Two-Dimensional Conformal Field Theory

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1. Two-dimensional conformal field theories
   - Quantum field theories in mathematics
   - A definition and early conjectures
   - Problems and a mathematical program
2. The major problems solved
   - The geometry of vertex operator algebras
   - Intertwining operators and vertex tensor categories
   - Modular invariance and Verlinde formula
   - Rigidity and modularity
   - Full and open-closed conformal field theories
3. Unsolved problems
   - Higher-genus theories and locally convex completions
   - Nonrational conformal field theories

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Part 1

Two-dimensional conformal field theories

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Early formulations

- The early mathematical study of quantum field theories started in 1950's. It tried to put the "operator-valued fields" on a rigorous mathematical foundation using functional analysis. The Wightman axioms, the Osterwalder-Schrader theorem and the Haag-Kastler axiomatic system all appeared during this period.
- Important works on the construction of theories satisfying some of these axioms were done by I. Segal, Jaffe, Glimm and others in 1970's.
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Geomrtric formulations

- Starting from 1980's, motivated by the path integral approach in physics, Kontsevich, G. Segal and Atiyah proposed that quantum field theories are functors from geometric categories to categories formed by Hilbert spaces.
- Such a definition and the subsequent construction and study are very successful in the case of topological quantum field theories. The main reason for this success is that the Hilbert space for a topological quantum field theory is typically finite dimensional.
- For nontrivial nontopological quantum field theories, the Hilbert space must be infinite dimensional. The construction and study of these theories are much more difficult.

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Mathematical problems

- One of the Millennium Prize Problems: Establish rigorously the existence of the quantum Yang-Mills theory and prove that there is a mass gap in this theory.
- Many mathematical conjectures arising from string theory are in fact arising from two-dimensional superconformal field theories or higher-dimensional supersymmetric Yang-Mills theories.
- To understand these mathematical conjectures completely, we need to construct the correspoding quantum field theories.
- The simplest nontopological quantum field theories are two-dimensional conformal field theories. Many conjectures in mathematics have been derived based on the stronger conjectures that the corresponding two-dimensional conformal field theories exist.

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- 1988, G. Segal: Definitions of modular functor and weakly conformal field theory.
- A two-dimensional conformal field theory in the sense of Kontsevich-Segal is
  - a locally convex topological vector space H,
  - a nondegenerate hermitian form,
  - a projective functor from the category whose morphisms are Riemann surfaces with parametrized boundaries to the category of tensor powers of H and traceclass maps,

  satisfying additional but natural conditions.
- From now on, for simplicity, we omit the words "two-dimensional" so that conformal field theories mean two-domensional conformal field theories.

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  - Verlinde conjecture: For a rational conformal field theory, the modular transformaton S on the space of vacuum characters associated to $\tau \mapsto -1/\tau$ diagonaizes the matrices formed by fusion rules.
  - Verlinde formula for fusion rules: Express fusion rules in terms of matrix elements of the modular transformaton S.
- 1988, Moore and Seiberg:
  - Obtained Moore-Seiberg polynomial equations using conjectures on operator product expansion and modular invariance for intertwining operators.
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  - Obtained knot and three-manifold invariants from conjectures for rational conformal field theories.
  - Conjectured that in the case of Wess-Zumino-Witten models, these invariants should be the same as those from the corresponding Chern-Simons theories.
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The mathematical problems

- **Problem 1**: Formulate precisely and prove the conjectures of Verlinde, Moore-Seiberg and Witten.
- **Problem 2**: Give a construction of conformal field theories satisfying the axioms of Kontsevich and Segal, or at least prove the existence of such conformal field theories. In particular, give a construction of the Wess-Zumino-Witten models and the minimal models, or at least prove the existence of these theories.
- **Problem 3**: Study the moduli space of conformal field theories.
- **Problem 4**: Construct the $N = 2$ superconformal field theories associated to Calabi-Yau manifolds, prove Gepner's conjecture, turn the ideas of Green-Plesser into a mathematical construction of mirror symmetry.

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A long term program

- If there exists a conformal field theory satisfying the definition of Kontsevich and G. Segal, then the space of meromorphic fields should form an algebraic structure called vertex operator algebra in mathematics and chiral algebra in physics.
- The first part of the program: Construct and study conformal field theories using the representation theory of vertex operator algebras.
- The second part of the program: Study the moduli space of conformal field theories using the cohomology theory and deformation theory of vertex operator algebras.

