Two-dimensional conformal field theory

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Chapter 3

Modules

We give the definitions of various notions of modules for grading-restricted vertex algebras and vertex operator algebras. As for algebras, we use the duality property as the main axiom. Basic properties of these modules are proved. As for algebras, some of these properties are in fact the main axioms for suitable modules in other papers and books.

We give a construction theorem for lower-bounded generalized modules for grading-restricted vertex algebras. Then basic examples of such modules for the Heisenberg vertex operator algebras, affine vertex operator algebras $V(\ell, 0)$ and Virasoro vertex operator algebras $V(c, 0)$, are given by verifying the conditions needed to apply this theorem. (More examples will be added later.)

3.1 Definitions and basic properties

Roughly speaking, a module for a grading-restricted vertex algebra V is $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ equipped with a vertex operator map $Y_W : V \otimes W \to W((x))$ and an operator $L_W(-1)$ satisfying all the axioms for a grading-restricted vertex algebra that still make sense. But we also have to consider more general types of modules.

Definition 3.1.1. Let V be a grading-restricted vertex algebra. A *generalized V-module* is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ equipped with a *vertex operator map*

$$\begin{aligned} Y_W : V \otimes W &\to W((x)), \\ u \otimes w &\mapsto Y_W(u, x)w \end{aligned}$$

satisfying the following axioms:

1. Axioms for the grading: There are operators $L_W(0)$, $L_W(0)_S$ and $L_W(0)_N$ on W such that $L_W(0) = L_W(0)_S + L_W(0)_N$, $L_W(0)_S v = nv$ for $v \in W_{[n]}$, $L_W(0)_N$ is nilpotent (for $w \in W$, there exists $K \in \mathbb{N}$ such that $(L_W(0)_N)^K w = 0$), and

$$[L_W(0), Y_W(v, x)] = x\frac{d}{dx}Y_W(v, x) + Y_W(L_V(0)v, x)$$

   for $v \in V$.

2. *Identity property*: Let 1_W be the identity operator on W. Then $Y_W(\mathbf{1}, z) = 1_W$.
3. $L(-1)$-*derivative property*: There exists $L_W(-1) : W \to W$ such that for $u \in V$,

$$\frac{d}{dz} Y_W(u, z) = Y_W(L_V(-1)u, z) = [L_W(-1), Y_W(u, z)].$$

4. *Duality*: For $u_1, u_2 \in V$, $w \in W$ and $w' \in W'$, the series

$$\begin{gathered}\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w\rangle,\\ \langle w', Y_W(u_2, z_2)Y_W(u_1, z_1)w\rangle,\\ \langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w\rangle\end{gathered}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

A *lower-bounded generalized V-module* is a generalized V-module $(W, Y_W, L_W(0), L_W(-1))$ such that $W_{[n]} = 0$ when $\Re n$ is sufficiently negative. A *grading-restricted generalized V-module* is a lower-bounded generalized V-module $(W, Y_W, L_W(0), L_W(-1))$ such that $\dim W_{[n]} < \infty$. An *ordinary V-module* or simply a *V-module* is a grading-restricted generalized V-module $(W, Y_W, L_W(0), L_W(-1))$ such that $L_W(0)_N = 0$. When V is a Möbius vertex algebra, a *lower-bounded generalized V-module* or *grading-restricted generalized V-module* or an *ordinary V-module* is asuch a V-module when V is viewed as a grading-restricted generalized V-module together with a operator $L_W(1)$ of weight 1 on V satisfying satisfying

$$\begin{aligned}&[L_W(1), L_W(-1)] = 2L_W(0),\\ &[L_W(1), Y_W^g(v, x)] = Y_W^g(L_V(1)v, x) + 2xY_W^g(L_V(0)v, x) + x^2 Y_W^g(L_V(-1)v, x)\end{aligned}$$

for $v \in V$. When V is a vertex operator algebra, a *lower-bounded generalized V-module* or *grading-restricted generalized V-module* or an *ordinary V-module* is such a V-module when V is viewed as a grading-restricted vertex algebra such that $L(0) = \mathrm{Res}_x x Y_W(\omega, x)$ and $L_W(-1) = \mathrm{Res}_x Y_W(\omega, x)$.

All the properties for grading-restricted vertex algebras and vertex operator algebras that still make sense also hold for generalized V-modules. Here we state these properties without proofs. The proofs are the same as those for grading-restricted vertex algebras and vertex operator algebras and are left as exercises.

Operator product expansion For $u_1, u_2 \in V$, there exists $N \in \mathbb{N}$ such that $Y_V(u_1, x)u_2 = \sum_{n \leq N} (Y_V)_n(u_1)u_2 x^{-n-1}$ (see Subsection 5.1). Then

$$\begin{aligned}Y_W(u_1, z_1)Y_W(u_2, z_2) &= \sum_{n \leq N} Y_W((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1}\\ &\sim \sum_{n=0}^{N} Y_W((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1}.\end{aligned}$$

The Jacobi identity For $u_1, u_2 \in V$,

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u_1,x_1)Y_W(u_2,x_2) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_W(u_2,x_2)Y_W(u_1,x_1)$$
$$= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W(Y_V(u_1,x_0)u_2,x_2). \tag{3.1.1}$$

Commutator formula For $u_1, u_2 \in V$,

$$Y_W(u_1,x_1)Y_W(u_2,x_2) - Y_W(u_2,x_2)Y_W(u_1,x_1)$$
$$= \mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W(Y_V(u_1,x_0)u_2,x_2). \tag{3.1.2}$$

Associator formula For $u_1, u_2 \in V$,

$$Y_W(Y_V(u_1,x_0)u_2,x_2) - Y_W(u_1,x_0+x_2)Y_W(u_2,x_2)$$
$$= -\mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_W(u_2,x_2)Y_W(u_1,x_1). \tag{3.1.3}$$

Weak commutativity For $u_1, u_2 \in V$, and $N \in \mathbb{N}$ such that $x^N Y_V(u_1,x)u_2 \in V[[x]]$,

$$(x_1-x_2)^N Y_W(u_1,x_1)Y_W(u_2,x_2) = (x_1-x_2)^N Y_W(u_2,x_1)Y_W(u_1,x_1). \tag{3.1.4}$$

Weak associativity For $u_1 \in V$ and $w \in W$, and $N \in \mathbb{N}$ such that $x^N Y_W(u_1,x)w \in W[[x]]$,

$$(x_0+x_2)^N Y_W(Y_V(u_1,x_0)u_2,x_2)w = (x_0+x_2)^N Y_W(u_1,x_0+x_2)Y_W(u_2,x_2)w \tag{3.1.5}$$

for $u_2 \in V$.

Virasoro operators Let V be a vertex operator algebra with the conformal element ω. Write $Y_W(\omega,x) = \sum_{n\in\mathbb{Z}} L_W(n)x^{-n-2}$. Then

$$[L_W(m), L_W(n)] = (m-n)L_W(m+n) + \frac{c}{12}(m^3-m)\delta_{m+n,0}$$

for $m, n \in \mathbb{Z}$.

Exercise 3.1.2. Prove these properties for generalized V-modules

Definition 3.1.3. A *generalized V-submodule* of a generalized V-module W is a generalized V-module (W_0, Y_{W_0}) such that W_0 is a graded subspace of W and $Y_{W_0} = Y_W|_{V\otimes W_0}$. A generalized V-module W is said to be *irreducible* if there is no nonzero proper V-submodule of W. *Lower-bounded generalized V-submodules*, *grading-restricted generalized V-submodules*, *(ordinary) V-submodules* and the corresponding *irreducible* ones are defined in the obvious way.

3.2 A construction theorem for modules

In this section, we give a construction theorem for modules. This construction theorem is in fact the untwisted special case of Theorem 4.3 in [H4].

In the construction of grading-restricted vertex algebras in Section 2.5, we start with a grading-restricted vector space V, a set of generating fields $\phi^i(x)$ for $i \in I$, a vacuum $\mathbf{1}$ and an operator $L_V(-1)$ on V satisfying the five conditions in the section. For modules, we will start with a $\mathbb{C}$-graded vector space W, a set of generating fields $\phi^i_W(x)$ on W for $i \in I$ (here I is the same index set as the one for the algebra), an operator $L_W(-1)$ on W satisfying some conditions. But for modules, these are not enough. We have to introduce a set of what we will call "generator fields" $\psi^a_W(x)$ for $a \in A$.

Here we explain why we need these generator fields. In the proof of Lemma 2.5.6 in Section 4, we have to express an element v on which the generating fields act using these generating fields acting on the vacuum. But for a module, the generating fields $\phi^i_W(x)$ act on W and it is impossible to express $w \in W$ using these generating fields acting on the vacuum. To give a construction theorem for modules similar to the construction theorem for grading-restricted vertex algebras in Section 2.5, we have to introduce some fields which give elements in W when applied to the vacuum in the algebra V.

Note that in the properties of modules in the preceding subsection, there is no skew symmetry since it does not make sense. In fact, skew-symmetry is what motivates the definition of what we call generator fields. Here we first give this motivation.

Let W be a generalized V-module. In [FHL], a linear map

$$\begin{aligned} Y_{WV}^W : W \otimes V &\to W((x)),\\ w \otimes v &\mapsto Y_{WV}^W(w, x)v \end{aligned}$$

is defined by $Y_{WV}^W(w, x)v = e^{xL_W(-1)}Y_W(v, -x)w$ for $v \in V$ and $w \in W$. Replacing x_0 and x_2 in the weak associativity (4.1.9) by x_1 and $-x_2$, we obtain

$$(x_1 - x_2)^N Y_W(Y_V(u_1, x_1)u_2, -x_2)w = (x_1 - x_2)^N Y_W(u_1, x_1 - x_2)Y_W(u_2, -x_2)w \qquad (3.2.6)$$

for $u_1, u_2 \in V$ and $w \in W$, where N is any nonnegative integer such that $x^N Y_W(u_1, x)w \in W[[x]]$. Applying $e^{x_2L_W(-1)}$ to both sides of (3.2.6) from the left, using the definition of Y_{WV}^W and the $L(-1)$-conjugation property $e^{x_2L_W(-1)}Y_W(u_1, x_1 - x_2) = Y_W(u_1, x_1)e^{x_2L_W(-1)}$ (obtained by exponentiating the $L(-1)$-derivative property $[L_W(-1), Y_W(u_1, x_1)] = \frac{d}{dx}Y_W(u_1, x_1)$ and the formal Taylor's theorem) and then replacing u_1 by v and removing u_2, we see that (3.2.6) becomes

$$(x_1 - x_2)^N Y_{WV}^W(w, x_2)Y_V(v, x_1) = (x_1 - x_2)^N Y_W(v, x_1)Y_{WV}^W(w, x_2). \qquad (3.2.7)$$

Note that (3.2.7) is of the same form as the weak commutativity (4.1.8) but with $Y_W(u_2, x_2)$ replaced by $Y_{WV}^W(w, x_2)$.

Let M be a subspace of W such that W is spanned by coefficients of formal series of the form $Y(v, x)w$ for $v \in V$ and $w \in W$. Then we say that W *is generated by* M and M

is a set of *generators of* W. We also call $Y_{WV}^{W}(w,x)$ for $w \in M$ a *generator fields* of W. Let $\phi(x) = Y_V(v,x)$ be a generating field of V. We use $\psi(x)$ to denote the generator field $Y_{WV}^{W}(w,x)$. Then the weak commutativity (3.2.7) becomes

$$(x_1 - x_2)^N \phi(x_1)\psi(x_2) = (x_1 - x_2)^N \psi(x_2)\phi(x_1).$$

We shall use generator fields and this weak commutativity as additional data and properties in the construction theorem for modules.

Let V be a grading-restricted vertex algebra. We assume that V is a grading-restricted vertex algebra generated by $\phi^i(x) = Y_V(\phi^i_{-1}\mathbf{1}, x)$ for $i \in I$, where $\phi^i(x)$ is homogeneous with respect to weights, ϕ^i_{-1} is the constant term of $\phi^i(x)$, and $\phi^i_{-1}\mathbf{1} = \lim_{x\to 0}\phi^i(x)\mathbf{1}$ (see Section 2.5). Let wt ϕ^i be the weight of $\phi^i_{-1}\mathbf{1}$.

We shall give a construction of lower-bounded generalized V-modules. The construction is based on the following data and assumptions:

Data 3.2.1. (a) Let $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ be a $\mathbb{C}$-graded vector space such that $W_{[n]} = 0$ if $\Re(n)$ is sufficiently negative.

(b) Let

$$\begin{aligned} \phi^i_W : W &\to W((x)) \\ w &\mapsto \phi^i_W(x)w = \sum_{n\in\mathbb{Z}} (\phi^i_W)_n w x^{-n-1} \end{aligned}$$

for $i \in I$ (the same index set I for the generating fields for V) be linear maps called *generating field maps*.

(c) Let

$$\begin{aligned} \psi^a_W : V &\to W((x)) \\ v &\mapsto \psi^a_W(x)v = \sum_{n\in\mathbb{Z}} (\psi^a_W)_n v x^{-n-1} \end{aligned}$$

for $a \in A$ be linear maps called *generator field maps*.

(d) Let $L_W(0), L_W(-1)$ be linear operators on W.

Assumption 3.2.2. The data given in Data 3.2.1 satisfy the following properties:

1. There exist semisimple and nilpotent operators $L_W(0)_S$ and $L_W(0)_N$ on W such that $L_W(0) = L_W(0)_S + L_W(0)_N$. For $w \in W_{[n]}$, $L(0)w = nw$. For $i \in I$,

$$[L_W(0), \phi^i_W(x)] = x\frac{d}{dx}\phi^i_W(x) + (\mathrm{wt}\phi^i)\phi^i_W(x).$$

For $a \in A$, there exists wt $\psi_W^a \in \mathbb{C}$ and, when $L_W(0)_N \psi_W^a(x) \neq 0$, there exists $L_W(0)_N(a) \in A$ such that

$$L_W(0)\psi_W^a(x) - \psi_W^a(x)L_V(0) = x\frac{d}{dx}\psi_W^a(x) + (\text{wt } \psi_W^a)\psi_W^a(x) + \psi_W^{L_W(0)_N(a)}(x),$$

where $\psi_W^{L_W(0)_N(a)}(x) = 0$ when $L_W(0)_N\psi_W^a(x) = 0$.

