Two-dimensional conformal field theory

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Chapter 2

Vertex operator algebras

In this chapter, we give the definitions of grading-restricted vertex algebra and vertex operator algebra (grading-restricted conformal vertex algebra). The main difference between the definitions given in this book and many other papers books on vertex (operator) algebras is that we use the duality property (rationality, commutativity and associativity) as the main axioms. The duality property can be easily generalized to intertwining operators but the main axioms used in [B], [FLM], [FHL], [LL], [K] and [FB] cannot. Since the main objects studied in this book are intertwining operators, it is more appropriate to start with grading-restricted vertex algebras and vertex operator algebras by using the duality property as the main axioms.

Basic properties of vertex operator algebras are proved. Some of these properties are in fact the main axioms for vertex (operator) algebras used in other papers and books.

One of the main goal of this chapter is to give quickly some basic examples of vertex operator algebras. To do this, we first give a construction theorem of grading-restricted vertex algebras and vertex operator algebras. Then basic examples of vertex operator algebras, including the Heisenberg vertex operator algebras, the lattice vertex operator algebras, affine vertex operator algebras and Virasoro vertex operator algebras, are given by verifying the conditions needed to apply this theorem.

2.1 Meromorphic fields in a conformal field theory

In this section, we introduce meromorphic fields in a conformal field theory and prove some basic properties, including in particular a duality property. These properties, especially the duality property, will be used in the definition of vertex operator algebras in Section 2.3 below.

Let (H, ϕ) be a conformal field theory satisfying Definition 1.1.4. Since the purpose of this section is to give a motivation of the axioms for vertex operator algebras, for simplicity, we shall assume that ϕ is in fact a functor from $\mathcal{B}$ to $\mathcal{H}$ instead of $\mathcal{P}(\mathcal{H})$. In fact, it can be proved that from ϕ, we can always find a map from the set $\mathrm{Mor}(\mathcal{B}_{0,SL(2,\mathbb{C})})$ of morphisms in $\mathcal{B}$ given by Riemann sphere $\mathbb{C} \cup \{\infty\}$ with circles in $\mathbb{C}$ as boundary components and with elements of $SL(2,\mathbb{C})$ as boundary parametrizations to morphisms in $\mathcal{H}$. Since our discussion below uses only morphisms in $\mathrm{Mor}(\mathcal{B}_{0,SL(2,\mathbb{C})})$, our discussion below also works for an arbitrary conformal field theory satisfying Definition 1.1.4.

Consider two objects of the category $\mathcal{B}$: The first is the object containing 3 copies of S^1 and the second is the object containing one copy of S^1. Let $a_0, a_1, a_2, a_3 \in \mathbb{C}^\times$ and $z_1, z_2 \in \mathbb{C}$ satisfying $|a_0| > |z_i| + |a_i|, |a_3|$, for $i = 1, 2$, $|a_1| + |a_2| < |z_1 - z_2|$, $|a_i| + |a_3| < |z_i|$ for $i = 1, 2$. Then we have a morphism $[\Sigma_{z_1,z_2;a_0,a_1,a_2}] \in \mathrm{Mor}(\mathcal{B}_{0,SL(2,\mathbb{C})})$ from the first object to the second object given by the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with the negatively oriented ordered boundary components given by circles of radii $|a_1|, |a_2|, |a_3|$ centered at $z_1, z_2, 0$, respectively, and with the parametrizations of the boundary components given by the maps $e^{i\theta} \mapsto z_1 + a_1 e^{i\theta}$, $e^{i\theta} \mapsto z_2 + a_2 e^{i\theta}$, $e^{i\theta} \mapsto a_3 e^{i\theta}$, respectively, and with the positively oriented boundary component given by the circle of radius $|a_0|$ centered at 0 and with the parametrizations of the boundary component given by the map $e^{i\theta} \mapsto a_0 e^{i\theta}$.

For $v, w_0, w_1, w_2 \in H$ and fixed z_2, a_0, a_2, a_3, we have a function

$$(w_0, \phi([\Sigma_{z_1,z_2;a_0,a_1,a_2,a_3}])(v \otimes w_1 \otimes w_2)) \tag{2.1.1}$$

of z_1 and a_1. We say that $v \in H$ is a *meromorphic state of weight* $n \in \mathbb{Z}$ if for $w_0, w_1, w_2 \in H$ and fixed z_2, a_0, a_2, a_3, (2.1.1) as a function of z_1 is analytic and can be analytically extended to a meromorphic function of z_1 on $\mathbb{C} \cup \{\infty\}$ with the only possible poles $z_1 = 0, z_2, \infty$ and as a function of a_1 is proportional to a_1^n. For $n \in \mathbb{Z}$, it is clear that all meromorphic state of weight n form a subspace $V_{(n)}$ of H. Let $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$.

We now define a map $Y_V : \mathbb{C}^\times \to \mathrm{Hom}(V \otimes V, \overline{V})$ as follows: For $z \in \mathbb{C}^\times$, a_0, a_1, a_2 satisfying $|a_0| > |z| + |a_1|, |a_2|$, $|a_1| + |a_2| < |z_1|$, let $[\Sigma_{z;a_0,a_1,a_2}]$ be the morphism from the object of $\mathcal{B}$ containing two copies of S^1 to the object of $\mathcal{B}$ containing one copy of S^1 given by the Riemann sphere with boundary components similar to those in $[\Sigma_{z_1,z_2;a_0,a_1,a_2}]$. For $u \in V_{(n)}$, $v \in V_{(l)}$ and $w \in H$, we have a function $(w, \phi([\Sigma_{z;1,a_1,a_2}])(u \otimes v))$. Since $u \in V_{(n)}$, this function can be analytically extended to a meromorphic function of z on $\mathbb{C} \cup \{\infty\}$ with the only possible poles at $z = 0, \infty$ and is also proportional to a_1^n. Since $v \in V_{(l)}$, this function is also proportional to a_2^l. We define $(w, Y_V(u, z)v)$ to be the analytic extension of $a_1^{-n} a_2^{-l} (w, \phi([\Sigma_{z;1,a_1,a_2}])(u \otimes v))$. Note that by definition, $(w, Y_V(u, z)v)$ is a meromorphic function on $\mathbb{C} \cup \{\infty\}$ with the only poles at 0 and ∞, that is, a Laurent polynomial of z. The linear map $Y_V(u, z)$ is called the *meromorphic field associated to* u.

For $w \in H$, $v_1 \in V_{(k_1)}, v_2 \in V_{(k_2)}, v_3 \in V \in V_{(k_3)}$,

$$a_1^{-k_1} a_2^{-k_2} a_3^{-k_3} (w, \phi([\Sigma_{z_1,z_2;1,a_1,a_2,a_3}])(v_1 \otimes v_2 \otimes v_3)) \tag{2.1.2}$$

is a meromorphic function of z_1, z_2 on $\mathbb{C} \cup \{\infty\}$ with the only possible poles $z_1 = 0, \infty$, $z_2 = 0, \infty$ and $z_1 = z_2$ and is independent of a_1, a_2, a_3. Then on the region $|z_1| > |z_2| > 0$, (2.1.2) can be expanded as a Laurent series in z_1, z_2. From the axioms for conformal field theory, this Laurent series is equal to

$$(w, Y_V(v_1, z_1) Y_V(v_2, z_2) v_3). \tag{2.1.3}$$

In other words, the Laurent series (2.1.3) is absolutely convergent on the region $|z_1| > |z_2| > 0$ to (2.1.2), which is a rational function of z_1, z_2 on $\mathbb{C}$ with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$. This is called the *rationality of products of meromorphic fields*.

For the same $w \in H$, $v_1 \in V_{(k_1)}, v_2 \in V_{(k_2)}, v_3 \in V \in V_{(k_3)}$, we have also proved that for $\sigma \in S_m$,

$$(w, Y_V(v_2, z_2)Y_V(v_1, z_1)v_3) \tag{2.1.4}$$

is absolutely convergent on the region $|z_2| > |z_1| > 0$ to

$$a_2^{-k_2} \cdots a_1^{-k_1} a_3^{-k_3}(w, \phi([\Sigma_{z_2, z_1; 1, a_2, a_1, a_3}])(v_2 \otimes v_1 \otimes v_3)). \tag{2.1.5}$$

Since (H, ϕ) is an algebra over the PROP $\mathcal{B}$, (2.1.5) is equal to (2.1.4). Thus we have proved that (2.1.3) and (2.1.4) are absolutely convergent on the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively, to the common rational function (2.1.2) or (2.1.5) with the only possible poles $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$. This is called the *commutativity of the meromorphic fields*.

On the other hand, the rational function (2.1.2) can also be expanded on the region $|z_2| > |z_1 - z_2| > 0$ as a Laurent series in z_2 and $z_1 - z_2$. Also from the axioms for conformal field theory, this Laurent series is equal to

$$(w, Y_V(Y_V(v_1, z_1 - z_2)v_2, z_2)v_3). \tag{2.1.6}$$

In another words, the Laurent series (2.1.6) is absolutely convergent on the region $|z_2| > |z_1 - z_2| > 0$ to (2.1.2). This is called the *rationality of iterates of meromrophic fields*. We have proved that (2.1.3) and (2.1.6) are absolutely convergent on the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to the common rational function (2.1.2) with the only possible poles $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$. This is called the *associativity of the meromorphic fields*.

The rationality of products and iterates, the commutativity and associativity of meromorphic fields together are called the *duality property of meromrophic fields*.

We derive the duality property of meromorphic fields from the definition of meromorphic fields and axioms for conformal field theories. But one of the most difficult problem is to construct a conformal field theory satisfying all the axioms. The approach of the representation theory of vertex operator algebras is to construct conformal field theories using the following steps: (1) Construct and study vertex operator algebras, which are defined using the duality property or some equivalent properties as the main axiom. (2) Construct and study modules for vertex operator algebras. (3) Study intertwining operators among modules and use them to construct to construct all chiral correlation functions, modular functors and chiral weakly conformal field theories. (4) Putting chiral weakly conformal field theories and anti-chiral conformal field theories together to obtainfull conformal field theories.

In the remaining part of this chapter, we define, construct and study vertex operator algebras.

2.2 Motivating example: Fock spaces and meromorphic fields

We have introduced meromorphic fields from conformal field theories. Roughly speaking, vertex operator algebras are algebras of meromorphic fields. Since we still do not have an example of conformal field theories, it is useful to look at some examples of meromorphic fields before we give the abstract definition of vertex operator algebra. In this section, we use the Fock space for the Heisenberg algebra to give examples of meromorphic fields. We also give the vacuums and the conformal elements in this example. But we will verify the axioms satisfied by these data in Section 2.6

Let $\mathfrak{h}$ be a finite-dimensional inner product vector space over $\mathbb{R}$ with the inner product $(\cdot,\cdot)$. Let $\hat{\mathfrak{h}} = \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\mathbf{k}$ be the Heisenberg algebra associated to $\mathfrak{h}$ with the commutator formula given by

$$[a\otimes t^m, b\otimes t^n] = m(a,b)\delta_{m+n,0}\mathbf{k},$$
$$[a\otimes t^m, \mathbf{k}] = 0$$

for $a,b\in\mathfrak{h}$ and $m,n\in\mathbb{Z}$. Note that $\mathfrak{h}$ is a vector space over $\mathbb{R}$ while $\hat{\mathfrak{h}}$ is a vector space over $\mathbb{C}$. Let $\hat{\mathfrak{h}}_+ = \mathfrak{h}\otimes t\mathbb{C}[t]$, $\hat{\mathfrak{h}}_- = \mathfrak{h}\otimes t^{-1}\mathbb{C}[t^{-1}]$ and $\hat{\mathfrak{h}}_0 = \mathfrak{h}\otimes t^0\oplus\mathbb{C}\mathbf{k}$. These are all Lie subalgebras of $\hat{\mathfrak{h}}$.

Let $\hat{\mathfrak{h}}_+$ act on the one-dimensional space $\mathbb{C}$ as 0 and $\mathbf{k}$ acts on $\mathbb{C}$ as 1. Then $\mathbb{C}$ becomes an $\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0$-module. Then we have the induced $\hat{\mathfrak{h}}$-module $U(\hat{\mathfrak{h}})\otimes_{U(\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0)}\mathbb{C}$. By the Poincaré-Birkhoff-Witt theorem, $U(\hat{\mathfrak{h}})\otimes_{U(\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0)}\mathbb{C}$ is linearly isomorphic to $U(\hat{\mathfrak{h}}_-)\otimes_{\mathbb{C}}\mathbb{C}\simeq S(\hat{\mathfrak{h}}_-)$. In particular, $S(\hat{\mathfrak{h}}_-)$ is equipped with an $\hat{\mathfrak{h}}$-module structure under this linear isomorphism. The $\hat{\mathfrak{h}}$-module $S(\hat{\mathfrak{h}}_-)$ is a Fock space.

The $\mathbb{Z}$-grading on $\hat{\mathfrak{h}}_-$ gives a $\mathbb{Z}$-grading on $S(\hat{\mathfrak{h}}_-)$ so that $S(\hat{\mathfrak{h}}_-) = \coprod_{n\in\mathbb{Z}} S(\hat{\mathfrak{h}}_-)_{(n)}$. It is easy to verify that this grading on $S(\hat{\mathfrak{h}}_-)$ is grading-restricted; in fact, it is easy to verify that $S(\hat{\mathfrak{h}}_-)_{(n)} = 0$ when $n<0$ and $\dim S(\hat{\mathfrak{h}}_-)_{(n)} < \infty$. For $n\in\mathbb{N}$, the nonzero elements of $S(\hat{\mathfrak{h}}_-)_{(n)}$ are said to have *weight* n. The $\hat{\mathfrak{h}}$-module $S(\hat{\mathfrak{h}}_-)$ is spanned by elements of the form $(a_1\otimes t^{-n_1})\cdots(a_k\otimes t^{-n_k})$ for $a_1,\dots,a_k\in\mathfrak{h}$ and $n_1,\dots,n_k\in\mathbb{Z}_+$. The weight of this element is $n_1+\cdots+n_k$.

We denote $1\in S(\hat{\mathfrak{h}}_-)$ by $\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$. For $a\in\mathfrak{h}$ and $n\in\mathbb{Z}$, we denote the action of $a\otimes t^n$ on $S(\hat{\mathfrak{h}}_-)$ by $a(n)$. Then $S(\hat{\mathfrak{h}}_-)$ is spanned by elements of the form

$$a_1(-n_1)\cdots a_k(-n_k)\mathbf{1} = (a_1\otimes t^{-n_1})\cdots(a_k\otimes t^{-n_k})$$

for $a_1,\dots,a_k\in\mathfrak{h}$ and $n_1,\dots,n_k\in\mathbb{Z}_+$.

The $\hat{\mathfrak{h}}$-module structure on $S(\hat{\mathfrak{h}}_-)$ can also be obtained explicitly as follows: For $a\in\mathfrak{h}$ and $n\in\mathbb{Z}$, we define the action of $a(n)$ on $S(\hat{\mathfrak{h}}_-)$ by

$$a(n)(a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}) = a(n)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}$$

when $n < 0$ for $a_1, \dots, a_k \in \mathfrak{h}$ and $n_1, \dots, n_k \in \mathbb{Z}_+$,

$$a(n)(a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}) = \sum_{i=1}^{k} a_1(-n_1)\cdots a_{i-1}(-n_{i-1})[a(n), a_i(-n_i)]a_{i+1}\cdots a_k(-n_k)\mathbf{1}$$

when $n \geq 0$ and

$$\mathbf{k}(a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}) = a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}.$$

Then it is easy to verify that $S(\hat{\mathfrak{h}}_-)$ with this action of $\hat{\mathfrak{h}}$ is an $\hat{\mathfrak{h}}$-module.

For $a \in \mathfrak{h}$, let $a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$. Then $a(x)$ is a linear map from $S(\hat{\mathfrak{h}}_-)$ to $S(\hat{\mathfrak{h}}_-)[[x, x^{-1}]]$. For $v \in S(\hat{\mathfrak{h}}_-)$ and $v' \in S(\hat{\mathfrak{h}}_-)'$ (the graded dual of $S(\hat{\mathfrak{h}}_-)$), it is easy to see that $\langle v', a(x)v\rangle$ is a Laurent polynomial in x. If we substitute a complex variable z for the formal variable x, $\langle v', a(z)v\rangle$ is in fact a meromorphic function on the sphere $\mathbb{C} \cup \{\infty\}$ with the only possible poles 0 and ∞. Thus $a(z)$ is in fact an example of meromorphic fields on $\mathbb{C} \cup \{\infty\}$. In fact, more meromorphic fields can be constructed from these "generating meromorphic fields."

To give more meromorphic fields explicitly, we first introduce a normal ordering operation on products of the operators of the form $a(n)$ for $a \in \mathfrak{h}$, $n \in \mathbb{N}$. Consider the product $a_1(-n_1)\cdots a_k(-n_k)$, where $a_1, \dots, a_k \in \mathfrak{h}$ and $n_1, \dots, n_k \in \mathbb{Z}$. We define normal ordering ${}^\circ_\circ a_1(n_1)\cdots a_k(n_k){}^\circ_\circ$ of $a_1(n_1)\cdots a_k(n_k)$ to be the operator obtained by taking the product of these operators in an order such that $a_i(n_i)$ with $n_i \in \mathbb{N}$ are always to the right of those $a_i(n_i)$ with $n_i \in -Z_+$. Note that $a_i(n_i)$ with $n_i \in \mathbb{N}$ commute with themselves. So the normal ordered product can also be defined by taking the product these operators in an order such that $a_i(n_i)$ with $n_i \in \mathbb{N}$ are always to the right of all the other $a_i(n_i)$. More precisely, we can also define ${}^\circ_\circ a_1(n_1)\cdots a_k(n_k){}^\circ_\circ$ as foillwos: We can always find $\sigma \in S_k$ such that $n_{\sigma(1)}, \dots, n_{\sigma_l} < 0$, $n_{\sigma(l+1)}, \dots, n_{\sigma_k} \geq 0$. We define

$${}^\circ_\circ a_1(n_1)\cdots a_k(n_k){}^\circ_\circ = a_{\sigma(1)}(n_{\sigma(1)})\cdots a_{\sigma(k)}(n_{\sigma(k)}).$$

Note that though σ is not unique, ${}^\circ_\circ a_1(n_1)\cdots a_k(n_k){}^\circ_\circ$ is indeed well defined.