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A long term program

- If there exists a conformal field theory satisfying the definition of Kontsevich and G. Segal, then the space of meromorphic fields should form an algebraic structure called vertex operator algebra in mathematics and chiral algebra in physics.
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A long term program

- Rational conformal field theories have been constructed from modules and intertwining operators for vertex operator algebras, except that there are still some conjectures involving higher-genus Riemann surfaces to be proved. We believe that several classes of non-rational conformal field theories can also be constructed using the representation theory of vertex operator algebras.
- A cohomology theory and a deformation theory for vertex operator algebras and conformal field theories are being developed. We believe that the moduli space of conformal field theories can be studied using these theories.
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Part 2

The major problems solved

Outline

Vertex operator algebras

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- 1986, Borcherds: Vertex algebras. Examples from even lattices. Insight on the moonshine module.
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- Vertex operator algebras have a purely algebraic definition. They are analogues of commutative associative algebras and Lie algebras, but involve additional variables. Here I omit this definition.
- Mathematicians have developed systematic methods to construct vertex operator algebras.
- Examples of vertex operator algebras were constructed in physics and mathematics from representations of Heisenberg algebras, Clifford algebras, affine Lie algebras, Virasoro algebra, superconformal algebras and $\mathcal{W}$-algebras. They were also constructed from lattices (corresponding to certain types of tori). One remarkable example is the moonshine module whose automorphism group is the largest sporadic finite simple group, the Monster.

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- However, they did not solve the problem of giving a geometric formulation of the conformal element, the Virasoro algebra or, especially, the central charge for a vertex operator algebra. This geometric formulation is necessary if we want to construct conformal field theories in the sense of Kontsevich and G. Segal.

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The first major problem in the program

- 1988, Segal: The central charge of a conformal field theory should be interpreted as twice the power of the determinant line bundle over the moduli space of Riemann surfaces with parametrized boundaries.
- The first major problem to be solved in this program: From the works above, one can easily make a conjecture on what the geometric formulation of a vertex operator algebra (including the conformal element, the Virasoro algebra and the central charge) should be. Prove that the purely algebraic formulation of a vertex operator algebra is equivalent to this infinite-dimensional analytic and geometric formulation. This turned out to be a very difficult problem.

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The solution

- 1991, H.: Solved this problem completely.
- Main hard part: Prove that certain formal series obtained from vertex operators and the Virasoro operators are expansions of certain analytic functions coming from genus-zero Riemann surfaces and the determinant line bundle. This was done by using a theorem of Fischer and Grauert in the deformation theory of complex manifolds and the holomorphicity of the sewing isomorphisms for the determinant lines.
- Geometric definition: A vertex operator algebra of central charge c is roughly speaking a meromorphic representation of the $c/2$-th power of the determinant line bundle over the moduli space of Riemann sphere with punctures and local coordinates vanishing at the punctures, equipped with the natural sewing operation.

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Intertwining operators

- The graded dimension (vacuum character) of a vertex operator algebra in general is not modular invariant and thus is not enough to construct a genus-one conformal field theory. For affine Lie algebras and the Virasoro algebra, one needs to use all modules to obtain a modular invariant vector space. The modular invariance requirement forces us to consider modules for the vertex operator algebra, not just the algebra itself.
- Consequently, we have to study "vertex operators" among different modules. These "vertex operators" were called chiral vertex operators by Moore and Seiberg and intertwining operators by Frenkel, Lepowsky and myself.

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The operator product expansion conjecture of Moore and Seiberg

1988, Moore and Seiberg: "Consider the operator product expansion:

$$\Phi_{\alpha,a}(z_1)\Phi_{\beta,b}(z_2) = \sum_k \sum_{c\in V^l_{kr};d\in V^k_{ij}} F_{pk}\begin{bmatrix} i & j \\ l & r \end{bmatrix}^{cd}_{ab} \times \sum_{K\in\mathcal{H}_k} \xi^{\alpha,\beta}_{p,K,d}\begin{bmatrix} i & j \\ l & r \end{bmatrix}(z_1,z_2,z_3)\Phi_{K,c}(z_3) \quad (3.9)$$

.... This expansion is an asymptotic expansion which is believed to have a finite radius of convergence. It is valid for $z_1 \sim z_2 \sim z_3$."

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The second major problem

- There was no explanation or even discussion as to why this operator product expansion must hold. It was used by Moore and Seiberg as an additional hypothesis, **not a result**. Mathematically, it was clearly a conjecture.
- This operator product expansion is in fact equivalent to the associativity for intertwining operators:

$$\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2) = \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)$$

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- This was the second major problem to be solved. Since intertwining operators are in general multivalued and form vector spaces, the usual purely algebraic method used to study vertex operator algebras and modules does not work. It was necessary to develop a new method.