2. For $i \in I$ and $a \in A$,

$$[L_W(-1), \phi_W^i(x)] = \frac{d}{dx}\phi_W^i(x)$$

and

$$L_W(-1)\psi_W^a(x) - \psi_W^a(x)L_V(-1) = \frac{d}{dx}\psi_W^a(x).$$

3. For $a \in A$, $\psi_W^a(x)\mathbf{1} \in W[[x]]$ and the constant term $(\psi_W^a)_{-1}\mathbf{1} = \lim_{x\to 0}\psi_W^a(x)\mathbf{1}$ is homogeneous.

4. The vector space W is spanned by elements of the form

$$(\phi_W^{i_1})_{n_1}\cdots(\phi_W^{i_k})_{n_k}(\psi_W^a)_n v$$

for $i_1, \ldots, i_k \in I$, $a \in A$, $n, n_1, \ldots, n_k \in \mathbb{Z}$, and $v \in V$.

5. For $i, j \in I$, there exists $M_{ij} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{M_{ij}}[\phi_W^i(x_1), \phi_W^j(x_2)] = 0.$$

6. For $i \in I$ and $a \in A$, there exists $M_{ia} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{M_{ia}}\phi_W^i(x_1)\psi_W^a(x_2) = (x_1 - x_2)^{M_{ia}}\psi_W^a(x_2)\phi^i(x_1).$$

We have the following results analogous to Proposition 2.5.2:

Proposition 3.2.3. *Assume that the data given in Data 3.2.1 satisfy only the parts on ϕ_W^i for $i \in I$ of Conditions 1–4 in Assumption 3.2.2. Then Conditions 5 and 6 in Assumption 3.2.2 are equivalent to the following three conditions:*

7. *For $w' \in W'$, $w \in W$, $i_1, \ldots, i_{k+l} \in I$ and $a \in A$, the series*

$$\langle w', \phi_W^{i_1}(z_1)\cdots\phi_W^{i_k}(z_k)\psi_W^a(z)\phi^{i_1}(z_{k+1})\cdots\phi^{i_k}(z_{k+l})v\rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0 > 0$ to a rational function

$$R(\langle w', \phi_W^{i_1}(z_1)\cdots\phi_W^{i_k}(z_k)\psi_W^a(z)\phi^{i_1}(z_{k+1})\cdots\phi^{i_k}(z_{k+l})v\rangle)$$

in $z_1, \ldots, z_k, z, z_{k+1}, \ldots, z_{k+l}$ with the only possible poles at $z_i = 0$ for $i = 1, \ldots, k+l$, $z = 0$, $z_j = z_m$ for $j \neq m$ and $z_j = z$ for $j = 1, \ldots, k+l$. In addition, the order of the pole $z_j = z_m$ depends only on $\phi_W^{i_j}$ and $\phi_W^{i_m}$, the order of the pole $z_j = z$ depends only on $\phi_W^{i_j}$ and ψ_W^a, the order of the pole $z_j = 0$ depends only $\phi_W^{i_j}$ and v and the order of $z = 0$ depends only on ψ_W^a and v.

8. *For $w \in V$, $w' \in V'$, $i_1, i_2 \in I$,*

$$R(\langle w', \phi_W^{i_1}(z_1)\phi_W^{i_2}(z_2)w\rangle) = R(\langle w', \phi_W^{i_2}(z_2)\phi_W^{i_1}(z_1)w\rangle).$$

9. *For $v \in V$, $w' \in W'$, $i \in I$ and $a \in A$,*

$$R(\langle w', \phi_W^{i}(z_1)\psi_W^{a}(z_2)v\rangle) = R(\langle w', \psi_W^{a}(z_2)\phi^{i}(z_1)w\rangle).$$

The proof of this proposition is essentially the same as the proof of Proposition 2.5.2. We leave the proof as an exercise.

Exercise 3.2.4. Prove Proposition 3.2.3.

Proposition 3.2.5. *The space V, the fields ϕ^i for $i \in I$, $L_V(-1)$ and $\mathbf{1}$ have the following properties:*

10. *For $a \in \mathbb{C}$, $i \in I$ and $a \in A$,*

$$\begin{aligned} e^{cL_W(0)}\phi_W^i(x)e^{-cL_W(0)} &= e^{c(\text{wt }\phi^i)}\phi_W^i(e^c x),\\ e^{cL_W(0)}\psi_W^a(x)e^{-cL_V(0)} &= e^{c(\text{wt }\phi^i)}\psi_W^i(e^c x). \end{aligned}$$

11. *For $i_1, \dots, i_k \in I$, $n_1, \dots, n_k \in \mathbb{Z}$, $a \in A$, $n \in \mathbb{Z}$ and $v \in V$,*

$$\begin{aligned} &L_W(0)(\phi_W^{i_1})_{n_1}\cdots(\phi_W^{i_k})_{n_k}(\psi_W^a)_n v\\ &\quad= \sum_{j=1}^{k}(\phi_W^{i_1})_{n_1}\cdots(\phi_W^{i_{j-1}})_{n_{j-1}}\cdot\\ &\qquad\qquad \cdot\left((-n_j-1)(\phi_W^{i_j})_{n_j} + (\text{wt }\phi^{i_j})(\phi_W^{i_j})_{n_j}\right)(\phi_W^{i_{j+1}})_{n_{j+1}}\cdots(\phi_W^{i_k})_{n_k}(\psi_W^a)_n v\\ &\qquad + (\phi_W^{i_1})_{n_1}\cdots(\phi_W^{i_k})_{n_k}\left((-n-1)(\psi_W^a)_n + (\text{wt }\psi_W^a)(\psi_W^a)_n + (\psi_W^{L_W(0)_N(a)})_n\right)v\\ &\qquad + (\phi_W^{i_1})_{n_1}\cdots(\phi_W^{i_k})_{n_k}(\psi_W^a)_n L_V(0)v \end{aligned}$$

and

$$\begin{aligned} &L_W(-1)(\phi_W^{i_1})_{n_1}\cdots(\phi_W^{i_k})_{n_k}(\psi_W^a)_n v\\ &\quad= \sum_{j=1}^{k}(\phi_W^{i_1})_{n_1}\cdots(\phi_W^{i_{j-1}})_{n_{j-1}}(-n_j(\phi_W^{i_j})_{n_j-1})(\phi_W^{i_{j+1}})_{n_{j+1}}\cdots(\phi_W^{i_k})_{n_k}(\psi_W^a)_n v\\ &\qquad + (\phi_W^{i_1})_{n_1}\cdots(\phi_W^{i_k})_{n_k}(-n(\psi_W^a)_{n-1})v\\ &\qquad + (\phi_W^{i_1})_{n_1}\cdots(\phi_W^{i_k})_{n_k}(\psi_W^a)_n L_V(-1)v \end{aligned}$$

12. *For $c \in \mathbb{C}$, $z \in \mathbb{C}^\times$ satisfying $|z| > |a|$, $i \in I$ and $a \in A$, $e^{cL_W(-1)}\phi_W^i(z)e^{-cL_W(-1)} = \phi_W^i(z+c)$ and $e^{cL_W(-1)}\psi_W^i(z)e^{-cL_V(-1)} = \psi_W^i(z+c)$.*

13. *The operator* $L_W(-1)$ *has weight* 1 *and its adjoint* $L_W(-1)'$ *as an operator on* W' *has weight* -1. *In particular,* $e^{zL_W(-1)'}w' \in W'$ *for* $z \in \mathbb{C}$ *and* $w' \in W'$.

14. *For* $v \in V$, $w' \in W'$ *and* $\sigma \in S_k$,

$$R(\langle w', \varphi_1(z_1)\cdots\varphi_k(z_k)v\rangle) = R(\langle w', \varphi_{\sigma(1)}(z_{\sigma(1)})\cdots\varphi_{\sigma(k)}(z_{\sigma(k)})v\rangle),$$

where one of φ_j *is* ψ^a_W *for some* $a \in A$ *and the others are either in* $\{\phi^i\}_{i\in I}$ *when they are to the right of* ψ^a_W *or in* $\{\phi^i_W\}_{i\in I}$ *when they are to the left of* ψ^a_W.

These properties follows easily from Conditions 1–9 in Assumption 3.2.2 and Proposition 3.2.3. We also leave the proof of Proposition 3.2.5 as an exercise.

Exercise 3.2.6. Prove Proposition 3.2.5.

We define a vertex operator map

$$\begin{aligned} Y_W : V\otimes W &\to W((x)) \\ v\otimes w &\mapsto Y_W(v,x)w \end{aligned}$$

by

$$\begin{aligned} \langle w', &Y_W(\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1}, z)w\rangle \\ &= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1}\cdots\xi_k^{m_k}R\left(\langle w', \phi^{i_1}_W(\xi_1+z)\cdots\phi^{i_k}_W(\xi_k+z)w\rangle\right) \end{aligned} \tag{3.2.8}$$

for $i_1,\dots,i_k \in I$, $m_1,\dots,m_k \in \mathbb{Z}$, $w \in W$ and $w' \in W'$. But as in the definition of the vertex operator map for the algebra V given by (2.5.42), we first have to prove that the vertex operator map Y_W is well defined. In fact, only in the proof of this well-definedness we need the generator fields ψ^a_W for $a \in A$.

Lemma 3.2.7. *If*

$$\sum_{p=1}^{q}\lambda_p\phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1} = 0,$$

then

$$\sum_{p=1}^{q}\lambda_p\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}R\left(\langle w', \phi^{i_1^p}_W(\xi_1+z)\cdots\phi^{i_k^p}_W(\xi_k+z)w\rangle\right) = 0$$

for $w \in W$ *and* $w' \in W'$.

Proof. By Condition 4 in Assumption 3.2.2, we can take w to be of the form

$$(\phi^{j_1}_W)_{n_1}\cdots(\phi^{j_l}_W)_{n_l}(\psi^a_W)_n v$$

for $v \in V$. But by Condition 4 in Section 2.5, we can take v to be of the form $\phi^{q_1}_{r_1} \cdots \phi^{q_s}_{r_s}\mathbf{1}$. Moreover, in this case,

$$
\begin{aligned}
&R(\langle v', \phi^{i_1^p}(z_1) \cdots \phi^{i_k^p}(z_k)(\phi^{j_1}_W)_{n_1} \cdots (\phi^{j_l}_W)_{n_l}(\psi^a_W)_n \phi^{q_1}_{r_1} \cdots \phi^{q_s}_{r_s}\mathbf{1}\rangle)\\
&\quad = \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_l=0}\mathrm{Res}_{\zeta}\mathrm{Res}_{\eta_1=0} \cdots \mathrm{Res}_{\eta_s=0}\zeta_1^{n_1} \cdots \zeta_l^{n_l}\zeta^n\eta_1^{r_1} \cdots \eta_s^{r_s} \cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1^p}(z_1) \cdots \phi^{i_k^p}(z_k)\psi^a_W(\zeta)\phi^{j_1}(\zeta_1) \cdots \phi^{j_l}(\zeta_l)\phi^{q_1}(\zeta_1) \cdots \phi^{q_s}(\zeta_l)\mathbf{1}\rangle).
\end{aligned}
$$

Then the other steps of the proof is the same as that of (2.5.6), except that here we use Property 14 in Proposition 3.2.5. We leave these steps as an exercise. □

Exercise 3.2.8. Finish the proof of Lemma 3.2.7.

Theorem 3.2.9. *The graded vector space W together with $Y_W : V \otimes W \to W((x))$ defined by (3.2.8) is a lower-bounded generalized V-module. Moreover, (W, Y_W) is the unique structure of a lower-bounded generalized V-module on W such that $Y_W(\phi^i_{-1}\mathbf{1}, x) = \phi^i(x)$ for $i \in I$.*

Note that since the definition of Y_W does not use ψ^a_W, the proof of Theorem 3.2.9 also does not need ψ^a_W. Thus the proof of Theorem 3.2.9 is the same as that of Theorem 2.5.7, except that ϕ^i are replaced by ϕ^i_W. We leave the proof as an exercise.

Exercise 3.2.10. Prove Theorem 3.2.9.

3.3 Examples: Modules for Heisenberg vertex operator algebras

Recall in Section 2, we have the group algebra $\mathbb{C}[L]$ for a lattice. We now consider the group algebra $\mathbb{C}[\mathfrak{h}]$. We use the same notation to denote e^α to denote the the basis element $\alpha \in \mathfrak{h}$ in $\mathbb{C}[\mathfrak{h}]$ but note that now α does not have to be in a lattice L. Note that for each $\alpha \in \mathfrak{h}$, $\mathbb{C}e^\alpha$ is a subspace of $\mathbb{C}[\mathfrak{h}]$.

Let $M(1, \alpha) = S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}e^\alpha$. We define the action of the Heisenberg algebra $\hat{\mathfrak{h}}$ on $M(1, \alpha)$ in the same way as in Section 2; $a \otimes t^n$ for $n \neq 0$ acts only on $S(\hat{\mathfrak{h}}_-)$ and $a \otimes t^0$ acts only on e^α by $a(0)e^\alpha = (a, \alpha)e^\alpha$, and $\mathbf{k}$ acts on $M(1, \alpha)$ as 1. This explains our notation: 1 in $M(1, \alpha)$ is used to denote that the center $\mathbf{k}$ acts as 1. Then as in Section 2, it is easy to verify that $M(1, \alpha)$ becomes an $\hat{\mathfrak{h}}$-module. As in Section 2, we use $a(n)$ to denote the action of $a \otimes t^n$ on $M(1, \alpha)$. Then $M(1, \alpha)$ is spanned by elements of the form $a_1(-n_1) \cdots a_k(-n_k)e^\alpha$ for $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathfrak{h}$ and $n_1, \ldots, n_k \in \mathbb{Z}_+$.

We define the weight of $a_1(-n_1) \cdots a_k(-n_k)e^\alpha$ to be $n_1 + \cdots + n_k + \frac{1}{2}(\alpha, \alpha)$. In particular, the weight of e^α is $\frac{1}{2}(\alpha, \alpha)$. Then $M(1, \alpha) = \coprod_{n \in \frac{1}{2}(\alpha,\alpha)+\mathbb{N}} M_{[n]}(\alpha)$ is a space graded by $\frac{1}{2}(\alpha, \alpha) + \mathbb{N}$. The same argument as for $S(\hat{\mathfrak{h}}_-)$ shows that $M(1, \alpha)$ is grading restricted, that is, $M_{[n]}(\alpha) = 0$ when $n - \frac{1}{2}(\alpha, \alpha) < 0$ and $\dim M_{[n]}(\alpha) < \infty$.

