $w_3 \in W_3$. Thus $w_3' = 0$, proving that r is injective. Given $w_3' \in M_3'$, we extend it to an element $\bar{w}_3' \in W_3'$ by $\langle \bar{w}_3', \tilde{w}_3 + w_3\rangle = \langle w_3', w_3\rangle$ for $\tilde{w}_3 \in C_{\min(p,q)}(W_3)$ and $w_3 \in M_3$. By definition, $\bar{w}_3' \in C_{\min(p,q)}(W_3)^{\perp}$ and $r(\bar{w}_3') = w_3'$, proving r is surjective. We have proved that r is a linear isomorphism. Therefore we need only prove that $\dim C_{\min(p,q)}(W_3)^{\perp} \le \dim M_1 \dim M_2$.

Let $\mathcal{Y}$ be a surjective twisted intertwining operator of type $\binom{W_3}{W_1W_2}$. Then we have

$$\mathcal{Y}(w_1, x)w_2 = \sum_{k=0}^{K}\sum_{n\in\mathbb{C}} \mathcal{Y}_{n,k}(w_1)w_2 x^{-n-1}(\log x)^k$$

for $w_1 \in W_1$ and $w_2 \in W_2$. For homogeneous $w_1 \in W_1$ and $w_2 \in W_2$, we have

$$\mathrm{wt}\,\mathcal{Y}_{n,k}(w_1)w_2 = \mathrm{wt}\,w_1 - n - 1 + \mathrm{wt}\,w_2.$$

Hence for homogeneous $w_1 \in W_1$, $w_2 \in W_2$, and $w_3' \in W_3'$,

$$\langle w_3', \mathcal{Y}(w_1, x)w_2\rangle = \sum_{k=0}^{K} \langle w_3', \mathcal{Y}_{\mathrm{wt}\,w_1+\mathrm{wt}\,w_2-\mathrm{wt}\,w_3'-1,k}(w_1)w_2\rangle x^{-\mathrm{wt}\,w_1-\mathrm{wt}\,w_2+\mathrm{wt}\,w_3'}(\log x)^k.$$

Fix $z \in \mathbb{C}^{\times}$. Recall from Section 2 that we use $\log z$ to denote $\log|z| + i\arg z$, where $0 \le \arg z < 2\pi$ and the 0-th value $\mathcal{Y}(w_1, z)w_2 = \mathcal{Y}^0(w_1, z)w_2$ of $\mathcal{Y}(w_1, x)w_2$ at $x = z$. Then

$$\langle w_3', \mathcal{Y}(w_1, z)w_2\rangle = \sum_{k=0}^{K} \langle w_3', \mathcal{Y}_{\mathrm{wt}\,w_1+\mathrm{wt}\,w_2-\mathrm{wt}\,w_3'-1,k}(w_1)w_2\rangle e^{(-\mathrm{wt}\,w_1-\mathrm{wt}\,w_2+\mathrm{wt}\,w_3')\log z}(\log z)^k$$

is well defined. We define a linear map $f : C_{\min(p,q)}(W_3)^{\perp} \to (M_1 \otimes M_2)^*$ by

$$(f(w_3'))(w_1 \otimes w_2) = \langle w_3', \mathcal{Y}(w_1, z)w_2\rangle$$

for $w_1 \in M_1$, $w_2 \in M_2$ and $w_3' \in C_{\min(p,q)}(W_3)^{\perp}$. To prove

$$\dim C_{\min(p,q)}(W_3)^{\perp} \le \dim M_1 \dim M_2 = \dim(M_1 \otimes M_2)^*,$$

we need only prove that f is injective.

Assume that $f(w_3') = 0$ for an element $w_3' \in C_{\min(p,q)}(W_3)^{\perp}$. Then by the definition of f, for $w_1 \in M_1$, $w_2 \in M_2$,

$$\langle w_3', \mathcal{Y}(w_1, z)w_2\rangle = 0 \tag{4.2}$$

We now prove (4.2) for all $w_1 \in W_1$, $w_2 \in W_2$.

Since W_1 and W_2 are quasi-finite-dimensional generalized V-modules, we have $W_1 = \coprod_{m\in\mathbb{C}}(W_1)_{[m]}$ and $W_2 = \coprod_{n\in\mathbb{C}}(W_2)_{[n]}$ such that for $N \in \mathbb{N}$,

$$\dim\left(\coprod_{\Re(m)\le N}(W_1)_{[m]}\right), \dim\left(\coprod_{\Re(n)\le N}(W_2)_{[n]}\right) < \infty.$$

In fact, there exist $N_1^0 \in \mathbb{Z}$ such that $(W_1)_{[m]} = 0$ for $\Re(m) < N_1^0$ and there exists $m \in \mathbb{C}$ satisfying $N_1^0 \leq \Re(m) < N_1^0 + 1$ such that $(W_1)_{[m]} \neq 0$. Similarly we have such an N_2^0 for W_2. We want to prove (4.2) for all $w_1 \in \coprod_{\Re(m) \leq N_1^0 + N_1} (W_1)_{[m]}$ and $w_2 \in \coprod_{\Re(n) \leq N_2^0 + N_2} (W_1)_{[n]}$ for $N_1, N_2 \in \mathbb{N}$. We use induction on $N_1 + N_2$ to prove (4.2).

When $N_1 + N_2 = 0$, $N_1 = N_2 = 0$. Since $W_1 = C_p(W_1) \oplus M_1$ and $W_2 = C_q(W_2) \oplus M_2$, there exist homogeneous $u^{(k)} \in V_{g_1}^{[\alpha_k]} \cap V_+, v^{(l)} \in V_{g_2}^{[\beta_l]} \cap V_+$, $\tilde{w}_1^{(k)} \in (W_1)_{[p_k]}$, $\tilde{w}_2^{(l)} \in (W_2)_{[q_l]}$, for $k = 1, \ldots, s$, $l = 1, \ldots, t$, $\tilde{w}_1 \in M_1 \cap (W_1)_{[N_1^0]}$, and $\tilde{w}_2 \in M_2 \cap (W_2)_{[N_2^0]}$ such that

$$w_1 = \sum_{k=1}^{s} u^{(k)}_{\alpha_k - p} \tilde{w}_1^{(k)} + \tilde{w}_1, \tag{4.3}$$

$$w_2 = \sum_{l=1}^{t} v^{(l)}_{\beta_l - q} \tilde{w}_2^{(l)} + \tilde{w}_2. \tag{4.4}$$

We must have $u^{(k)}_{\alpha_k - p} \tilde{w}_1^{(k)} \in (W_1)_{[N_1^0]}$, since $w_1 \in (W_1)_{[N_1^0]}$. On the other hand, we have $u^{(k)}_{\alpha_k - p} \tilde{w}_1^{(k)} \in (W_1)_{[\mathrm{wt}\, u^{(k)} - \alpha_k + p - 1 + p_k]}$. So we obtain $\mathrm{wt}\, u^{(k)} - \alpha_k + p - 1 + p_k = N_1^0$, or equivalently, $p_k = N_1^0 - \mathrm{wt}\, u^{(k)} + \alpha_k - p + 1$. Since $\mathrm{wt}\, u^{(k)} \in \mathbb{Z}_+$ and $\Re(\alpha_k \in [0,1)$, we have $\Re(p_k) = N_1^0 - \mathrm{wt}\, u^{(k)} + \Re(\alpha_k) - p + 1 < N_1^0$. Hence $(W_1)_{[p_k]} = 0$ and therefore $u^{(k)}_{\alpha_k - p} \tilde{w}_1^{(k)} = 0$ for $k = 1, \ldots, s$. Similarly, we can prove $v^{(l)}_{\beta_l - q} \tilde{w}_2^{(l)} = 0$. Thus $w_1 = \tilde{w}_1 \in M_1$ and $w_2 = \tilde{w}_2 \in M_2$. In this case, (4.2) is true.