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- The proof of the associativity for intertwining operators also gave immediately natural associativity isomorphisms for the tensor product bifunctors constructed by Lepowsky and me. The coherence for the associativity isomorphisms follows easily from a characterization of the associativity isomorphisms.
- The braiding isomorphism can be obtained easily from the skew-symmetry of intertwining operators. In particular, the category of modules has a natural structure of a braided tensor category.
- In fact, since the tensor product bifunctor constructed by Lepowsky and me depends on a sphere with punctures and local coordinates vanishing at the punctures, what we obtain is what we call a "vertex tensor category."

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The modular invariance conjecture of Moore and Seiberg

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The third major problem

- By stating that the conformal blocks for the two-point function are given by the traces above, Moore and Seiberg in fact assumed the modular invaraince of the space spanned by these traces. This modular invariance was used as an additional hypothesis, **not a result**.
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Zhu's theorem

- In 1990, in his important Ph.D. thesis work, Zhu proved a partial result on the modular invariance conjecture of Moore and Seiberg.
- This partial result stated that when the vertex operator algebra satisfies a positive energy condition, a complete reducibility condition and a condition now called C_2-cofiniteness condition, the q-traces of products of n suitable modified vertex operators for irreducible modules can be analytically extended to meromorphic doubly-periodic functions on the plane with periods 1 and $\tau = (\log q)/2\pi i$ and span a vector space that is invariant under the action of the full modular group $SL(2, \mathbb{Z})$.

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The failure of Zhu's method

- 2000, Miyamoto: Generalized Zhu's partial result to the partial result for one intertwining operator and *n* vertex operators for modules, using Zhu's method.
- Unfortunately, the method developed by Zhu cannot be used or adapted to prove the (full) modular invariance conjecture of Moore and Seiberg mentioned above.
- The reason that Zhu's method cannot be used or adapted is the following: Zhu's method uses the commutator formula for vertex operators (acting on modules) to reduce the construction of genus-one *n*-point functions to the construction of genus-one one-point functions. But there is no commutator formula for general intertwining operators.

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- Prove that the q-traces of products of intertwining operators satisfy certain systems of modular invariant differential equations with regular singular points.
- Prove that the q-traces of products of intertwining operators are absolutely convergent and have genus-one associativity for intertwining operators (or genus-one operator product expansions) using the systems of differential equations and the associativity for intertwining operators.
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- The rigidity of braided tensor category structure on the category of modules for a vertex operator algebra was an open problem for many years.
- Another closely related hard open problem (let's call it modularity) was the nondegenracy property and the identification of the S-matrix obtained from the ribbon tensor category structure with the action of the modular transformation associated to $\tau \mapsto -1/\tau$ on the space spanned by the graded dimension.
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- In the case of Wess-Zumino-Witten models, Finkelberg's tensor-category-equivalence theorem together with Kazhdan-Lusztig's rigidity theorem for negative levels had been thought to prove the rigidity for almost all (but not all) cases. But it turned out that Finkelberg's paper had a gap and it required either the Verlinde formula proved by Faltings, Teleman and me or, alternatively, the rigidity proved by me, to fill the gap and prove the equivalence theorem (again, for almost all, but not all, cases).
- Note that, as is mentioned above, my proof of the Verlinde formula or my proof of the rigidity needs the full modular invariance theorem.
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   - A definition and early conjectures
   - Problems and a mathematical program
2. The major problems solved
   - The geometry of vertex operator algebras
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The sixth major problem

- The results discussed above are for chiral conformal field theories. Chiral conformal field theories are not enough. We need to put chiral and antichiral conformal field theories together in a suitable way to construct full conformal field theories. Here antichiral conformal field theories are just some chiral conformal field theories that will become the antichiral parts of full conformal field theories.
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- The problems and solutions discussed above are all for closed conformal field theories. We also need to construct open-closed conformal field theories.
- One needs to construct open-string vertex operator agebras. Modules for open-string vertex algebras in fact correspond exactly to the important D-branes introduced and studied by string theorists.
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Part 3

Unsolved problems

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- The major problem to be solved in the higher-genus case is a convergence problem similar to the convergence problem for products and iterates of intertwining operators in the genus-zero case and the convergence problem for traces of products and iterates of intertwining operators.
- To prove this convergence, one needs to prove some conjectures on certain types of functions on the infinite-dimensional Teichmüller spaces and moduli spaces of Riemann surfaces with parametrized boundaries.
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Locally convex completions

- In 1998 and 2000, I constructed topological completions of vertex operator algebras and modules. This construction can be generalized easily to construct topological completions of the state spaces of the genus-zero chiral and full conformal field theories discussed above. But to obtain the full topological completions, we need first construct higher-genus theories.
- On the other hand, in the case that the state space has a natural inner product, there is another completion given by the corresponding norm.
- **Conjecture (H.)**: The topological completion obtained from higher-genus theory using the method giving topological completions of vertex operator algebras is the same as the topological completion obtained from the norm given by the inner product.

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Outline

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Thanks!