Let L be an even positive definite lattice. For $\alpha \in L$, $M(1, \alpha)$ is in fact a $\hat{\mathfrak{h}}$-submodule of the vertex operator algebra $V_L = S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}[L]$. Then we have a vertex operator map Y_{V_L}

for V_L. By definition, $Y_{V_L}(u,x)v \in M(1,\alpha)$ for $u \in S(\hat{\mathfrak{h}}_-)\otimes\mathbb{C} \simeq S(\hat{\mathfrak{h}}_-)$ and $v \in M(1,\alpha)$. We define $Y_{M(1,\alpha)} = Y_{V_L}|_{S(\hat{\mathfrak{h}}_-)\otimes M(1,\alpha)}$. Then since V_L is a vertex operator algebra, all the axioms for $(M(1,\alpha), Y_{M(1,\alpha)}$ being a $S(\hat{\mathfrak{h}}_-)$-module are satisfied.

For $\alpha \in \mathfrak{h}$ but not in any even positive lattice, we cannot use grading-restricted vertex algebras associated to a lattice. This construction of $S(\hat{\mathfrak{h}}_-)$-module does not work . For such α, we will give the construction of an $S(\hat{\mathfrak{h}}_-)$-module structure on $M(1,\alpha)$ after we prove a construction theorem for modules. In this construction theorem, we shall need some formal series of operators called generator fields. These can be obtained from generalizations of the vertex operators $Y_{V_L}(e^\alpha, x)$ in Section 2. Here we define these generator fields and give their properties.

For $\alpha \in \mathfrak{h}$, we define $\psi^\alpha(x) : S(\hat{\mathfrak{h}}_-) \to M(1,\alpha)$ by

$$\begin{aligned}
&\psi^\alpha(x)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}\\
&\quad= \exp\left(-\sum_{n\in-\mathbb{Z}_+}\frac{\alpha(n)}{n}x^{-n}\right)\exp\left(-\sum_{n\in\mathbb{Z}_+}\frac{\alpha(n)}{n}x^{-n}\right)a_1(-n_1)\cdots a_k(-n_k)e^\alpha
\end{aligned}$$

for $a_1,\dots,a_k \in \mathfrak{h}$ and $n_1,\dots,n_k \in \mathbb{Z}_+$. In the case that $\alpha \in L$ for a even positive definite lattice,

$$\psi^\alpha(x)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1} = Y_{V_L}(e^\alpha, x)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}.$$

Note that (2.7.82)–(2.7.81) still hold for $\alpha \in \mathfrak{h}$. Also by definition,

$$\begin{aligned}
&a(0)\psi^\alpha(x)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}\\
&\quad= (a,\alpha)\exp\left(-\sum_{n\in-\mathbb{Z}_+}\frac{\alpha(n)}{n}x^{-n}\right)\exp\left(-\sum_{n\in\mathbb{Z}_+}\frac{\alpha(n)}{n}x^{-n}\right)a_1(-n_1)\cdots a_k(-n_k)e^\alpha\\
&\quad= (a,\alpha)\psi^\alpha(x)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1} + \psi^\alpha(x)a(0)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}.
\end{aligned}$$

From these formulas, by the same calculations as in Section 2, we obtain the following weak commutativity between $a(x_1)$ and $\psi^\alpha(x_2)$ for $a, \alpha \in \mathfrak{h}$:

$$(x_1 - x_2)a(x_1)\psi^\alpha(x_2) = (x_1 - x_2)\psi^\alpha(x_2)a(x_1).$$

We now give the theorem for $M(1,\alpha)$ but we prove only the irreducibility here. The the proof of the (ordinary) module structure can be obtained by verifying the properties in Assumption 3.2.2 and is left as an exercise. On the other hand, the (ordinary) module structure can also be obtained as a special case of the construction of modules for the affine vertex operator algebra $V(\ell, 0)$ in the next subsection (see Corollary 3.4.10).

Theorem 3.3.1. *For $\alpha \in \mathfrak{h}$, $M(1,\alpha)$ has a structure of irreducible (ordinary) $S(\hat{\mathfrak{h}}_-)$-module.*

Proof. We take $I = \mathfrak{h}$ and $\phi^a_{M(1,\alpha)}(x) = a(x)$ for $a \in \mathfrak{h}$. We also $A = \{\alpha\}$ and $\psi^a_{M(1,\alpha)}(x) = \psi^\alpha(x)$. Then one can verify that $M(1,\alpha)$, $a(x)$ for $a \in \mathfrak{h}$ and ψ^α satisfy Assumption 3.2.2 (see Exercise 3.3.2). By Theorem 3.2.9, $M(1,\alpha)$ has a structure of (ordinary) $S(\hat{\mathfrak{h}}_-)$-module.

Next we show that $M(1,\alpha)$ is in fact irreducible. Assume that W_0 is a nonzero $S(\hat{\mathfrak{h}}_-)$-submodule of $M(1,\alpha)$. From the commutator formula (4.1.6 with $u_1 = a(-1)\mathbf{1}$ and $u_2 = b(-1)\mathbf{1}$, we obtain

$$[a_W(m), b_W(n)] = m(a,b)\delta_{m+n,0}$$

for $m, n \in \mathbb{Z}$, where $a_W(m)$ and $b_W(n)$ are given by $Y_W(a(-1)\mathbf{1}, x) = \sum_{m\in\mathbb{Z}} a(m)x^{-m-1}$ and $Y_W(b(-1)\mathbf{1}, x) = \sum_{n\in\mathbb{Z}} b(n)x^{-n-1}$. In particular, we see that W is an $\hat{\mathfrak{h}}$-module. Since W is a $S(\hat{\mathfrak{h}}_-)$-submodule of $M(1,\alpha)$, this $\hat{\mathfrak{h}}$-module structure must be induced from the $\hat{\mathfrak{h}}$-module structure on $M(1,\alpha)$. So W is a $\hat{\mathfrak{h}}$-submodule of $M(1,\alpha)$.

Since $M(1,\alpha)$ is grading restricted, W as a submodle must also be grading-restricted. Since W is nonzero, there exists homogeneous $w \in W$ such that any element of W of weight less than wt w is 0. In particular, $\hat{\mathfrak{h}}_+$ annihilates w since elements of $\hat{\mathfrak{h}}_+$ lowers the weights. Since $M(1,\alpha) = S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}e^\alpha$, we see that the only elements annihilated by $\hat{\mathfrak{h}}_+$ are those in $\mathbb{C}e^\alpha$. Thus $w = \lambda e^\alpha$ for some $\lambda \in \mathbb{C}$. Then $W = M(1,\alpha)$. So W cannot be proper. □

Exercise 3.3.2. Prove that $M(1,\alpha)$ has a structure of (ordinary) $S(\hat{\mathfrak{h}}_-)$-module by verifying the properties in Assumption 3.2.2 directly.

3.4 Examples: Modules for the affine vertex operator algebras

We construct modules for the affine vertex (operator) algebras $V = V(\ell, 0)$ and $V = L(\ell, 0)$ in this section.

Let M be a finite-dimensional module for $\mathfrak{g}$. For $a \in \mathfrak{g}$, $m \in M$, and $n > 0$, define actions of $a(0)$, $a(n)$, and $\mathbf{k}$ on $w \in M$ to be the action of a, 0, and multiplication by ℓ, respectively. Then M becomes a module for $\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$. We have the induced $\hat{\mathfrak{g}}$-module

$$V(\ell, M) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)} M.$$

We will often omit the tensor product symbol when writing elements of $V(\ell, M)$ so that for $w \in M$, $1 \otimes w$ will be written as w.

Recall the Casimir element

$$\Omega = \sum_{i=1}^{\dim \mathfrak{g}} u^i u^i,$$

where $\{u^i : 1 \leq i \leq \dim \mathfrak{g}\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to the form $(\cdot,\cdot)$. Since M is a $\mathfrak{g}$-module, Ω acts on M. We denote the action of ω on M by Ω_M and this is the Casimir operator on M introduced in Section 7. Since M is finite dimensional, it is a finite direct sum of irreducible $\mathfrak{g}$-modules by Theorem C.22. On each irreducible $\mathfrak{g}$-submodule, by Proposition C.19, Ω_M commutes with the action of elements of $\mathfrak{g}$ and thus must be proportional to the identity operator on this submodule. So M is a direct sum of eigenspaces of Ω_M. For an element w in an eigenspace of Ω_M, we define its weight, denoted by wt w, to be the eigenvalue of Ω_M associated to the eigenvector w divided by $2(\ell + h^\vee)$. We then define $L(0)$ on $\widetilde{W}$ by defining

$$L_{\widetilde{W}}(0)a_1(n_1)\cdots a_k(n_k)w = (-n_1 - \cdots - n_k + \text{wt } w)w.$$

Our generating fields are the maps $a(x) : V(\ell, M) \to V(\ell, M)((x))$ for $a \in \mathfrak{g}$, where for $w \in V(\ell, M)$,

$$a(x)w = \sum_{n\in\mathbb{Z}} a(n)wx^{-n-1},$$

and $a(n)$ is the action of $a \otimes t^n \in \hat{\mathfrak{g}}$ on $V(\ell, M)$.

We will show that $V(\ell, M)$ has a natural structure of a $V(\ell, 0)$-module. But instead of constructing the $V(\ell, 0)$-module structure directly, we will first construct a $V(\ell, 0)$-module starting from M using the method in Section 5 of [H4] and [H5] and then show that the $V(\ell, 0)$-module that we have constructed is equivalent to $V(\ell, M)$ as $\hat{\mathfrak{g}}$-module.

In Subsections 3.4.1 and 3.4.2, we construct $V(\ell, 0)$-modules when $V(\ell, 0)$ is viewed as a grading-restricted vertex algebra and when $V(\ell, 0)$ is viewed as a vertex operator algebra, respectively.

3.4.1 Modules for the grading-restricted vertex algebra $V(\ell, 0)$

Let

$$\widetilde{M} = U(\hat{\mathfrak{g}}) \otimes (M \otimes \mathbb{C}[t, t^{-1}]) \otimes V(\ell, 0).$$

Then $\widetilde{M}$ is a (left) $U(\hat{\mathfrak{g}})$-module. We will also often omit the tensor product symbol when we write down elements of $\widetilde{M}$. By the Poincaré-Birkhoff-Witt theorem, $\widetilde{M}$ is spanned by elements of the form

$$\mathbf{k}^r a_1(n_1)\cdots a_k(n_k)(w \otimes t^n)v \tag{3.4.9}$$

for $a_i \in \mathfrak{g}$, $n_i, n \in \mathbb{Z}$, $r \in \mathbb{N}$, $w \in M$, and $v \in V(\ell, 0)$. For $w \in M$ and $n \in \mathbb{Z}$, we also have a linear map $\psi^w_n : V(\ell, 0) \to \widetilde{M}$ given by $\psi^w_n(v) = (w \otimes t^n)v$ for $v \in V(\ell, 0)$.

For homogeneous $v \in V(\ell, 0)$ and $w \in M$, we define the weight of an element of the form (3.4.9) to be $-n_1 - \cdots - n_k + \text{wt } w - n - 1 + \text{wt } v$. The weights of such elements give $\widetilde{M}$ a grading. Using the weights, we also obtain an operator $L_{\widetilde{M}}(0)$ on $\widetilde{M}$ defined by $L_{\widetilde{M}}(0)w = (\text{wt } w)w$ for a homogeneous element $w \in \widetilde{M}$. We also define another operator $L_{\widetilde{M}}(-1)$ on $\widetilde{M}$, motivated by Part 2 of Assumption 3.2.2, by using the commutator formulas

$$\begin{aligned}
&[L_{\widetilde{M}}(-1), \mathbf{k}] = 0,\\
&[L_{\widetilde{M}}(-1), a(n)] = -na(n-1),\\
&L_{\widetilde{M}}(-1)\psi^w_n - \psi^w_n L_{V(\ell,0)}(-1) = -n\psi^w_{n-1},
\end{aligned} \tag{3.4.10}$$

for $a \in \mathfrak{g}$, $n \in \mathbb{Z}$, $w \in M$ and the operator $L_{V(\ell,0)}(-1)$ on $V(\ell, 0)$.

For $a \in \mathfrak{g}$, let $a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$ be the formal series with the operators $a(n)$ on $\widetilde{M}$ as the coefficients. For $w \in M$, let $\psi^w(x) = \sum_{n\in\mathbb{Z}} \psi^w_n x^{-n-1}$ be the formal series with the linear maps ψ^w_n from $V(\ell, 0)$ to $\widetilde{M}$ as the coefficients.

We shall construct a lower-bounded generalized $V(\ell, 0)$-module on a quotient of $\widetilde{M}$. Let J_1 be the $U(\hat{\mathfrak{g}})$-submodule of $\widetilde{M}$ generated by the following elements:

1. $\psi_n^w \mathbf{1}$ for $w \in M$ and $n \in \mathbb{N}$.
2. $a(0)\psi_n^w \mathbf{1} - \psi_n^{aw}\mathbf{1}$ for $a \in \mathfrak{g}$, $w \in M$ and $n \in -\mathbb{Z}_+$.
3. $\mathbf{k}w - \ell w$ for $w \in \widetilde{M}$.
4. The coefficients

$$(a(m)\psi_n^w - \psi_n^w a(m) - a(m-1)\psi_{n+1}^w + \psi_{n+1}^w a(m-1))v$$

for $m, n \in \mathbb{Z}$ of the formal series

$$(x_1 - x_2)a(x_1)\psi^w(x_2)v - (x_1 - x_2)\psi^w(x_2)a(x_1)v$$

for $a \in \mathfrak{g}$, $w \in M$, and $v \in V(\ell, 0)$.

Let $\widehat{M} = \widetilde{M}/J_1$. We shall use the same notations as those for elements of $\widetilde{M}$ to denote elements of $\widehat{M}$. Since J_1 is generated by $L_{\widetilde{M}}(0)$-homogeneous elements, J_1 is graded by the eigenspaces of $L_W(0)$. Then $L_{\widetilde{M}}(0)$ induces an operator $L_{\widehat{M}}(0)$ on the quotient $\widehat{M}$ such that $\widehat{M}$ is graded by the eigenspaces of $L_{\widehat{M}}(0)$. We also have the following result defining an operator $L_{\widehat{M}}(-1)$ on $\widehat{M}$:

Proposition 3.4.1. *We have $L_{\widetilde{M}}(-1)J_1 \subset J_1$. In particular, $L_{\widetilde{M}}(-1)$ induces to an operator $L_{\widehat{M}}(-1)$ on $\widehat{M}$.*

Proof. For $w \in M$, $L_{\widetilde{M}}(-1)\psi_0^w \mathbf{1} = \psi_0^w L_{V(\ell,0)}(-1)\mathbf{1} = 0 \in J_1$. For $w \in M$ and $n \neq 0$, $L_{\widetilde{M}}(-1)\psi_n^w \mathbf{1} = \psi_n^w L_{V(\ell,0)}(-1)\mathbf{1} - n\psi_{n-1}^w \mathbf{1} = -n\psi_{n-1}^w \mathbf{1} \in J_1$.