Assume that when $N_1 + N_2 < m \in \mathbb{N}$, (4.2) is true. In the case $N_1 + N_2 = m$, we consider $w_1 \in (W_1)_{[n_1]}$ and $w_2 \in (W_2)_{[n_2]}$ for $n_1, n_2 \in \mathbb{C}$ such that $\Re(n_1) \leq N_1^0 + N_1$, $\Re(n_2) \leq N_2^0 + N_2$, and $N_0^1 + N_2^0 + m - 1 < \Re(n_1 + n_2) \leq N_0^1 + n_2^0 + N_1 + N_2$. We still have (4.3) and (4.4) for homogeneous $u^{(k)} \in V_{g_1}^{[\alpha_k]} \cap V_+, v^{(l)} \in V_{g_2}^{[\beta_l]} \cap V_+$, $\tilde{w}_1^{(k)} \in (W_1)_{[p_k]}$, $\tilde{w}_2^{(l)} \in (W_2)_{[q_l]}$, for $k = 1, \ldots, s$, $l = 1, \ldots, t$, $\tilde{w}_1 \in M_1 \cap (W_1)_{[n_1]}$, and $\tilde{w}_2 \in M_2 \cap (W_2)_{[n_2]}$. In fact, we can always find $u^{(k)} \in V_{g_1,g_2}^{[\alpha_k, \hat{\beta}_k]} \cap V_+, v^{(l)} \in V_{g_1,g_2}^{[\hat{\alpha}_l, \beta_l]} \cap V_+$ and the corresponding $\tilde{w}_1^{(k)}$, $\tilde{w}_2^{(l)}$, $\tilde{w}_1$, and $\tilde{w}_2$ such that (4.3) and (4.4) hold. In this case, similarly to the proof above in the case $N_1 = N_2 = 0$, we have $\mathrm{wt}\, u^{(k)} - \alpha_k + p - 1 + p_k = n_1$, or equivalently, $\mathrm{wt}\, u^{(k)} + p_k = n_1 + \alpha_k - p + 1$. Similarly, we also have $\mathrm{wt}\, v^{(l)} - \beta_l + q - 1 + q_l = n_2$ or equivalently, $\mathrm{wt}\, v^{(l)} + q_l = n_2 + \beta_l - q + 1$.

We need to use (2.4). We first rewrite (2.4) as

$$\begin{aligned}
&\sum_{k \in \mathbb{N}} \sum_{l \in \mathbb{N}} \binom{\alpha_1 - n_1}{k} (-x)^{k+l} \left(\binom{\mathcal{N}_{g_1}}{l} x^{-\mathcal{N}_{g_2}} v \right)_{\alpha_1 + \alpha_2 - n_1 - n_2 - k - l} \mathcal{Y}(w_1, x) \\
&\quad - \sum_{k \in \mathbb{N}} \sum_{l \in \mathbb{N}} \binom{\alpha_1 - n_1}{k} e^{\pi \mathbf{i} (\alpha_1 - n_1 - k - l)} x^{\alpha_1 - n_1 - k - l} \cdot \\
&\qquad\qquad \cdot \mathcal{Y}(w_1, x) \left(e^{\pi \mathbf{i} \mathcal{N}_{g_1}} \binom{\mathcal{N}_{g_1}}{l} x^{\mathcal{N}_{g_1} - \mathcal{N}_{g_2}} v \right)_{\alpha_2 - n_2 + k + l} \\
&= \sum_{k \in \mathbb{N}} \sum_{l \in \mathbb{N}} \binom{\alpha_2 - n_2}{k} x^{\alpha_2 - n_2 - k - l} \mathcal{Y} \left(\left(\binom{\mathcal{N}_{g_2}}{l} v \right)_{\alpha_1 - n_1 + k + l} w_1, x \right)
\end{aligned} \tag{4.5}$$

by replacing v by $x^{-\mathcal{N}_{g_2}}v$ in (2.4). Then using (4.5) with v, α_1, α_2, n_1, n_2, w_1, and w_2 in (4.5) taken to be $u^{(k)}$, α_k, $\hat{\beta}_k$, p, 1, $\tilde{w}_1^{(k)}$, respectively, and changing the summation indices from k, l to i, j, we obtain

$$\begin{aligned}
&\langle w_3', \mathcal{Y}(u^{(k)}_{\alpha_k-p}\tilde{w}_1^{(k)}, x)w_2\rangle\\
&\quad=\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\binom{\alpha_k-p}{i}(-1)^{i+j}x^{i+j-\hat{\beta}_k+1}\cdot\\
&\qquad\qquad\cdot\left\langle w_3', \left(\binom{\mathcal{N}_{g_1}}{j}x^{-\mathcal{N}_{g_2}}u^{(k)}\right)_{\alpha_k+\hat{\beta}_k-p-1-i-j}\mathcal{Y}(\tilde{w}_1^{(k)}, x)w_2\right\rangle\\
&\qquad-\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}, j+i\neq 0}\binom{\hat{\beta}_k-1}{i}x^{-i-j}\left\langle w_3', \mathcal{Y}\left(\left(\binom{\mathcal{N}_{g_2}}{j}u^{(k)}\right)_{\alpha_k-p+i+j}\tilde{w}_1^{(k)}, x\right)w_2\right\rangle\\
&\qquad-\sum_{i\in\mathbb{N}}\binom{\alpha_k-p}{i}e^{\pi\mathbf{i}(\alpha_k-p-i-j)}x^{\alpha_k-\hat{\beta}_k-p+1-i-j}\cdot\\
&\qquad\qquad\cdot\langle w_3', \mathcal{Y}(\tilde{w}_1^{(k)}, x)\left(e^{\pi\mathbf{i}\mathcal{N}_{g_1}}\binom{\mathcal{N}_{g_1}}{j}x^{\mathcal{N}_{g_1}-\mathcal{N}_{g_2}}u^{(k)}\right)_{\hat{\beta}_k-1+i+j}w_2\rangle \qquad (4.6)
\end{aligned}$$

We now prove that each term in the right-hand side of (4.6) is 0.

Since $\alpha_k+\hat{\beta}_k=\sigma(\alpha_k,\hat{\beta}_k)+\epsilon(\alpha_k,\hat{\beta}_k)$, $\alpha_k+\hat{\beta}_k-p-1-i-j=\sigma(\alpha_k,\hat{\beta}_k)+\epsilon(\alpha_k,\hat{\beta}_k)-p-1-i-j$, where by definition, $\sigma(\alpha_k,\hat{\beta}_k)\in P_V^{g_1g_2}$ and $\epsilon(\alpha_k,\hat{\beta}_k)-p-1-i-j\leq -p$. So

$$\left(\binom{\mathcal{N}_{g_1}}{j}x^{-\mathcal{N}_{g_2}}u^{(k)}\right)_{\alpha_k+\hat{\beta}_k-p-1-i-j}\mathcal{Y}(\tilde{w}_1^{(k)}, x)w_2\in C_p(V)\{x\}[\log x]\subset C_{\min(p,q)}(V)\{x\}[\log x].$$

Since $w_3'\in C_{\min(p,q)}(W_3)^{\perp}$, we see that the first term in the right-hand side of (4.6) is 0.

Since $\binom{\mathcal{N}_{g_2}}{j}u^{(k)}$ is also homogeneous with respect to the weight grading and its weight is equal to $\mathrm{wt}\, u^{(k)}$, we have

$$\left(\binom{\mathcal{N}_{g_2}}{j}u^{(k)}\right)_{\alpha_k-p+i+j}\tilde{w}_1^{(k)}\in (W_1)_{[\mathrm{wt}\, u^{(k)}-\alpha_k+p-1-i+j+p_k]}$$

for $i, j\in\mathbb{N}$, $i+j\neq 0$. We have also proved above $\mathrm{wt}\, u^{(k)}-\alpha_k+p-1+p_k=n_1$. So we have

$$\Re(\mathrm{wt}\, u^{(k)}-\alpha_k+p-1-i+p_k+n_2)=\Re(n_1+n_2)-i<\Re(n_1+n_2)\leq N_0^1+N_2^0+m$$

for $i\in\mathbb{Z}_+$. By the induction assumption,

$$\left\langle w_3', \mathcal{Y}\left(\left(\binom{\mathcal{N}_{g_2}}{j}u^{(k)}\right)_{\alpha_k-p+i+j}\tilde{w}_1^{(k)}, x\right)w_2\right\rangle=0.$$

Hence the second term in the right-hand side of (4.6) is 0.