For $a_1, \dots, a_k \in \mathfrak{h}$ and $n_1, \dots, n_k \in \mathbb{Z}_+$, we define

$$Y_{S(\hat{\mathfrak{h}}_-)}(a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}, x) = {}^\circ_\circ \frac{1}{(n_1-1)!}\frac{d^{n_1-1}}{dx^{n_1-1}}a_1(x)\cdots\frac{1}{(n_k-1)!}\frac{d^{n_k-1}}{dx^{n_k-1}}a_k(x){}^\circ_\circ. \tag{2.2.7}$$

This is a linear map from $S(\hat{\mathfrak{h}}_-)$ to $S(\hat{\mathfrak{h}}_-)[[x, x^{-1}]]$. We also obtain a linear map

$$Y_{S(\hat{\mathfrak{h}}_-)} : S(\hat{\mathfrak{h}}_-) \otimes S(\hat{\mathfrak{h}}_-) \to S(\hat{\mathfrak{h}}_-)[[x, x^{-1}]]$$
$$u \otimes v \mapsto Y_{S(\hat{\mathfrak{h}}_-)}(u, x)v$$

called the vertex operator map for $S(\hat{\mathfrak{h}}_-)$.

Exercise 2.2.1. For $v \in S(\hat{\mathfrak{h}}_-)$ and $v' \in S(\hat{\mathfrak{h}}_-)'$, show that

$$\langle v', Y_{S(\hat{\mathfrak{h}}_-)}(a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}, x)v\rangle$$

is a Laurent polynomial in x.

By Exercise 2.2.1,

$$\langle v', Y_{S(\hat{\mathfrak{h}}_-)}(a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}, z)v\rangle$$

is a meromorphic function on the sphere $\mathbb{C}\cup\{\infty\}$ with the only possible poles 0 and ∞. Thus $Y_{S(\hat{\mathfrak{h}}_-)}(a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}, z)$ is also an example of meromorphic fields on the sphere $\mathbb{C}\cup\{\infty\}$.

When we multiply these meromorphic fields in a suitable way, applying the products to an element $v \in S(\hat{\mathfrak{h}}_-)$ and paring with an element $v' \in S(\hat{\mathfrak{h}}_-)'$, we will obtain meromorphic functions in several variables. This is in fact one part of the main axiom in the definition of vertex operator algebra in next section. We will discuss this precisely in Section 2.6 on the Heisenberg vertex operator algebras.

Since we are studying conformal field theories, there must be conformal symmetry in our theory. In general, there are conformal anomalies for the conformal symmetries in conformal field theories. So more precisely, we must have actions of the Virasoro algebra. For a conformal field theory, the action of the Virasoro algebra is given by what is called a stress energy tensor in physics.

Let $\{u^i\}_{i=1}^{\dim\mathfrak{h}}$ be an orthonormal basis of $\mathfrak{h}$. The stress-energy tensor in this case is defined to be the series

$$T(x) = \frac{1}{2}\sum_{i=1}^{\dim\mathfrak{h}} {}^{\circ}_{\circ}u^i(x)u^i(x){}^{\circ}_{\circ}.$$

From the definition of the vertex operator above, we also have

$$T(x) = Y_{S(\hat{\mathfrak{h}}_-)}(\omega, x),$$

where

$$\omega = \frac{1}{2}\sum_{i=1}^{\dim\mathfrak{h}} u^i(-1)^2\mathbf{1}.$$

Since $T(x)$ is a Laurent series with operators on $S(\hat{\mathfrak{h}}_-)$ as coefficients, we have

$$T(x) = \sum_{n\in\mathbb{Z}} L_{S(\hat{\mathfrak{h}}_-)}(n)x^{-n-2}.$$

Then the coefficients $L_{S(\hat{\mathfrak{h}}_-)}(n)$ satisfy the Virasoro commutator relations. We will prove this in Section 2.6.

We now have a quadruple $(S(\hat{\mathfrak{h}}_-), Y_{S(\hat{\mathfrak{h}}_-)}, \mathbf{1}, \omega)$. This quadruple is an example of vertex operator algebras and the triple $(S(\hat{\mathfrak{h}}_-), Y_{S(\hat{\mathfrak{h}}_-)}, \mathbf{1})$ is in fact an example of a grading-restricted vertex algebra. See Section 2.6 for the precise construction and proofs.

2.3 Definition

In this section, with the motivating example in the preceding section in mind, we give definitions of grading-restricted vertex algebra, quasi-vertex operator algebra or Möbius vertex algebra, conformal element and vertex operator algebra.

For a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, let $V' = \coprod_{n\in\mathbb{Z}} V_{(n)}^*$ be its graded dual space and $\overline{V} = \prod_{n\in\mathbb{Z}} V_{(n)}$ be its algebraic completion. On V and V', we use the topology given by the dual pair (V, V'). For $n \in \mathbb{N}$, a sequence (or more generally a net) $\{f_n\}$ in $\mathrm{Hom}(V \otimes \cdots \otimes V, \overline{V})$ is convergent to $f \in \mathrm{Hom}(V \otimes \cdots \otimes V, \overline{V})$ if for $v_1, \dots, v_n \in V$ and $v' \in V'$, $\langle v', f_n(v_1 \otimes \cdots \otimes v_n)\rangle$ is convergent to $\langle v', f(v_1 \otimes \cdots \otimes v_n)\rangle$. In particular, analytic maps from a region in $\mathbb{C}$ to $\mathrm{Hom}(V^{\otimes n}, \overline{V})$ make sense. For a $\mathbb{C}$-graded vector space, we use the same notations and definition of convergence.

We give the definition of grading-restricted vertex algebra first.

Definition 2.3.1. A *grading-restricted vertex algebra* is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, equipped with a linear map

$$\begin{aligned} Y_V : V \otimes V &\to V[[x, x^{-1}]], \\ u \otimes v &\mapsto Y_V(u, x)v, \end{aligned}$$

or equivalently, an analytic map

$$\begin{aligned} Y_V : \mathbb{C}^\times &\to \mathrm{Hom}(V \otimes V, \overline{V}), \\ z &\mapsto Y_V(\cdot, z)\cdot : u \otimes v \mapsto Y_V(u, z)v \end{aligned}$$

called the *vertex operator map* and a *vacuum* $\mathbf{1} \in V_{(0)}$ satisfying the following axioms:

1. Axioms for the grading: (a) *Grading-restriction condition*: When n is sufficiently negative, $V_{(n)} = 0$ and $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$. (b) *$L(0)$-commutator formula*: Let $L_V(0) : V \to V$ be defined by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, x)] = x\frac{d}{dx}Y_V(v, x) + Y_V(L_V(0)v, x) \qquad (2.3.8)$$

   for $v \in V$.

2. Axioms for the vacuum: (a) *Identity property*: Let 1_V be the identity operator on V. Then $Y_V(\mathbf{1}, x) = 1_V$. (b) *Creation property*: For $u \in V$, $\lim_{x\to 0} Y_V(u, x)\mathbf{1}$ exists and is equal to u.

3. *$L(-1)$-derivative property* and *$L(-1)$-commutator formula*: Let $L_V(-1) : V \to V$ be the operator given by

$$L_V(-1)v = \lim_{x\to 0} \frac{d}{dx} Y_V(v, x)\mathbf{1}$$

   for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx}Y_V(v, x) = Y_V(L_V(-1)v, x) = [L_V(-1), Y_V(v, x)]. \qquad (2.3.9)$$

4. *Duality*: For $u_1, u_2, v \in V$ and $v' \in V'$, the series

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle, \tag{2.3.10}$$
$$\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle, \tag{2.3.11}$$
$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle, \tag{2.3.12}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

Remark 2.3.2. In Definition 2.3.1, the duality property can be stated separately as three axioms, that is, the *rationality* (the convergence of (2.3.10), (2.3.11) and (2.3.12) to rational functions in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively), the *commutativity* (the statement that the rational functions to which (2.3.10) and (2.3.11) converge are equal) and the *associativity* (the statement that the (2.3.10) and (2.3.12) are equal in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$). These axioms are not independent. In fact, the associativity follows from the rationality and commutativity (see [FHL]) and the commutativity also follows from the rationality and associativity (see [H2]).

Definition 2.3.3. A *Möbius vertex algebra* or a *quasi-vertex operator algebra* is a grading-restricted vertex algebra $(V, Y_V, \mathbf{1})$ together with an operator $L_V(1)$ of weight 1 on V satisfying

$$\begin{aligned}
[L_V(-1), L_V(1)] &= -2L_V(0),\\
[L_V(1), Y_V(v, x)] &= Y_V(L_V(1)v, x) + 2xY_V(L_V(0)v, x) + x^2Y_V(L_V(-1)v, x)
\end{aligned} \tag{2.3.13}$$

for $v \in V$.

Definition 2.3.4. Let V_1 and V_2 be grading-restricted vertex algebras. A homomorphism from V_1 to V_2 is a grading-preserving linear map $g : V_1 \to V_2$ such that $gY_{V_1}(u, x)v = Y_{V_2}(gu, x)gv$. An isomorphism from V_1 to V_2 is an invertible homomorphism from V_1 to V_2. When $V_1 = V_2 = V$, an isomorphism from V to V is called an automorphism of V.

Definition 2.3.5. Let $(V, Y_V, \mathbf{1})$ be a grading-restricted vertex algebra. A *conformal element* of V is an element $\omega \in V$ satisfying the following axioms:

1. There exists $c \in \mathbb{C}$ such that $Y_V(\omega, x)\omega$ is equal to $L_V(-1)\omega x^{-1} + 2\omega x^{-2} + \frac{c}{2}\mathbf{1}x^{-4}$ plus a V-valued power series in x.

2. $L_V(-1) = \mathrm{Res}_x Y_V(\omega, x)$ and $L_V(0) = \mathrm{Res}_x xY_V(\omega, x)$ (Res_x being the operation of taking the coefficient of x^{-1} of a Laurent series).

A grading-restricted vertex algebra equipped with a conformal element is called a *vertex operator algebra* (or, more consistently, *grading-restricted conformal vertex algebra*).

2.4 Basic properties

In this section, we prove basic properties of vertex operator algebras. Some of them have been used as the main axioms for vertex operator algebras in many papers and books.

2.4.1 Operator product expansion

Let V be a grading-restricted vertex algebra. For $u_1, u_2, v \in V$ and $v' \in V'$, by definition,

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle$$

is absolutely convergent in the region $|z_1| > |z_2| > 0$ and

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

is absolutely convergent in the region $|z_2| > |z_1 - z_2| > 0$. Since $(V')^*$ is canonically isomorphic to $\overline{V} = \prod_{n\in\mathbb{Z}} V_{(n)}$, $Y_V(u_1, z_1)Y_V(u_2, z_2)v$ (when $|z_1| > |z_2| > 0$) and $Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v$ (when $|z_2| > |z_1 - z_2| > 0$) as elements of $(V')^*$ will be viewed as elements of $\overline{V}$. Since v is arbitrary, $Y_V(u_1, z_1)Y_V(u_2, z_2)$ (when $|z_1| > |z_2| > 0$) and $Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)$ (when $|z_2| > |z_1 - z_2| > 0$) are maps from V to $\overline{V}$. Then we obtain the *associativity*

$$Y_V(u_1, z_1)Y_V(u_2, z_2) = Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2) \tag{2.4.14}$$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. Since $Y_V(u_1, x) \in V((x))$, we have $Y_V(u_1, x)u_2 = \sum_{n\in\mathbb{Z}}(Y_V)_n(u_1)u_2x^{-n-1}$ for $(Y_V)_n(u_1)u_2 \in V$ and there exists $N \in \mathbb{N}$ such that $(Y_V)_n(u_1)u_2 = 0$ for $n > N$. Then we have

$$Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2) = \sum_{n\leq N} Y_V((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1}.$$

in the region $|z_2| > |z_1 - z_2| > 0$. From this expansion and (2.4.14), we obtain

$$Y_V(u_1, z_1)Y_V(u_2, z_2) = \sum_{n\leq N} Y_V((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1} \tag{2.4.15}$$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. The formula (2.4.15) is called the *operator product expansion* of the fields or vertex operators $Y_V(u_1, z_1)$ and $Y_V(u_2, z_2)$. The terms that are singular in the right-hand side of (2.4.15) are

$$\sum_{n=0}^{N} Y_V((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1}.$$

These singular terms are the only useful terms in the calculations of the commutators of the fields or vertex operators $Y_V(u_1, z_1)$ and $Y_V(u_2, z_2)$. So physicists usually write the operator product expansion with only these singular terms as

$$Y_V(u_1, z_1)Y_V(u_2, z_2) \sim \sum_{n=0}^{N} Y_V((Y_V)_n(u_1)u_2, z_2)(z_1 - z_2)^{-n-1}. \tag{2.4.16}$$

2.4.2 The Jacobi identity

Let $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$ be the formal delta function. Then we have the basic property of $\delta(x)$: For any formal Laurent series $f(x)$ with coefficients in a vector space such that $f(x)\delta(x)$ and $f(1)$ is well defined,

$$f(x)\delta(x) = f(1)\delta(x).$$

By definition, we have

$$\delta(x) = (1-x)^{-1} - (-x+1)^{-1},$$

where use the convention that $(1-x)^{-1}$ and $(-x+1)^{-1}$ are the binomial expansions of $(1-x)^{-1}$ in the nonnegative and negative, respectively, powers of x. We also have

$$x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = (x_1-x_2)^{-1} - (-x_2+x_1)^{-1},$$

where we also use the convention that $(x_1-x_2)^{-1}$ and $(-x_2+x_1)^{-1}$ are the binomial expansions of $(x_1-x_2)^{-1}$ in the nonnegativepowers in x_2 and in the nonnegative powers in x_1, respectively. In the future, we will always use the convention that a formal binomial expression is expanded in the nonnegative powers of the second formal variable.

We need to consider the following three formal delta functions:

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right),\ x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right), x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right).$$

(Note that here we have used the binomial expansion convention above.) In these formal expressions, we always expand a binomial as a formal Laurent series in nonnegative powers in the second formal variable. It is easy to check directly that the following identity holds:

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right), \tag{2.4.17}$$

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right). \tag{2.4.18}$$

Let V be a grading-restricted vertex algebra. For $u_1, u_2, v \in V$ and $v' \in V'$, the duality property says that (2.3.10), (2.3.11) and (2.3.12) are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. This common rational function can be written explicitly as $\frac{f(z_1,z_2)}{z_1^r z_2^s (z_1-z_2)^t}$, where $f(z_1, z_2)$ is a polynomial in z_1 and z_2 and $r, s, t \in \mathbb{N}$. We multiply the Laurent polynomial $\frac{f(x_1,x_2)}{x_1^r x_2^s x_0^t}$ in the formal variable x_0, x_1 and x_2 to both sides of (2.4.17) to obtain the identity

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\frac{f(x_1,x_2)}{x_1^r x_2^s x_0^t} - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\frac{f(x_1,x_2)}{x_1^r x_2^s x_0^t} = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\frac{f(x_1,x_2)}{x_1^r x_2^s x_0^t}. \tag{2.4.19}$$

Using the basic property of the formal delta function, we can rewrite (2.4.19) as

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\frac{f(x_1,x_2)}{x_1^r x_2^s (x_1-x_2)^t} - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\frac{f(x_1,x_2)}{x_1^r x_2^s(-x_2+x_1)^t}$$
$$= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\frac{f(x_1,x_2)}{(x_2+x_0)^r x_2^s x_0^t}. \quad (2.4.20)$$

Note that $\frac{1}{(x_1-x_2)^t}$ in $\frac{f(x_1,x_2)}{x_1^r x_2^s(x_1-x_2)^t}$ is expanded in nonnegative powers of x_2. We already know that (2.3.10) is absolutely convergent in the region $|z_1|>|z_2|>0$ to $\frac{f(z_1,z_2)}{z_1^r z_2^s(z_1-z_2)^t}$. In other words, the expansion of $\frac{f(z_1,z_2)}{z_1^r z_2^s(z_1-z_2)^t}$ as a Laurent series in z_1 and z_2 in the region $|z_1|>|z_2|>0$ is exactly (2.3.10). This is the same as saying that $\frac{f(x_1,x_2)}{x_1^r x_2^s(x_1-x_2)^t}$ as a formal Laurent series in x_1 and x_2 obtained by expanding $\frac{1}{(x_1-x_2)^t}$ in nonnegative powers of x_2 is exactly

$$\langle v', Y_V(u_1,x_1)Y_V(u_2,x_2)v\rangle. \quad (2.4.21)$$

So we can replace $\frac{f(z_1,z_2)}{z_1^r z_2^s(z_1-z_2)^t}$ in (2.4.20) by (2.4.21). Similarly, we can replace $\frac{f(x_1,x_2)}{x_1^r x_2^s(-x_2+x_1)^t}$ and $\frac{f(x_1,x_2)}{(x_2+x_0)^r x_2^s x_0^t}$ in (2.4.19) by

$$\langle v', Y_V(u_2,x_2)Y_V(u_1,x_1)v\rangle$$

and

$$\langle v', Y_V(Y_V(u_1,x_0)u_2,x_2)v\rangle,$$

respectively. Thus we obtain

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\langle v', Y_V(u_1,x_1)Y_V(u_2,x_2)v\rangle - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\langle v', Y_V(u_2,x_2)Y_V(u_1,x_1)v\rangle$$
$$= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\langle v', Y_V(Y_V(u_1,x_0)u_2,x_2)v\rangle.$$

Since v' and v are arbitrary, we obtain the following Jacobi identity:

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_V(u_1,x_1)Y_V(u_2,x_2) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_V(u_2,x_2)Y_V(u_1,x_1)$$
$$= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_1,x_0)u_2,x_2). \quad (2.4.22)$$

The Jacobi identity is the main axiom for vertex operator algebras in [FLM] and [FHL].

Exercise 2.4.1. Prove that the Jacobi identity can be used to replace the duality property in the definition of grading-restricted vertex algebra.

On the other hand, the Jacobi identity can be used to define more general vertex algebras than what we are interested in this book, for example, vertex algebras without gradings.