For $a \in \mathfrak{g}$, $w \in M$ and $n \in \mathbb{Z}_+$, we have

$$\begin{aligned} L_{\widetilde{M}}(-1)(a(0)\psi_{-n}^w \mathbf{1} - \psi_{-n}^{aw}\mathbf{1}) &= L_{\widetilde{M}}(-1)a(0)\psi_{-n}^w \mathbf{1} - L_{\widetilde{M}}(-1)\psi_{-n}^{aw}\mathbf{1} \\ &= a(0)L_{\widetilde{M}}(-1)\psi_{-n}^w \mathbf{1} - n\psi_{-n-1}^{aw}\mathbf{1} \\ &= n(a(0)\psi_{-n-1}^w \mathbf{1} - \psi_{-n-1}^{aw}\mathbf{1}) \\ &\in J_1. \end{aligned}$$

Finally, for $m, n \in \mathbb{Z}$, $a \in \mathfrak{g}$, $w \in M$, and $v \in V(\ell, 0)$, we have

$$\begin{aligned} &L_{\widetilde{M}}(-1)((a(m)\psi_n^w - \psi_n^w a(m) - a(m-1)\psi_{n+1}^w + \psi_{n+1}^w a(m-1))v) \\ &\quad = L_{\widetilde{M}}(-1)a(m)\psi_n^w v - L_{\widetilde{M}}(-1)\psi_n^w a(m)v - L_{\widetilde{M}}(-1)a(m-1)\psi_{n+1}^w v \\ &\qquad + L_{\widetilde{M}}(-1)\psi_{n+1}^w a(m-1)v \\ &\quad = a(m)L_{\widetilde{M}}(-1)\psi_n^w v - ma(m-1)\psi_n^w v - \psi_n^w L_{\widetilde{M}}(-1)a(m)v + n\psi_{n-1}^w a(m)v \\ &\qquad - a(m-1)L_{\widetilde{M}}(-1)\psi_{n+1}^w v + (m-1)a(m-2)\psi_{n+1}^w v + \psi_{n+1}^w L_{\widetilde{M}}(-1)a(m-1)v \end{aligned}$$

$$\begin{aligned}
&\quad - (n+1)\psi^w_n a(m-1)v \\
&= -na(m)\psi^w_{n-1}v - ma(m-1)\psi^w_n v + m\psi^w_n a(m-1)v + n\psi^w_{n-1}a(m)v \\
&\quad + (n+1)a(m-1)\psi^w_n v + (m-1)a(m-2)\psi^w_{n+1}v - (m-1)\psi^w_{n+1}a(m-2)v \\
&\quad - (n+1)\psi^w_n a(m-1)v \\
&= -n(a(m)\psi^w_{n-1} - \psi^w_{n-1}a(m) - a(m-1)\psi^w_n + \psi^w_n a(m-1))v \\
&\quad - (m-1)(a(m-1)\psi^w_n - \psi^w_n a(m-1) - a(m-2)\psi^w_{n+1} + \psi^w_{n+1}a(m-2))v \\
&\in J_1
\end{aligned}$$

□

We need the following lemma:

Lemma 3.4.2. *In $\widehat{M}$,*

$$a(n)\psi^w_m\mathbf{1} = a(0)\psi^w_{m+n}\mathbf{1}, \tag{3.4.11}$$

for $a \in \mathfrak{g}, n \in \mathbb{N}, w \in M$, and $m \in \mathbb{Z}$.

Proof. When $n = 0$, the two sides of (3.4.11) are the same. So we need only prove (3.4.11) for $n \in \mathbb{Z}_+$. We have

$$a(n)\psi^w_m\mathbf{1} = a(n-1)\psi^w_{m+1}\mathbf{1} + \psi^w_m a(n)\mathbf{1} - \psi^w_{m+1}a(n-1)\mathbf{1} = a(n-1)\psi^w_{m+1}\mathbf{1}.$$

Using this formula and induction, we obtain (3.4.11). □

Now we verify the properties in Assumption 3.2.2 in this case.

Proposition 3.4.3. *Let $\{w_i\}_{i=1}^{\dim M}$ be a basis of M such that w_i for $i = 1, \dots, \dim M$ are eigenvectors of Ω_M with eigenvalues* wt w_i. *The space $\widehat{M}$, the series $a(x)$ for $a \in \mathfrak{g}$, the series $\psi^{w_i}(x)$ for $i = 1, \dots, \dim M$, and the operators $L_{\widehat{M}}(0)$ and $L_{\widehat{M}}(-1)$ have the following properties:*

1. *For $w \in \widehat{M}_{[n]}$, $L_{\widehat{M}}(0)w = nw$. For $a \in \mathfrak{h}$, $[L_{\widehat{M}}(0), a(x)] = x\frac{d}{dx}a(x) + a(x)$. For $i = 1, \dots, \dim M$,*

$$L_{\widehat{M}}(0)\psi^{w_i}(x) - \psi^{w_i}(x)L_{\widehat{M}}(0) = x\frac{d}{dx}\psi^{w_i}(x) + (\text{wt } w_i)\psi^{w_i}(x).$$

2. *For $a \in \mathfrak{h}$, $[L_{\widehat{M}}(-1), a(x)] = \frac{d}{dx}a(x)$. For $i = 1, \dots, \dim M$,*

$$L_{\widehat{M}}(-1)\psi^{w_i}(x) - \psi^{w_i}(x)L_{V(\ell,0)}(-1) = \frac{d}{dx}\psi^{w_i}(x).$$

3. *For $i = 1, \dots, \dim M$, $\psi^{w_i}(x)\mathbf{1} \in \widehat{M}[[x]]$ and the constant term $\psi^{w_i}_{-1}\mathbf{1} = \lim_{x\to 0}\psi^{w_i}(x)\mathbf{1}$ is an eigenvector of weight* wt w_i.

4. *The vector space* $\widehat{M}$ *is spanned by elements of the form*
$$a_1(-n_1)\cdots a_k(-n_k)\psi_m^{w_i}\mathbf{1} \tag{3.4.12}$$
for $a_1,\dots,a_k\in\mathfrak{g}$, $n_1,\dots,n_k,m\in\mathbb{Z}_+$, *and* $i=1,\dots,\dim M$. *In particular,* $\widehat{M}$ *is grading restricted.*

5. *For* $a,b\in\mathfrak{g}$,
$$(x_1-x_2)^2[a(x_1),b(x_2)]=0.$$

6. *For* $a\in\mathfrak{h}$, $i=1,\dots,\dim M$,
$$(x_1-x_2)a(x_1)\psi^{w_i}(x_2)-(x_1-x_2)\psi^{w_i}(x_2)a(x_1).$$

Proof. Properties 1 and 2 follow from the definitions of the operators $L_{\widehat{M}}(0)$ and $L_{\widehat{M}}(-1)$.

Since $\psi_m^{w_i}\mathbf{1}=0$ for $m\in\mathbb{N}$, we have $\psi^{w_i}(x)\mathbf{1}\in\widehat{M}[[x]]$. By definition, $\mathrm{wt}\ \psi_{-1}^{w_i}\mathbf{1}=\mathrm{wt}\ w_i-(-1)-1=\mathrm{wt}\ w_i$. These prove Property 3.

Properties 5 and 6 follow from the commutator formulas for the affine Lie algebra $\hat{\mathfrak{g}}$ and the definition J_1 and $\widehat{M}$.

We still need to prove Property 4. Recall that $\widetilde{M}$ is spanned by elements of the form
$$\mathbf{k}^p a_1(n_1)\cdots a_k(n_k)\psi_m^w a_{k+1}(n_{k+1})\cdots a_{k+l}(n_{k+l})\mathbf{1}$$
for $p\in\mathbb{Z}_+$, $a_1,\dots,a_{k+l}\in\mathfrak{g}$, $n_1,\dots,n_{k+l},m\in\mathbb{Z}$, $w\in M$, and $v\in V(\ell,0)$. By the definition of $\widehat{M}$, we see that it is spanned by elements of the form
$$a_1(n_1)\cdots a_k(n_k)\psi_m^w a_{k+1}(n_{k+1})\cdots a_{k+l}(n_{k+l})\mathbf{1} \tag{3.4.13}$$
for $a_1,\dots,a_{k+l}\in\mathfrak{g}$, $n_1,\dots,n_{k+l},m\in\mathbb{Z}$, $w\in M$, and $v\in V(\ell,0)$.

As linear maps from $V(\ell,0)$ to $\widehat{M}$,
$$\psi_m^w a(n)=a(n)\psi_m^w+\psi_{m-1}^w a(n+1)-a(n+1)\psi_{m-1}^w.$$
Using this equation, we can write (3.4.13) as a sum of terms still of the form (3.4.13) but with a smaller conformal weight for the part $a_{k+1}(n_{k+1})\cdots a_{k+l}(n_{k+l})\mathbf{1}\in V(\ell,0)$. By induction, we see that (3.4.13) is a sum of terms of the form
$$a_1(n_1)\cdots a_k(n_k)\psi_m^w\mathbf{1}$$
for $a_1,\dots,a_k\in\mathfrak{g}$, $n_1,\dots,n_k\in\mathbb{Z}, m\in-\mathbb{Z}_+$, and $w\in M$.

By the Poincarë-Birkhoff-Witt theorem, we need only consider elements of the form
$$a_1(-n_1)\cdots a_k(-n_k)a_{k+1}(0)\cdots a_{k+l}(0)a_{k+l+1}(m_1)\cdots a_{k+l+p}(m_p)\psi_m^w\mathbf{1},$$
where $n_1,\cdots,n_k,m_1,\cdots,m_p\in\mathbb{Z}_+$, $m\in-\mathbb{Z}_+$, and $w\in M$. Using Proposition 3.4.2, this is equal to
$$\begin{aligned}&a_1(n_1)\cdots a_k(n_k)a_{k+1}(0)\cdots a_{k+l}(0)a_{k+l+1}(0)\cdots a_{k+l+p}(0)\psi_{m+m_1+\cdots+m_p}^w\mathbf{1}\\&\quad=a_1(n_1)\cdots a_k(n_k)\psi_{m+m_1+\cdots+m_p}^{a_{k+1}\cdots a_{k+l}a_{k+l+1}\cdots a_{k+l+p}w}\mathbf{1}.\end{aligned}$$
This is 0 in $\widehat{M}$ unless $m+m_1+\cdots+m_p\in-\mathbb{Z}_+$. We have proved Property 4. □

By Theorem 3.2.9, we obtain:

Theorem 3.4.4. *The graded space* $\widehat{M}$ *equipped with the vertex operator map* $\mathcal{Y}_{\widehat{M}} : V(\ell,0) \otimes \widehat{M} \to \widehat{M}((x))$ *given by*

$$\begin{aligned}\langle w', &Y_{\widehat{M}}(a_1(-m_1)\cdots a_k(-m_k)\mathbf{1}, z)w\rangle \\ &= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{-m_1}\cdots\xi_k^{-m_k}R\left(\langle w', a_1(\xi_1+z)\cdots a_k(\xi_k+z)w\rangle\right)\end{aligned} \tag{3.4.14}$$

for $a_1,\dots,a_k \in \mathfrak{g}$*,* $m_1,\dots,m_k \in \mathbb{Z}_+$*,* $w \in \widehat{M}$*, and* $w' \in \widehat{M}'$ *is an ordinary* $V(\ell,0)$*-module when* $V(\ell,0)$ *is viewed as a grading-restricted vertex algebra.*

Since $\widehat{M}$ is a $V(\ell,0)$-module, as in Section 3.2, we have a linear map

$$\begin{aligned}Y_{\widehat{M}V(\ell,0)}^{\widehat{M}} : \widehat{M}\otimes V(\ell,0) &\to \widehat{M}((x))\\ w\otimes v &\mapsto Y_{\widehat{M}V(\ell,0)}^{\widehat{M}}(w,x)v\end{aligned}$$

defined by

$$Y_{\widehat{M}V(\ell,0)}^{\widehat{M}}(w,x)v = e^{xL_{\widehat{M}}(-1)}Y_{\widehat{M}}(v,-x)w$$

for $v \in V(\ell,0)$ and $w \in \widehat{M}$.

