Addentum to the proof of Proposition 1.4 in [H]

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Replacing the first sentence "Conversely, given any element of $H^2_{\frac{1}{2}}(V, W)$, let $\Phi \in C^2_{\frac{1}{2}}(V, W)$ be a representative of this element. " in the last paragraph on Pagd 14 of [H] by the following:

Conversely, given any element of $H^2_{\frac{1}{2}}(V, W)$, let $\Phi \in C^2_{\frac{1}{2}}(V, W)$ be a representative of this element satisfying $(\Phi(v_1 \otimes \mathbf{1}))(z_1, z_2) = 0$. We need to show that we can indeed find such a representative. Let $\widetilde{\Phi}$ be an arbitrary representative of this element. Then

$$\begin{aligned}
&\langle w', ((\delta^2_{\frac{1}{2}}\widetilde{\Phi})(v_1 \otimes v_2 \otimes v_3))(z_1, z_2, z_3)\rangle \\
&\quad = R\left(\langle w', Y(v_1, z_1)(\widetilde{\Phi}(v_2 \otimes v_3))(z_2, z_3)\rangle + \langle w', (\widetilde{\Phi}(v_1 \otimes Y(v_2, z_2 - z_3)v_3))(z_1, z_3)\rangle\right) \\
&\quad\quad - R\left(\langle w', (\widetilde{\Phi}(Y(v_1, z_1 - z_2)v_2 \otimes v_3))(z_2, z_3)\rangle + \langle w', Y(v_3, z_3)(\widetilde{\Phi}(v_1 \otimes v_2))(z_1, z_2)\rangle\right).
\end{aligned}$$

Since $\delta^2_{\frac{1}{2}}\widetilde{\Phi} = 0$, we obtain

$$\begin{aligned}
&R\left(\langle w', Y(v_1, z_1)(\widetilde{\Phi}(v_2 \otimes v_3))(z_2, z_3)\rangle + \langle w', (\widetilde{\Phi}(v_1 \otimes Y(v_2, z_2 - z_3)v_3))(z_1, z_3)\rangle\right) \\
&\quad - R\left(\langle w', (\widetilde{\Phi}(Y(v_1, z_1 - z_2)v_2 \otimes v_3))(z_2, z_3)\rangle + \langle w', Y(v_3, z_3)(\widetilde{\Phi}(v_1 \otimes v_2))(z_1, z_2)\rangle\right) \\
&= 0.
\end{aligned}$$

Let $v_2 = v_3 = \mathbf{1}$ and use $Y(v_1, z_1 - z_2)\mathbf{1} = e^{(z_1 - z_2)L(-1)}v_1$ and

$$(\widetilde{\Phi}(e^{(z_1 - z_2)L(-1)}v_1 \otimes \mathbf{1}))(z_2, z_3) = (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, z_3).$$

We obatin

$$\begin{aligned}
&R\left(\langle w', Y(v_1, z_1)(\widetilde{\Phi}(\mathbf{1} \otimes \mathbf{1}))(z_2, z_3)\rangle + \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, z_3)\rangle\right) \\
&\quad - R\left(\langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, z_3)\rangle + \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, z_2)\rangle\right) \\
&= 0. \qquad (1)
\end{aligned}$$

By the $L(-1)$-derivative property,

$$\langle w', Y(v_1, z_1)(\widetilde{\Phi}(\mathbf{1} \otimes \mathbf{1}))(z_2, z_3)\rangle = \langle w', Y(v_1, z_1)(\widetilde{\Phi}(\mathbf{1} \otimes \mathbf{1}))(0, 0)\rangle,$$

$$\langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, z_3)\rangle = \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle,$$
$$\langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, z_3)\rangle = \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle,$$
$$\langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, z_2)\rangle = \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle.$$