Since $w_2 \in (W_2)_{[n_2]}$ and the coefficients of $e^{\pi \mathbf{i} \mathcal{N}_{g_1}}\binom{\mathcal{N}_{g_1}}{j} x^{\mathcal{N}_{g_1}-\mathcal{N}_{g_2}} u^{(k)}$ are homogeneous with weight $\mathrm{wt}\, u^{(k)}$, we have

$$\left(e^{\pi \mathbf{i} \mathcal{N}_{g_1}}\binom{\mathcal{N}_{g_1}}{j} x^{\mathcal{N}_{g_1}-\mathcal{N}_{g_2}} u^{(k)}\right)_{\hat{\beta}_k-1+i+j} w_2 \in (W_2)_{[\mathrm{wt}\, u^{(k)}-\hat{\beta}_k-i-j+n_2]}.$$

We also know that $\tilde{w}_1^{(k)} \in (W_1)_{[p_k]}$. Using $\mathrm{wt}\, u^{(k)}+p_k = n_1+\alpha_k-p+1$ proved above, $\Re(\alpha_k), \Re(\hat{\beta}_k) \in [0,1)$, and $p \in 2+\mathbb{N}$, we have

$$\begin{aligned}\Re(p_k+\mathrm{wt}\, u^{(k)}-\hat{\beta}_k-i+n_2) &= \Re(n_1+n_2+\alpha_k-p+1-\hat{\beta}_k-i)\\ &< \Re(n_1+n_2)\\ &\le N_0^1+N_2^0+m.\end{aligned}$$

Then by the induction assumption,

$$\langle w_3', \mathcal{Y}(\tilde{w}_1^{(k)}, x)\left(e^{\pi \mathbf{i} \mathcal{N}_{g_1}}\binom{\mathcal{N}_{g_1}}{j} x^{\mathcal{N}_{g_1}-\mathcal{N}_{g_2}} u^{(k)}\right)_{\hat{\beta}_k-1+i+j} w_2\rangle = 0$$

and thus the third term in the right-hand side of (4.6) is 0.

We have proved that each term in the right-hand side of (4.6) is 0. So the left-hand side of (4.6) is also 0.

Similarly, we first rewrite (2.4) as

$$\begin{aligned}&\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\binom{\alpha_1-n_1}{k}(-x)^{k+l}\left(\binom{\mathcal{N}_{g_1}}{j}e^{\pi \mathbf{i}\mathcal{N}_{g_1}}x^{-\mathcal{N}_{g_1}}v\right)_{\alpha_1+\alpha_2-n_1-n_2-k-l}\mathcal{Y}(w_1,x)\\ &\quad-\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\binom{\alpha_1-n_1}{k}e^{\pi \mathbf{i}(\alpha_1-n_1-k-l)}x^{\alpha_1-n_1-k-l}\mathcal{Y}(w_1,x)\left(\binom{\mathcal{N}_{g_1}}{j}v\right)_{\alpha_2-n_2+k+l}\\ &\quad=\sum_{k\in\mathbb{N}}\sum_{l\in\mathbb{N}}\binom{\alpha_2-n_2}{k}x^{\alpha_2-n_2-k-l}\mathcal{Y}\left(\left(\binom{\mathcal{N}_{g_2}}{j}x^{\mathcal{N}_{g_2}}e^{\pi \mathbf{i}\mathcal{N}_{g_1}}x^{-\mathcal{N}_{g_1}}v\right)_{\alpha_1-n_1+k+l}w_1,x\right)\end{aligned}\tag{4.7}$$

by replacing v in (2.4) by $e^{\pi \mathbf{i}\mathcal{N}_{g_1}}x^{-\mathcal{N}_{g_1}}v$. Then using (4.7) with v, α_1, α_2, n_1, n_2, and w_1 taken to be $v^{(l)}$, $\hat{\alpha}_l$, β_l, 1, q, w_1, respectively, we have

$$\begin{aligned}&\langle w_3', \mathcal{Y}(w_1,x)v^{(l)}_{\beta_l-q}\tilde{w}_2^{(l)}\rangle\\ &\quad=\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\binom{\hat{\alpha}_l-1}{i}(-1)^{i+j}e^{\pi \mathbf{i}(\hat{\alpha}_l-1)}x^{i+j-\hat{\alpha}_l+1}\cdot\\ &\qquad\qquad\cdot\left\langle w_3', \left(\binom{\mathcal{N}_{g_1}}{j}e^{\pi \mathbf{i}\mathcal{N}_{g_1}}x^{-\mathcal{N}_{g_1}}v^{(l)}\right)_{\hat{\alpha}_l+\beta_l-1-q-i-j}\mathcal{Y}(w_1,x)\tilde{w}_2^{(l)}\right\rangle\\ &\quad-\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\binom{\beta_l-q}{i}e^{\pi \mathbf{i}(\hat{\alpha}_l-1)}x^{-\hat{\alpha}_l+\beta_l+1-q-i-j}\cdot\end{aligned}$$

$$\cdot \left\langle w_3', \mathcal{Y}\left(\left(\binom{\mathcal{N}_{g_2}}{j} x^{\mathcal{N}_{g_2}} e^{\pi \mathbf{i} \mathcal{N}_{g_1}} x^{-\mathcal{N}_{g_1}} v^{(l)}\right)_{\hat{\alpha}_l - 1 + i + j} w_1, x\right) \tilde{w}_2^{(l)} \right\rangle$$

$$- \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}, j+i \neq 0} \binom{\hat{\alpha}_l - 1}{i} e^{\pi \mathbf{i}(i+j)} x^{-i-j} \left\langle w_3', \mathcal{Y}(w_1, x) \left(\binom{\mathcal{N}_{g_1}}{j} v^{(l)}\right)_{\beta_l - q + i + j} \tilde{w}_2^{(l)} \right\rangle \quad (4.8)$$

We now prove that the right-hand side of (4.8) is 0.

Since $\hat{\alpha}_l + \beta_l = \sigma(\hat{\alpha}_l, \beta_l) + \epsilon(\hat{\alpha}_l, \beta_l)$, $\hat{\alpha}_l + \beta_l - 1 - q - i = \sigma(\hat{\alpha}_l, \beta_l) + \epsilon(\hat{\alpha}_l, \beta_l) - 1 - q - i$, where by definition, $\sigma(\hat{\alpha}_l, \beta_l) \in P_V^{g_1 g_2}$ and $\epsilon(\hat{\alpha}_l, \beta_l) - 1 - q - i \leq -q$. So

$$\begin{aligned} \left(\binom{\mathcal{N}_{g_1}}{j} e^{\pi \mathbf{i} \mathcal{N}_{g_1}} x^{-\mathcal{N}_{g_1}} v^{(l)}\right)_{\hat{\alpha}_l + \beta_l - 1 - q - i - j} \mathcal{Y}(w_1, x) \tilde{w}_2^{(l)} &\in C_q(V)\{x\}[\log x] \\ &\subset C_{\min(p,q)}(V)\{x\}[\log x]. \end{aligned}$$

Since $w_3' \in C_{\min(p,q)}(W_3)^{\perp}$, we see that the first term in the right-hand side of (4.8) is 0.

Since $w_1 \in (W_1)_{[n_1]}$ and the coefficients of $\binom{\mathcal{N}_{g_2}}{j} x^{\mathcal{N}_{g_2}} e^{\pi \mathbf{i} \mathcal{N}_{g_1}} x^{-\mathcal{N}_{g_1}} v^{(l)}$ are homogeneous with weight $\mathrm{wt}\, v^{(l)}$, we have

$$\left(\binom{\mathcal{N}_{g_2}}{j} x^{\mathcal{N}_{g_2}} e^{\pi \mathbf{i} \mathcal{N}_{g_1}} x^{-\mathcal{N}_{g_1}} v^{(l)}\right)_{\hat{\alpha}_l - 1 + i + j} w_1 \in (W_1)_{[\mathrm{wt}\, v^{(l)} - \hat{\alpha}_l - i - j + n_1]}.$$

We also know that $\tilde{w}_2^{(l)} \in (W_2)_{[q_l]}$. Using $\mathrm{wt}\, v^{(l)} + q_l = n_2 + \beta_l - q + 1$ proved above, $\Re(\hat{\alpha}_l), \Re(\beta_l) \in [0, 1)$, and $q \in 2 + \mathbb{N}$, we have

$$\begin{aligned} \Re(\mathrm{wt}\, v^{(l)} - \hat{\alpha}_l - i - j + n_1 + q_l) &= \Re(n_1 + n_2 - \hat{\alpha}_l + \beta_l - q + 1 - i - j) \\ &< \Re(n_1 + n_2) \\ &\leq N_0^1 + N_2^0 + m. \end{aligned}$$

Then by the induction assumption,

$$\left\langle w_3', \mathcal{Y}\left(\left(\binom{\mathcal{N}_{g_2}}{j} x^{\mathcal{N}_{g_2}} e^{\pi \mathbf{i} \mathcal{N}_{g_1}} x^{-\mathcal{N}_{g_1}} v^{(l)}\right)_{\hat{\alpha}_l - 1 + i + j} w_1, x\right) \tilde{w}_2^{(l)} \right\rangle = 0$$

and thus the second term in the right-hand side of (4.8) is 0.