Proposition 3.4.5. *For* $w \in M$*,* $\psi^w(x) = Y_{\widehat{M}V(\ell,0)}^{\widehat{M}}(\psi_{-1}^w\mathbf{1}, x)$*.*

Proof. For $w \in M$,

$$Y_{\widehat{M}V(\ell,0)}^{\widehat{M}}(\psi_{-1}^w\mathbf{1},x)\mathbf{1} = e^{xL_{\widehat{M}}(-1)}Y_{\widehat{M}}(\mathbf{1},-x)\psi_{-1}^w\mathbf{1} = e^{xL_{\widehat{M}}(-1)}\psi_{-1}^w\mathbf{1}. \tag{3.4.15}$$

On the other hand, for $k \in \mathbb{Z}_+$,

$$L_{\widehat{M}}(-1)\psi_{-k}^w\mathbf{1} = \psi_{-1}^w L_{\widehat{M}}(-1)\mathbf{1} + k\psi_{-k-1}^w\mathbf{1} = k\psi_{-k-1}^w\mathbf{1}.$$

Then we have

$$L_{\widehat{M}}(-1)^k\psi_{-1}^w\mathbf{1} = k!\psi_{-k-1}^w\mathbf{1}$$

for $k \in \mathbb{Z}_+$. Thus we have

$$\psi^w(x)\mathbf{1} = \sum_{k\in\mathbb{N}}\psi_{-k-1}^w\mathbf{1}x^k = \sum_{k\in\mathbb{N}}\frac{x^kL_{\widehat{M}}(-1)^k}{k!}\psi_{-1}^w\mathbf{1} = e^{xL_{\widehat{M}}(-1)}\psi_{-1}^w\mathbf{1}. \tag{3.4.16}$$

From (3.4.15) and (3.4.16), we obtain

$$\psi^w(x)\mathbf{1} = Y_{\widehat{M}V(\ell,0)}^{\widehat{M}}(\psi_{-1}^w\mathbf{1},x)\mathbf{1}. \tag{3.4.17}$$

For $a \in \mathfrak{g}$ and $n \in \mathbb{Z}_+$,

$$a(n)\psi_{-1}^w\mathbf{1} = \psi_{-1}^w a(n)\mathbf{1} - a(n-1)\psi_0^w\mathbf{1} + \psi_0^w a(n-1)\mathbf{1} = 0.$$

Then we have

$$xY_{\widehat{M}}(a(-1)\mathbf{1},x)\psi^w_{-1}=\sum_{n\in\mathbb{Z}}a(n)\psi^w_{-1}x^{-n}=\sum_{n\in-\mathbb{N}}a(n)\psi^w_{-1}x^{-n}\in\widehat{M}[[x]].$$

So by (3.2.7), we have

$$(x_1-x)a(x_1)Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x)v=(x_1-x)Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x)a(x_1)v$$

for $v\in V(\ell,0)$. Repeatedly using this weak commutativity, we obtain

$$\begin{aligned}&(x_1-x)\cdots(x_k-x)a_1(x_1)\cdots,a_k(x_k)Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x)\mathbf{1}\\&\quad=(x_1-x)\cdots(x_k-x)Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x)a_1(x_1)\cdots,a_k(x_k)\mathbf{1}\end{aligned}$$

for $a_1,\dots,a_k\in\mathfrak{g}$. Since $\psi^w(x)$ satisfies the same weak commutativity, we also have

$$\begin{aligned}&(x_1-x)\cdots(x_k-x)a_1(x_1)\cdots,a_k(x_k)\psi^w(x)\mathbf{1}\\&\quad=(x_1-x)\cdots(x_k-x)\psi^w(x)a_1(x_1)\cdots,a_k(x_k)\mathbf{1}.\end{aligned}$$

Then by (3.4.17), we obtain

$$\begin{aligned}&(x_1-x)\cdots(x_k-x)\psi^w(x)a(x_1)\cdots a_k(x_k)\mathbf{1}\\&\quad=(x_1-x)\cdots(x_k-x)a(x_1)\cdots,a_k(x_k)\psi^w(x)\mathbf{1}\\&\quad=(x_1-x)\cdots(x_k-x)a(x_1)\cdots a_k(x_k)Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x)\mathbf{1}\\&\quad=(x_1-x)\cdots(x_k-x)Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x)a(x_1)\cdots,a_k(x_k)\mathbf{1}.\end{aligned}\tag{3.4.18}$$

Since both

$$\psi^w(x)a_1(x_1)\cdots,a_k(x_k)\mathbf{1}$$

and

$$Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x)a_1(x_1)\cdots,a_k(x_k)\mathbf{1}$$

are in $\widehat{M}((x))((x_1))\cdots((x_k))$, we can multiply both sides of (3.4.18) by

$$(-x+x_1)^{-1}\cdots(-x+x_k)^{-1}\in\mathbb{C}((x))((x_1))\cdots((x_k))$$

to obtain

$$\psi^w(x)a(x_1)\cdots a_k(x_k)\mathbf{1}=Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x)a(x_1)\cdots a_k(x_k)\mathbf{1}.\tag{3.4.19}$$

Taking coefficients of powers of $x_1,\dots,x_k$ in both sides of (3.4.19), we obtain

$$\psi^w(x)a_1(n_1)\cdots a_k(n_k)\mathbf{1}=Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x)a_1(n_1)\cdots,a_k(n_k)\mathbf{1}$$

for $k\in\mathbb{N}$, $a_1,\dots,a_k\in\mathfrak{g}$ and $n_1,\dots,n_k\in\mathbb{Z}$. Since $V(\ell,0)$ is spanned by elements of the form $a(n_1)\cdots a_k(n_k)$, we obtain $\psi^w(x)=Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x)$. □

The V-module $\widehat{M}$ is constructed as a quotient $\hat{\mathfrak{g}}$-module. We now identify $\widehat{M}$ explicitly. Let $\Lambda(M) = \mathbb{C}[x] \otimes M$. We define operators $L_{\Lambda(M)}(0)$ and $L_{\Lambda(M)}(-1)$ on $\Lambda(M)$ by $L_{\Lambda(M)}(0)(x^m \otimes w) = m(x^m \otimes w) + x \otimes \Omega w$ and $L_{\Lambda(M)}(-1)(x^m \otimes w) = x^{m+1} \otimes w$ for $m \in \mathbb{N}$ and $w \in M$. Then $\Lambda(M)$ is graded by eigenspaces of $L_{\Lambda(M)}(0)$ and is spanned by elements of the form $L_{\Lambda(M)}(-1)^m(1 \otimes w)$ for $w \in W$. We shall identify the subspace $\mathbb{C} \otimes M$ of $\Lambda(M)$ with M so that $1 \otimes w$ can be written as w and $L_{\Lambda(M)}(-1)^m(1 \otimes w)$ can be written simply as $L_{\Lambda(M)}(-1)^m w$. Let $\mathfrak{g}_0$ acts on $\Lambda(M)$ by $a(0)L_{\Lambda(M)}(-1)^m w = L_{\Lambda(M)}(-1)^m aw$ and $\mathbf{k}L_{\Lambda(M)}(-1)^m w = \ell L_{\Lambda(M)}(-1)^m w$ for $m \in \mathbb{N}$ and $w \in M$. Let $\hat{\mathfrak{g}}_+$ act on $M = \mathbb{C} \otimes M$ as 0. We define an action of $\hat{\mathfrak{g}}_+$ on $\Lambda(M)$ using the commutator formula $[a(n), L_{\Lambda(M)}(-1)] = na(n-1)$ for $n \in \mathbb{Z}_+$ and the actions of $\hat{\mathfrak{g}}_+$ and $\mathfrak{g}_0$ on $M = \mathbb{C} \otimes M$. Let $\mathbf{k}$ act on $\Lambda(M)$ as ℓ. Then $\Lambda(M)$ becomes a $\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0$-module. Using the $\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0$-module $\Lambda(M)$, we obtain the induced $\hat{\mathfrak{g}}$-module $\widetilde{\Lambda}(M) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ + \hat{\mathfrak{g}}_0)} \Lambda(M)$. Moreover, we have operators $L_{\widetilde{\Lambda}(M)}(0)$ and $L_{\widetilde{\Lambda}(M)}(-1)$ on $\widetilde{\Lambda}(M)$ defined using the commutator formulas $[L_{\widetilde{\Lambda}(M)}(0), a(n)] = -na(n)$ and $[L_{\widetilde{\Lambda}(M)}(-1), a(n)] = -na(n-1)$ and the operator $L_{\Lambda(M)}(0)$ and $L_{\Lambda(M)}(-1)$. The space $\widetilde{\Lambda}(M)$ is graded by the eigenspaces of $L_{\widetilde{\Lambda}(M)}(0)$.

Proposition 3.4.6. *As a grading-restricted $\hat{\mathfrak{g}}$-module, $\widehat{M}$ is equivalent to $\widetilde{\Lambda}(M)$. Moreover, the equivalence between $\widehat{M}$ and $\widetilde{\Lambda}(M)$ commutes with the actions of $L_{\widehat{M}}(0)$, $L_{\widehat{M}}(-1)$, $L_{\widetilde{\Lambda}(M)}(0)$, and $L_{\widetilde{\Lambda}(M)}(-1)$. In particular, Theorem 3.4.4 with $\widehat{M}$ replaced by $\widetilde{\Lambda}(M)$ holds.*

Proof. By the Poincaré-Birkhoff-Witt theorem, $U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ + \hat{\mathfrak{g}}_0)} \Lambda(M)$ is linearly isomorphic to $U(\hat{\mathfrak{g}}_-) \otimes \Lambda(M)$. Then $U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ + \hat{\mathfrak{g}}_0)} \Lambda(M)$ is spanned by elements of the form $a_1(-n_1) \cdots a_k(-n_k)L_{\Lambda(M)}(-1)^m w_i$ for $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}_+$, $m \in \mathbb{N}$, and $i = 1, \dots, \dim M$. Let $f : U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ + \hat{\mathfrak{g}}_0)} \Lambda(M) \to \widehat{M}$ be the linear map defined by

$$f(a_1(-n_1) \cdots a_k(-n_k)L_{\Lambda(M)}(-1)^m w_i) = a_1(-n_1) \cdots a_k(-n_k)L_{\widehat{M}}(-1)^m \psi^{w_i}_{-1}\mathbf{1}$$

for $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}_+$, $m \in \mathbb{N}$, and $i = 1, \dots, \dim M$. The map f is well defined since the only relations among elements of the form $a_1(n_1) \cdots a_k(n_k) \otimes x^m \otimes w_i$ are given by the relations in $U\hat{\mathfrak{g}}_-)$, which are also satisfied by their images under f. By the definition of J_1 and $\widehat{M}$, f is also injective because the only relations among elements of the form $a_1(n_1) \cdots a_k(n_k)L_{\widehat{M}}(-1)^m \psi^{w_i}_{-1}\mathbf{1}$ are also given by the relations in $U\hat{\mathfrak{g}}_-)$.

By Property 4 in Proposition 3.4.3, $\widehat{M}$ is spanned by elements of the form (3.4.12). From (3.4.10), we obtain

$$\psi^w_m\mathbf{1} = -\frac{1}{m+1}L_{\widehat{M}}(-1)\psi^w_{m+1}$$

for $m \in -\mathbb{Z}_+ - 1$ and $w \in M$. From this formula and the fact that $\widehat{M}$ is spanned by elements of the form (3.4.12), we see that $\widehat{M}$ is spanned by elements of the form $a_1(-n_1) \cdots a_k(-n_k)L_{hatM}(-1)^m \psi^{w_i}_{-1}\mathbf{1}$ for $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}_+$, $m \in \mathbb{N}$, and $i = 1, \dots, \dim M$. In particular, f is surjective and thus is a linear isomorphism. From the definition of f, we see that f is a $\hat{\mathfrak{g}}$-module map and commutes with the actions of $L_{\widehat{M}}(0)$, $L_{\widehat{M}}(-1)$, $L_{\widetilde{\Lambda}(M)}(0)$, and $L_{\widetilde{\Lambda}(M)}(-1)$. $\square$

3.4.2 Modules for the vertex operator algebra $V(\ell, 0)$ for $\ell \neq -h^\vee$

In the case $\ell \neq -h^\vee$, $V(\ell, 0)$ is in fact a vertex operator algebra. We give an ordinary $V(\ell, 0)$-module structure on a quotient of $\widehat{M}$ given in the preceding subsection when $V(\ell, 0)$ is viewed as a vertex operator algebra.

The $V(\ell, 0)$-module $\widehat{M}$ when $V(\ell, 0)$ is viewed as a grading-restricted vertex algebra is not a $V(\ell, 0)$-module when $V(\ell, 0)$ is viewed as a vertex operator algebra. The reason is that $L_{\widehat{M}}(0)$ and $L_{\widehat{M}}(-1)$ are not equal to $\mathrm{Res}_x x Y_{\widehat{M}}(\omega.x)$ and $\mathrm{Res}_x Y_{\widehat{M}}(\omega.x)$, respectively. As in [**?**], we give a a $V(\ell, 0)$-module structure on a further quotient $\overparen{M}$ of $\widehat{M}$.

For simplicity, we denote $\mathrm{Res}_x x Y_{\widehat{M}}(\omega.x)$ and $\mathrm{Res}_x Y_{\widehat{M}}(\omega.x)$ by L_0 and L_{-1}, respectively, only in this subsection. Let J_2 be the $\hat{\mathfrak{g}}$-submodule of $\widehat{M}$ generated by the elements of the form

$$L_{-1}\psi_n^w \mathbf{1} - L_{\widehat{M}}(-1)\psi_n^w \mathbf{1}$$

for $w \in M$ and $n \in -\mathbb{Z}_+$. Since the vertex operator map $Y_{\widehat{M}}$ for $\widehat{M}$ is defined using the action of $\hat{\mathfrak{g}}$, we see that J_2 is closed under the action of the vertex operator map $Y_{\widehat{M}}$. Since ψ_{-1}^w is of weight wt w and both L_1 and $L_{\widehat{M}}(-1)$ are of weight 1, J_2 is a graded subspace of $\widehat{M}$ and is thus invariant under $L_{\widehat{M}}(0)$. To give J_2 a $V(\ell, 0)$-module structure when $V(\ell, 0)$ is viewed as a grading-restricted vertex algebra, we still need the following result:

Proposition 3.4.7. *The $\hat{\mathfrak{g}}$-submodule J_2 of $\widehat{M}$ is invariant under $L_{\widehat{M}}(-1)$.*

Proof. For $w \in M$ and $n \in -\mathbb{Z}_+$, we have

$$\begin{aligned}
&L_{\widehat{M}}(-1)(L_{-1}\psi_n^w \mathbf{1} - L_{\widehat{M}}(-1)\psi_n^w \mathbf{1})\\
&\quad = \mathrm{Res}_x L_{\widehat{M}}(-1) Y_{\widehat{M}}(\omega, x)\psi_n^w \mathbf{1} - L_{\widehat{M}}(-1)^2 \psi_n^w \mathbf{1}\\
&\quad = \mathrm{Res}_x Y_{\widehat{M}}(\omega, x) L_{\widehat{M}}(-1)\psi_n^w \mathbf{1} + \mathrm{Res}_x \frac{d}{dx} Y_{\widehat{M}}(\omega, x)\psi_n^w \mathbf{1} + n L_{\widehat{M}}(-1)\psi_{n-1}^w \mathbf{1}\\
&\quad = L_{-1} L_{\widehat{M}}(-1)\psi_n^w \mathbf{1} + n L_{\widehat{M}}(-1)\psi_{n-1}^w \mathbf{1}\\
&\quad = -n L_{-1}\psi_{n-1}^w \mathbf{1} + n L_{\widehat{M}}(-1)\psi_{n-1}^w \mathbf{1}\\
&\quad = -n(L_{-1}\psi_{n-1}^w \mathbf{1} - L_{\widehat{M}}(-1)\psi_{n-1}^w \mathbf{1})\\
&\quad \in J_2.
\end{aligned}$$

□

Let $\overparen{M} = \widehat{M}/J_2$. Since we have proved that J_2 is a $V(\ell, 0)$-submodule of $\widehat{M}$, $\overparen{M}$ is a $V(\ell, 0)$-module when $V(\ell, 0)$ is viewed as a grading-restricted vertex algebra. In particular, the vertex operator map $Y_{\widehat{M}}$, the operators $L_{\widehat{M}}(0)$, $L_{\widehat{M}}(-1)$, the series $a(x)$ and $\psi^x(x)$ for $a \in \mathfrak{g}$, $w \in M$ on $\widehat{M}$ induces a vertex operator map $Y_{\overparen{M}}$, operators $L_{\overparen{M}}(0)$ and $L_{\overparen{M}}(-1)$, respectively. For simplicity, we still use the notations for those for elements of $\widehat{M}$ and operators without subscript $\widehat{M}$ on $\widehat{M}$ to denote the corresponding elements of $\overparen{M}$ and the corresponding operators on $\overparen{M}$. In particular, we still use L_0, L_{-1}, $a(n)$, ψ_n^w, $a(x)$, and $\psi^x(x)$ for $a \in \mathfrak{g}$, $w \in M$ to denote the corresponding operators and series of operator coefficients on $\widehat{M}$. But note that in $\widehat{M}$, $L_{-1}\psi^W_{-1}\mathbf{1} = L_{\widehat{M}}(-1)\psi^W_{-1}\mathbf{1}$ and all the relations derived from this relation and properties of $V(\ell,0)$-modules hold.