2.4.3 Skew-symmetry

Replacing u_1, u_2, x_1, x_2 and x_0 in (2.4.22) by u_2, u_1, x_2, x_1 and $-x_0$, respectively, we obtain

$$-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_V(u_2,x_2)Y_V(u_1,x_1)+x_0^{-1}\delta\left(\frac{-x_1+x_2}{-x_0}\right)Y_V(u_2,x_1)Y_V(u_2,x_1)$$
$$=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_V(Y_V(u_2,-x_0)u_1,x_1). \tag{2.4.23}$$

Since the left-hand sides of (2.4.23) and (2.4.22) are equal, the right-hand sides are also equal. So we obtain

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_1,x_0)u_2,x_2)=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_V(Y_V(u_2,-x_0)u_1,x_1). \tag{2.4.24}$$

Using (2.4.18) and the basic property of the formal delta function in the right-hand side of (2.4.24), we see that (2.4.24) becomes

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_1,x_0)u_2,x_2)=x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_V(Y_V(u_2,-x_0)u_1,x_2+x_0). \tag{2.4.25}$$

From the $L(-1)$-derivative property

$$\frac{d}{dx_2}Y_V(Y_V(u_2,-x_0)u_1,x_2)=Y_V(L_V(-1)Y_V(u_2,-x_0)u_1,x_2),$$

we obtain

$$\frac{d^n}{dx_2^n}Y_V(Y_V(u_2,-x_0)u_1,x_2)=Y_V(L_V(-1)^nY_V(u_2,-x_0)u_1,x_2) \tag{2.4.26}$$

for $n\in\mathbb{N}$. For $f(x_2)\in V((x_2))$, we have the formal Taylor's theorem

$$f(x_2+x_0)=\sum_{n\in\mathbb{N}}\frac{x_0^n}{n!}\frac{d^n}{dx_2^n}f(x_2). \tag{2.4.27}$$

Applying both sides of (2.4.25) to $\mathbf{1}$, using the formal Taylor's theorem (2.4.27) with $f(x_2)=Y_V(Y_V(u_2,-x_0)u_1,x_2)\mathbf{1}$, using (2.4.26), taking Res_{x_1}, letting $x_2=0$ and then replacing x_0 by x, we obtain the skew-symmetry

$$Y_V(u_1,x)u_2=\sum_{n\in\mathbb{N}}\frac{x^n}{n!}L_V(-1)^nY_V(u_2,-x)u_1=e^{xL_V(-1)}Y_V(u_2,-x)u_1. \tag{2.4.28}$$

2.4.4 Commutator and associator formula

For a formal Laurent series $f(x)$ in x, we use $\mathrm{Res}_xf(x)$ to denote the coefficient of x^{-1} term in $f(x)$. Now taking Res_{x_0} on both sides of the Jacobi identity (2.4.22), we obtain the commutator formula for vertex operators:

$$Y_V(u_1,x_1)Y_V(u_2,x_2)-Y_V(u_2,x_2)Y_V(u_1,x_1)$$

$$= \mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_V(Y_V(u_1,x_0)u_2,x_2). \tag{2.4.29}$$

Taking Res_{x_1} on both sides of the Jacobi identity (2.4.22) and using (2.4.18) and the basic property of the formal delta function, we have

$$\begin{aligned}
&Y_V(Y_V(u_1,x_0)u_2,x_2)\\
&\quad= \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) Y_V(u_1,x_1)Y_V(u_2,x_2)\\
&\quad\quad - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2,x_2)Y_V(u_1,x_1)\\
&\quad= \mathrm{Res}_{x_1} x_1^{-1}\delta\left(\frac{x_0+x_2}{x_1}\right) Y_V(u_1,x_1)Y_V(u_2,x_2)\\
&\quad\quad - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2,x_2)Y_V(u_1,x_1)\\
&\quad= \mathrm{Res}_{x_1} x_1^{-1}\delta\left(\frac{x_0+x_2}{x_1}\right) Y_V(u_1,x_0+x_2)Y_V(u_2,x_2)\\
&\quad\quad - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2,x_2)Y_V(u_1,x_1)\\
&\quad= Y_V(u_1,x_0+x_2)Y_V(u_2,x_2)\\
&\quad\quad - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2,x_2)Y_V(u_1,x_1).
\end{aligned} \tag{2.4.30}$$

Moving the first term in the right-hand side of (2.4.30) to the left-hand side, we obtain the associator formula for vertex operators:

$$\begin{aligned}
&Y_V(Y_V(u_1,x_0)u_2,x_2) - Y_V(u_1,x_0+x_2)Y_V(u_2,x_2)\\
&\quad= -\mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y_V(u_2,x_2)Y_V(u_1,x_1).
\end{aligned} \tag{2.4.31}$$

2.4.5 Weak commutativity and weak associativity

Since $Y_V(u_1,x_0)u_2$ is a formal Laurent series with only finitely many negative powers in x_0, there exists $N \in \mathbb{Z}_+$ such that $x_0^N Y_V(u_1,x_0)u_2 \in V[[x_0]]$. Multiplying $(x_1-x_2)^N$ to the right-hand side of the commutator formula (2.4.29), using the basic property of the formal delta function and using the fact that Res_{x_0} of a formal power series is 0, we obtain

$$\begin{aligned}
&\mathrm{Res}_{x_0}(x_1-x_2)^N x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_V(Y_V(u_1,x_0)u_2,x_2)\\
&\quad= \mathrm{Res}_{x_0} x_0^N x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_V(Y_V(u_1,x_0)u_2,x_2)\\
&\quad= 0.
\end{aligned} \tag{2.4.32}$$

Thus $(x_1 - x_2)^N$ multiplied to the left-hand side of (2.4.29) is also 0. So we obtained the weak commutativity:

$$(x_1 - x_2)^N Y_V(u_1, x_1) Y_V(u_2, x_2) = (x_1 - x_2)^N Y_V(u_2, x_2) Y_V(u_1, x_1). \tag{2.4.33}$$

Similarly, since $Y_V(u_1, x_1)v$ is a formal Laurent series with only finitely many negative powers in x_1, there exists $N \in \mathbb{Z}_+$ such that $x_1^N Y_V(u_1, x_1)v \in V[[x_1]]$. Multiplying $(x_0+x_2)^N$ to the right-hand side of the associator formula (2.4.31), applying the result to v, using the basic property of the formal delta function and using the fact that Res_{x_1} of a formal power series is 0, we obtain

$$\begin{aligned} -\mathrm{Res}_{x_1}&(x_0 + x_2)^N x_0^{-1} \delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_V(u_2, x_2) Y_V(u_1, x_1)v \\ &= -\mathrm{Res}_{x_1} x_1^N x_0^{-1} \delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_V(u_2, x_2) Y_V(u_1, x_1)v \\ &= 0. \end{aligned} \tag{2.4.34}$$

Thus $(x_1 - x_2)^N$ multiplied to the left-hand side of (2.4.29) and then applied to v is also 0. So we obtained the weak associativity:

$$(x_0 + x_2)^N Y_V(Y_V(u_1, x_0)u_2, x_2)v = (x_0 + x_2)^N Y_V(u_1, x_0 + x_2) Y_V(u_2, x_2)v. \tag{2.4.35}$$

Weak commutativity and weak associativity can also be obtained directly from the duality property in the definition of grading-restricted vertex algebra.

2.4.6 Conformal element and Virasoro operators

Let ω be a conformal element of V (see Definition 2.3.5). Then

$$Y_V(\omega, x)\omega = L_V(-1)\omega x^{-1} + 2\omega x^{-2} + \frac{c}{2}\mathbf{1}x^{-4} + G(x), \tag{2.4.36}$$

where $G(x) \in V[[x]]$. Using the commutator formula (2.4.29) with $u_1 = u_2 = \omega$, we obtain

$$\begin{aligned} &Y_V(\omega, x_1) Y_V(u_2, x_2) - Y_V(\omega, x_2) Y_V(u_1, x_1) \\ &\quad = \mathrm{Res}_{x_0} x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) Y_V(Y_V(\omega, x_0)\omega, x_2) \\ &\quad = \mathrm{Res}_{x_0} x_0^{-1} x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) Y_V(L_V(-1)\omega, x_2) + 2\mathrm{Res}_{x_0} x_0^{-2} x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) Y_V(\omega, x_2) \\ &\qquad + \frac{c}{2}\mathrm{Res}_{x_0} x_0^{-4} x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) Y_V(\mathbf{1}, x_2) + \mathrm{Res}_{x_0} x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) Y_V(G(x_0), x_2) \\ &\quad = x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) \frac{\partial}{\partial x_2} Y_V(\omega, x_2) + 2x_1^{-1} \frac{\partial}{\partial x_2} \delta\left(\frac{x_2}{x_1}\right) Y_V(\omega, x_2) + \frac{c}{12} x_1^{-1} \frac{\partial^3}{\partial x_2^3} \delta\left(\frac{x_2}{x_1}\right), \end{aligned} \tag{2.4.37}$$

where in the last equality, we have used the $L(-1)$-derivative property, the formal Taylor's theorem (2.4.27) applied to $x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)$ and the fact $G(x_0) \in V[[x_0]]$.

Writing

$$Y(\omega, x) = \sum_{\in\mathbb{Z}} L_V(n)x^{-n-2}$$

and then taking the coefficients of $x_1^{-m-2}x_2^{-n-2}$ in (2.4.37), we obtain the Virasoro relations

$$\begin{aligned}
&L_V(m)L_V(n) - L_V(n)L_V(m)\\
&\quad = \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} x_1^{m+1}x_2^{n+1}(Y_V(\omega, x_1)Y_V(u_2, x_2) - Y_V(\omega, x_2)Y_V(u_1, x_1))\\
&\quad = \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} x_1^{m+1}x_2^{n+1}x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\frac{\partial}{\partial x_2}Y_V(\omega, x_2)\\
&\qquad + 2\mathrm{Res}_{x_1}\mathrm{Res}_{x_2} x_1^{m+1}x_2^{n+1}x_1^{-1}\frac{\partial}{\partial x_2}\delta\left(\frac{x_2}{x_1}\right)Y_V(\omega, x_2)\\
&\qquad + \frac{c}{12}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2} x_1^{m+1}x_2^{n+1}x_1^{-1}\frac{\partial^3}{\partial x_2^3}\delta\left(\frac{x_2}{x_1}\right)\\
&\quad = \mathrm{Res}_{x_2} x_2^{m+n+2}\frac{\partial}{\partial x_2}Y_V(\omega, x_2) + 2(m+1)\mathrm{Res}_{x_2} x_2^{m+n+1}Y_V(\omega, x_2)\\
&\qquad + \frac{c}{12}(m+1)m(m-1)\mathrm{Res}_{x_2} x_2^{m+n-1}\\
&\quad = (-m-n-2)L_V(m+n) + 2(m+1)L_V(m+n) + \frac{c}{12}(m+1)m(m-1)\delta_{m+n,0}\\
&\quad = (m-n)L_V(m+n) + \frac{c}{12}(m^3-m)\delta_{m+n,0}.
\end{aligned} \tag{2.4.38}$$

It is also easy to see by reversing the proof above that if the Virasoro relations (2.4.38) holds, then (2.4.36) holds. Thus we can replace (2.4.36) in Definition 2.3.5 by (2.4.38).

2.4.7 $L(-1)$-, $L(0)$-, $L(1)$-conjugation formulas

For a grading-restricted vertex operator algebra V, we have two special operators $L_V(0)$ and $L_V(-1)$ on V. We have the commutator formulas (2.3.9) and (2.3.8).

For $v \in V$, applying $[L_V(-1), \cdot]$ to the vertex operator $Y_V(v, x)$ n times, multiplying the result by $\frac{y^n}{n!}$, and then taking the sum over $\mathbb{N}$, we obtain $e^{yL_V(-1)}Y_V(v, x)e^{-L_V(-1)}$. But by (2.3.9), applying $[L_V(-1), \cdot]$ to $Y_V(v, x)$ n times is the same as applying $L_V(-1)^n$ to v in $Y_V(v, x)$ to obtain $Y_V(L_V(-1)^n v, x)$. Multiplying $Y_V(L_V(-1)^n v, x)$ by $\frac{y^n}{n!}$ and then taking the sum over $\mathbb{N}$, we obtain $Y_V(e^{yL_V(-1)}v, x)$. On the other hand, applying $L_V(-1), \cdot]$ to $Y_V(v, x)$ n times is also the same as taking the derivative of $Y_V(v, x)$ n times, that is, $\frac{d^n}{dx^n}Y_V(v, x)$. Multiplying $\frac{d^n}{dx^n}Y_V(v, x)$ by $\frac{y^n}{n!}$ and then taking the sum over $\mathbb{N}$, we obtain $\sum_{n\in\mathbb{N}}\frac{y^n}{n!}\frac{d^n}{dx^n}Y_V(v, x)$, which can be written as $Y_V(v, x+y)$ by Taylor's theorem (2.4.27). By the calculations above in both two case, we obtain the $L(-1)$-conjugation formula

$$e^{yL_V(-1)}Y_V(v, x)e^{-L_V(-1)} = Y_V(e^{yL_V(-1)}v, x) = Y_V(v, x+y) \tag{2.4.39}$$

for $v \in V$.

For homogeneous $v \in V$ and $n \in \mathbb{Z}$, using (2.3.8), we have

$$\begin{aligned}
&[L_V(0), \mathrm{Res}_x x^n Y_V(v,x)]\\
&\quad = \mathrm{Res}_x x^n [L_V(0), Y_V(v,x)]\\
&\quad = \mathrm{Res}_x x^{n+1} \frac{d}{dx} Y_V(v,x) + \mathrm{Res}_x x^n Y_V(L_V(0)v, x)\\
&\quad = \mathrm{Res}_x \frac{d}{dx} x^{n+1} Y_V(v,x) - \mathrm{Res}_x (n+1) x^n Y_V(v,x) + (\mathrm{wt}\ v)\mathrm{Res}_x x^n Y_V(v,x)\\
&\quad = (\mathrm{wt}\ v - n - 1)\mathrm{Res}_x x^n Y_V(v,x).
\end{aligned}$$

Then for homogeneous $u, v \in V$, we have

$$\begin{aligned}
L_V(0)\mathrm{Res}_x x^n Y_V(v,x)u &= \mathrm{Res}_x x^n Y_V(v,x) L_V(0)u + (\mathrm{wt}\ v - n - 1)\mathrm{Res}_x x^n Y_V(v,x)u\\
&= (\mathrm{wt}\ v - n - 1 + \mathrm{wt}\ u)\mathrm{Res}_x x^n Y_V(v,x)u.
\end{aligned}$$

So $\mathrm{wt}\ (\mathrm{Res}_x x^n Y_V(v,x)u) = (\mathrm{wt}\ v - n - 1 + \mathrm{wt}\ u)$. Then we have

$$\begin{aligned}
x^{L_V(0)}\mathrm{Res}_z z^n Y_V(v,z) x^{-L_V(0)} u &= (x^{L_V(0)}\mathrm{Res}_z z^n Y_V(v,z)u) x^{-\mathrm{wt}\ u}\\
&= x^{\mathrm{wt}\ v - n - 1}\mathrm{Res}_z z^n Y_V(v,z)u\\
&= \mathrm{Res}_z z^n x^{-n-1} Y_V(x^{L_V(0)}v, z)\\
&= \mathrm{Res}_\zeta \zeta^n Y_V(x^{L_V(0)}v, x\zeta)u,
\end{aligned}$$

where z and ζ are another two dummy formal variables and, in the last step, we change the variable from z to $\zeta = x^{-1}z$. Since u is an arbitrary homogeneous element of V, we obtain

$$x^{L_V(0)}\mathrm{Res}_z z^n Y_V(v,z) x^{-L_V(0)} = \mathrm{Res}_y y^n Y_V(x^{L_V(0)}v, xy).$$

Multiplying both sides by y^{-n-1}, then taking sum over $n \in \mathbb{Z}$, and using

$$\sum_{n\in\mathbb{Z}} \mathrm{Res}_z z^n y^{-n-1} f(z) = \mathrm{Res}_z y^{-1}\delta\left(\frac{z}{y}\right) f(z) = f(y)$$

for $f(y) \in \mathbb{C}[[y, y^{-1}]]$, we obtain the $L(0)$-conjugation formula

$$x^{L_V(0)} Y_V(v,y) x^{-L_V(0)} = Y_V(x^{L_V(0)}v, xy) \tag{2.4.40}$$

for $v \in V$.

For a Möbius vertex algebra or a vertex operator algebra V, we have an operator $L_V(V)(1)$ on V. In this case, we also have the commutator formula (2.3.13).

For $v \in V$, applying $[L_V(1), \cdot]$ to the vertex operator $Y_V(v,x)$ n times, multiplying the result by $\frac{y^n}{n!}$, and then taking the sum over $\mathbb{N}$, we obtain $e^{yL_V(1)}Y_V(v,x)e^{-L_V(1)}$. But by (2.3.13), applying $[L_V(1), \cdot]$ to $Y_V(v,x)$ n times is the same as applying $(L_V(1) + 2xL_V(0) + x^2 L_V(-1))^n$ to v in $Y_V(v,x)$ to obtain $Y_V((L_V(1) + 2xL_V(0) + x^2 L_V(-1))^n v, x)$. Multiplying

$Y_V((L_V(1) + 2xL_V(0) + x^2 L_V(-1))^n v, x)$ by $\frac{y^n}{n!}$ and then taking the sum over $\mathbb{N}$, we obtain $Y_V(e^{y(L_V(1)+2xL_V(0)+x^2L_V(-1))}v, x)$.

We need the identity

$$e^{y(L_V(1)+2xL_V(0)+x^2L_V(-1))} = e^{yx^2(1-yx)^{-1}L_V(-1)}e^{y(1-yx)L_V(1)}(1 - yx)^{-2L_V(0)}. \tag{2.4.41}$$

This identity is equivalent to the identity

$$e^{y(L_V(1)+2xL_V(0)+x^2L_V(-1))}(1 - yx)^{2L_V(0)}e^{-y(1-yx)L_V(1)e^{-yx^2(1-yx)^{-1}L_V(-1)}} = 1_V.$$

To prove this identity, we show that the derivative of the left-hand side is 0 so that the left-hand side must be a constant. Since when $x = 0$, the left-hand side is 1_V, we see that the left-hand-side is 1_V. The details of this proof is left as an exercise. Using (2.4.41) and (2.4.39), we have

$$\begin{aligned}
Y_V(e^{y(L_V(1)+2xL_V(0)+x^2L_V(-1))}v, x) &= Y_V(e^{yx^2(1-yx)^{-1}L_V(-1)}e^{y(1-yx)L_V(1)}(1 - yx)^{-2L_V(0)}v, x)\\
&= Y_V(e^{y(1-yx)L_V(1)}(1 - yx)^{-2L_V(0)}v, x + yx^2(1 - yx)^{-1})\\
&= Y_V(e^{y(1-yx)L_V(1)}(1 - yx)^{-2L_V(0)}v, x(1 - yx)^{-1}).
\end{aligned}$$

Thus we obtain the $L(1)$-conjugation property

$$e^{yL_V(1)}Y_V(v, x)e^{-L_V(1)} = Y_V(e^{y(1-yx)L_V(1)}(1 - yx)^{-2L_V(0)}v, x(1 - yx)^{-1}) \tag{2.4.42}$$

for $v \in V$.

Exercise 2.4.2. Prove the identity (2.4.41).