Since $\binom{\mathcal{N}_{g_1}}{j} v^{(l)}$ is homogeneous with weight equal to $\mathrm{wt}\, v^{(l)}$,

$$\left(\binom{\mathcal{N}_{g_1}}{j} v^{(l)}\right)_{\beta_l - q + i + j} \tilde{w}_2^{(l)} \in (W_2)_{[\mathrm{wt}\, v^{(l)} - \beta_l + q - 1 - i - j + q_l]}$$

for $i, j \in \mathbb{N}$, $i + j \neq 0$. We have proved above $\mathrm{wt}\, v^{(l)} - \beta_l + q - 1 + q_l = n_2$. So we have

$$\Re(n_1 + \mathrm{wt}\, v^{(l)} - \beta_l + q - 1 - i - j + q_l) = \Re(n_1 + n_2) - i - j < \Re(n_1 + n_2) \leq N_0^1 + N_2^0 + m$$

for $i, j \in \mathbb{N}$, $i + j \neq 0$. By the induction assumption,

$$\left\langle w_3', \mathcal{Y}(w_1, x)\left(\binom{\mathcal{N}_{g_1}}{j}v^{(l)}\right)_{\beta_l - q + i + j}\tilde{w}_2^{(l)}\right\rangle = 0.$$

Thus the third term in the right-hand side of (4.8) is 0.

We have proved that the right-hand side of (4.8) is 0. So we obtain the left-hand side of (4.8) is 0.

Now we have

$$\begin{aligned}\langle w_3', \mathcal{Y}(w_1, z)w_2\rangle &= \sum_{k=1}^{s}\langle w_3', \mathcal{Y}(u^{(k)}_{\alpha_k - 2}\tilde{w}_1^{(k)}, x)w_2\rangle + \langle w_3', \mathcal{Y}(\tilde{w}_1, z)w_2\rangle\\ &= \langle w_3', \mathcal{Y}(\tilde{w}_1, z)w_2\rangle\\ &= \sum_{l=1}^{t}\langle w_3', \mathcal{Y}(\tilde{w}_1, z)v^{(l)}_{-1}\tilde{w}_2^{(l)}\rangle + \langle w_3', \mathcal{Y}(\tilde{w}_1, z)\tilde{w}_2\rangle\\ &= \langle w_3', \mathcal{Y}(\tilde{w}_1, z)\tilde{w}_2\rangle\\ &= 0,\end{aligned}$$

proving (4.2) for all $w_1 \in W_1$ and $w_2 \in W_2$.

Using the $L(-1)$-derivative property for $\mathcal{Y}$, we have

$$\langle w_3', \mathcal{Y}(w_1, \xi)w_2\rangle = \langle w_3', \mathcal{Y}(e^{(\xi - z)L_{W_1}(-1)}w_1, z)w_2\rangle = 0$$

on the region $|\xi - z| < |z|$. But $\langle w_3', \mathcal{Y}(w_1, \xi)w_2\rangle$ is an analytic function of ξ. This analytic function equal to 0 on a region means that it is equal to 0 on its domain. So we obtain $\langle w_3', \mathcal{Y}(w_1, \xi)w_2\rangle = 0$ on the region $\xi \neq 0$. Thus $\langle w_3', \mathcal{Y}_{n,k}(w_1)w_2\rangle = 0$ for $w_1 \in W_1$, $w_2 \in W_2$, $n \in \mathbb{C}$, $k = 1, \dots, K$. Since $\mathcal{Y}$ is surjective, we must have $w_3' = 0$, proving the injectivity of f. □

Remark 4.2. The inequality (4.1) is an analogue of the (untwisted) Nahm inequality (3.8) in Theorem 3.5 in [H8]. It is an ananlogue but not a generalization because $p, q \neq 1$ in (4.1). In the case that g_1 and g_2 are of finite orders and $p = q = 2$, (4.1) has been proved in [YZ].

Remark 4.3. By Proposition 3.5, "a generalized surjective product W_3" in Theorem 4.1 can be replaced by "a weak surjective product W_3."

5 $P(z)$-tensor products of C_n-cofinite grading-restricted generalized twisted V-modules

In this section, we consider two categories of grading-restricted generalized twisted V-modules for a Möbius vertex algebra or a quasi-vertex operator algebra V (see [FHL] and [HLZ1]) and construct $P(z)$-tensor product bifunctors for these categories.

In this section, V is a Möbius vertex algebra. Note that automorphisms of V also commute with the operator $L_V(1)$. Let G be an abelian group of automorphisms of V and $n \in 2+\mathbb{N}$. Let $\mathcal{C}_n^G$ be the category of C_n-cofinite grading-restricted generalized g-twisted V-modules for $g \in G$ and let $\widetilde{\mathcal{C}}_n^G$ be the category of C_n-cofinite grading-restricted generalized g-twisted V-modules with G-actions for $g \in G$. In Subsection 5.1, for $z \in \mathbb{C}^\times$, we construct a $P(z)$-tensor product bifunctor for the category $\mathcal{C}_n^G$. Then in Subsection 5.2, we construct a $P(z)$-tensor product bifunctor for the category $\widetilde{\mathcal{C}}_n^G$.

In the case that V is C_2-cofinite and of positive energy, by the main result in [Ta1], every finitely-generated lower-bounded generalized twisted V-module is C_n-cofinite for $n \in 2+\mathbb{N}$. On the other hand, by Theorem 3.3, for $n \in 2+\mathbb{N}$, every C_n-cofinite lower-bounded g-twisted generalized V-module is finitely generated. So in this case, the category $\mathcal{C}_n^G$ (or $\widetilde{\mathcal{C}}_n^G$) is the same as the category of finitely-generated lower-bounded generalized twisted V-modules (or the category of finitely-generated lower-bounded generalized twisted V-modules with G-actions). Note that in this case, $\mathcal{C}_n^G$ and $\widetilde{\mathcal{C}}_n^G$ are independent of n.

5.1 $P(z)$-tensor product bifunctor for $\mathcal{C}_n^G$

In this subsection, we construct a $P(z)$-tensor product bifunctor for $\mathcal{C}_n^G$. We first recall twisted $P(z)$-intertwining maps in [DH].

Definition 5.1. Let g_1, g_2 be commuting automorphisms of V, W_1, W_2, W_3 generalized g_1-, g_2-, g_1g_2-twisted V-modules, respectively, and $z \in \mathbb{C}^\times$. A *twisted $P(z)$-intertwining map of type* $\binom{W_3}{W_1W_2}$ is a linear map $I : W_1 \otimes W_2 \to \overline{W}_3$ given by $I(w_1 \otimes w_2) = \mathcal{Y}(w_1, z)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$, where $\mathcal{Y}$ is a twisted intertwining operator of type $\binom{W_3}{W_1W_2}$.

Next we recall the definition of $P(z)$-tensor product in [DH] when the category is $\mathcal{C}_n^G$.

Definition 5.2. Let $g_1, g_2 \in G$ and let W_1 and W_2 be grading-restricted generalized g_1- and g_2-twisted V-modules, respectively, in the category $\mathcal{C}_n^G$. A *$P(z)$-product of W_1 and W_2 in $\mathcal{C}_n^G$* is a pair (W_3, I) consisting of a C_n-cofinite grading-restricted generalized g_1g_2-twisted V-module W_3 and a twisted $P(z)$-intertwining map I of type $\binom{W_3}{W_1W_2}$. A *$P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}_n^G$* is a $P(z)$-product $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ satisfying the following universal property: For any $P(z)$-product (W_3, I) of W_1 and W_2, there exists a unique module map $f : W_1 \boxtimes_{P(z)} W_2 \to W_3$ such that we have the commutative diagram

$$\begin{tikzcd}
W_1 \otimes W_2 \arrow[r, "I"] \arrow[d, "\boxtimes_{P(z)}"'] & \overline{W}_3 \\
\overline{W_1 \boxtimes_{P(z)} W_2} \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

where $\bar{f}$ is the natural extension of f to $\overline{W_1 \boxtimes_{P(z)} W_2}$.