We are ready to prove the following result:

Theorem 3.4.8. *In the case $\ell + h^\vee \neq 0$, the graded space $\widehat{M}$ equipped with the vertex operator map $Y_{\widehat{M}}$ is an ordinary $V(\ell,0)$-module when $V(\ell,0)$ is viewed as a vertex operator algebra.*

Proof. We need only prove $L_{\widehat{M}}(0) = L_0$ and $L_{\widehat{M}}(-1) = L_{-1}$. To prove these, we prove that L_{-1} and L_{-1} satisfy the defining properties for $L_{\widehat{M}}(0)$ and $L_{\widehat{M}}(-1)$, respectively.

By definition, we have $[L_{-1}, \mathbf{k}] = 0$. For $a \in \mathfrak{g}$, by the commutator formula for the vertex operator map $Y_{\widehat{M}}$, we have

$$\begin{aligned}
[a(x_2), Y_{\widehat{M}}(\omega, x_1)] &= [Y_{\widehat{M}}(a(-1)\mathbf{1}, x_2), Y_{\widehat{M}}(\omega, x_1)]\\
&= \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_{\widehat{M}}(Y_{V(\ell,0)}(a(-1)\mathbf{1}, x_0)\omega, x_2)\\
&= \sum_{n\in\mathbb{Z}} \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) x_0^{-n-1} Y_{\widehat{M}}(a(n)\omega, x_2)\\
&= \sum_{n\in\mathbb{N}} \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) x_0^{-n-1} Y_{\widehat{M}}(a(n)\omega, x_2)\\
&= \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) x_0^{-2} Y_{\widehat{M}}(a(-1)\mathbf{1}, x_2)\\
&= -\frac{\partial}{\partial x_1} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) a(x_2),
\end{aligned}$$

where we have used $a(n)\omega = 0$ for $n > 2$ and (2.8.122)–(2.8.124). From this formula, we obtain $[L_{-1}, a(n)] = -na(n-1)$ for $n \in \mathbb{Z}$.

The vertex operator map $Y_{\widehat{M}}$ has the associator formula

$$\begin{aligned}
&Y_{\widehat{M}}(Y_{V(\ell,0)}(u, x_0)v, x_2)w\\
&\quad = Y_{\widehat{M}}(u, x_0 + x_2)Y_{\widehat{M}}(v, x_2)w - \mathrm{Res}_{x_0} x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_{\widehat{M}}(v, x_2)Y_{\widehat{M}}(u, x_1)w
\end{aligned}$$

for $u, v \in V(\ell,0)$ and $w \in \widehat{M}$. Multiplying both sides from left by $e^{-x_2 L_{\widehat{M}}(-1)}$, we obtain

$$\begin{aligned}
&e^{-x_2 L_{\widehat{M}}(-1)} Y_{\widehat{M}}(Y_{V(\ell,0)}(u, x_0)v, x_2)w\\
&\quad = e^{-x_2 L_{\widehat{M}}(-1)} Y_{\widehat{M}}(u, x_0 + x_2)Y_{\widehat{M}}(v, x_2)w\\
&\qquad - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) e^{-x_2 L_{\widehat{M}}(-1)} Y_{\widehat{M}}(v, x_2)Y_{\widehat{M}}(u, x_1)w.
\end{aligned}$$

This formula together with the definition of $Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}$ and the $L(-1)$-conjugation formula gives

$$Y_{\widehat{M}}(u, x_0)Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(w, -x_2)v - Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(w, -x_2)Y_{V(\ell,0)}(u, x_0)v$$

$$= \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(Y_{\widehat{M}}(u,x_1)w,-x_2)v.$$

Replacing x_0, $-x_2$, x_1 by x_1, x_2, x_0, respectively, taking $u=\omega$, w to be $\psi^w_{-1}\mathbf{1}$ for $w \in M$, and using Lemma 3.4.5, we obtain

$$Y_{\widehat{M}}(\omega,x_1)\psi^w(x_2)-\psi^w(x_2)Y_{V(\ell,0)}(\omega,x_1) = \mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(Y_{\widehat{M}}(\Omega,x_0)\psi^w_{-1}\mathbf{1},x_2). \tag{3.4.20}$$

We also need the $L(-1)$-derivative property for $Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}$. From the commutator formula for $L_{\widehat{M}}(-1)$ and $Y_{\widehat{M}}$ and the $L(-1)$-derivative property for $Y_{\widehat{M}}$, we have

$$L_{\widehat{M}}(-1)Y_{\widehat{M}}(v,-x)w - Y_{\widehat{M}}(v,-x)L_{\widehat{M}}(-1)w = -\frac{d}{dx}Y_{\widehat{M}}(v,-x)w$$

for $v \in V(\ell,0)$ and $w \in \widehat{M}$. Multiplying both sides by $e^{xL_{\widehat{M}}(-1)}$, we obtain

$$L_{\widehat{M}}(-1)e^{xL_{\widehat{M}}(-1)}Y_{\widehat{M}}(v,-x)w - e^{xL_{\widehat{M}}(-1)}Y_{\widehat{M}}(v,-x)L_{\widehat{M}}(-1)w = -e^{xL_{\widehat{M}}(-1)}\frac{d}{dx}Y_{\widehat{M}}(v,-x)w.$$

Using the definition of $Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}$ and this formula, we obtain the $L(-1)$-derivative property for $Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}$:

$$\begin{aligned}
\frac{d}{dx}Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(w,x)v &= \frac{d}{dx}e^{xL_{\widehat{M}}(-1)}Y_{\widehat{M}}(v,-x)w\\
&= L_{\widehat{M}}(-1)e^{xL_{\widehat{M}}(-1)}Y_{\widehat{M}}(v,-x)w + e^{xL_{\widehat{M}}(-1)}\frac{d}{dx}Y_{\widehat{M}}(v,-x)w\\
&= e^{xL_{\widehat{M}}(-1)}Y_{\widehat{M}}(v,-x)L_{\widehat{M}}(-1)w\\
&= Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(L_{\widehat{M}}(-1)w,x)v.
\end{aligned}$$

Applying $\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_2^n$ to both sides of (3.4.8), evaluating Res_{x_1} and Res_{x_0}, using $L_{-1}\psi^W_{-1}\mathbf{1} = L_{\widehat{M}}(-1)\psi^W_{-1}\mathbf{1}$, the $L(-1)$-derivative property for $Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}$ and Proposition 3.4.5, and then evaluating Res_{x_2}, we obtain

$$\begin{aligned}
L_{-1}\psi^w_n - \psi^w_n L_{V(\ell,0)}(-1) &= \mathrm{Res}_{x_2}\mathrm{Res}_{x_0}x_2^n Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(Y_{\widehat{M}}(\omega,x_0)\psi^w_{-1}\mathbf{1},x_2)\\
&= \mathrm{Res}_{x_2}x_2^n Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(L_{-1}\psi^w_{-1}\mathbf{1},x_2)\\
&= \mathrm{Res}_{x_2}x_2^n Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(L_{\widehat{M}}(-1)\psi^w_{-1}\mathbf{1},x_2)\\
&= \mathrm{Res}_{x_2}x_2^n \frac{d}{dx_2}Y^{\widehat{M}}_{\widehat{M}V(\ell,0)}(\psi^w_{-1}\mathbf{1},x_2)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Res}_{x_2} x_2^n \frac{d}{dx_2}\psi^w(x_2)\\
&= -n\psi^w_{n-1}.
\end{aligned}
$$

We have proved that L_{-1} satisfies the same defining properties as $L_{\widehat{M}}(-1)$. Thus we have proved $L_{\widehat{M}}(-1) = L_{-1}$.

We still need to prove $L_{\widehat{M}}(0) = L_0$ The same calculations as those from (2.8.118)–(2.8.126) gives

$$
[a(m), L_0] = [a(m), \mathrm{Res}_{x_2} x_2 Y_{\widehat{M}}(\omega, x_2)] = ma(m). \tag{3.4.21}
$$

The same calculation as (2.8.106) gives

$$
\begin{aligned}
&Y_{\widehat{M}}(u^i(-1)^2\mathbf{1}, x_2)\\
&\quad= \left(\sum_{m\in\mathbb{N}} u^i(-m-1)x_2^m\right) u^i(x_2) + u^i(x_2)\left(\sum_{m\in-\mathbb{Z}_+} u^i(-m-1)x_2^m\right).
\end{aligned} \tag{3.4.22}
$$

Applying both sides of (3.4.22) to $\psi^w_{-1}\mathbf{1}$ for $w \in M$ and using $u^i(n)\psi^w_{-1}\mathbf{1} = 0$ for $n > 0$, we obtain

$$
\begin{aligned}
&Y_{\widehat{M}}(u^i(-1)^2\mathbf{1}, x_2)\psi^w_{-1}\mathbf{1}\\
&\quad= \left(\sum_{m\in\mathbb{N}} u^i(-m-1)x_2^m\right) u^i(x_2)\psi^w_{-1}\mathbf{1} + u^i(x_2)\left(\sum_{m\in-\mathbb{Z}_+} u^i(-m-1)x_2^m\right)\psi^w_{-1}\mathbf{1}\\
&\quad= \left(\sum_{m\in\mathbb{N}} u^i(-m-1)x_2^m\right) \sum_{n\in-\mathbb{N}} u^i(n)x_2^{-n-1}\psi^w_{-1}\mathbf{1} + u^i(x_2)u^i(0)x_2^{-1}\psi^w_{-1}\mathbf{1}.
\end{aligned} \tag{3.4.23}
$$

Taking the coefficients of x_2^{-2} on both sides of (3.4.23), we obtain

$$
\mathrm{Res}_{x_2} x_2 Y_{\widehat{M}}(u^i(-1)^2\mathbf{1}, x_2)\psi^w_{-1}\mathbf{1} = u^i(0)u^i(0)\psi^w_{-1}\mathbf{1} = \psi^{u^iu^iw}_{-1}\mathbf{1}. \tag{3.4.24}
$$

Summing over $i = 1, \dots, \dim\mathfrak{g}$ on both sides of (3.4.24) and then dividing the results by $2(\ell + h^\vee)$, we obtain

$$
\mathrm{Res}_{x_2} x_2 Y_{\widehat{M}}(\omega, x_2)\psi^w_{-1}\mathbf{1} = (\mathrm{wt}\ w)\psi^w_{-1}\mathbf{1} = L_{\widehat{M}}(0)\psi^w_{-1}\mathbf{1}.
$$

Thus we have proved that

$$
\begin{aligned}
[a(m), L_0] &= [a(m), L_{\widehat{M}}(0)],\\
L_0\psi^w_{-1}\mathbf{1} &= L_{\widehat{M}}(0)\psi^w_{-1}\mathbf{1}.
\end{aligned}
$$

From these formulas, we obtain $L_{\widehat{M}}(0) = L_0$. □

We have identified the V-module $\widehat{M}$ with $\widetilde{\Lambda}(M) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0)} \Lambda(M)$ using the map f in the preceding subsection. We now identify $\widehat{M}$ explicitly. To do this, we need to give $f^{-1}(J_2) \subset \widetilde{\Lambda}(M)$ explicitly. We know that $\widehat{M}$ is spanned by elements of the form $a_1(n_1) \cdots a_k(n_k) L_{hatM}(-1)^m \psi_{-1}^{w_i}\mathbf{1}$ for $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}$, $m \in \mathbb{N}$, and $i = 1, \dots, \dim M$. Since J_2 is the $\hat{\mathfrak{g}}$-module generated by the elements of the form $\mathrm{Res}_x Y_{\widehat{M}}(\omega, x)\psi_n^w\mathbf{1} - L_{\widehat{M}}(-1)\psi_n^w\mathbf{1}$ and $\mathrm{Res}_x Y_{\widehat{M}}(\omega, x)$ is equal to a linear combination of the actions of elements of $U(\hat{\mathfrak{g}})$, we see that $\widehat{M} = \widehat{M}/J_2$ is spanned elements of the form $a_1(n_1) \cdots a_k(n_k)\psi_{-1}^{w_i}\mathbf{1} + J_2$ for $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}$, and $i = 1, \dots, \dim M$. Then $\widetilde{\Lambda}(M)/f^{-1}(J_2)$ is is spanned by elements of the form $a_1(n_1) \cdots a_k(n_k)\mathbf{1} + f^{-1}(J_2)$ for $a_1, \dots, a_k \in \mathfrak{g}$, $n_1, \dots, n_k \in \mathbb{Z}$, and $i = 1, \dots, \dim M$. But elements of the form $a_1(n_1) \cdots a_k(n_k)\mathbf{1} + f^{-1}(J_2)$ span a $\hat{\mathfrak{g}}$-module equivalent to $V(\ell, M) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0)} M$ (see the introduction of this section). Thus we obtain the following result:

Proposition 3.4.9. *As a grading-restricted $\hat{\mathfrak{g}}$-module, $\widehat{M}$ is equivalent to $V(\ell, M) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0)} M$. In particular, Theorem 3.4.8 with $\widehat{M}$ replaced by $V(\ell, M)$ holds.*

As an application of Theorem 3.4.8 and Proposition 3.4.9, we obtain the $S(\hat{\mathfrak{h}}_-)$-module structure on $M(1, \alpha)$ in Theorem 3.3.1 as a consequence.