2.5 A construction theorem

Though we have motivated the definitions of grading-restricted vertex algebra and vertex operator algebra in Section 2.2, it is still not easy to understand them without the full details of some main examples. In this section, we give a construction theorem which reduces the verification of the axioms for grading-restricted vertex algebras to the verification of some simple properties. We will then give the main examples grading-restricted vertex algebras and vertex operator algebras in the next four sections using this construction theorem and the results in Subsection 2.4.6.

Let $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ be a $\mathbb{Z}$-graded vector space such that $V_{(n)} = 0$ for n sufficiently negative and $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$. Since $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$, we have $\overline{V} = (V')^*$. Elements of $V_{(n)}$ is said to have *weight* n. Let $L_V(0) : V \to V$ be the operator defined by the grading on V, that is, by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then for $a \in \mathbb{C}$, the operator $e^{aL_V(0)}$ on V defined by $e^{aL_V(0)}v = e^{an}v$ for $v \in V_{(n)}$ has a natural extension to $\overline{V}$. For $n \in \mathbb{Z}$, we use π_n to denote the projection from V or $\overline{V}$ to $V_{(n)}$.

An operator O on V satisfying $[L_V(0), O] = nO$ is said to have *weight* n. Similarly for operators on the graded dual V' of V.

Lemma 2.5.1. *Let* $\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n x^{-n-1} \in (\text{End } V)[[x.x^{-1}]]$*. If there exists* wt $\phi \in \mathbb{Z}$ *such that*

$$[L_V(0), \phi(x)] = x\frac{d}{dx}\phi(x) + (\text{wt } \phi)\phi(x),$$

then $\phi_n \in \text{Hom}(V, V)$ *is homogeneous of weight* wt $\phi - n - 1$*. In particular, for* $v \in V$*,* $\phi(x)v$ *as a Laurent series in* x *has only finitely many negative power terms and for* $v' \in V'$*,* $\langle v', \phi(x)\cdot\rangle$ *as a Laurent series with coefficients in* V' *has only finitely many positive powers of* x*.*

Proof. Taking the coefficients of the bracket formula for $L_V(0)$ and $\phi(z)$, we obtain that ϕ_n is of weight wt $\phi - n - 1$. Since $V_{(n)} = 0$ for n sufficiently negative and the weight of ϕ_n is wt $\phi - n - 1$, for $v \in V$, $\phi(x)v$ has only finitely many negative power terms and for $v' \in V'$, $\langle v', \phi(x)\cdot\rangle$ as a Laurent series with coefficients in V' has only finitely many positive powers of x. □

Let $\phi^i(x) \in (\text{End } V)[[x, x^{-1}]]$ for $i \in I$ and $\mathbf{1} \in V_{(0)}$. Write $\phi^i(x) = \sum_{n\in\mathbb{Z}} \phi^i_n x^{-n-1}$ for $i \in I$. Assume that $\phi^i(x) \in (\text{End } V)[[x, x^{-1}]]$ for $i \in I$ and $\mathbf{1} \in V_{(0)}$ satisfy the following conditions:

1. For $i \in I$, there exists wt $\phi^i \in \mathbb{Z}$ such that $[L_V(0), \phi^i(x)] = x\frac{d}{dx}\phi^i(x) + (\text{wt } \phi^i)\phi^i(x)$.

2. There exists an operator $L_V(-1)$ on V such that $L_V(-1)\mathbf{1} = 0$ and $[L_V(-1), \phi^i(x)] = \frac{d}{dx}\phi^i(x)$ for $i \in I$.

3. For $i \in I$, $\phi^i(x)\mathbf{1} \in V[[x]]$.

4. The vector space V is spanned by elements of the form $\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k}\mathbf{1}$ for $i_1, \dots, i_k \in I$ and $n_1, \dots, n_k \in \mathbb{Z}$.

5. For $i, j \in I$, there exists $N_{ij} \in \mathbb{Z}_+$ such that

$$(x_1 - x_2)^{N_{ij}}\phi^i(x_1)\phi^j(x_2) = (x_1 - x_2)^{N_{ij}}\phi^j(x_2)\phi^i(x_1). \qquad (2.5.43)$$

Proposition 2.5.2. *Let* $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ *be a* $\mathbb{Z}$*-graded vector space,* ϕ^i *for* $i \in I$ *linear maps from* V *to* $V[[x, x^{-1}]]$*, or equivalently, analytic maps from* $\mathbb{C}^\times$ *to* $\text{Hom}(V, \overline{V})$*,* $L_V(-1)$ *an operator on* V *and* $\mathbf{1} \in V_{(0)}$*. Assume that they satisfy Conditions 1–4. Then Condition 5 is equivalent to the following two properties:*

6. *For* $v' \in V'$*,* $v \in V$ *and* $i_1, \dots, i_k \in I$*, the series* $\langle v', \phi^{i_1}(z_1) \cdots \phi^{i_k}(z_k)v\rangle$ *(a Laurent series in* $z_1, \dots, z_k$ *with complex coefficients) is absolutely convergent in the region* $|z_1| > \cdots > |z_k| > 0$ *to a rational function* $R(\langle v', \phi^{i_1}(z_1) \cdots \phi^{i_k}(z_k)v\rangle)$ *in* $z_1, \dots, z_k$ *with the only possible poles at* $z_i = 0$ *for* $i = 1, \dots, k$ *and* $z_j = z_l$ *for* $j \neq l$*. In addition, the order of the pole* $z_j = z_l$ *is independent of* ϕ^{i_n} *for* $n \neq j, l$*,* v *and* v' *and the order of the pole* $z_j = 0$ *is independent of* ϕ^{i_n} *for* $n \neq j$ *and* v'*.*

7. For $v \in V$, $v' \in V'$, $i_1, i_2 \in I$,

$$R(\langle v', \phi^{i_1}(z_1)\phi^{i_2}(z_2)v\rangle) = R(\langle v', \phi^{i_2}(z_2)\phi^{i_1}(z_1)v\rangle).$$

Proof. We leave the proof that Properties 6 and 7 imply Condition 5 as an exercise.

Now we assume that Condition 5 holds. Consider the Laurent series

$$\prod_{1\le p<q\le k}(x_p - x_q)^{N_{i_p i_q}}\langle v', \phi^{i_1}(x_1)\cdots\phi^{i_k}(x_k)v\rangle. \tag{2.5.44}$$

For $1 \le l \le k$, using (2.5.43), the Laurent series (2.5.44) is equal to

$$\prod_{1\le p<q\le k}(x_p - x_q)^{N_{i_p i_q}}\langle v', \phi^{i_1}(x_1)\cdots\phi^{i_{l-1}}(x_{l-1})\phi^{i_{l+1}}(x_{l+1})\cdots\phi^{i_k}(x_k)\phi^{i_l}(x_l)v\rangle. \tag{2.5.45}$$

By Lemma 2.5.1, (2.5.45) has only finitely many negative power terms in x_l. So the same is true for (2.5.44). On the other hand, using (2.5.43) again, (2.5.44) is equal to

$$\prod_{1\le p<q\le k}(x_p - x_q)^{N_{i_p i_q}}\langle v', \phi^{i_l}(x_l)\phi^{i_1}(x_1)\cdots\phi^{i_{l-1}}(x_{l-1})\phi^{i_{l+1}}(x_{l+1})\cdots\phi^{i_k}(x_k)v\rangle. \tag{2.5.46}$$

By Lemma 2.5.1 again, (2.5.46) has only finitely many positive power terms in x_l. So the same is true for (2.5.44). Thus (2.5.44) must be a Laurent polynomial in x_l. Since this is true for $1 \le l \le k$, (2.5.44) is a Laurent polynomial in $x_1, \dots, x_k$.

For fixed $1 \le p < q \le k$, the expansion coefficients of

$$\langle v', \phi(x_1)\cdots\phi(x_k)v\rangle \tag{2.5.47}$$

as Laurent series in x_l for $l \ne p, q$ are of the form

$$\langle v', \phi^{i_1}_{n_1}\cdots\phi^{i_{p-1}}_{n_{p-1}}\phi^{i_p}(x_p)\phi^{i_{p+1}}_{n_{p+1}}\cdots\phi^{i_{q-1}}_{n_{q-1}}\phi^{i_q}(x_q)\phi^{i_{q+1}}_{n_{q+1}}\cdots\phi^{i_k}_{n_k}v\rangle \tag{2.5.48}$$

for $n_l \in \mathbb{Z}$, $l \ne p, q$. Clearly (2.5.48) contains only finitely many negative powers in x_q and finitely many positive powers in x_p. But we have shown that when multiplied by $(x_p - x_q)^{N_{pq}}$, it becomes a Laurent polynomial. Thus (2.5.48) must be the product of a Laurent polynomial in x_p and x_q and the expansion of $(x_p - x_q)^{-N_{pq}}$ as a Laurent series in nonnegative powers of x_q. Since p and q are arbitrary, we see that (2.5.47) with $x_1, \dots, x_k$ substituted by $z_1, \dots, z_k$ is equal to the product of a Laurent polynomial in $z_1, \dots, z_k$ and the expansion of $\prod_{1\le p<q\le k}(z_p - z_q)^{-N_{pq}}$ in the region $|z_1| > \cdots > |z_k| > 0$. This is Property 6. Property 7 follows immediately from Property 6 in the case $k = 2$ and (2.5.43). □

Exercise 2.5.3. Prove that Properties 6 and 7 imply Condition 5.

Proposition 2.5.4. *The space V, the fields ϕ^i for $i \in I$, $L_V(-1)$ and $\mathbf{1}$ have the following properties:*

8. *For* $a \in \mathbb{C}$ *and* $i \in I$, $e^{aL_V(0)}\phi^i(x)e^{-aL_V(0)} = e^{a(\mathrm{wt}\ \phi^i)}\phi^i(e^a x)$.

9. $L_V(-1)\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1} = \sum_{j=1}^{k}\phi^{i_1}_{n_1}\cdots\phi^{i_{j-1}}_{n_{j-1}}(-n_j\phi^{i_j}_{n_j-1})\phi^{i_{j+1}}_{n_{j+1}}\cdots\phi^{i_k}_{n_k}\mathbf{1}$.

10. *For* $a \in \mathbb{C}$, $z \in \mathbb{C}^\times$ *satisfying* $|z| > |a|$ *and* $i \in I$, $e^{aL_V(-1)}\phi^i(z)e^{-aL_V(-1)} = \phi^i(z+a)$.

11. *The operator* $L_V(-1)$ *has weight* 1 *and its adjoint* $L_V(-1)'$ *as an operator on* V' *has weight* -1 *(the weight of an operator on* V' *is defined in the same way as that of an operator on* V*). In particular,* $e^{zL_V(-1)'}v' \in V'$ *for* $z \in \mathbb{C}$ *and* $v' \in V'$.

12. *For* $v \in V$, $v' \in V'$ *and* $\sigma \in S_k$,

$$R(\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)v\rangle) = R(\langle v', \phi^{i_{\sigma(1)}}(z_{\sigma(1)})\cdots\phi^{i_{\sigma(k)}}(z_{\sigma(k)})v\rangle).$$

Proof. These properties follow immediately from Conditions 1–5 and Properties 6 and 7. We leave the details as an exercise. □

Exercise 2.5.5. Prove Properties 8–12 using Conditions 1–5 and Properties 6 and 7.

We now define a vertex operator map. We first give the motivation of this definition. The vertex operator map we want to define is a map

$$\begin{aligned} Y_V : \mathbb{C}^\times &\to \mathrm{Hom}(V\otimes V, \overline{V}),\\ z &\mapsto Y_V(\cdot, z)\cdot : u\otimes v \mapsto Y_V(u,z)v. \end{aligned}$$

We define $Y_V(\phi^i_{-1}\mathbf{1}, z)v = \phi^i(z)v$ for $i \in I$ and $v \in V$. The vertex operator map should satisfy the rationality and associativity property. In particular, we should have

$$R(\langle v', Y_V(\phi^{i_1}(\xi_1)\cdots\phi^{i_k}(\xi_k)\mathbf{1}, z)v\rangle) = R(\langle v', \phi^{i_1}(\xi_1+z)\cdots\phi^{i_k}(\xi_k+z)v\rangle)$$

for $i_1, \ldots, i_k \in I$, $v \in V$ and $v' \in V'$.

Motivated by this associativity formula, we define the vertex operator map as follows: For $v' \in V'$, $v \in V$, $i_1, \ldots, i_k \in I$, $m_1, \ldots, m_k \in \mathbb{Z}$, we define Y_V by

$$\begin{aligned} &\langle v', Y_V(\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1}, z)v\rangle \\ &\quad = \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1}\cdots\xi_k^{m_k}R(\langle v', \phi^{i_1}(\xi_1+z)\cdots\phi^{i_k}(\xi_k+z)v\rangle). \end{aligned} \quad (2.5.49)$$

Note that for a meromorphic function $f(\xi)$, $\mathrm{Res}_{\xi=0}f(\xi)$ means expanding $f(\xi)$ as a Laurent series in $0 < |\xi| < r$ for r sufficiently small so that no other poles are in this disk and then taking the coefficient of ξ^{-1}. We can also expand $f(\xi)$ as a Laurent series in a different region. In general, the coefficient of ξ^{-1} in this Laurent series might be different from $\mathrm{Res}_{\xi=0}f(\xi)$. Also note that the order to take these residues is important. Different orders in general give vertex operators for different elements.

Since $\overline{V} = (V')^*$, for fixed $\phi^{i_1}_{m_1} \cdots \phi^{i_k}_{m_k}\mathbf{1}, v \in V$, the formula above indeed gives an element

$$Y_V(\phi^{i_1}_{m_1} \cdots \phi^{i_k}_{m_k}\mathbf{1}, z)v \in \overline{V},$$

which in turn gives

$$Y_V(\phi^{i_1}_{m_1} \cdots \phi^{i_k}_{m_k}\mathbf{1}, x)v \in V[[x, x^{-1}]].$$

Since there might be relations among elements of the form $\phi^{i_1}_{m_1} \cdots \phi^{i_k}_{m_k}\mathbf{1}$, we first have to show that the definition above indeed gives a well-defined map from $\mathbb{C}^\times$ to $\mathrm{Hom}(V \otimes V, \overline{V})$. Let ϕ^0 be the map from $\mathbb{C}^\times$ to $\mathrm{Hom}(V, \overline{V})$ given by $\phi^0(z) = 1_V$. Let wt $\phi^0 = 0$. Then Conditions 1 to 5 and Properties 6 to 12 above still hold for ϕ^i, $i \in \tilde{I} = I \cup \{0\}$. Then any relation among such elements can always be written as

$$\sum_{p=1}^{q} \lambda_p \phi^{i_1^p}_{m_1^p} \cdots \phi^{i_k^p}_{m_k^p}\mathbf{1} = 0$$

for some $i_j^p \in \tilde{I}$ and $m_j^p \in \mathbb{Z}$, $p = 1, \dots, q$, $j = 1, \dots, k$.

Lemma 2.5.6. *If*

$$\sum_{p=1}^{q} \lambda_p \phi^{i_1^p}_{m_1^p} \cdots \phi^{i_k^p}_{m_k^p}\mathbf{1} = 0,$$

then

$$\sum_{p=1}^{q} \lambda_p \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1^p} \cdots \xi_k^{m_k^p} R(\langle v', \phi^{i_1^p}(\xi_1 + z) \cdots \phi^{i_k^p}(\xi_k + z)v\rangle) = 0$$

for $v \in V$ and $v' \in V'$.

Proof. By Condition 4, we can take v to be of the form $\phi^{j_1}_{n_1} \cdots \phi^{j_l}_{n_l}\mathbf{1}$. Moreover, in this case,

$$\begin{aligned}
&R(\langle v', \phi^{i_1^p}(z_1) \cdots \phi^{i_k^p}(z_k)\phi^{j_1}_{n_1} \cdots \phi^{j_l}_{n_l}\mathbf{1}\rangle) \\
&\quad = \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_l=0} \zeta_1^{n_1} \cdots \zeta_k^{n_l} R(\langle v', \phi^{i_1^p}(z_1) \cdots \phi^{i_k^p}(z_k)\phi^{j_1}(\zeta_1) \cdots \phi^{j_l}(\zeta_l)\mathbf{1}\rangle).
\end{aligned}$$

Then

$$\begin{aligned}
&\mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1^p} \cdots \xi_k^{m_k^p} R(\langle v', \phi^{i_1^p}(\xi_1 + z) \cdots \phi^{i_k^p}(\xi_k + z)v\rangle) \\
&\quad = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1^p} \cdots \xi_k^{m_k^p} \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_l=0} \zeta_1^{n_1} \cdots \zeta_l^{n_l} \cdot \\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1^p}(\xi_1 + z) \cdots \phi^{i_k^p}(\xi_k + z)\phi^{j_1}(\zeta_1) \cdots \phi^{j_l}(\zeta_l)\mathbf{1}\rangle) \\
&\quad = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1^p} \cdots \xi_k^{m_k^p} \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_l=0} \zeta_1^{n_1} \cdots \zeta_l^{n_l} \cdot \\
&\qquad\qquad \cdot R(\langle v', \phi^{j_1}(\zeta_1) \cdots \phi^{j_l}(\zeta_l)\phi^{i_1^p}(\xi_1 + z) \cdots \phi^{i_k^p}(\xi_k + z)\mathbf{1}\rangle) \\
&\quad = \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1^p} \cdots \xi_k^{m_k^p} \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_l=0} \zeta_1^{n_1} \cdots \zeta_l^{n_l} \cdot \\
&\qquad\qquad \cdot R(\langle e^{zL_V(-1)'}v', \phi^{j_1}(\zeta_1 - z) \cdots \phi^{j_l}(\zeta_l - z)\phi^{i_1^p}(\xi_1) \cdots \phi^{i_k^p}(\xi_k)\mathbf{1}\rangle)
\end{aligned}$$

$$= \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_l=0} \zeta_1^{n_1} \cdots \zeta_l^{n_l} \cdot \\ \cdot R(\langle e^{zL_V(-1)'} v', \phi^{j_1}(\zeta_1 - z) \cdots \phi^{j_l}(\zeta_l - z) \phi^{i_1^p}_{m_1^p} \cdots \phi^{i_k^p}_{m_k^p} \mathbf{1} \rangle).$$

Thus

$$\sum_{p=1}^{q} \lambda_p \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1^p} \cdots \xi_k^{m_k^p} R(\langle v', \phi^{i_1^p}(\xi_1 + z) \cdots \phi^{i_k^p}(\xi_k + z) v \rangle)$$
$$= \sum_{p=1}^{q} \lambda_p \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_l=0} \zeta_1^{n_1} \cdots \zeta_l^{n_l} \cdot \\ \cdot R(\langle e^{zL_V(-1)'} v', \phi^{j_1}(\zeta_1 - z) \cdots \phi^{j_l}(\zeta_l - z) \phi^{i_1^p}_{m_1^p} \cdots \phi^{i_k^p}_{m_k^p} \mathbf{1} \rangle)$$
$$= \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_l=0} \zeta_1^{n_1} \cdots \zeta_l^{n_l} \cdot \\ \cdot R\left(\left\langle e^{zL_V(-1)'} v', \phi^{j_1}(\zeta_1 - z) \cdots \phi^{j_l}(\zeta_l - z) \left(\sum_{p=1}^{q} \lambda_p \phi^{i_1^p}_{m_1^p} \cdots \phi^{i_k^p}_{m_k^p} \mathbf{1} \right) \right\rangle \right)$$
$$= 0,$$

proving the lemma. □

From this lemma, we see that the vertex operator map Y_V is well defined. We are now ready to formulate and prove the main result of this section.