In [DH], a construction of a $P(z)$-tensor product bifunctor using twisted intertwining maps is given based on some assumptions (see Assumption 4.4 in [DH]) as mentioned above in the introduction. But the first two assumptions in Assumption 4.4 in [DH] do not hold for the category $\mathcal{C}_n^G$. On the other hand, many of the results and their proofs in [DH] still hold in the general setting in our case. We now give a construction of the $P(z)$-tensor product bifunctor for the category $\mathcal{C}_n^G$ by using the results in [DH] that still hold, the results in Sections 3 and 4, and the method in [H8]. Note that for a lower-bounded generalized twisted V-module, we still have the contragredient of this module.

We recall the construction in [DH] in the case that the category is $\mathcal{C}_n^G$, even though Assumption 4.4 in [DH] does hold for the category $\mathcal{C}_n^G$. Given a $P(z)$-product (W_3, I) of W_1 and W_2 in $\mathcal{C}_n^G$, for $w_3' \in W_3'$, we have an element $\lambda_{I,w_3'} \in (W_1 \otimes W_2)^*$ defined by

$$\lambda_{I,w_3'}(w_1 \otimes w_2) = \langle w_3', I(w_1 \otimes w_2)\rangle \tag{5.1}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. Let $W_1 \boxslash_{P(z)} W_2$ be the subspace of $(W_1 \otimes W_2)^*$ spanned by $\lambda_{I,w_3'}$ for all $P(z)$-products (W_3, I) and $w_3' \in W_3'$. We define $(W_1 \boxslash_{P(z)} W_2)_{[n]}^{[\alpha]}$ to be the subspace of $W_1 \boxslash_{P(z)} W_2$ spanned by $\lambda_{I,w_3'}$ for all $P(z)$-products (W_3, I) and $w_3' \in w_3' \in (W_3')_{[n]}^{[\alpha]} \subset W_3'$ for $n \in \mathbb{C}$ and $\alpha \in P_{W_3'}^g$. Then

$$W_1 \boxslash_{P(z)} W_2 = \coprod_{n\in\mathbb{C}} \coprod_{\alpha+\mathbb{Z}\in\mathbb{C}/\mathbb{Z}} (W_1 \boxslash_{P(z)} W_2)_{[n]}^{[\alpha]}.$$

We define a vertex operator map

$$Y_{W_1 \boxslash_{P(z)} W_2} : V \otimes (W_1 \boxslash_{P(z)} W_2) \to (W_1 \boxslash_{P(z)} W_2)\{x\}[\log x]$$

by

$$Y_{W_1 \boxslash_{P(z)} W_2}(v, x)\lambda_{I,w_3'} = \lambda_{I, Y_{W_3'}(v,x)w_3'} \tag{5.2}$$

for $v \in V$ and $\lambda_{I,w_3'} \in W_1 \boxslash_{P(z)} W_2$. We recall the following result in [DH]:

Proposition 5.3 (Proposition 4.3 in [DH]). *The pair $(W_1 \boxslash_{P(z)} W_2, Y^{g_1 g_2}_{W_1 \boxslash_{P(z)} W_2})$ is a generalized $(g_1 g_2)^{-1}$-twisted V-module.*

Note that even though we have not shown that $W_1 \boxslash_{P(z)} W_2$ is grading-restricted yet, its contragredient $(W_1 \boxslash_{P(z)} W_2)'$ is still well defined. Let $W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxslash_{P(z)} W_2)'$. The results and proofs in [DH] in fact shows that $\boxtimes_{P(z)}$ gives a functor from $\mathcal{C}_n^G \times \mathcal{C}_n^G$ to the category of generalized g-twisted V-module for $g \in G$. We want to show in particular that the image of this functor is in $\mathcal{C}_n^G$.

We have the candidate $W_1 \boxtimes_{P(z)} W_2$ for our $P(z)$-tensor product twisted module. To obtain a $P(z)$-tensor product of W_1 and W_2, we also need a $P(z)$-intertwining map $\boxtimes_{P(z)}$. We recall the construction of $\boxtimes_{P(z)}$ in [DH].

The construction of $\boxtimes_{P(z)}$ in [DH] is given after Assumption 4.4 in [DH]. But this construction works without Assumption 4.4. In fact, Assumption 4.4 in [DH] contains three assumptions. In our case, the first and second assumptions in Assumption 4.4 in [DH] do not hold. But we will still be able to prove below that $W_1 \boxtimes_{P(z)} W_2$ is a $P(z)$-tensor product of W_1 and W_2 in the category $\mathcal{C}_n^G$.

We need Proposition 4.5 in [DH]. Let W be a generalized $(g_1g_2)^{-1}$-twisted V-module and $f : W \to W_1 \boxbslash_{P(z)} W_2$ a V-module map. Note that in [DH], W is required to be in the category $\mathcal{C}$ considered there. But what we know now is that $W_1 \boxtimes_{P(z)} W_2$ is a generalized $(g_1g_2)^{-1}$-twisted V-module. In this section, we will use Proposition 4.5 in [DH] to prove that $W_1 \boxtimes_{P(z)} W_2$ is C_n-cofinite grading-restricted generalized $(g_1g_2)^{-1}$-twisted V-module. So we cannot assume that W is an object of $\mathcal{C}_n^G$. But the construction and Proposition 4.5 in [DH] works even when we do not know whether W is an object of $\mathcal{C}_n^G$.

Since the double contragredient of W might not be equivalent to W, we cannot view elements of W'' as elements of W. But we know that an element of W' is given by its pairing with all elements of W. We write the pairing between W and W' by $\langle w, w'\rangle$ instead of $\langle w', w\rangle$ for $w \in W$ and $w' \in W'$. We define a linear map

$$\begin{aligned} f' : W_1 \otimes W_2 &\to \overline{W'} = \prod_{n\in\mathbb{C}} W^*_{[n]} \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}_f(w_1, z)w_2 \end{aligned}$$

by

$$\langle w, f'(w_1 \otimes w_2)\rangle = (f(w))(w_1 \otimes w_2) \tag{5.3}$$

for $w_1 \in W_1$, $w_2 \in W_2$ and $w \in W$. We then define another linear map

$$\begin{aligned} \mathcal{Y}_f : W_1 \otimes W_2 &\to W'\{x\}[\log x] \\ w_1 \otimes w_2 &\mapsto f'(w_1 \otimes w_2) \end{aligned}$$

by

$$\mathcal{Y}_f(w_1, x)w_2 = x^{L_{W'}(0)}e^{-(\log z)L_{W_3}(0)}\mathcal{Y}_f(x^{-L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1, z)x^{-L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2 \tag{5.4}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. Note that by our notation, $f' = \mathcal{Y}_f(\cdot z)\cdot$.

Proposition 5.4 (Proposition 4.5 in [DH]). *The linear map $f' = \mathcal{Y}_f(\cdot z)\cdot$ given by* (5.3) *and $\mathcal{Y}_f : W_1 \otimes W_2 \to W'\{x\}[\log x]$ given by* (5.3) *and* (5.4) *are a twisted $P(z)$-intertwining map and a twisted intertwining operator, respectively, of type $\binom{W'}{W_1W_2}$. In particular, in the case that $W = W_1 \boxbslash_{P(z)} W_2$ and $f = 1_{W_1 \boxbslash_{P(z)} W_2} : W \to W_1 \boxbslash_{P(z)} W_2$ is the identity map, we obtain a twisted $P(z)$-intertwining map $1'_{W_1 \boxbslash_{P(z)} W_2}$ and a twisted intertwining operator $\mathcal{Y}_{1_{W_1 \boxbslash_{P(z)} W_2}}$ of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1W_2}$.*

We denote the $P(z)$-intertwining map of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1W_2}$ in Proposition 5.4 by $\boxtimes_{P(z)}$. Let

$$w_1 \boxtimes_{P(z)} w_{2n} = \boxtimes_{P(z)}(w_1 \otimes w_2) = \mathcal{Y}(w_1, z)w_2 \in \overline{W_1 \boxtimes_{P(z)} W_2}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. The element $w_1 \boxtimes_{P(z)} w_2$ is the tensor product of the elements w_1 and w_2. By (5.3), we have

$$\lambda(w_1 \otimes w_2) = \langle \lambda, w_1 \boxtimes_{P(z)} w_2\rangle \tag{5.5}$$

for $\lambda \in W_1 \boxbackslash_{P(z)} W_2$, $w_1 \in W_1$ and $w_2 \in W_2$.