Moreover, $\langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle$ and $\langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle$ are Laurent polynomials in z_1 so that

$$\begin{aligned}
&R\left(\langle w', Y(v_1, z_1)(\widetilde{\Phi}(\mathbf{1} \otimes \mathbf{1}))(0, 0)\rangle + \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle\right)\\
&\quad - R\left(\langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle + \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle\right)\\
&= \langle w', Y(v_1, z_1)(\widetilde{\Phi}(\mathbf{1} \otimes \mathbf{1}))(0, 0)\rangle + \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle\\
&\quad - \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle + \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle\\
&= \langle w', Y(v_1, z_1)(\widetilde{\Phi}(\mathbf{1} \otimes \mathbf{1}))(0, 0)\rangle - \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle.
\end{aligned}$$

Then by (1), we obtain

$$\langle w', Y(v_1, z_1)(\widetilde{\Phi}(\mathbf{1} \otimes \mathbf{1}))(0, 0)\rangle = \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle. \tag{2}$$

Let $\Gamma : V \to \widetilde{W}_{z_1}$ be defined by

$$(\Gamma(v_1))(z_1) = (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0).$$

Since $\widetilde{\Phi} \in C^2_{\frac{1}{2}}(V, W)$, we see that $\Gamma \in C^1(V, W)$. By definition and (2),

$$\begin{aligned}
&\langle w', ((\delta\Gamma)(v_1 \otimes v_2))(z_1, z_2)\rangle\\
&\quad = R(\langle w', Y(v_1, z_1)(\Gamma(v_2))(z_2)\rangle) - R(\langle w', (\Gamma(Y(v_1, z_1 - z_2)v_2(z_2)\rangle)\\
&\qquad + R(\langle w', Y(v_2, z_2)(\Gamma(v_1))(z_1)\rangle)\\
&\quad = R(\langle w', Y(v_1, z_1)(\widetilde{\Phi}(v_2 \otimes \mathbf{1}))(z_2, 0)\rangle) - R(\langle w', ((\widetilde{\Phi}(Y(v_1, z_1 - z_2)v_2 \otimes \mathbf{1})(z_2, 0)\rangle)\\
&\qquad + R(\langle w', Y(v_2, z_2)(\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle).
\end{aligned}$$

Let $v_2 = \mathbf{1}$. Then using the formulas above, we obtain

$$\begin{aligned}
&\langle w', ((\delta(\Gamma)(v_1 \otimes \mathbf{1}))(z_1, z_2)\rangle\\
&\quad = R(\langle w', Y(v_1, z_1)(\widetilde{\Phi}(\mathbf{1} \otimes \mathbf{1}))(z_2, 0)\rangle) - R(\langle w', ((\widetilde{\Phi}(Y(v_1, z_1 - z_2)\mathbf{1} \otimes \mathbf{1})(z_2, 0)\rangle)\\
&\qquad + R(\langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle)\\
&\quad = \langle w', Y(v_1, z_1)(\widetilde{\Phi}(\mathbf{1} \otimes \mathbf{1}))(0, 0)\rangle - \langle w', ((\widetilde{\Phi}(v_1, \otimes \mathbf{1})(z_1, 0)\rangle\\
&\qquad + \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle\\
&\quad = \langle w', (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\rangle.
\end{aligned}$$

Since w' is arbitrary, we have

$$((\delta(\Gamma)(v_1 \otimes \mathbf{1}))(z_1, z_2) = (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0).$$

Now let $\Phi = \widetilde{\Phi} - \delta\Gamma$. Then Φ is a representative of the same cohomology class as $\widetilde{\Phi}$. Moreover

$$\begin{aligned}(\Phi(v_1 \otimes \mathbf{1}))(z_1, z_2) &= (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, z_2) - ((\delta(\Gamma)(v_1 \otimes \mathbf{1}))(z_1, z_2)\\ &= (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0) - (\widetilde{\Phi}(v_1 \otimes \mathbf{1}))(z_1, 0)\\ &= 0.\end{aligned}$$

This proves the existenec of such a representative Φ.

References

[H] Y.-Z. Huang, First and second cohomologies of grading-restricted vertex algebras, *Comm. Math. Phys.* **327** (2014), 261-278.

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