Theorem 2.5.7. *Let $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ be a $\mathbb{Z}$-graded vector space, ϕ^i for $i \in I$ linear maps from V to $V[[x, x^{-1}]]$, or equivalently, maps from $\mathbb{C}^\times$ to $\mathrm{Hom}(V, \overline{V})$, $L_V(-1)$ an operator on V and $\mathbf{1} \in V_{(0)}$. Assume that they satisfy Conditions 1–5. Then the triple $(V, Y_V, \mathbf{1})$ is a grading-restricted vertex algebra generated by $\phi^i_{-1}\mathbf{1}$ for $i \in I$. Moreover, this is the unique grading-restricted vertex algebra structure on V with the vacuum $\mathbf{1}$ such that $Y(\phi^i_{-1}\mathbf{1}, z) = \phi^i(z)$ for $i \in I$.*

Proof. The vertex operator map Y_V is clearly analytic. The grading-restriction axiom is by assumption satisfied. The $L(-1)$-bracket formula follows from Condition 2 and the definition of Y_V. The identity property and the creation property also follow from of the definition of Y_V.

Let $L_V(0)'$ be the adjoint operator of $L_V(0)$. For $v' \in V'$, $v \in V$, $i_1, \dots, i_k \in I$ and $n_1, \dots, n_k \in \mathbb{Z}$, $a \in \mathbb{C}^\times$

$$\langle v', a^{L_V(0)} Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k} \mathbf{1}, z) a^{-L_V(0)} v \rangle$$
$$= \langle a^{L_V(0)'} v', Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k} \mathbf{1}, z) a^{-L_V(0)} v \rangle$$
$$= \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{n_1} \cdots \xi_k^{n_k} R(\langle a^{L_V(0)'} v', \phi^{i_1}(\xi_1 + z) \cdots \phi^{i_k}(\xi_k + z) a^{-L_V(0)} v \rangle)$$
$$= \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{n_1} \cdots \xi_k^{n_k} R(\langle v', a^{L_V(0)} \phi^{i_1}(\xi_1 + z) \cdots \phi^{i_k}(\xi_k + z) a^{-L_V(0)} v \rangle)$$
$$= \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{n_1} \cdots \xi_k^{n_k} a^{\mathrm{wt}\ \phi^{i_1} + \cdots \mathrm{wt}\ \phi^{i_k}} R(\langle v', \phi^{i_1}(a\xi_1 + az) \cdots \phi^{i_k}(a\xi_k + az) v \rangle)$$
$$= \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_k=0} \zeta_1^{n_1} \cdots \zeta_k^{n_k} a^{\mathrm{wt}\ \phi^{i_1} + \cdots \mathrm{wt}\ \phi^{i_k} - k - n_1 - \cdots - n_k} \cdot$$

$$\cdot R(\langle v', \phi^{i_1}(\zeta_1+az)\cdots\phi^{i_k}(\zeta_k+az)v\rangle)$$
$$= \langle v', Y_V(a^{L_V(0)}\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, az)v\rangle).$$

This formula implies the $L(0)$-bracket formula.

From Condition 2 and the definition of Y_V, we obtain

$$\frac{d}{dz}Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z) = [L_V(-1), Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z)].$$

From Property 9 and the definition of Y_V, we obtain

$$\frac{d}{dz}Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z) = Y_V(L_V(-1)\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z).$$

Applying both sides of this formula to $\mathbf{1}$, taking the limit $z \to 0$ and then using the creation property, we obtain

$$L_V(-1)\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1} = \lim_{z\to 0}\frac{d}{dz}Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z)\mathbf{1}.$$

The $L(-1)$-derivative property is proved.

Let $\{e_n\}_{n\in\mathbb{Z}}$ be a homogeneous basis of V and $\{e'_n\}_{n\in\mathbb{Z}}$ its dual basis in V'. Then we have

$$\begin{aligned}
&\langle v', Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1)Y_V(\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle\\
&\quad=\sum_{n\in\mathbb{Z}}\langle v', Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1)e_n\rangle\langle e'_n, Y_V(\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle\\
&\quad=\sum_{n\in\mathbb{Z}}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad\cdot R(\langle v', \phi^{i_1}(\zeta_1+z_1)\cdots\phi^{i_k}(\zeta_k+z_1)e_n\rangle)R(\langle e'_n, \phi^{j_1}(\xi_1+z_2)\cdots\phi^{j_l}(\xi_l+z_2)v\rangle)\\
&\quad=\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad\cdot\sum_{n\in\mathbb{Z}}R(\langle v', \phi^{i_1}(\zeta_1+z_1)\cdots\phi^{i_k}(\zeta_k+z_1)e_n\rangle)R(\langle e'_n, \phi^{j_1}(\xi_1+z_2)\cdots\phi^{j_l}(\xi_l+z_2)v\rangle).
\end{aligned} \tag{2.5.50}$$

By Property 6, when $|z_1| > \cdots > |z_{k+l}| > 0$,

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}R(\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)e_n\rangle)R(\langle e'_n, \phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle)\\
&\quad=\sum_{n\in\mathbb{Z}}\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)e_n\rangle\langle e'_n, \phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle\\
&\quad=\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle
\end{aligned} \tag{2.5.51}$$

is absolutely convergent to the rational function

$$R(\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle) \tag{2.5.52}$$

in $z_1, \ldots, z_{k+l}$. On the other hand, since the only possible poles of (2.5.52) are $z_i - z_j = 0$ for $i \neq j$ and $z_i = 0$, there is a unique expansion of such a rational function in the region $|z_1|, \ldots, |z_k| > |z_{k+1}|, \ldots, |z_{k+l}| > 0$, $z_i \neq z_j$ for $i \neq j$, $i, j = 1, \ldots, k$ and $i, j = k+1, \ldots, k+l$ such that each term is a product of two rational functions, one in $z_1, \ldots, z_k$ and the other in $z_{k+1}, \ldots, z_{k+l}$. Since the left-hand side of (2.5.51) is a series of the same form and is absolutely convergent in the region $|z_1| > \cdots > |z_{k+l}| > 0$ to (2.5.52), it must be absolutely convergent in the larger region $|z_1|, \ldots, |z_k| > |z_{k+1}|, \ldots, |z_{k+l}| > 0$, $z_i \neq z_j$ for $i \neq j$, $i, j = 1, \ldots, k$ and $i, j = k+1, \ldots, k+l$ to (2.5.52).

Substituting $\zeta_i + z_1$ for z_i for $i = 1, \ldots, k$ and $\xi_j + z_2$ for z_{k+j} for $j = 1, \ldots, l$, we see that

$$\sum_{n \in \mathbb{Z}} R(\langle v', \phi^{i_1}(\zeta_1 + z_1) \cdots \phi^{i_k}(\zeta_k + z_1) e_n \rangle) R(\langle e'_n, \phi^{j_1}(\xi_1 + z_2) \cdots \phi^{j_l}(\xi_l + z_2) v \rangle)$$

is absolutely convergent to

$$R(\langle v', \phi^{i_1}(\zeta_1 + z_1) \cdots \phi^{i_k}(\zeta_k + z_1) \phi^{j_1}(\xi_1 + z_2) \cdots \phi^{j_l}(\xi_l + z_2) v \rangle)$$

when $|\zeta_1 + z_1|, \ldots, |\zeta_k + z_1| > |\xi_1 + z_2|, \ldots, |\xi_l + z_2| > 0$, $\zeta_i \neq \zeta_j$ for $i, j = 1, \ldots, k$ and $\xi_i \neq \xi_j$ for $i, j = 1, \ldots, l$. When $|z_1| > |z_2| > 0$, we can always find sufficiently small neighborhood of 0 such that when $\zeta_1, \ldots, \zeta_k, \xi_1, \ldots, \xi_l$ are in this neighborhood, $|\zeta_1 + z_1|, \ldots, |\zeta_k + z_1| > |\xi_1 + z_2|, \ldots, |\xi_l + z_2| > 0$ holds. Thus we see that when $|z_1| > |z_2| > 0$, the right-hand side of (2.5.50) is absolutely convergent to

$$\begin{aligned} &\mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_k=0} \zeta_1^{n_1} \cdots \zeta_k^{n_k} \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_l=0} \xi_1^{m_1} \cdots \xi_l^{m_l} \cdot \\ &\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1 + z_1) \cdots \phi^{i_k}(\zeta_k + z_1) \phi^{j_1}(\xi_1 + z_2) \cdots \phi^{j_l}(\xi_l + z_2) v \rangle) \end{aligned} \tag{2.5.53}$$

This is a rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. In particular, the left-hand side of (2.5.50), that is,

$$\langle v', Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k} \mathbf{1}, z_1) Y_V(\phi^{j_1}_{m_1} \cdots \phi^{j_l}_{m_l} \mathbf{1}, z_2) v \rangle, \tag{2.5.54}$$

is absolutely convergent in the region $|z_1| > |z_2| > 0$ to this rational function.

We have proved the rationality of the product of two vertex operators. We are ready to prove the commutativity. The calculation above also shows that

$$\langle v', Y_V(\phi^{j_1}_{m_1} \cdots \phi^{j_l}_{m_l} \mathbf{1}, z_2) Y_V(\phi^{i_1}_{n_1} \cdots \phi^{i_k}_{n_k} \mathbf{1}, z_1) v \rangle \tag{2.5.55}$$

is absolutely convergent to the rational function

$$\begin{aligned} &\mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_l=0} \xi_1^{m_1} \cdots \xi_l^{m_l} \mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_k=0} \zeta_1^{n_1} \cdots \zeta_k^{n_k} \cdot \\ &\qquad \cdot R(\langle v', \phi^{j_1}(\xi_1 + z_2) \cdots \phi^{j_l}(\xi_l + z_2) \phi^{i_1}(\zeta_1 + z_1) \cdots \phi^{i_k}(\zeta_k + z_1) v \rangle), \end{aligned} \tag{2.5.56}$$

in the regions $|z_2| > |z_1| > 0$, respectively. By Property 12, the rational functions (2.5.53) and (2.5.56) are equal. Thus (2.5.54) and (2.5.55) are absolutely convergent in the regions

$|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively, to a common rational function with the only possible poles at $z_1 = z_2$, $z_1 = 0$ and $z_2 = 0$.

We now prove the associativity. For $i_1, \dots, i_k, j_1, \dots, j_l \in I$, $m_1, \dots, m_l \in \mathbb{Z}$, $v \in V$ and $v' \in V'$, using the expansion of $\phi^{i_1}(\xi_1), \dots, \phi^{i_k}(\xi_k)$ and the definition of Y_V, we have

$$\begin{aligned}
&\langle v', Y_V(\phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z)v\rangle\\
&\quad= \sum_{p_1,\dots,p_k\in\mathbb{Z}} \langle v', Y_V(\phi^{i_1}_{p_1}\cdots\phi^{i_k}_{p_k}\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z)v\rangle z_1^{-p_1-1}\cdots z_k^{-p_k-1}\\
&\quad= \sum_{p_1,\dots,p_k\in\mathbb{Z}} \mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{p_1}\cdots\zeta_k^{p_k}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1+z)\cdots\phi^{i_k}(\zeta_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle)z_1^{-p_1-1}\cdots z_k^{-p_k-1}.
\end{aligned} \tag{2.5.57}$$

We now expand

$$R(\langle v', \phi^{i_1}(\zeta_1+z)\cdots\phi^{i_k}(\zeta_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle)$$

as a Laurent series $\sum_{l\in\mathbb{Z}} f_l(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)\zeta_k^{-l-1}$ in ζ_k in the region $|z|, |\zeta_1|, \dots, |\zeta_{k-1}| > |\zeta_k| > |\xi_1|, \dots, |\xi_l|$, where $f_l(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)$ are rational functions in $\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l$ and z. Then in the region that the Laurent series expansion holds, we have

$$\begin{aligned}
&\sum_{p_k\in\mathbb{Z}} \mathrm{Res}_{\zeta_k=0}\zeta_k^{p_k}\left(\sum_{l\in\mathbb{Z}} f_l(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)\zeta_k^{-l-1}\right) z_k^{-p_k-1}\\
&\quad= \sum_{p_k\in\mathbb{Z}} f_{p_k}(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)z_k^{-p_k-1}\\
&\quad= R(\langle v', \phi^{i_1}(\zeta_1+z)\cdots\phi^{i_{k-1}}(\zeta_{k-1}+z)\phi^{i_k}(z_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle).
\end{aligned} \tag{2.5.58}$$

Repeating this step for the variables $\zeta_{k-1}, \dots, \zeta_1$, we see that the right-hand side of (2.5.57) is equal to the expansion of

$$\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}R(\langle v', \phi^{i_1}(z_1+z)\cdots\phi^{i_k}(z_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle) \tag{2.5.59}$$

as a Laurent series in $z_1 \dots, z_k$ in the region $|z| > |z_1| > \cdots > |z_k| > 0$. Thus the left-hand side of (2.5.57) is absolutely convergent to (2.5.59) in the region for this Laurent series expansion. In particular, in the region $|z| > |z_1| > \cdots > |z_k| > 0$,

$$\begin{aligned}
&\langle v', Y_V(\phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z)v\rangle\\
&\quad= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(z_1+z)\cdots\phi^{i_k}(z_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle).
\end{aligned} \tag{2.5.60}$$

Now we have

$$\langle v', Y_V(Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \langle v', Y_V(e_n, z_2)v\rangle \langle e_n', Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1 - z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle\\
&= \sum_{n\in\mathbb{Z}} \langle v', Y_V(e_n, z_2)v\rangle \mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\cdot\\
&\qquad\qquad \cdot R(\langle e_n', \phi^{i_1}(\zeta_1 + z_1 - z_2)\cdots\phi^{i_k}(\zeta_k + z_1 - z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle). \quad (2.5.61)
\end{aligned}
$$

But by (2.5.60), in the region $|z_2| > |\zeta_1 + z_1 - z_2| > \cdots > |\zeta_k + z_1 - z_2| > 0$, we have

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} \langle v', Y_V(e_n, z_2)v\rangle \langle e_n', \phi^{i_1}(\zeta_1 + z_1 - z_2)\cdots\phi^{i_k}(\zeta_k + z_1 - z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle\\
&\quad = \langle v', Y_V(\phi^{i_1}(\zeta_1 + z_1 - z_2)\cdots\phi^{i_k}(\zeta_k + z_1 - z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle\\
&\quad = \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1 + z_1)\cdots\phi^{i_k}(\zeta_k + z_1)\phi^{j_1}(\xi_1 + z_2)\cdots\phi^{j_l}(\xi_l + z_2)v\rangle).
\end{aligned}
\tag{2.5.62}
$$

The right-hand side of (2.5.62) is a rational function in $\zeta_1, \ldots, \zeta_k$, z_1 and z_2 with the only possible poles $\zeta_i - \zeta_j = 0$, for $i \neq j$, $\zeta_i + z_1 = 0$, $\zeta_i + z_1 - z_2 = 0$ and $z_2 = 0$. There is a unique expansion of such a rational function in the region $|z_2| > |\zeta_1 + z_1 - z_2|, \ldots, |\zeta_k + z_1 - z_2| > 0$, $\zeta_i \neq \zeta_j$ for $i \neq j$, $i, j = 1, \ldots, k$, such that each term is a product of two rational functions, one in z_2 and the other in $\zeta_1, \ldots, \zeta_k$ and z_1. Since

$$
\sum_{n\in\mathbb{Z}} \langle v', Y_V(e_n, z_2)v\rangle R(\langle e_n', \phi^{i_1}(\zeta_1 + z_1 - z_2)\cdots\phi^{i_k}(\zeta_k + z_1 - z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle)
$$

is a series of the same form and is equal to the left-hand side of (2.5.62) in the region $|z_2| > |\zeta_1 + z_1 - z_2| > \cdots > |\zeta_k + z_1 - z_2| > 0$, it must be absolutely convergent to the right-hand side of (2.5.62) in the larger region $|z_2| > |\zeta_1 + z_1 - z_2|, \ldots, |\zeta_k + z_1 - z_2| > 0$. Thus we obtain

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} \langle v', Y_V(e_n, z_2)v\rangle R(\langle e_n', \phi^{i_1}(\zeta_1 + z_1 - z_2)\cdots\phi^{i_k}(\zeta_k + z_1 - z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle)\\
&\quad = \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1 + z_1)\cdots\phi^{i_k}(\zeta_k + z_1)\phi^{j_1}(\xi_1 + z_2)\cdots\phi^{j_l}(\xi_l + z_2)v\rangle)
\end{aligned}
\tag{2.5.63}
$$

in the region $|z_2| > |\zeta_1 + z_1 - z_2|, \ldots, |\zeta_k + z_1 - z_2| > 0$. Thus when $|z_2| > |z_1 - z_2| > 0$, the right-hand side of (5.3.14) is absolutely convergent to

$$
\begin{aligned}
&\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1 + z_1)\cdots\phi^{i_k}(\zeta_k + z_1)\phi^{j_1}(\xi_1 + z_2)\cdots\phi^{j_l}(\xi_l + z_2)v\rangle),
\end{aligned}
\tag{2.5.64}
$$

which is proved above to be equal to the left hand side of (2.5.50) in the region $|z_1| > |z_2| > 0$. The associativity is proved.

To prove the uniqueness, we need only show that any grading-restricted vertex superalgebra structure on V with the vacuum $\mathbf{1}$ must have the vertex operator map defined by (2.5.49). But this is clear from the motivation that we discussed before the definition (2.5.49) of the vertex operator map Y_V. □

We call the grading-restricted vertex algebra given in Theorem 2.5.7 the *grading-restricted vertex algebra generated by* ϕ^i, $i \in I$. The maps ϕ^i, $i \in I$, are called the *generating fields* of the grading-restricted vertex algebra V.

Remark 2.5.8. In the proof of Theorem 2.5.7, we gave a proof of the associativity using the definition (2.5.49) of the vertex operators. But the associativity can also be obtained by using Proposition 3.6.1 in [FHL].