We also need a result on lower-bounded generalized surjective product of two C_n-cofinite grading-restricted generalized V-modules.

Proposition 5.5. *Let g_1 and g_2 be commuting automorphisms of V and W_1 and W_2 C_n-cofinite grading-restricted generalized g_1- and g_2-twisted V-modules, respectively. Then the lower-bounded generalized surjective products of W_1 and W_2 are uniformly lower bounded, that is, there exists $N_0 \in \mathbb{Z}$ depending only on W_1 and W_2 such that for any lowr-bounded generalized surjective product W_3 of W_1 and W_2, $(W_3)_{[m]} = 0$ when $\Re(m) < N_0$.*

Proof. We consider the set of $\dim(W_3/C_1(W_3))$ for all lower-bounded generalized surjective product W_3 of W_1 and W_2. By Theorem 4.1, this set of nonnegative integers is bounded from above by $\dim(W_1/C_1(W_1)) \dim(W_2/C_1(W_2))$. Hence there must be a maximum of this set. Let $W_3^{\max}$ be a lower-bounded generalized surjective product of W_1 and W_2 such that $\dim W_3/C_1(W_3)$ is the maximum of the set. Since $W_3^{\max}$ is lower-bounded, there exists $N_0 \in \mathbb{Z}$ such that $(W_3^{\max})_{[n]} = 0$ when $\Re(n) < N_0$.

Given any lower-bounded generalized surjective product W_3 of W_1 and W_2, we have a surjective intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$. For any fixed $z \in \mathbb{C}^\times$, we have a $P(z)$-intertwining map $I_{\mathcal{Y}} = \mathcal{Y}(\cdot, z)\cdot$ of type $\binom{W_3}{W_1 W_2}$. The $P(z)$-intertwining map $I_{\mathcal{Y}}$ gives a V-module map $I'_{\mathcal{Y}}$ from W_3' to the generalized $(g_1 g_2)^{-1}$-twisted V-module $W_1 \boxbackslash_{P(z)} W_2$ defined by

$$(I'_{\mathcal{Y}}(w_3'))(w_1 \otimes w_2) = \langle w_3', I_{\mathcal{Y}}(w_1 \otimes w_2) \rangle = \langle w_3', \mathcal{Y}(w_1, z) w_2 \rangle$$

for $w_1 \in W_1$, $w_2 \in W_2$, and $w_3' \in W_3'$. Since the intertwining operator $\mathcal{Y}$ is surjective, this V-module map is injective. The image of W_3' under this V-module map is a lower-bounded generalized V-submdoule of $W_1 \boxbackslash_{P(z)} W_2$. The same is also true for $W_3^{\max}$. Let W be the sum of the image of W_3' under the V-module map from W_3' to $W_1 \boxbackslash_{P(z)} W_2$ and the image of $(W_3^{\max})'$ under the V-module map from $(W_3^{\max})'$ to $W_1 \boxbackslash_{P(z)} W_2$. Then W is also a lower-bounded generalized V-submodule of $W_1 \boxbackslash_{P(z)} W_2$. Let $J : W \to W_1 \boxbackslash_{P(z)} W_2$ be the inclusion map. Then by Proposition 5.4, we have a twisted intertwining operator $\mathcal{Y}_J$ of type $\binom{W'}{W_1 W_2}$. Since J is injective, $\mathcal{Y}_{J'}$ is surjective. Since W is lower bounded, W' is also lower bounded. By Theorem 4.1, W' is a C_n-cofinite grading-restricted generalized $g_1 g_2$-twisted V-module and $\dim W'/C_n(W') \leq \dim W_3^{\max}/C_1(W_3^{\max})$.

We now prove $(W')_{[m]} = 0$ when $\Re(m) < N_0$. By definition, we have an injective V-module map from $(W_3^{\max})'$ to W. Its adjoint is a surjective V-module map $f : W' \to W_3^{\max}$. Since f is a V-module map, $f(C_n(W')) \subset C_n(W_3^{\max})$. Then the map f induces a surjective linear map $\bar{f} : W'/C_n(W') \to W_3^{\max}/C_n(W_3^{\max})$. In particular, we have $\dim(W'/C_n(W')) \geq \dim(W_3^{\max}/C_n(W_3^{\max}))$. But we already have $\dim(W'/C_n(W')) \leq \dim(W_3^{\max}/C_n(W_3^{\max}))$. So we obtain

$$\dim(W'/C_n(W')) = \dim(W_3^{\max}/C_n(W_3^{\max})).$$

Thus $\bar{f}$ is injective and $\ker \bar{f} = 0$. Since $\bar{f}$ is induced from f, we obtain $\ker f \subset C_n(W')$.

If $(W')_{[m]} \neq 0$ for some $m \in \mathbb{C}$ satisfying $\Re(m) < N_0$, then we can find $m_0 \in \mathbb{C}$ such that $\Re(m_0) < N_0$, $(W')_{m_0]} \neq 0$, and $(W')_{[m]]} = 0$ for $\Re(m) < m_0 - 1$ since W' is lower-bounded. Let $w' \in W^*_{[m_0]}$. Then $f(w') \in (W_3^{\max})_{[m_0]}$. Since $\Re(m_0) < N_0$, we have $f(w') = 0$ from the definition of N_0. Then $w' \in \ker f \subset C_n(W')$. Then w' is a linear combination of elements of the form $v_{\alpha-n}\tilde{w}'$ for homogeneous $v \in V^{[\alpha]}$ and homogeneous $\tilde{w}' \in W'$. But $\mathrm{wt}\, v_{\alpha-n}\tilde{w}' = \mathrm{wt}\, v - \alpha + n - 1 + \mathrm{wt}\, \tilde{w}'$. Since w' is a linear combination of such elements, we must have $m_0 = \mathrm{wt}\, v_{\alpha-n}\tilde{w}' = \mathrm{wt}\, v - \alpha + n - 1 + \mathrm{wt}\, \tilde{w}'$. Then we have $\mathrm{wt}\, \tilde{w}' = m_0 - \mathrm{wt}\, v + \alpha - n + 1 < m_0 - 1$. Since $(W')_{[m]]} = 0$ for $\Re(m) < m_0 - 1$, we obtain $\tilde{w}' = 0$ and therefore $w' = 0$. Thus we obtain $(W')_{m_0]} = 0$. Contradiction. So we have proved that $(W')_{[m]} = 0$ when $\Re(m) < N_0$. But $(W')_{[m]} = 0$ when $\Re(m) < N_0$ implies $W_{[m]} = 0$ when $\Re(m) < N_0$.