Corollary 3.4.10. *For $\alpha \in \mathfrak{h}$, $M(1, \alpha)$ has a structure of $S(\hat{\mathfrak{h}}_-)$-module.*

Proof. We take $\mathfrak{g}$ to be the abelian Lie algebra $\mathfrak{h}$ with the inner product on $\mathfrak{h}$ as the positive definite invariant bilinear form and the level ℓ to be 1. Since $\mathfrak{h}$ is abelian, $h^\vee = 0$. Then $\ell + h^\vee = 1 \neq 0$. So in this case, the vertex operator algebra $V(1, 0)$ is in fact isomorphic to $S(\hat{\mathfrak{h}}_-)$. By Theorem 3.4.8 and Proposition 3.4.9, we obtain a $S(\hat{\mathfrak{h}}_-)$-module structure on $V(1, \mathbb{C}e^\alpha) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0)} \mathbb{C}e^\alpha$. But $V(1, \mathbb{C}e^\alpha) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0)} \mathbb{C}e^\alpha$ is linearly isomorphic to $U(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}e^\alpha = S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}e^\alpha = M(1, \alpha)$. It is easy to see that the vertex operator maps for $V(1, \mathbb{C}e^\alpha)$ and for $M(1, \alpha)$ are the same under the linear isomorphism between them. So we have proved that $M(1, \alpha)$ is an $S(\hat{\mathfrak{h}}_-)$-module. □

3.5 Examples: Modules for the Virasoro vertex operator algebras

In this section, we use Theorem 3.2.9 to construct modules for the vertex operator algebra $V(c, 0)$ for $c \in \mathbb{C}$.

Fix $c, h \in \mathbb{C}$. Let $\mathbb{C}w_{c,h}$ be a one dimensional vector space over $\mathbb{C}$ with a basis element $w_{c,h}$. Let $\mathbf{c}$ acts on $\mathbb{C}w_{c,h}$ as the multiplication by c, Vir_+ on $\mathbb{C}w_{c,h}$ as 0 and $L_0 w_{c,h} = h w_{c,h}$. Then $\mathbb{C}$ becomes a module for the subalgebra $\mathrm{Vir}_+ \oplus \mathrm{Vir}_0$. Let $U(\mathrm{Vir})$ be the universal enveloping algebra of Vir. The we have the induced Vir-module $M(c, h) = U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}_+ \oplus \mathrm{Vir}_0)} \mathbb{C}w_{c,h}$, which by the Poincaré-Birkhoff-Witt theorem is linearly isomorphic to $U(\mathrm{Vir}_-) \otimes w_{c,h}$. We denote the action of L_n for $n \in \mathbb{Z}$ on $M(c, h)$ by $L(n)$ and $1 \otimes_{U(\mathrm{Vir}_+ \oplus \mathrm{Vir}_0)} w_{c,h}$ still by $w_{c,h}$.

Then $M(c,h)$ is spanned by elements of the form $L(-n_1)\cdots L(-n_k)w_{c,h}$ for $n_1,\dots,n_k\in\mathbb{Z}_+$ and the action of Vir on $M(c,h)$ is given by

$$L(-n)(L(-n_1)\cdots L(-n_k)w_{c,h}) = L(-n)L(-n_1)\cdots L(-n_k)w_{c,h}$$

for $n\in\mathbb{Z}_+$ and

$$\begin{aligned}
&L(n)(L(-n_1)\cdots L(-n_k)w_{c,h})\\
&\quad=\sum_{i=1}^{k} L(-n_1)\cdots L(-n_{i-1})[L(n),L(-n_i)]L(-n_{i+1})\cdots L(-n_k)w_{c,h}\\
&\qquad+L(-n_1)\cdots L(-n_k)L(n)w_{c,h}\\
&\quad=\sum_{i=1}^{k}(n+n_i)L(-n_1)\cdots L(-n_{i-1})L(n-n_i)L(-n_{i+1})\cdots L(-n_k)w_{c,h}\\
&\qquad+\sum_{i=1}^{k}\frac{c}{12}(n^3-n)\delta_{n-n_i,0}L(-n_1)\cdots L(-n_{i-1})L(-n_{i+1})\cdots L(-n_k)w_{c,h}\\
&\qquad+\delta_{n,0}hL(-n_1)\cdots L(-n_k)w_{c,h}
\end{aligned}$$

for $n\in\mathbb{N}$.

The Vir-module $M(c,h)$ is the Verma module of the central charge c with the lowest conformal weight h. Every Vir-module of central charge c generated by a lowest conformal weight vector of weight h is a quotient of $M(c,h)$.

For $c,h\in\mathbb{C}$, we now give the Vir-module $M(c,h)$ a (ordinary) $V(c,0)$-module structure. The Vir-module $M(c,h)$ is spanned by elements of the form $L(-n_1)\cdots L(-n_k)w_{c,h}$ for $n_1,\dots,n_k\in\mathbb{Z}_+$. We define the weight of $L(-n_1)\cdots L(-n_k)w_{c,h}$ to be $n_1+\cdots+n_k+h$. For $n\in h+\mathbb{N}$, let $M_{[n]}(c,0)$ be the subspace of $M(c,h)$ consisting all elements of weight n. Then we have

$$M(c,h)=\coprod_{n\in\mathbb{N}} M_{[h+n]}(c,h).$$

As in the case of $M(c,0)$, we also have the stress energy tensor

$$T_{c,h}(x)=\sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$$

acting on $M(c,h)$.

As in the case for modules for affine vertex (operator) algebras, we do not give a direct construction of a $V(c,0)$-module structure on $M(c,h)$. Instead, we construct a $V(c,0)$-module using the method in [H4] and then prove that this $V(c,0)$-module is equivalent to $M(c,h)$.

Let

$$\widetilde{M}_{c,h}=U(\mathrm{Vir})\otimes(\mathbb{C}w_{c,h}\otimes\mathbb{C}[t,t^{-1}])\otimes V(c,0).$$

Then $\widetilde{M}$ is a Vir-module. We will omit the tensor product symbol when we write down elements of $\widetilde{M}$. By the Poincaré-Birkhoff-Witt theorem, $\widetilde{M}_{c,h}$ is spanned by elements of the form

$$\mathbf{c}^r L(n_1)\cdots L(n_k)(w_{c,h}\otimes t^n)v \tag{3.5.25}$$

for $n_i, n \in \mathbb{Z}$, $r \in \mathbb{N}$, and $v \in V(c,0)$. For $n \in \mathbb{Z}$, we also have a linear map $\psi_n^{c,h} : V(c,0) \to \widetilde{M}_{c,h}$ given by $\psi_n^{c,h}(v) = (w_{c,h}\otimes t^n)v$ for $v \in V(c,0)$.

For homogeneous $v \in V(c,h)$, we define the weight of an element of the form (3.5.25) to be $-n_1 - \cdots - n_k + h - n - 1 + \mathrm{wt}\ v$. Then $\widetilde{M}_{c,h}$ has a grading. Let $T_{c,h}(x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$ be the formal series with the operators $L(n)$ on $\widetilde{M}_{c,h}$ as the coefficients. Let $\psi^{c,h}(x) = \sum_{n\in\mathbb{Z}} \psi_n^{c,h}x^{-n-1}$ be the formal series with the linear maps ψ_n^w from $V(c,h)$ to $\widetilde{M}_{c,h}$ as the coefficients.

We shall construct a lower-bounded generalized $V(c,h)$-module on a quotient of $\widetilde{M}_{c,h}$. Let J_1 be the $U(\mathrm{Vir})$-submodule of $\widetilde{M}_{c,h}$ generated by the following elements:

1. $\psi_n^{c,h}\mathbf{1}$ for $n \in \mathbb{N}$.

2. $L(0)\psi_n^{c,h}\mathbf{1} - (h-n-1)\psi_n^{c,h}\mathbf{1}$ for $n \in -\mathbb{Z}_+$.

3. $\mathbf{c}w - cw$ for $w \in \widetilde{M}_{c,h}$.

4. The coefficients

   $$L(m)\psi_n^{c,h} - \psi_n^{c,h}L(m) - 2L(m+1)\psi_{n-1}^{c,h} + 2\psi_{n-1}^{c,h}L(m+1) + L(m+2)\psi_{n-2}^{c,h} - \psi_{n-2}^{c,h}L(m+2)$$

   for $m, n \in \mathbb{Z}$ of the formal series

   $$(x_1-x_2)^2 T_{c,h}(x_1)\psi^{c,h}(x_2)v - (x_1-x_2)\psi^{c,h}(x_2)T(x_1))v$$

   for $v \in V(c,h)$, where $T(x_1) = Y_{V(c,0)}(\omega, x)$.

5. $L(-1)\psi_n^{c,h} - \psi_n^w L(-1) + n\psi_{n-1}^w$ for $n \in \mathbb{Z}$.

Let $\widehat{M}_{c,h} = \widetilde{M}_{c,h}/J_1$. We shall use the same notations as those for elements of $\widetilde{M}_{c,h}$ to denote elements of $\widehat{M}$. Since J_1 is generated by $L0)$-homogeneous elements, J_1 and $\widehat{M}_{c,h}$ are both graded by the eigenspaces of $L(0)$. We have the following result verifying in Assumption 3.2.2 for $\widehat{M}_{c,h}$, the series $T_{c,h}(x)$ and $\psi^{c,h}(x)$ on $\widehat{M}_{c,h}$:

Proposition 3.5.1. *The space $\widehat{M}_{c,h}$, the series $T_{c,h}(x)$ of operators on $\widehat{M}_{c,h}$ and the series of maps from $V(c,h)$ to $\widehat{M}_{c,h}$ have the following properties:*

1. *For $w \in (\widehat{M}_{c,h})_{[n]}$, $L_{\widehat{M}_{c,h}}(0)w = nw$. We also have $[L_(0), T_{c,h}(x)] = x\frac{d}{dx}T_{c,h}(x) + T_{c,h}(x)$ and $L(0)\psi^{c,h}(x) - \psi^{c,h}(x)L(0) = x\frac{d}{dx}\psi^{c,h}(x) + h\psi^{c,h}(x)$.*

2. *We have $[L(-1), T_{c,h}(x)] = \frac{d}{dx}T_{c,h}(x)$ and $L(-1)\psi^{w_i}(x) - \psi^{w_i}(x)L-1) = \frac{d}{dx}\psi^{w_i}(x)$.*

3. *We have* $\psi^{c,h}(x)\mathbf{1} \in \widehat{M}_{c,h}[[x]]$ *and the constant term* $\psi^{c,h}_{-1}\mathbf{1} = \lim_{x\to 0}\psi^{c,h}(x)\mathbf{1}$ *is an eigenvector of of* $L(0)$ *weight* h.

4. *The vector space* $\widehat{M}_{c,h}$ *is spanned by elements of the form*
$$L(-n_1)\cdots L(-n_k)\psi^{c,h}_{-1}\mathbf{1}$$
for $n_1, \dots, n_k \in \mathbb{Z}_+$. *In particular,* $\widehat{M}_{c,h}$ *is grading restricted.*

5. *We have*
$$(x_1-x_2)^4 T_{c,h}(x_1)T_{c,h}(x_2) = (x_1-x_2)^4 T_{c,h}(x_2)T_{c,h}(x_1).$$

6. *We have*
$$(x_1-x_2)^2 T_{c,h}(x_1)\psi^{c,h}(x_2) - (x_1-x_2)\psi^{c,h}(x_2)T(x_1)).$$

Proof. Properties 1, 2, and 3 follow from the definition of J_1. Properties 5 and 6 follow from the commutator relations of the Virasoro operators and the definition of J_1, respectively.

We need only prove Property 4. We know that $\widehat{M}_{c,h}$ is spanned by elements of the form $L(-n_1)\cdots L(-n_k)\psi^{c,h}_m v$ for $n_1, \dots, n_k \in \mathbb{Z}_+$, $m \in \mathbb{Z}$, and $v \in V(c,0)$. Since $V(c,0)$ is spanned by $L(-m_1)\cdots L(-m_l)\mathbf{1}$ for $m_1, \dots, m_l \in -\mathbb{Z}_+$, we see that $\widehat{M}_{c,h}$ is spanned by elements of the form

$$L(-n_1)\cdots L(-n_k)\psi^{c,h}_m L(-m_1)\cdots L(-m_l)\mathbf{1} \tag{3.5.26}$$

for $n_1, \dots, n_k, m_1, \dots, m_l \in \mathbb{Z}_+$, $m \in \mathbb{Z}$. From the definition of J_1, we have

$$L(m)\psi^{c,h}_n - \psi^{c,h}_n L(m) = 2L(m+1)\psi^{c,h}_{n-1} - 2\psi^{c,h}_{n-1}L(m+1) - L(m+2)\psi^{c,h}_{n-2} + \psi^{c,h}_{n-2}L(m+2)$$

for $m, n \in \mathbb{Z}$. Using this formula, we can write (3.5.26) as a linear combination of elements still of the form (3.5.26) but with a smaller conformal weight for the part $L(-m_1)\cdots L(-m_l)\mathbf{1} \in V(c,h)$. Then by induction, we see that (3.5.26) can be written as a linear combination of elements of the from

$$L(-n_1)\cdots L(-n_k)\psi^{c,h}_m\mathbf{1} \tag{3.5.27}$$

for $n_1, \dots, n_k \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$. But by the definition of J_1, $\psi^{c,h}_m\mathbf{1} = 0$ for $m \in \mathbb{N}$. So (3.5.26) is a linear combination of (3.5.27) for $n_1, \dots, n_k \in \mathbb{Z}_+$ and $m \in -\mathbb{Z}_+$. On the other hand, by the definition of J_1 again, we have

$$\psi^w_{n-1}\mathbf{1} = -\frac{1}{n}L(-1)\psi^{c,h}_n\mathbf{1} + \frac{1}{n}\psi^w_n L(-1)\mathbf{1} = -\frac{1}{n}L(-1)\psi^{c,h}_n\mathbf{1}$$

for $n \in -\mathbb{Z}_+$. Then by induction, we see that (3.5.27) is a linear combination of elements of the form $L(-n_1)\cdots L(-n_k)\psi^{c,h}_{-1}\mathbf{1}$ for $n_1, \dots, n_k \in \mathbb{Z}_+$, proving Property 4. □

By Proposition 3.5.1, Theorem 3.2.9, and the definition of the action of $L(0)$ and $L(-1)$ on $\widehat{M}_{c,h}$, we obtain:

Theorem 3.5.2. *The graded space* $\widehat{M}_{c,h}$ *equipped with the vertex operator map* $Y_{\widehat{M}_{c,h}}$: $V(c,0)\otimes \widehat{M}_{c,h}\to \widehat{M}_{c,h}((x))$ *given by*

$$\begin{aligned}\langle w',&Y_{\widehat{M}_{c,h}}(L(-m_1)\cdots L(-m_k)\mathbf{1},z)w\rangle\\ &=\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{-m_1+1}\cdots\xi_k^{-m_k+1}R\left(\langle w',T_{c,h}(\xi_1+z)\cdots T_{c,h}(\xi_k+z)w\rangle\right)\end{aligned}\tag{3.5.28}$$

for $m_1,\dots,m_k\in\mathbb{Z}_+$, $w\in\widehat{M}_{c,h}$, *and* $w'\in\widehat{M}'_{c,h}$ *is an ordinary* $V(c,0)$*-module when* $V(c,0)$ *is viewed as a vertex operator algebra.*

Proof. By Proposition 3.5.1 and Theorem 3.2.9, The graded space $\widehat{M}_{c,h}$ equipped with the vertex operator map $Y_{\widehat{M}_{c,h}}$ is an ordinary V-module when $V(c,0)$ is viewed as a grading-restricted vertex algebra.