Proof of Theorem 2.6.3. By Proposition 2.6.2, Conditions 1–5 needed in Theorem 2.5.7 are satisfied by $S(\hat{\mathfrak{h}}_-)$, $a(x)$ for $a \in \mathfrak{h}$, $L_{S(\hat{\mathfrak{h}}_-)}(0)$ and $L_{S(\hat{\mathfrak{h}}_-)}(-1)$. By Theorem 2.5.7, Theorems 2.6.3 is proved. ∎

Proof of Theorem 2.8.4. By Proposition 2.7.2, Conditions 1–5 needed in Theorem 2.5.7 are satisfied by V_L, $a(x)$ for $a \in \mathfrak{h}$, $Y_{V_L}(e^\alpha, x)$ for $\alpha \in L$, $L_{V_L}(0)$ and $L_{V_L}(-1)$. By Theorem 2.5.7, Theorems 2.8.4 is proved. ∎

2.6 Examples: Heisenberg vertex operator algebras (for free bosons)

We now give the full construction of the Heisenberg vertex operator algebras started in Section 2.2. The Heisenberg vertex operator algebras are the vertex operator algebras for the free boson theories. The reader is referred to Section 2.2 for the basic material on the Heisenberg algebra $\hat{\mathfrak{h}}$ associated to a finite-dimensional inner product space $\mathfrak{h}$ over $\mathbb{R}$, the Fock space $S(\hat{\mathfrak{h}}_-)$, the vertex operator map $Y_{S(\hat{\mathfrak{h}}_-)}$, the vacuum $\mathbf{1}$ and the conformal element ω. But here we will give the vertex operator map using the definition in the preceding section. We will leave it as an exercise to show that these two definitions are indeed the same.

Proposition 2.6.1. *For $a, b \in \mathfrak{h}$, we have*

$$[a(x_1), b(x_2)] = -(a,b)\frac{\partial}{\partial x_1} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = (a,b)\left((x_1-x_2)^{-2} - (-x_2+x_1)^{-2}\right). \qquad (2.6.65)$$

Proof. Note that by definition, $\mathbf{k}$ acts on $S(\hat{\mathfrak{h}}_-)$ as $1 \in \mathbb{C}$. For $a, b \in \mathfrak{h}$,

$$[a(x_1), b(x_2)] = \sum_{m,n\in\mathbb{Z}} [a(m), b(n)] x_1^{-m-1} x_2^{-n-1}$$

$$
\begin{aligned}
&= \sum_{m,n\in\mathbb{Z}} (a,b) m\delta_{m+n,0} x_1^{-m-1} x_2^{-n-1} \\
&= -(a,b) \sum_{n\in\mathbb{Z}} n x_1^{n-1} x_2^{-n-1} \\
&= -(a,b) \frac{\partial}{\partial x_1} \sum_{n\in\mathbb{Z}} x_1^{n} x_2^{-n-1} \\
&= -(a,b) \frac{\partial}{\partial x_1} \left((x_1-x_2)^{-1} - (-x_2+x_1)^{-1}\right) \\
&= -(a,b) \frac{\partial}{\partial x_1} x_2^{-1} \delta\left(\frac{x_1}{x_2}\right) \\
&= (a,b) \left((x_1-x_2)^{-2} - (-x_2+x_1)^{-2}\right),
\end{aligned}
$$

proving (2.6.65). $\square$

Though we have introduced the Virasoro operators $L_{S(\hat{\mathfrak{h}}_-)}(n)$ in Section 2.2, here we give different definitions of $L_{S(\hat{\mathfrak{h}}_-)}(0)$ and $L_{S(\hat{\mathfrak{h}}_-)}(-1)$. We will show later that these operators are the same as those given in Section 2.2. Let $L_{S(\hat{\mathfrak{h}}_-)}(0)$ be the operator on $S(\hat{\mathfrak{h}}_-)$ giving the grading on $S(\hat{\mathfrak{h}}_-)$, that is, $L_{S(\hat{\mathfrak{h}}_-)}(0)v = nv$ for $v \in S(\hat{\mathfrak{h}}_-)_{(n)}$. We denote $1 \in S(\hat{\mathfrak{h}}_-)$ by $\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$. Then $S(\hat{\mathfrak{h}}_-)$ is spanned by elements of the form

$$a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$$

for $a_1, \dots, a_k \in \mathfrak{h}$ and $n_1, \dots, n_k \in \mathbb{Z}_+$. We define an operator $L_{S(\hat{\mathfrak{h}}_-)}(-1)$ on $S(\hat{\mathfrak{h}}_-)$ by

$$
\begin{aligned}
&L_{S(\hat{\mathfrak{h}}_-)}(-1) a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)} \\
&\qquad = \sum_{i=1}^{k} n_i a_1(-n_1)\cdots a_{i-1}(-n_{i-1}) a_i(-n_i-1) a_{i+1}(-n_{i+1})\cdots a_k(-n_k)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}.
\end{aligned}
$$

Proposition 2.6.2. *The series $a(x)$ for $a \in \mathfrak{h}$ and the operators $L_{S(\hat{\mathfrak{h}}_-)}(0)$ and $L_{S(\hat{\mathfrak{h}}_-)}(-1)$ have the following properties:*

1. *For $a \in \mathfrak{h}$, $[L_{S(\hat{\mathfrak{h}}_-)}(0), a(x)] = x\frac{d}{dx}a(x) + a(x)$.*

2. *$L_{S(\hat{\mathfrak{h}}_-)}(-1)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)} = 0$, $[L_{S(\hat{\mathfrak{h}}_-)}(-1), a(x)] = \frac{d}{dx}a(x)$ for $a \in \mathfrak{h}$.*

3. *For $a \in \mathfrak{h}$, $a(x)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)} \in S(\hat{\mathfrak{h}}_-)[[x]]$. Moreover, $\lim_{x\to 0} a(x)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)} = a(-1)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$.*

4. *The vector space $S(\hat{\mathfrak{h}}_-)$ is spanned by elements of the form*

$$a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}$$

for $a_1, \dots, a_k \in \mathfrak{h}$ and $n_1, \dots, n_k \in \mathbb{Z}_+$.

5. For $a, b \in \mathfrak{h}$,
$$(x_1 - x_2)^2 a(x_1)b(x_2) = (x_1 - x_2)^2 b(x_2)a(x_1).$$

Proof. Properties 1–4 are easily verified using the definitions. Property 5 follows from Proposition 2.6.1. □

From Proposition 2.6.2, the conditions needed for all the results obtained in the preceding section are satisfied. By Property 6 in Proposition 2.5.2 we see that for $a_1, \dots, a_k \in \mathfrak{h}$, $v \in S(\hat{\mathfrak{h}}_-)$, and $v' \in S(\hat{\mathfrak{h}}_-)'$,
$$\langle v', a_1(z_1) \cdots a_k(z_k)v\rangle$$
is the expansion in the region $|z_1| > \cdots > |z_k| > 0$ of a rational function denoted by
$$R(\langle v', a_1(z_1) \cdots a_k(z_k)v\rangle)$$
in $z_1, \dots, z_k$ with the only possible poles at $z_i = 0$ and $z_i - z_j = 0$ for $i, j = 1, \dots, k$. Applying Theorem 2.5.7 to our case, we obtain:

Theorem 2.6.3. *The $\mathbb{Z}$-graded vector space $S(\hat{\mathfrak{h}}_-)$ equipped with the the vertex operator map $Y_{S(\hat{\mathfrak{h}}_-)}$ defined by*
$$\begin{aligned}\langle v', &Y_{S(\hat{\mathfrak{h}}_-)}(\alpha_1(-n_1) \cdots \alpha_k(-n_k)\mathbf{1}_{S(\hat{\mathfrak{h}}_-)}, z)v\rangle \\ &= \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0}\xi_1^{-n_1} \cdots \xi_k^{-n_k}\xi_{k+1}^{-1}R(\langle v', a_1(\xi_1 + z) \cdots a_k(\xi_k + z)v\rangle)\end{aligned} \tag{2.6.66}$$
for $a_1, \dots, a_k \in \mathfrak{h}$, $v \in S(\hat{\mathfrak{h}}_-)$ and $v' \in S(\hat{\mathfrak{h}}_-)'$ and the vacuum $\mathbf{1}$ is a grading-restricted vertex algebra. Moreover, this is the unique grading-restricted vertex algebra structure on $S(\hat{\mathfrak{h}}_-)$ with the vacuum $\mathbf{1}$ such that $Y(a(-1)\mathbf{1}, z) = a(x)$ for $a \in \mathfrak{h}$.

The vertex operator map for the grading-restricted vertex algebra $S(\hat{\mathfrak{h}}_-)$ is given by (2.6.66). But in Section 2.2, the vertex operator map is given by (2.2.7).

Exercise 2.6.4. Prove that the vertex operator map given by (2.2.7) is the same as the one given by (2.6.66).

We now give the Virasoro operators explicitly. Recall the the stress-energy tensor
$$T(x) = \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}} {}^{\circ}_{\circ} u^i(x)u^i(x) {}^{\circ}_{\circ}$$
given in Section 2.2. By definition
$$\begin{aligned}T(x) &= \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}} {}^{\circ}_{\circ} u^i(x)u^i(x) {}^{\circ}_{\circ} \\ &= \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} {}^{\circ}_{\circ} u^i(k)u^i(l) {}^{\circ}_{\circ} x^{-k-l-2}\end{aligned}$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} \sum_{k\in\mathbb{Z}} {}^{\circ}_{\circ} u^i(k)u^i(n-k) {}^{\circ}_{\circ} x^{-n-2}.$$

Taking the coefficients of $T(x)$, we obtai

$$L_{S(\hat{\mathfrak{h}}_-)}(n) = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} \sum_{k\in\mathbb{Z}} {}^{\circ}_{\circ} u^i(k)u^i(n-k) {}^{\circ}_{\circ} = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} \sum_{k\in -\mathbb{Z}_+} u^i(k)u^i(n-k) + \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} \sum_{k\in\mathbb{N}} u^i(n-k)u^i(k) \tag{2.6.67}$$

for $n \in \mathbb{Z}$.

We now calculate $T(x)\omega$.

Lemma 2.6.5. *We have*

$$T(x)\omega = L_{S(\hat{\mathfrak{h}}_-)}(-1)\omega x^{-1} + 2\omega x^{-1} + \frac{\dim \mathfrak{h}}{2} x^{-4} + G(x), \tag{2.6.68}$$

where $G(x) \in S(\hat{\mathfrak{h}}_-)[[x]]$.

Proof. By definition,

$$\begin{aligned}
&{}^{\circ}_{\circ} u^i(x)u^i(x) {}^{\circ}_{\circ} u^j(-1)^2\mathbf{1} \\
&\quad = \sum_{k,l\in\mathbb{Z}} {}^{\circ}_{\circ} u^i(k)u^i(l) {}^{\circ}_{\circ} u^j(-1)^2\mathbf{1} x^{-k-l-2} \\
&\quad = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{N}} u^i(k)u^i(l)u^j(-1)^2\mathbf{1} x^{-k-l-2} + \sum_{k\in\mathbb{Z}} \sum_{l\in -\mathbb{Z}_+} u^i(l)u^i(k)u^j(-1)^2\mathbf{1} x^{-k-l-2} \\
&\quad = \sum_{k\in\mathbb{Z}} \sum_{l=0}^{2} u^i(k)u^i(l)u^j(-1)^2\mathbf{1} x^{-k-l-2} + 2\delta_{ij} \sum_{k<-2} u^i(k)u^j(-1)\mathbf{1} x^{-k-3} \\
&\qquad + \sum_{k=0}^{2} \sum_{l\in -\mathbb{Z}_+} u^i(l)u^i(k)u^j(-1)^2\mathbf{1} x^{-k-l-2} + \sum_{k\in -\mathbb{Z}_+} \sum_{l\in -\mathbb{Z}_+} u^i(l)u^i(k)u^j(-1)^2\mathbf{1} x^{-k-l-2}.
\end{aligned} \tag{2.6.69}$$

Note that the last term in the right-hand side of (2.6.69) is in $S(\hat{\mathfrak{h}}_-)[[x]]$. So we need only calculate the first two terms. By the commutator relations for the Heisenberg algebra, we have

$$\begin{aligned}
&u^i(0)u^j(-1)^2\mathbf{1} = 0, \\
&u^i(1)u^j(-1)^2\mathbf{1} = 2(u^i, u^j)u^j(-1)\mathbf{1} = 2\delta_{ij}u^j(-1)\mathbf{1}, \\
&u^i(0)u^j(-1)^2\mathbf{1} = 0.
\end{aligned}$$

Then

$$\sum_{k\in\mathbb{Z}} \sum_{l=0}^{2} u^i(k)u^i(l)u^j(-1)^2\mathbf{1} x^{-k-l-2}$$

$$
\begin{aligned}
&= 2\delta_{ij} \sum_{k\in\mathbb{Z}} u^i(k)u^j(-1)\mathbf{1}x^{-k-3} \\
&= 2\delta_{ij} \sum_{k=-2}^{1} u^i(k)u^j(-1)\mathbf{1}x^{-k-3} + 2\delta_{ij} \sum_{k<-2} u^i(k)u^j(-1)\mathbf{1}x^{-k-3} \\
&= 2\delta_{ij} \sum_{k=-2}^{1} u^i(k)u^j(-1)\mathbf{1}x^{-k-3} + 2\delta_{ij} \sum_{k<-2} u^i(k)u^j(-1)\mathbf{1}x^{-k-3} \\
&= 2\delta_{ij}u^i(-2)u^j(-1)\mathbf{1}x^{-1} + 2\delta_{ij}u^i(-1)u^j(-1)\mathbf{1}x^{-2} + 2\delta_{ij}^2\mathbf{1}x^{-4} \\
&\quad + 2\delta_{ij} \sum_{k<-2} u^i(k)u^j(-1)\mathbf{1}x^{-k-3}
\end{aligned} \tag{2.6.70}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{2} \sum_{l\in-\mathbb{Z}_+} u^i(l)u^i(k)u^j(-1)^2\mathbf{1}x^{-k-l-2} \\
&\quad = 2\delta_{ij} \sum_{l\in-\mathbb{Z}_+} u^i(l)u^j(-1)\mathbf{1}x^{-l-3} \\
&\quad = 2\delta_{ij}u^i(-2)u^j(-1)\mathbf{1} + 2\delta_{ij}u^i(-1)u^j(-1)\mathbf{1} + 2\delta_{ij} \sum_{l<-2} u^i(l)u^j(-1)\mathbf{1}x^{-l-3}
\end{aligned} \tag{2.6.71}
$$

Note that the last terms in both (2.6.70) and (2.6.71) are in $S(\hat{\mathfrak{h}}_-)[[x]]$. Sustituting (2.6.70) and (2.6.71) into (2.6.69), taking sum over $i, j = 1, \ldots, \dim \mathfrak{h}$, dividing both sides by 4 and using the definition of ω and $L_{S(\hat{\mathfrak{h}}_-)}(-1)$, we obtain

$$
\begin{aligned}
T(x)\omega &= \frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{h}} \delta_{ij}u^i(-2)u^j(-1)\mathbf{1}x^{-1} + \frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{h}} \delta_{ij}u^i(-1)u^j(-1)\mathbf{1}x^{-2} + \frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{h}} \delta_{ij}^2\mathbf{1}x^{-4} \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{h}} \delta_{ij} \sum_{k<-2} u^i(k)u^j(-1)\mathbf{1}x^{-k-3} + \frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{h}} \delta_{ij}u^i(-2)u^j(-1)\mathbf{1} \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{h}} 2\delta_{ij}u^i(-1)u^j(-1)\mathbf{1} + \frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{h}} \delta_{ij} \sum_{l<-2} u^i(l)u^j(-1)\mathbf{1}x^{-l-3} \\
&\quad + \sum_{k\in-\mathbb{Z}_+} \sum_{l\in-\mathbb{Z}_+} u^i(l)u^i(k)u^j(-1)^2\mathbf{1}x^{-k-l-2} \\
&= \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} 2u^i(-2)u^i(-1)\mathbf{1}x^{-1} + 2\frac{1}{2} \sum_{i,j=1}^{\dim \mathfrak{h}} u^i(-1)u^i(-1)\mathbf{1}x^{-2} + \frac{\dim \mathfrak{h}}{2}\mathbf{1}x^{-4} + G(x) \\
&= L_{S(\hat{\mathfrak{h}}_-)}(-1)\omega x^{-1} + 2\omega x^{-1} + \frac{\dim \mathfrak{h}}{2}x^{-4} + G(x),
\end{aligned}
$$

where

$$
\begin{aligned}
G(x) = &\frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} \delta_{ij} \sum_{k<-2} u^i(k)u^j(-1)\mathbf{1}x^{-k-3} + \frac{1}{2}\sum_{i,j=1}^{\dim\mathfrak{h}} \delta_{ij} \sum_{l<-2} u^i(l)u^j(-1)\mathbf{1}x^{-l-3}\\
&+ \sum_{k\in -\mathbb{Z}_+} \sum_{l\in -\mathbb{Z}_+} u^i(l)u^i(k)u^j(-1)^2\mathbf{1}x^{-k-l-2}\\
\in\, & S(\hat{\mathfrak{h}}_-)[[x]].
\end{aligned}
$$

□

From (2.6.68) and Subsection 2.4.6, we obtain:

Theorem 2.6.6. *The element ω is a conformal element of the grading-restricted vertex algebra $S(\hat{\mathfrak{h}}_-)$. In particular, $S(\hat{\mathfrak{h}}_-)$ is a vertex operator algebra (see Definition 2.3.5).*

Proof. By the definition of $Y_{S(\hat{\mathfrak{h}}_-)}$, we have

$$u^i(-1)^2\mathbf{1} = \mathrm{Res}_{x_0}x_0^{-1}u^i(x_0)u^i(-1)\mathbf{1} = \mathrm{Res}_{x_0}x_0^{-1}Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_0)u^i(-1)\mathbf{1}.$$