Since the image $I'_{\mathcal{Y}}(W_3')$ of W_3' under $I'_{\mathcal{Y}}$ in $W_1 \boxbackslash_{P(z)} W_2$ is in W, we also have $(I'_{\mathcal{Y}}(W_3'))_{[m]} = 0$ when $\Re(m) < N_0$. Since $\mathcal{Y}$ is surjective, $I'_{\mathcal{Y}}$ is injective. So we also have $(W_3')_{[m]} = 0$ when $\Re(m) < N_0$. Thus we obtain $(W_3)_{[m]} = 0$ when $\Re(m) < N_0$. ☐

Theorem 5.6. *Let $g_1, g_2 \in G$, W_1, W_2, W_3 grading-restricted generalized g_1-, g_2-, g_1g_2-twisted V-modules, respectively, in the category $\mathcal{C}_n^G$ and $z \in \mathbb{C}^\times$. Then $W_1 \boxbackslash_{P(z)} W_2$ is a grading-restricted generalized $(g_1g_2)^{-1}$-twisted V-module and $W_1 \boxtimes_{P(z)} W_2$ is a C_n-cofinite grading-restricted generalized g-twisted V-module. Moreover, $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ is a $P(z)$-tensor product of W_1 and W_2 in the category $\mathcal{C}_n^G$.*

Proof. For any $P(z)$-product (W_3, I), we have a module map $I' : W_3' \to W_1 \boxbackslash_{P(z)} W_2$ given by $(I'(w_3'))(w_1 \otimes w_2) = \langle w_3', I(w_1 \otimes w_2)\rangle$ for $w_3' \in W_3'$, $w_1 \in W_1$, and $w_2 \in W_2$. Consider the image $I'(W_3) \in W_1 \boxbackslash_{P(z)} W_2$ of W_3' under I'. By Proposition 5.4, the inclusion map $J : I'(W_3) \to W_1 \boxbackslash_{P(z)} W_2$ gives a $P(z)$-intertwining operator $\mathcal{Y}_J$ of type $\binom{I'(W_3)'}{W_1 W_2}$. Since J is injective, $\mathcal{Y}_J$ is surjective By Proposition 5.5, $(I'(W_3)')_{[m]} = 0$ when $\Re(m) < N_0$, where $N_0 \in \mathbb{Z}$ depends only on W_1 and W_2 and its existence is given by Proposition 5.5. Hence $I'(W_3)$ has the same property, that is, $I'(W_3)_{[m]} = 0$ when $\Re(m) < N_0$.

By definition, $W_1 \boxbackslash_{P(z)} W_2$ is in fact the sum of $I'(W_3)$ for all $P(z)$-products of the form (W_3, I). From the property $I'(W_3)_{[m]} = 0$ when $\Re(m) < N_0$, the sum of all such $I'(W_3)$ satisfies the same property, that is, $(W_1 \boxbackslash_{P(z)} W_2)_{[m]} = 0$ when $\Re(m) < N_0$. Thus $W_1 \boxbackslash_{P(z)} W_2$ is in fact lower-bounded. So $W_1 \boxtimes_{P(z)} W_2$ is also lower-bounded.

The same argument shows that the twisted intertwining operator $\mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}$ given in Proposition 5.4 is surjective. By Theorem 4.1, we see that $W_1 \boxtimes_{P(z)} W_2$ is C_n-cofinite. Then by Proposition 3.3, $W_1 \boxtimes_{P(z)} W_2$ is quasi-finite-dimensional, and, in particular, is grading-restricted. As the contragredient of $W_1 \boxtimes_{P(z)} W_2$, $W_1 \boxbackslash_{P(z)} W_2$ is also grading-restricted.

Finally, we prove that $W_1 \boxtimes_{P(z)} W_2$ together with the twisted $P(z)$-intertwining map $\boxtimes_{P(z)}$ is a $P(z)$-tensor product of W_1 and W_2. Given a $P(z)$-product (W_3, I), we have a module map $I' : W_3' \to W_1 \boxbackslash_{P(z)} W_2$ defined above. Then the adjoint $\eta : W_1 \boxtimes_{P(z)} W_2 \to W_3$ of I' is also a module map. Let $\bar{\eta} : \overline{W_1 \boxtimes_{P(z)} W_2} \to \overline{W}_3$ be the natural extension of η Then by definitions, we have

$$\langle w_3', \bar{\eta}(w_1 \boxtimes_{P(z)} w_2)\rangle = \langle I'(w_3'), w_1 \boxtimes_{P(z)} w_2)\rangle = (I'(w_3'))(w_1 \otimes w_2) = \langle w_3', I(w_1 \otimes w_2)\rangle \quad (5.6)$$

for $w_1 \in W_1$, $w_2 \in W_2$, and $w_3' \in W_3'$. So we obtain $\bar{\eta}(w_1 \boxtimes_{P(z)} w_2) = I(w_1 \otimes w_2)$. Since $\bar{\eta}(w_1 \boxtimes_{P(z)} w_2)$ can be rewritten as $(\bar{\eta} \circ \boxtimes_{P(z)})(w_1 \otimes w_2)$, we obtain $\bar{\eta} \circ \boxtimes_{P(z)} = I$. The uniqueness of η follows from the definition of η. We have proved the universal property for $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$. So $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ is a $P(z)$-tensor product of W_1 and W_2. $\square$

Theorem 5.6 in fact assigns each object (W_1, W_2) in the category $\mathcal{C}_n^G \times \mathcal{C}_n^G$ an object $W_1 \boxtimes_{P(z)} W_2$ in $\mathcal{C}_n^G$. As in [DH], we can also assign a morphism (f_1, f_2) in $\mathcal{C}_n^G \times \mathcal{C}_n^G$ a morphism $f_1 \boxtimes_{P(z)} f_2$ in $\mathcal{C}_n^G$ in the same way by using the universal property of a $P(z)$-tensor product. We then also obtain the following result whose proof is the same as the proof of Theorem 4.7 in [DH]:

Theorem 5.7. *The assignment, denoted by $\boxtimes_{P(z)}$, given by $(W_1, W_2) \mapsto W_1 \boxtimes_{P(z)} W_2$ and $(f_1, f_2) \mapsto f_1 \boxtimes_{P(z)} f_2$ is a functor from $\mathcal{C}_n^G \times \mathcal{C}_n^G$ to $\mathcal{C}_n^G$.*

5.2 $P(z)$-tensor product bifunctor for $\widetilde{\mathcal{C}}_n^G$

In this subsection, for $z \in \mathbb{C}^\times$, we give a $P(z)$-tensor product bifunctor for the category $\widetilde{\mathcal{C}}_n^G$ of C_n-cofinite grading-restricted generalized g-twisted V-modules with G-actions for $g \in G$.

Let $g_1, g_2 \in G$. Given C_n-cofinite grading-restricted generalized g_1- and g_2-twisted V-modules W_1 and W_2, respectively, for $g_1, g_2 \in G$. Since W_1 and W_2 are G-modules, $(W_1 \otimes W_2)^*$ is also a G-module. In particular, $(W_1 \otimes W_2)^*$ has an action of $(g_1 g_2)^{-1}$.

Given a $P(z)$-product (W_3, I) of W_1 and W_2 in $\widetilde{\mathcal{C}}_n^G$, for $w_3' \in W_3'$, we have $\lambda_{I, w_3'} \in (W_1 \otimes W_2)^*$ defined by (5.1). Let $W_1 \boxbslash_{P(z)} W_2$ be the subspace of $(W_1 \otimes W_2)^*$ spanned by $\lambda_{I, w_3'}$ for all $P(z)$-products (W_3, I) and $w_3' \in W_3'$. Note that in general, $W_1 \boxbslash_{P(z)} W_2$ here might be different from $W_1 \boxbslash_{P(z)} W_2$ in the case that W_1 and W_2 are viewed as grading-restricted generalized g-twisted V-modules without g and G-actions. For simplicity, we use the same notation $\boxbslash_{P(z)}$, but we will call them $W_1 \boxbslash_{P(z)} W_2$ in $\widetilde{\mathcal{C}}_n^G$ and $W_1 \boxbslash_{P(z)} W_2$ in $\mathcal{C}_n^G$ to distinguish them when it is necessary.

As discussed above, G acts on $(W_1 \otimes W_2)^*$. By the definition of twisted $P(z)$-intertwining map in the category $\widetilde{\mathcal{C}}_n^G$, for a $P(z)$-product (W_3, I) of W_1 and W_2 in $\widetilde{\mathcal{C}}_n^G$, and $w_3' \in (W_3')^{[\alpha]}$, we have

$$\begin{aligned}
(h\lambda_{I, w_3'})(w_1 \otimes w_2) &= \lambda_{I, w_3'}(h^{-1}(w_1 \otimes w_2)) \\
&= \lambda_{I, w_3'}(h^{-1} w_1 \otimes h^{-1} w_2) \\
&= \langle w_3', I(h^{-1} w_1 \otimes h^{-1} w_2) \rangle \\
&= \langle w_3', h^{-1} I(w_1 \otimes w_2) \rangle \\
&= \langle h w_3', I(w_1 \otimes w_2) \rangle \\
&= \lambda_{I, h w_3'}(w_1 \otimes w_2)
\end{aligned}$$

for $w_1 \in W_1$ and $w_2 \in W_2$. So we obtain $h\lambda_{I, w_3'} = \lambda_{I, h w_3'}$, which means that $h\lambda_{I, w_3'} \in W_1 \boxbslash_{P(z)} W_2$. Thus the G-action on $(W_1 \otimes W_2)^*$ induces a G-action on $W_1 \boxbslash_{P(z)} W_2$.