We still need to prove $\mathrm{Res}_x xY_{\widehat{M}_{c,h}}(\omega,x)=L(0)$ and $\mathrm{Res}_x Y_{\widehat{M}_{c,h}}(\omega,x)=L(-1)$. In this case, $\omega=L(-2)\mathbf{1}$. By (3.5.28), for $w\in\widehat{M}_{c,h}$, and $w'\in\widehat{M}'_{c,h}$,

$$\begin{aligned}\langle w',\mathrm{Res}_x xY_{\widehat{M}_{c,h}}(L(-2)\mathbf{1},x)w\rangle&=\mathrm{Res}_{z=0}z\langle w',Y_{\widehat{M}_{c,h}}(L(-2)\mathbf{1},z)w\rangle\\&=\mathrm{Res}_{z=0}z\mathrm{Res}_{\xi=0}\xi_1^{-1}R\left(\langle w',T_{c,h}(\xi_1+z)w\rangle\right)\\&=\mathrm{Res}_{z=0}z\mathrm{Res}_{\xi=0}\xi_1^{-1}\langle w',T_{c,h}(\xi_1+z)w\rangle\\&=\mathrm{Res}_{z=0}z\langle w',T_{c,h}(z)w\rangle\\&=\langle w',L(0)w\rangle.\end{aligned}$$

So we obtain $\mathrm{Res}_x xY_{\widehat{M}_{c,h}}(\omega,x)=L(0)$. Similarly, we have $\mathrm{Res}_x Y_{\widehat{M}_{c,h}}(\omega,x)=L(-1)$. ☐

We now identify $\widehat{M}_{c,h}$ explicitly.

Proposition 3.5.3. *As a grading-restricted* Vir*-module,* $\widehat{M}_{c,h}$ *is equivalent to* $M(c,h)$*. In particular, Theorem 3.4.8 with* $\widehat{M}$ *replaced by* $M(c,h)$ *holds.*

Proof. By the discussion in the beginning of this section, $M(c,h)=U(\mathrm{Vir})\otimes_{U(\mathrm{Vir}_+\oplus\mathrm{Vir}_0)}\mathbb{C}w_{c,h}$ is spanned by elements of the form $L(-n_1)\cdots L(-n_k)w_{c,h}$ for $n_1,\dots,n_k\in\mathbb{Z}_+$. Let f : $M(c,h)\to\widehat{M}_{c,h}$ be the linear map defined by

$$f(L(-n_1)\cdots L(-n_k)w_{c,h})=L(-n_1)\cdots L(-n_k)w_{c,h}\psi^{c,h}_{-1}\mathbf{1}$$

for $n_1,\dots,n_k\in\mathbb{Z}_+$. The map f is well defined since the only relations among elements of the form $a_1(n_1)\cdots a_k(n_k)\otimes x^m\otimes w_i$ are given by the relations in $U(\mathrm{Vir}_-)$, which are also satisfied by their images under f. By the definition of J_1 and $\widehat{M}_{c,h}$, f is also injective because the only relations among elements of the form $L(-n_1)\cdots L(-n_k)w_{c,h}\psi^{c,h}_{-1}\mathbf{1}$ are also given by the relations in $U(\mathrm{Vir}_-)$. By Property 4 in Proposition 3.5.1, we see that f is also surjective and thus is a linear isomorphism. From the definition of f, f is a Vir-module map. ☐

3.6 Contragredient modules

For modules for a Lie algebra, their dual space also have natural structures of modules for the same Lie algebra. For lower-bounded generalized modules for a Möbius vertex algebra, their graded dual spaces also have natural structures of lower-bounded generalized modules for the same Möbius vertex algebra. These module structures on the graded dual spaces are called "contragredient modules." In this subsection, we give the vertex operator map on the graded dual space of a lower-bounded generalized module for a Möbius vertex algebra and prove that the graded dual space together with this vertex operator map is a lower-bounded generalized module.

Let V be a Möbius vertex algebra and $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ a lower-bounded generalized V-module. We define a vertex operator map

$$\begin{aligned} Y_{W'} : V \otimes W' &\to W'[[x, x^{-1}]] \\ v \otimes w' &\mapsto Y'_W(v, x)w' \end{aligned}$$

by

$$\langle Y_{W'}(v, x)w', w\rangle = \langle w', Y_W(e^{xL_V(1)}(-x^{-2})^{L_V(0)}v, x^{-1})w\rangle$$

for $v \in V$, $w \in W$ and $w' \in W'$. Also, we define the operator $L_{W'}(0)$, $L_{W'}(-1)$, and $L_{W'}(1)$ on W' to be the adjoint operators of $L_W(0)$, $L_W(1)$, and $L_W(-1)$ on W.

Theorem 3.6.1. *The pair $(W', Y_{W'})$ is a lower-bounded generalized V-module.*

Proof. We prove only the duality property here. The proofs of all the other axioms are left as an exercise. We assume these other axioms have been proved. The commutator formula between $L_{W'}(0)$ and the vertex operator $Y_{W'}(v, x)$ for $v \in V$ in the Axioms for the grading for modules is needed in the proof of the duality below. We assume that this formula is proved.

Let $u, v \in V$, $w \in W$ and $w' \in W'$. Then we have

$$\begin{aligned} &\langle Y_{W'}(u, z_1)Y_{W'}(v, z_2)w', w\rangle \\ &\quad = \langle w', Y_W(e^{z_2L_V(1)}(-z_2^{-2})^{L_V(0)}v, z_2^{-1})Y_W(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})w\rangle. \end{aligned} \tag{3.6.29}$$

For homogeneous $u, v \in V$, $(-z_2^{-2})^{L_V(0)}v = (-z_2^{-2})^{\mathrm{wt}\ v}v$ and $(-z_1^{-2})^{L_V(0)}u = (-z_1^{-2})^{\mathrm{wt}\ v}v$. Also there exist $p, q \in \mathbb{Z}_+$ such that $L_V(1)^pu = L_V(1)^qv = 0$. So $e^{z_1L_V(1)}u \in V[z_2]$ and $e^{z_2L_V(1)}v \in V[z_1]$. Then $e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u \in V[z_1, z_1^{-1}]$ and $e^{z_2L_V(1)}(-z_2^{-2})^{L_V(0)}v \in V[z_2, z_2^{-1}]$. Using this fact and the duality property for W, we see that the right-hand side of (3.6.29) is absolutely convergent on the region $|z_2^{-1}| > |z_1^{-1}| > 0$, or equivalently, $|z_1| > |z_2| > 0$, to a rational function in z_1^{-1} and z_2^{-1} with the only possible poles $z_1^{-1}, z_2^{-1} = 0$ and $z_1^{-1} = z_2^{-1}$. But such a rational function is the same as a rational function in z_1 and z_2 with the only possible poles $z_1, z_2 = 0$ and $z_1 = z_2$.

We also have

$$\langle Y_{W'}(v, z_2)Y_{W'}(u, z_1)w', w\rangle$$

$$= \langle w', Y_W(e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})Y_W(e^{z_2 L_V(1)}(-z_2^{-2})^{L_V(0)}v, z_2^{-1})w\rangle. \tag{3.6.30}$$

By the fact that $e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u \in V[z_1, z_1^{-1}]$ and $e^{z_2 L_V(1)}(-z_2^{-2})^{L_V(0)}v \in V[z_2, z_2^{-1}]$ and the duality property for W, the right-hand side of (3.6.30) is absolutely convergent on the region $|z_1^{-1}| > |z_2^{-1}| > 0$, or equivalently, $|z_2| > |z_1| > 0$ to the same rational function that the right-hand side of (3.6.29) is convergent to.

Finally we have

$$\begin{aligned}&\langle Y_{W'}(Y_V(u, z_1 - z_2)v, z_2)w', w\rangle\\ &\quad = \langle w', Y_W(e^{z_2 L_V(1)}(-z_2^{-2})^{L_V(0)}Y_V(u, z_1 - z_2)v, z_2^{-1})w\rangle.\end{aligned} \tag{3.6.31}$$

Using the $L(0)$- and $L(1)$-conjugation formulas

$$\begin{aligned}y^{L_V(0)}Y_V(v, x)y^{-L_V(0)} &= Y_V(x^{L_V(0)}v, xy), \qquad (3.6.32)\\ e^{yL_V(1)}Y_V(v, x)e^{-yL_V(1)} &= Y_V(e^{y(1-yx)L_V(1)}(1 - yx)^{-2L_V(0)}v, x(1 - yx)^{-1})\end{aligned}$$

for $v \in V$, we see that on the region $|z_2| > |z_1 - z_2| > 0$ so that the expansion of $(1 + (z_1 - z_2)z_2^{-1})^p$ as a power series in $(z_1 - z_2)z_2^{-1}$ is absolutely convergent to $(1 + (z_1 - z_2)z_2^{-1})^p = (z_1 z_2^{-1})^p$ for $p \in \mathbb{Z}$, the right-hand side of (3.6.31) is equal to

$$\begin{aligned}&\langle w', Y_W(e^{z_2 L_V(1)}Y_V((-z_2^{-2})^{L_V(0)}u, -(z_1 - z_2)z_2^{-2})(-z_2^{-2})^{L_V(0)}v, z_2^{-1})w\rangle\\ &\quad = \langle w', Y_W(Y_V(e^{z_2(1+(z_1-z_2)z_2^{-1})L_V(1)}(1 + (z_1 - z_2)z_2^{-1})^{-2L_V(0)}(-z_2^{-2})^{L_V(0)}u,\\ &\qquad\qquad - (z_1 - z_2)z_2^{-2}(1 + (z_1 - z_2)z_2^{-1})^{-1})e^{z_2 L_V(1)}(-z_2^{-2})^{L_V(0)}v, z_2^{-1})w\rangle\\ &\quad = \langle w', Y_W(Y_V(e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1} - z_2^{-1})e^{z_2 L_V(1)}(-z_2^{-2})^{L_V(0)}v, z_2^{-1})w\rangle.\end{aligned} \tag{3.6.33}$$

By the fact that $e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u \in V[z_1, z_1^{-1}]$ and $e^{z_2 L_V(1)}(-z_2^{-2})^{L_V(0)}v \in V[z_2, z_2^{-1}]$ and the duality property for W, the right-hand side of (3.6.33) is absolutely convergent on the region $|z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$, or equivalently, $|z_1| > |z_1 - z_2| > 0$ to the same rational function that the right-hand side of (3.6.29) is convergent to. From (3.6.31) and (3.6.33), we see that the left-hand side of (3.6.31) is absolutely convergent on the region $|z_1|, |z_2| > |z_1 - z_2| > 0$ to the same rational function that the right-hand side of (3.6.29) is convergent to. But by the $L(0)$-conjugation formulas (3.6.32) for Y_V and $L(0)$-conjugation formula for $Y_{W'}$ (which can be proved using the commutator formula between $L_{W'}(0)$ and $Y_{W'}(v, x)$) and the definition of $L_{W'}(0)$, we see that the left-hand side of (3.6.31) is equal to

$$\langle Y_{W'}(Y_V(z_2^{-L_V(0)}u, (z_1 - z_2)z_2^{-1})z_2^{-L_V(0)}v, 1)z_2^{-L_{W'}(0)}w', z_2^{L_W(0)}w\rangle. \tag{3.6.34}$$

But (3.6.34) is a Laurent series in $(z_1 - z_2)z_2^{-1}$ with finitely many negative power terms and a polynomial in z_2. If (3.6.34) is absolutely convergent for $|(z_1 - z_2)z_2^{-1}| = r \in \mathbb{R}_+$, then it is absolutely convergent on the region $0 < |(z_1 - z_2)z_2^{-1}| < r$. We have proved that the left-hand side of (3.6.31) is absolutely convergent on the region $|z_1|, |z_2| > |z_1 - z_2| > 0$. So (3.6.34) is absolutely convergent on the same region. But for any $0 < r < 1$, we can find z_1, z_2 in this region such that $|(z_1 - z_2)z_2^{-1}| = r$. For example, we can take $z_2 = 1$, $z_1 = 1 + r$. Then

$z_1 - z_2 = r$. In this case, we have $|z_1| = 1 + r$, $|z_2| = 1 > r = |z_1 - z_2| > 0$ and $|(z_1 - z_2)z_2^{-1}| = r$. Thus (3.6.34) is absolutely convergent on the region $0 < |(z_1 - z_2)z_2^{-1}| < r$. Since r is an arbitrary positive number less that 1, we see that (3.6.34) and therefore also the left-hand side of (3.6.31) is absolutely convergent on the region $|z_2| > |z_1 - z_2| > 0$. Since on the region $|z_1|, |z_2| > |z_1 - z_2| > 0$, the left hand side of (3.6.31) is convergent to the the same rational function that the right-hand side of (3.6.29) is convergent to, it is also convergent to the same rational function on the region $|z_2| > |z_1 - z_2| > 0$. □

Exercise 3.6.2. Prove the other axioms for the contragredient V-module W'.