Using this and the first equality in (2.4.30), we have

$$
\begin{aligned}
&Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)^2\mathbf{1}, x_2)\\
&\quad = \mathrm{Res}_{x_0}x_0^{-1}Y_{S(\hat{\mathfrak{h}}_-)}(Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_0)u^i(-1)\mathbf{1}, x_2)\\
&\quad = \mathrm{Res}_{x_0}x_0^{-1}\mathrm{Res}_{x_1}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_1)Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_2)\\
&\qquad - \mathrm{Res}_{x_0}x_0^{-1}\mathrm{Res}_{x_1}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_2)Y_{S(\hat{\mathfrak{h}}_-)}(u^i(-1)\mathbf{1}, x_1)\\
&\quad = \mathrm{Res}_{x_1}(x_1-x_2)^{-1}u^i(x_1)u^i(x_2) - \mathrm{Res}_{x_1}(-x_2+x_1)^{-1}u^i(x_2)u^i(x_1)\\
&\quad = \left(\sum_{m\in\mathbb{N}} u^i(-m-1)x_2^m\right)u^i(x_2) + u^i(x_2)\left(\sum_{m\in -\mathbb{Z}_+} u^i(-m-1)x_2^m\right)\\
&\quad = {}^{\circ}_{\circ}u^i(x_2)u^i(x_2){}^{\circ}_{\circ}.
\end{aligned}
\tag{2.6.72}
$$

Uisng (2.6.72) and the definition of $T(x)$, we obtain $Y_{S(\hat{\mathfrak{h}}_-)}(\omega, x) = T(x)$. By (2.6.68),

$$Y_{S(\hat{\mathfrak{h}}_-)}(\omega, x)\omega = L_{S(\hat{\mathfrak{h}}_-)}(-1)\omega x^{-1} + 2\omega x^{-2} + \frac{\dim\mathfrak{h}}{2}\mathbf{1}x^{-4} + G(x).$$

The other property for $L_{S(\hat{\mathfrak{h}}_-)}(-1)$ and $L_{S(\hat{\mathfrak{h}}_-)}(0)$ can be verified using the formula (2.6.67). We omit the proofs here. □

We now give the operator production expansion of the vertex operators generating the vertex operator algebra $S(\hat{\mathfrak{h}}_-)$. By Theorem 2.6.3, $S(\hat{\mathfrak{h}}_-)$ has a structure of a grading-restricted vertex algebra. As usual, we use $Y_{S(\hat{\mathfrak{h}}_-)}$ to denote its vertex operator map. It is easy to see from the definition of the vertex operator map in the preceding section, we see that $Y_{S(\hat{\mathfrak{h}}_-)}(a(-1)\mathbf{1}, x) = a(x)$ for $a \in \mathfrak{h}$. Then the operator product expansion of $a(z_1)$ and $b(z_2)$ is

$$\begin{aligned} a(z_1)b(z_2) &= Y_{S(\hat{\mathfrak{h}}_-)}(a(-1)\mathbf{1}, z_1)Y_{S(\hat{\mathfrak{h}}_-)}(b(-1)\mathbf{1}, z_2) \\ &= Y_{S(\hat{\mathfrak{h}}_-)}(Y_{S(\hat{\mathfrak{h}}_-)}(a(-1)\mathbf{1}, z_1 - z_2)b(-1)\mathbf{1}, z_2) \end{aligned} \tag{2.6.73}$$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. But

$$\begin{aligned} Y_{S(\hat{\mathfrak{h}}_-)}&(a(-1)\mathbf{1}, z_1 - z_2)b(-1)\mathbf{1} = a(z_1 - z_2)b(-1)\mathbf{1} \\ &= \sum_{n\in\mathbb{Z}} a(n)b(-1)\mathbf{1}(z_1 - z_2)^{-n-1} \\ &= (a, b)\mathbf{1}(z_1 - z_2)^{-2} + \sum_{n\in-\mathbb{Z}_+} a(n)b(-1)\mathbf{1}(z_1 - z_2)^{-n-1}. \end{aligned} \tag{2.6.74}$$

Substituting the right-hand side of (2.6.74) into the right-hand side of (2.6.73), we obtain the explicit form

$$a(z_1)b(z_2) = (a, b)(z_1 - z_2)^{-2} + \sum_{n\in-\mathbb{Z}_+} Y_{S(\hat{\mathfrak{h}}_-)}(a(n)b(-1)\mathbf{1}, z_2)(z_1 - z_2)^{-n-1} \tag{2.6.75}$$

of the operator product expansion of $a(z_1)$ and $b(z_2)$ Since the only singular term in $z_1 - z_2$ in the right-hand side of (2.7.108) is $(a, b)(z_1 - z_2)^{-2}$, we obtain

$$a(z_1)b(z_2) \sim (a, b)(z_1 - z_2)^{-2}. \tag{2.6.76}$$

The last formula can also be calculated using the commutator formula (2.6.65). Apply both sides of (2.6.65) to $v \in S(\hat{\mathfrak{h}}_-)$ and then rewrite the resulting formula as

$$a(x_1)b(x_2)v - (a, b)(x_1 - x_2)^{-2}v = b(x_2)a(x_1)v - (a, b)(-x_2 + x_1)^{-2}v. \tag{2.6.77}$$

Note that the left-hand side (2.6.77) has only finitely many negative powers of x_2 and the right-hand side of (2.6.77) has only finitely many negative powers of x_1. Thus both sides of (2.6.77) have finitely many negative powers of both x_1 and x_2. Let $f(x_1, x_2)$ be this Laurent series with finitely many negative powers of x_1 and x_2. We can write $f(x_1, x_2)$ as $f(x_2+(x_1-x_2), x_2)$ and expand it as a Laurent series in x_2 and x_1-x_2 with only nonnegative powers of $x_1 - x_2$. We use $f(x_2 + (x_1 - x_2), x_2)$ to denote this expansion. So we obtain the operator product expansion

$$a(x_1)b(x_2)v = (a, b)(x_1 - x_2)^{-2}v + f(x_2 + (x_1 - x_2), x_2).$$

Since the expansion of $f(x_2 + (x_1 - x_2), x_2)$ contain only nonnegative powers of $x_1 - x_2$ and v is arbitrary, we obtain (2.6.76).

2.7 Examples: Lattice vertex operator algebras (for free bosons on tori)

Lattice vertex operator algebras play a special role in the representation theory of vertex operator algebras. The classical vertex operators were one class of operator series motivating the definition of vertex operator algebra. In this section, we give a full construction of the lattice vertex operator algebras.

Let L be a positive-definite even lattice of rank n with nondegenerate symmetric $\mathbb{Z}$-linear form $(\cdot,\cdot)$. Then $\mathfrak{h} = L\otimes_{\mathbb{Z}}\mathbb{R}$ is an n-dimensional vector space over $\mathbb{R}$ with a positive definite bilinear form still denoted by $(\cdot,\cdot)$. Then we have the Heisenberg algebra $\hat{\mathfrak{h}} = \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\mathbf{k}$. As in the preceding section, we also have the subalgebras $\hat{\mathfrak{h}}_+ = \mathfrak{h}\otimes t\mathbb{C}[t]$, $\hat{\mathfrak{h}}_- = \mathfrak{h}\otimes t^{-1}\mathbb{C}[t^{-1}]$ and $\hat{\mathfrak{h}}_0 = \mathfrak{h}\otimes t^0\oplus\mathbb{C}\mathbf{k}$. Given a module M for the abelian Lie algebra $\mathfrak{h}\otimes t^0 = \mathfrak{h}$, let $\hat{\mathfrak{h}}_+$ act on M as 0 and $\mathbf{k}$ acts on M as 1. Then M becomes an $\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0$-module. By the Poincaré-Birkhoff-Witt theorem, the induced module $U(\hat{\mathfrak{h}})\otimes_{U(\hat{\mathfrak{h}}_+\oplus\hat{\mathfrak{h}}_0)} M$ is linearly isomorphic to $U(\hat{\mathfrak{h}}_-)\otimes M = S(\hat{\mathfrak{h}}_-)\otimes M$. In particular, $S(\hat{\mathfrak{h}}_-)\otimes M$ is equipped with an $\hat{\mathfrak{h}}$-module structure under this linear isomorphism.

Fix a basis $\alpha_1,\dots,\alpha_r$ of L. We define a $\mathbb{Z}$-linear map $\epsilon : L\times L\to\mathbb{Z}$ by

$$\epsilon(\alpha_i,\alpha_j) = \begin{cases} (\alpha_i,\alpha_j) & i>j \\ 0 & i\le j \end{cases}$$

for $i,j = 1,\dots,r$. Let $\hat{L} = \{1,-1\}\times L$. Define a multiplication on $\hat{L}$ by

$$(\theta,\alpha)\cdot(\tau,\beta) = (\theta\tau(-1)^{\epsilon(\alpha,\beta)},\alpha+\beta)$$

for $\theta,\tau\in\{1,-1\}$ and $\alpha,\beta\in L$. We have a surjective map $^- : \hat{L}\to L$ defined by $\overline{(\theta,\alpha)} = \alpha$ for $\theta\in\{1,-1\}$ and $\alpha\in L$. We also have an injective map from $\{1,-1\}$ to $\hat{L}$ defined by $\theta\mapsto(\theta,0)$ for $\theta\in\{1,-1\}$. It is clear that these maps are homomorphisms of groups and we have the exact sequence

$$1\to\{1,-1\}\to\hat{L}\xrightarrow{-}L\to 1,$$

that is, $\hat{L}$ is a central extension of L by the group $\{1,-1\}$. The commutator map $c : L\times L\to\mathbb{Z}/2\mathbb{Z}$ of this central extension is given by $c(\alpha,\beta) = (\alpha,\beta)+2\mathbb{Z}$. We shall denote $(1,\alpha)\in\hat{L}$ by e_α for $\alpha\in L$ and $(\theta,0)$ by θ for $\theta\in\mathbb{Z}/2\mathbb{Z}$. Then

$$(\theta,\alpha) = \theta e_\alpha = e_\alpha\theta$$

and

$$e_\alpha e_\beta = (-1)^{\epsilon(\alpha,\beta)}e_{\alpha+\beta}.$$

Let $\mathbb{C}[L]$ be the group algebra of L. We shall use e^α to denote the element $\alpha\in L$ in $\mathbb{C}[L]$. Then in $\mathbb{C}[L]$, we have $e^\alpha e^\beta = e^{\alpha+\beta}$ for $\alpha,\beta\in L$. We have an action of the abelian Lie algebra $\mathfrak{h}\otimes t^0 = \mathfrak{h}$ on $\mathbb{C}[L]$ by $(a\otimes t^0)\cdot e^\alpha = (a,\alpha)e^\alpha$. Then $\mathbb{C}[L]$ is an $\mathfrak{h}\otimes t^0$-module. We also have an action of $\hat{L}$ on $\mathbb{C}[L]$ defined by $(\theta,\alpha)\cdot e^\beta = (\theta(-1)^{\epsilon(\alpha,\beta)})e^{\alpha+\beta}$ for $\theta\in\{1,-1\}$

and $\alpha, \beta \in L$. In particular, $e_\alpha \cdot e^\beta = (-1)^{\epsilon(\alpha,\beta)} e^{\alpha+\beta}$ for $\alpha \in L$ and $\theta \cdot e^\beta = \theta e^\beta$. It is clear that this action gives $\mathbb{C}[L]$ an $\hat{L}$-module structure.

Let $V_L = S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}[L]$. The grading on $\hat{\mathfrak{h}}_-$ gives a grading on $S(\hat{\mathfrak{h}}_-)$ called weight and we give a grading on $\mathbb{C}[L]$ also called weight by defining the weight of e^α to be $\frac{1}{2}(\alpha, \alpha)$. These gradings give a grading on V_L also called weight. Then we have $V_L = \coprod_{n\in\mathbb{Z}} (V_L)_{(n)}$ where $(V_L)_{(n)}$ is the homogeneous subspace of V_L of weight n. It is clear that V_L is grading-restricted since the gradings on $S(\hat{\mathfrak{h}}_-)$ and $\mathbb{C}[L]$ are both grading restricted. In fact since L is positive definite, we have $(V_L)_{(n)} = 0$ when $n < 0$.

As is discussed above, we have an $\hat{h}$-module structure on $V_L = S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}[L]$. Since $\mathbb{C}[L]$ is an $\hat{L}$-module, $V_L = S(\hat{\mathfrak{h}}_-) \otimes \mathbb{C}[L]$ is also an $\hat{L}$-module with $\hat{L}$ acts only on $\mathbb{C}[L]$. We denote the action of $a \otimes t^n$ on V_L by $a(n)$ for $a \in \mathfrak{h}$ and $n \in \mathbb{Z}$. For $a \in \mathfrak{h}$, let $a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$. For $\alpha \in L$, for simplicity, we shall also use e^α to denote $1 \otimes e^\alpha \in V_L$. For $\alpha \in L$ and a formal variable x, we define $x^\alpha \cdot (u \otimes e^\beta) = x^{(\alpha,\beta)}(u \otimes e^\beta)$ for $u \in S(\hat{\mathfrak{h}}_-)$ and $\beta \in L$. For $\alpha \in L$, let

$$Y_{V_L}(e^\alpha, x) = \exp\left(-\sum_{n<0} \frac{\alpha(n)}{n} x^{-n}\right) \exp\left(-\sum_{n>0} \frac{\alpha(n)}{n} x^{-n}\right) e_\alpha x^\alpha \in (\text{End } V_L)[[x, x^{-1}]].$$

For a vector space M, let $\int \cdot \, dx : M[[x]] \oplus x^{-2}M[[x^{-1}]] \to xM[[x]] \oplus x^{-1}M[[x^{-1}]]$ be the linear map given by the integrating the formal series in $M[[x]] \oplus x^{-2}M[[x^{-1}]]$ with constant terms being 0. Then the images of $M[[x]]$ and $x^{-2}M[[x^{-1}]]$ are $xM[[x]]$ and $x^{-1}M[[x^{-1}]]$, respectively. For $a \in \mathfrak{h}$, let $a(x)^+ = \sum_{n\in\mathbb{Z}_+} a(n)x^{-n-1}$ and $a(x)^- = \sum_{n\in\mathbb{Z}_-} a(n)x^{-n-1}$. Then $a(x) = a(x)^+ + a(x)^- + a(0)x^{-1}$. Also

$$\int a(x)^+ dx = -\sum_{n>0} \frac{a(n)}{n} x^{-n},$$
$$\int a(x)^- dx = -\sum_{n<0} \frac{a(n)}{n} x^{-n}.$$

Thus we have
$$Y_{V_L}(e^\alpha, x) = e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} e_\alpha x^\alpha$$
for $\alpha \in L$.

We need the following commutator formulas:

Proposition 2.7.1. *For $a, b \in \mathfrak{h}$, $\alpha, \beta \in L$, we have*

$$[a(x_1), b(x_2)] = (a, b)\left((x_1 - x_2)^{-2} - (-x_2 + x_1)^{-2}\right), \tag{2.7.78}$$
$$[a(x_1), Y_{V_L}(e^\alpha, x_2)] = (a, \alpha)\left((x_1 - x_2)^{-1} - (-x_2 + x_1)^{-1}\right) Y_{V_L}(e^\alpha, x_2), \tag{2.7.79}$$
$$\begin{aligned}[Y_{V_L}(e^\alpha, x_1), Y_{V_L}(e^\beta, x_2)] = &\left((x_1 - x_2)^{(\alpha,\beta)} - (-x_2 + x_1)^{(\alpha,\beta)}\right) \cdot \\ &\cdot e^{\int \alpha(x_1)^- dx_1} e^{\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^+ dx_1} e^{\int \beta(x_2)^+ dx_2} e_\alpha e_\beta x_1^\alpha x_2^\beta, \end{aligned} \tag{2.7.80}$$

where for $n \in \mathbb{Z}$, $(x_1 - x_2)^n$ and $(-x_2 + x_1)^n$ means their binomial expansions in nonnegative powers of x_2 and x_1, respectively.

Proof. The proof of (2.7.78) is completely the same as that of (2.6.65) above.

For $a \in \mathfrak{h}$ and $\alpha \in L$, taking $b = \alpha$ in (2.7.78), we obtain

$$[a(x_1)^{\pm}, \alpha(x_2)^{\pm}] = 0, \tag{2.7.81}$$

$$[a(0), \alpha(x_2)^{\pm}] = 0, \tag{2.7.82}$$

$$[a(x_1)^{+}, \alpha(x_2)^{-}] = \frac{(a,\alpha)}{(x_1 - x_2)^2}, \tag{2.7.83}$$

$$[a(x_1)^{-}, \alpha(x_2)^{+}] = -\frac{(a,\alpha)}{(x_2 - x_1)^2}. \tag{2.7.84}$$

Applying the map $\int \cdot \, dx_2$ to both sides of (2.7.81)–(2.7.84) and then switch the order of the commutators, we obtain

$$\left[\int \alpha(x_2)^{\pm} dx_2, a(x_1)^{\pm}\right] = 0, \tag{2.7.85}$$

$$\left[\int \alpha(x_2)^{\pm} dx_2, a(0)\right] = 0, \tag{2.7.86}$$

$$\left[\int \alpha(x_2)^{-} dx_2, a(x_1)^{+}\right] = -\frac{(a,\alpha)}{x_1 - x_2} + \frac{(a,\alpha)}{x_1}, \tag{2.7.87}$$

$$\left[\int \alpha(x_2)^{+} dx_2, a(x_1)^{-}\right] = -\frac{(a,\alpha)}{x_2 - x_1}, \tag{2.7.88}$$

where in (2.7.87), the term $-\frac{(a,\alpha)}{x_1}$ appears because the constant term of $\int \cdot \, dx_2$ must be 0. From (2.7.85)–(2.7.88), we obtain

$$\begin{aligned}
e^{\int \alpha(x_2)^{\pm} dx_2} a(x_1)^{\pm} e^{-\int \alpha(x_2)^{\pm} dx_2} &= a(x_1)^{\pm}, \\
e^{\int \alpha(x_2)^{\pm} dx_2} a(0) e^{-\int \alpha(x_2)^{\pm} dx_2} &= a(0), \\
e^{\int \alpha(x_2)^{-} dx_2} a(x_1)^{+} e^{-\int \alpha(x_2)^{-} dx_2} &= a(x_1)^{+} - \frac{(a,\alpha)}{x_1 - x_2} + \frac{(a,\alpha)}{x_1}, \\
e^{\int \alpha(x_2)^{+} dx_2} a(x_1)^{-} e^{-\int \alpha(x_2)^{+} dx_2} &= a(x_1)^{-} - \frac{(a,\alpha)}{x_2 - x_1},
\end{aligned}$$

or equivalently,

$$a(x_1)^{\pm} e^{\int \alpha(x_2)^{\pm} dx_2} = e^{\int \alpha(x_2)^{\pm} dx_2} a(x_1)^{\pm}, \tag{2.7.89}$$

$$a(0) e^{\int \alpha(x_2)^{\pm} dx_2} = e^{\int \alpha(x_2)^{\pm} dx_2} a(0), \tag{2.7.90}$$

$$a(x_1)^{+} e^{\int \alpha(x_2)^{-} dx_2} = e^{\int \alpha(x_2)^{-} dx_2} a(x_1)^{+} + \left(\frac{(a,\alpha)}{x_1 - x_2} - \frac{(a,\alpha)}{x_1}\right) e^{\int \alpha(x_2)^{-} dx_2}, \tag{2.7.91}$$

$$a(x_1)^{-} e^{\int \alpha(x_2)^{+} dx_2} = e^{\int \alpha(x_2)^{+} dx_2} a(x_1)^{-} + \frac{(a,\alpha)}{x_2 - x_1} e^{\int \alpha(x_2)^{+} dx_2}. \tag{2.7.92}$$

On the other hand, for $u \in S(\hat{\mathfrak{h}}_-)$ and $\beta \in L$,

$$
\begin{aligned}
&a(0)e_\alpha x_2^\alpha(u \otimes e^\beta)\\
&\quad = (-1)^{\epsilon(\alpha,\beta)} x_2^{(\alpha,\beta)}(a, \alpha+\beta)(u \otimes e^{\alpha+\beta})\\
&\quad = (-1)^{\epsilon(\alpha,\beta)} x_2^{(\alpha,\beta)}(a, \alpha)(u \otimes e^{\alpha+\beta}) + (-1)^{\epsilon(\alpha,\beta)} x_2^{(\alpha,\beta)}(a, \beta)(u \otimes e^{\alpha+\beta})\\
&\quad = (a,\alpha)e_\alpha x_2^\alpha(u \otimes e^\beta) + e_\alpha x_2^\alpha a(0)(u \otimes e^\beta).
\end{aligned}
$$

Thus we obtain

$$
a(0)e_\alpha x_2^\alpha = (a,\alpha)e_\alpha x_2^\alpha + e_\alpha x_2^\alpha a(0). \tag{2.7.93}
$$

Using (2.7.89)–(2.7.93), we have

$$
\begin{aligned}
&a(x_1)Y_{V_L}(e^\alpha, x_2)\\
&\quad = a(x_1)^+ e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha + a(x_1)^- e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha\\
&\qquad + a(0)x_1^{-1} e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha\\
&\quad = e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha a(x_1)^+ + \left(\frac{(a,\alpha)}{x_1 - x_2} - \frac{(a,\alpha)}{x_1}\right) e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha\\
&\qquad + e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha a(x_1)^- + \frac{(a,\alpha)}{x_2 - x_1} e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha\\
&\qquad + e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha a(0)x_1^{-1} + \frac{(a,\alpha)}{x_1} e^{\int \alpha(x_2)^- dx_2} e^{\int \alpha(x_2)^+ dx_2} e_\alpha x_2^\alpha\\
&\quad = Y_{V_L}(e^\alpha, x_2)a(x_1) + (a,\alpha)\left((x_1 - x_2)^{-1} - (-x_2 + x_1)^{-1}\right) Y_{V_L}(e^\alpha, x_2).
\end{aligned}
$$

This is (2.7.79).