Taking $h = (g_1 g_2)^{-1}$ in $h\lambda_{I, w_3'} = \lambda_{I, h w_3'}$, we obtain $((g_1 g_2)^{-1} \lambda_{I, w_3'}) = \lambda_{I, (g_1 g_2)^{-1} w_3'}$. Then we have

$$((g_1 g_2)^{-1} - e^{2\pi \mathbf{i} \alpha})^k \lambda_{I, w_3'} = \lambda_{I, (g_1 g_2)^{-1} - e^{2\pi \mathbf{i} \alpha})^k w_3'}$$

for $k \in \mathbb{Z}_+$. Since $w_3' \in (W_3')^{[\alpha]}$, there exists $k \in \mathbb{Z}_+$ such that $((g_1g_2)^{-1} - e^{2\pi \mathbf{i}\alpha})^k w_3' = 0$. So we obtain $((g_1g_2)^{-1} - e^{2\pi \mathbf{i}\alpha})^k \lambda_{I,w_3'} = 0$. Let $(W_1 \boxbackslash_{P(z)} W_2)^{[\alpha]}$ be the generalized eigenspace of the action of $(g_1g_2)^{-1}$ on $W_1 \boxbackslash_{P(z)} W_2$ with eigenvalue $e^{2\pi \mathbf{i}\alpha}$. We have proved that $\lambda_{I,w_3'} \in (W_1 \boxbackslash_{P(z)} W_2)^{[\alpha]}$. Since $W_1 \boxbackslash_{P(z)} W_2$ is spanned by elements of the form $\lambda_{I,w_3'}$, $W_1 \boxbackslash_{P(z)} W_2$ is a direct sum of generalized eigenspaces of the action of $(g_1g_2)^{-1}$ on $W_1 \boxbackslash_{P(z)} W_2$, that is,

$$W_1 \boxbackslash_{P(z)} W_2 = \coprod_{\alpha + \mathbb{Z} \in \mathbb{C}/\mathbb{Z}} (W_1 \boxbackslash_{P(z)} W_2)^{[\alpha]}.$$

We define a vertex operator map

$$Y_{W_1 \boxbackslash_{P(z)} W_2} : V \otimes (W_1 \boxbackslash_{P(z)} W_2) \to (W_1 \boxbackslash_{P(z)} W_2)\{x\}[\log x]$$

using the same formula (5.2) as in the case in the category $\mathcal{C}_n^G$. We consider the identity operator $1_{W_1 \boxbackslash_{P(z)} W_2} : W_1 \boxbackslash_{P(z)} W_2 \to W_1 \boxbackslash_{P(z)} W_2$ on $W_1 \boxbackslash_{P(z)} W_2$. By taking $f = 1_{W_1 \boxbackslash_{P(z)} W_2}$ in Proposition 5.4, we obtain a twisted $P(z)$-intertwining map $1'_{W_1 \boxbackslash_{P(z)} W_2}$ in the category $\mathcal{C}_n^G$. But note that $W_1 \boxbackslash_{P(z)} W_2$ in this subsection might be different from $W_1 \boxbackslash_{P(z)} W_2$ in the preceding subsection. This twisted $P(z)$-intertwining map $\boxtimes_{P(z)}$ in the category $\mathcal{C}_n^G$ might also be different from the twisted $P(z)$-intertwining map $\boxtimes_{P(z)}$ obtained in the last part of Proposition 5.4. For simplicity, we will use the same notation $\boxtimes_{P(z)}$ to denote $1'_{W_1 \boxbackslash_{P(z)} W_2}$. We also have $w_1 \boxtimes_{P(z)} w_2$ of $w_1 \in W_1$ and $w_2 \in W_2$ defined to be $\boxtimes_{P(z)}(w_1 \otimes w_2)$. Proposition 5.4 does not tell us whether $\boxtimes_{P(z)}$ is a twisted $P(z)$-intertwining map in the category $\widetilde{\mathcal{C}}_n^G$. But we have

$$\begin{aligned}
\langle \lambda, h(w_1 \boxtimes_{P(z)} w_2) \rangle &= \langle h^{-1}\lambda, w_1 \boxtimes_{P(z)} w_2 \rangle \\
&= (h^{-1}\lambda)(w_1 \otimes w_2) \\
&= \lambda(h(w_1 \otimes w_2)) \\
&= \lambda(hw_1 \otimes hw_2) \\
&= \langle \lambda, hw_1 \boxtimes_{P(z)} hw_2 \rangle
\end{aligned}$$

for $\lambda \in W_1 \boxbackslash_{P(z)} W_2$ in $\widetilde{\mathcal{C}}_n^G$, $w_1 \in W_1$, and $W_2 \in W_2$. Thus we obtain

$$h(w_1 \boxtimes_{P(z)} w_2) = hw_1 \boxtimes_{P(z)} hw_2$$

for $h \in G$, $w_1 \in W_1$ and $w_2 \in W_2$. So $\boxtimes_{P(z)}$ is in fact a twisted $P(z)$-intertwining map in the category $\widetilde{\mathcal{C}}_n^G$. In particular, the twisted intertwining operator $\mathcal{Y}_{1_{W_1 \boxbackslash_{P(z)} W_2}}$ is a twisted intertwining operator in the category $\widetilde{\mathcal{C}}_n^G$.

Theorem 5.8. *Let g_1, g_2 be commuting automorphisms of V, W_1, W_2, W_3 grading-restricted generalized g_1-, g_2-, g_1g_2-twisted V-modules, respectively, in the category $\widetilde{\mathcal{C}}_n^G$ and $z \in \mathbb{C}^\times$. Then $W_1 \boxbackslash_{P(z)} W_2$ is a grading-restricted generalized $(g_1g_2)^{-1}$-twisted V-module with a G-action and its contragredient $W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxbackslash_{P(z)} W_2)'$ is a C_n-cofinite grading-restricted generalized g-twisted V-module with a G-action. Moreover, $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ is a $P(z)$-tensor product of W_1 and W_2 in the category $\widetilde{\mathcal{C}}_n^G$.*

Proof. Since twisted $P(z)$-intertwining maps in the category $\widetilde{\mathcal{C}}_n^G$ are also twisted $P(z)$-intertwining maps in the category $\mathcal{C}_n^G$, $W_1 \boxbslash_{P(z)} W_2$ in this subsection is a generalized $(g_1g_1)^{-1}$-twisted V-submodule of what is denoted using the same notation $W_1 \boxbslash_{P(z)} W_2$ in the preceding subsection. By Theorem 5.6, $W_1 \boxbslash_{P(z)} W_2$ in the preceding subsection is grading-restricted, its generalized $(g_1g_1)^{-1}$-twisted V-submodule $W_1 \boxbslash_{P(z)} W_2$ in this subsection is also grading-restricted. Thus its contragredient $W_1 \boxtimes_{P(z)} W_2$ is also grading-restricted.

Since $1_{W_1 \boxbslash_{P(z)} W_2}$ is injective, the twisted intertwining operator $\mathcal{Y}_{1_{W_1 \boxbslash_{P(z)} W_2}}$ is surjective. Thus by Theorem 4.1, $W_1 \boxtimes_{P(z)} W_2$ is C_n-cofinite and is thus in $\widetilde{\mathcal{C}}_n^G$. Now the same argument as in the last part of the proof of Theorem 5.6 shows that $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ is a $P(z)$-tensor product of W_1 and W_2 in the category $\widetilde{\mathcal{C}}_n^G$. □

We can also define $f_1 \boxtimes_{P(x)} f_2$ for two morphisms f_1 and f_2 in $\widetilde{\mathcal{C}}_n^G$. Then the following result is obtained in the same way as Theorem 5.7:

Theorem 5.9. *The assignment, denoted by $\boxtimes_{P(z)}$, given by $(W_1, W_2) \mapsto W_1 \boxtimes_{P(z)} W_2$ and $(f_1, f_2) \mapsto f_1 \boxtimes_{P(z)} f_2$ is a functor from $\widetilde{\mathcal{C}}_n^G \times \widetilde{\mathcal{C}}_n^G$ to $\widetilde{\mathcal{C}}_n^G$.*

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