Taking $a \in \mathfrak{h}$ to be $\beta \in L \subset \mathfrak{h}$ in (2.7.88), we obtain

$$
[\beta(x_2)^-, e^{\int \alpha(x_1)^+ dx_1}] = \frac{(\alpha,\beta)}{x_1 - x_2} e^{\int \alpha(x_1)^+ dx_1}. \tag{2.7.94}
$$

Applying $-\int \cdot \, dx_2$ to both sides of (2.7.94), we obtain

$$
\begin{aligned}
&\left[-\int \beta(x_2)^- dx_2, e^{\int \alpha(x_1)^+ dx_1}\right]\\
&\quad = ((\alpha,\beta)\log(x_1 - x_2) - (\alpha,\beta)\log x_1)\, e^{\int \alpha(x_1)^+ dx_1}\\
&\quad = \log\left(1 - \frac{x_2}{x_1}\right)^{(\alpha,\beta)} e^{\int \alpha(x_1)^+ dx_1}.
\end{aligned} \tag{2.7.95}
$$

From (2.7.95), we obtain

$$
e^{-\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^+ dx_1} e^{\int \beta(x_2)^- dx_2} = \left(1 - \frac{x_2}{x_1}\right)^{(\alpha,\beta)} e^{\int \alpha(x_1)^+ dx_1},
$$

or equivalently,

$$e^{\int \alpha(x_1)^+ dx_1} e^{\int \beta(x_2)^- dx_2} = \left(1 - \frac{x_2}{x_1}\right)^{(\alpha,\beta)} e^{\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^+ dx_1}. \tag{2.7.96}$$

For $u \in S(\hat{\mathfrak{h}}_-)$ and $\gamma \in L$, we have

$$\begin{aligned} x_1^\alpha e_\beta(u \otimes e^\gamma) &= (-1)^{\epsilon(\beta,\gamma)} x_1^{(\alpha,\beta)+(\alpha,\gamma)}(u \otimes e^{\beta+\gamma}),\\ e_\beta x_1^\alpha(u \otimes e^\gamma) &= (-1)^{\epsilon(\beta,\gamma)} x_1^{(\alpha,\gamma)}(u \otimes e^{\beta+\gamma}). \end{aligned}$$

Therefore we obtain

$$x_1^\alpha e_\beta = x_1^{(\alpha,\beta)} e_\beta x_1^\alpha. \tag{2.7.97}$$

Using (2.7.96) and (2.7.97), we obtain

$$\begin{aligned} &Y_{V_L}(e^\alpha, x_1) Y_{V_L}(e^\beta, x_2)\\ &\quad = e^{\int \alpha(x_1)^- dx_1} e^{\int \alpha(x_1)^+ dx_1} e_\alpha x_1^\alpha e^{\int \beta(x_2)^- dx_2} e^{\int \beta(x_2)^+ dx_2} e_\beta x_2^\beta\\ &\quad = (x_1 - x_2)^{(\alpha,\beta)} e^{\int \alpha(x_1)^- dx_1} e^{\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^+ dx_1} e^{\int \beta(x_2)^+ dx_2} e_\alpha e_\beta x_1^\alpha x_2^\beta. \end{aligned} \tag{2.7.98}$$

From (2.7.98), we also obtain

$$\begin{aligned} &Y_{V_L}(e^\beta, x_2) Y_{V_L}(e^\alpha, x_1)\\ &\quad = (x_2 - x_1)^{(\alpha,\beta)} e^{\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^- dx_1} e^{\int \beta(x_2)^+ dx_2} e^{\int \alpha(x_1)^+ dx_1} e_\beta e_\alpha x_1^\alpha x_2^\beta. \end{aligned} \tag{2.7.99}$$

Since the commutator map of the central extension $\hat{L}$ is $c(\alpha, \beta) = (\alpha, \beta) + 2\mathbb{Z}$, we have

$$e_\beta e_\alpha = (-1)^{(\alpha,\beta)} e_\alpha e_\beta.$$

Thus the right-hand side of (2.7.99) is equal to

$$(-x_2 + x_1)^{(\alpha,\beta)} e^{\int \alpha(x_1)^- dx_1} e^{\int \beta(x_2)^- dx_2} e^{\int \alpha(x_1)^+ dx_1} e^{\int \beta(x_2)^+ dx_2} e_\alpha e_\beta x_1^\alpha x_2^\beta. \tag{2.7.100}$$

From (2.7.98)–(2.7.100), we obtain (2.7.80). □

Let $L_{V_L}(0)$ be the operator on V_L giving the grading on V_L, that is, $L_{V_L}(0)v = nv$ for $v \in (V_L)_{(n)}$. Note that V_L is spanned by elements of the form

$$a_1(-n_1) \cdots a_k(-n_k) e^\beta$$

for $a_1, \dots, a_k \in \mathfrak{h}$, $n_1, \dots, n_k \in \mathbb{Z}_+$ and $\beta \in L$. We define an operator $L_{V_L}(-1)$ on V_L by

$$\begin{aligned} &L_{V_L}(-1) a_1(-n_1) \cdots a_k(-n_k) e^\beta\\ &\qquad = \sum_{i=1}^k n_i a_1(-n_1) \cdots a_{i-1}(-n_{i-1}) a_i(-n_i - 1) a_{i+1}(-n_{i+1}) \cdots a_k(-n_k) e^\beta\\ &\qquad\quad + a_1(-n_1) \cdots a_k(-n_k) \beta(-1) e^\beta. \end{aligned}$$

We denote $e^0 \in V_L$ by $\mathbf{1}_{V_L}$.

Proposition 2.7.2. *The series $a(x)$ for $a \in \mathfrak{h}$, $Y_{V_L}(e^\alpha, x)$ for $\alpha \in L$ and the operators $L_{V_L}(0)$ and $L_{V_L}(-1)$ have the following properties:*

1. *For $a \in \mathfrak{h}$, $[L_{V_L}(0), a(x)] = x\frac{d}{dx}a(x) + a(x)$ and for $\alpha \in L$, $[L_{V_L}(0), Y_{V_L}(e^\alpha, x)] = x\frac{d}{dx}Y_{V_L}(e^\alpha, x) + \frac{1}{2}(\alpha, \alpha)Y_{V_L}(e^\alpha, x)$.*

2. *$L_V(-1)\mathbf{1}_{V_L} = 0$, $[L_V(-1), a(x)] = \frac{d}{dx}a(x)$ for $a \in \mathfrak{h}$ and $[L_V(-1), Y_{V_L}(e^\alpha, x)] = \frac{d}{dx}Y_{V_L}(e^\alpha, x)$ for $\alpha \in L$.*

3. *For $a \in \mathfrak{h}$ and $\alpha \in L$, $a(x)\mathbf{1}_{V_L}, Y_{V_L}(e^\alpha, x)\mathbf{1}_{V_L} \in V_L[[x]]$. Moreover, $\lim_{x\to 0} a(x)\mathbf{1}_{V_L} = a(-1)\mathbf{1}_{V_L}$ and $\lim_{x\to 0} Y_{V_L}(e^\alpha, x)\mathbf{1}_{V_L} = e^\alpha$.*

4. *The vector space V_L is spanned by elements of the form*

$$\begin{aligned} &a_1(-n_1)\cdots a_k(-n_k)e^\alpha \\ &\quad = a_1(-n_1)\cdots a_k(-n_k)e_\alpha\mathbf{1}_{V_L} \\ &\quad = \mathrm{Res}_{x_1}\cdots\mathrm{Res}_{x_k}x_1^{-n_1}\cdots x_k^{-n_k}x_{k+1}^{-1}a_1(x_1)\cdots a_k(x_k)Y_{V_L}(e^\alpha, x_{k+1})\mathbf{1}_{V_L} \end{aligned} \quad (2.7.101)$$

   for $a_1, \dots, a_k \in \mathfrak{h}$, $n_1, \dots, n_k \in \mathbb{Z}_+$ and $\alpha \in L$.

5. *For $a, b \in \mathfrak{h}$,*

$$(x_1 - x_2)^2 a(x_1)b(x_2) = (x_1 - x_2)^2 b(x_2)a(x_1).$$

   For $a \in \mathfrak{h}$ and $\alpha \in L$,

$$(x_1 - x_2)a(x_1)Y_{V_L}(e^\alpha, x_2) = (x_1 - x_2)Y_{V_L}(e^\alpha, x_2)a(x_1).$$

   For $\alpha, \beta \in L$,

$$(x_1 - x_2)^{-(\alpha,\beta)}Y_{V_L}(e^\alpha, x_1)Y_{V_L}(e^\beta, x_2) = (x_1 - x_2)^{-(\alpha,\beta)}Y_{V_L}(e^\beta, x_2)Y_{V_L}(e^\alpha, x_1)$$

   when $(\alpha, \beta) < 0$ and

$$Y_{V_L}(e^\alpha, x_1)Y_{V_L}(e^\beta, x_2) = Y_{V_L}(e^\beta, x_2)Y_{V_L}(e^\alpha, x_1)$$

   when $(\alpha, \beta) \geq 0$.

Proof. Property 1 can be verified by the definition of $L_{V_L}(0)$ and straightforward calculations.

The first two formulas in Property 2 can also be verified by the definition of $L_{V_L}(-1)$ and straightforward calculations. Here we prove the third equality. We first need several commutator formulas. For $\alpha \in L$, from $[L_V(-1), \alpha(x)] = \frac{d}{dx}\alpha(x)$ whose proof we omitted, we obtain

$$\begin{aligned} [L_{V_L}(-1), \alpha(x)^-] &= \frac{d}{dx}\int \alpha(x)^-, \\ [L_{V_L}(-1), \alpha(x)^+] &= \frac{d}{dx}\alpha(x)^+ - \alpha(0)x^{-2}. \end{aligned}$$

Applying $\int \cdot\, dx$ to both sides, we obtain

$$\left[L_{V_L}(-1), \int \alpha(x)^- dx\right] = \frac{d}{dx}\left(\int \alpha(x)^- dx\right) - \alpha(-1), \tag{2.7.102}$$

$$\left[L_{V_L}(-1), \int \alpha(x)^+ dx\right] = \frac{d}{dx}\int \alpha(x)^+ dx + \alpha(0)x^{-1}. \tag{2.7.103}$$

By the definition of $L_{V_L}(-1)$, for a product A of operators of the form $a(-m)$ for $a \in \mathfrak{h}$ and $m \in \mathbb{Z}_+$, $[L_{V_L}, A]$ is a linear combinations of products of the operators of the same form. In particular, $[L_{V_L}(-1), A]$ commutes with $e_\alpha x^\alpha$. For such a product A and $\beta \in L$,

$$\begin{aligned}
&L_{V_L}(-1)e_\alpha x^\alpha A e^\beta \\
&\quad = L_{V_L}(-1)A e_\alpha x^\alpha e^\beta \\
&\quad = [L_{V_L}(-1), A]e_\alpha x^\alpha e^\beta + A L_{V_L}(-1)e_\alpha x^\alpha e^\beta \\
&\quad = e_\alpha x^\alpha [L_{V_L}(-1), A]e^\beta + x^{(\alpha,\beta)}(-1)^{\epsilon(\alpha,\beta)} A L_{V_L}(-1)e^{\alpha+\beta} \\
&\quad = e_\alpha x^\alpha [L_{V_L}(-1), A]e^\beta + x^{(\alpha,\beta)}(-1)^{\epsilon(\alpha,\beta)} A(\alpha+\beta)(-1)e^{\alpha+\beta} \\
&\quad = e_\alpha x^\alpha [L_{V_L}(-1), A]e^\beta + x^{(\alpha,\beta)}(-1)^{\epsilon(\alpha,\beta)} A\alpha(-1)e^{\alpha+\beta} + x^{(\alpha,\beta)}(-1)^{\epsilon(\alpha,\beta)} A\beta(-1)\ e^{\alpha+\beta} \\
&\quad = e_\alpha x^\alpha [L_{V_L}(-1), A]e^\beta + \alpha(-1)e_\alpha x^\alpha A e^\beta + A e_\alpha x^\alpha \beta(-1)e^\beta \\
&\quad = e_\alpha x^\alpha [L_{V_L}(-1), A]e^\beta + \alpha(-1)e_\alpha x^\alpha A e^\beta + e_\alpha x^\alpha A L_{V_L}(-1)(u \otimes e^\beta) \\
&\quad = e_\alpha x^\alpha L_{V_L}(-1)A e^\beta + \alpha(-1)e_\alpha x^\alpha A e^\beta,
\end{aligned}$$

where we have used the fact that $S(\hat{\mathfrak{h}}_-)$ is a commutative algebra and e_α and x^α commute with A. So we obtain the commutator formula

$$[L_{V_L}(-1), e_\alpha x^\alpha] = \alpha(-1)e_\alpha x^\alpha. \tag{2.7.104}$$

For $u \in S(\hat{\mathfrak{h}}_-)$ and $\beta \in L$,

$$\begin{aligned}
\alpha(0)e_\alpha(u \otimes e^\beta) &= (-1)^{\epsilon(\alpha,\beta)}\alpha(0)(u \otimes e^{\alpha+\beta}) \\
&= (\alpha,\alpha)(-1)^{\epsilon(\alpha,\beta)}(u \otimes e^\beta) + (\alpha,\beta)(-1)^{\epsilon(\alpha,\beta)}(u \otimes e^\beta) \\
&= (\alpha,\alpha)e_\alpha(u \otimes e^\beta) + (\alpha,\beta)e_\alpha(u \otimes e^\beta) \\
&= (\alpha,\alpha)e_\alpha(u \otimes e^\beta) + e_\alpha\alpha(0)(u \otimes e^\beta),
\end{aligned}$$

which gives us the commutator formula

$$[\alpha(0), e_\alpha] = (\alpha,\alpha)e_\alpha. \tag{2.7.105}$$

Using the fact that $[\alpha(-1), \cdot]$ is a derivation on the algebra of operators on V_L as coefficients, we have

$$[\alpha(-1), e^{\int \alpha(x)^+ dx}] = e^{\int \alpha(x)^+ dx}\left[\alpha(-1), \int \alpha(x)^+ dx\right]$$

$$= e^{\int \alpha(x)^+ dx} [\alpha(-1), -\alpha(1)]\, x^{-1}$$
$$= e^{\int \alpha(x)^+ dx} (\alpha, \alpha) x^{-1}.$$

Note that both $[L_{V_L}(-1), \cdot]$ and $\frac{d}{dx}$ are derivations on the algebra of series in x with operators on V_L as coefficients. Using these properties, (2.7.102), (2.7.103) and (2.7.104) and the formula

$$\frac{d}{dx} x^\alpha = \alpha(0) x^\alpha x^{-1},$$

we have

$$\begin{aligned}
&[L_{V_L}(-1), Y(_{V_L}(e^\alpha, x)] \\
&\quad = [L_{V_L}(-1), e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} e_\alpha x^\alpha] \\
&\quad = [L_{V_L}(-1), e^{\int \alpha(x)^- dx}] e^{\int \alpha(x)^+ dx} e_\alpha x^\alpha + e^{\int \alpha(x)^- dx} [L_{V_L}(-1), e^{\int \alpha(x)^+ dx}] e_\alpha x^\alpha \\
&\quad\quad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} [L_{V_L}(-1), e_\alpha x^\alpha] \\
&\quad = e^{\int \alpha(x)^- dx} \left[L_{V_L}(-1), \int \alpha(x)^- dx \right] e^{\int \alpha(x)^+ dx} e_\alpha x^\alpha \\
&\quad\quad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \left[L_{V_L}(-1), \int \alpha(x)^+ dx \right] e_\alpha x^\alpha \\
&\quad\quad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \alpha(-1) e_\alpha x^\alpha \\
&\quad = e^{\int \alpha(x)^- dx} \left(\frac{d}{dx} \left(\int a(x)^- dx \right) - \alpha(-1) \right) e^{\int \alpha(x)^+ dx} e_\alpha x^\alpha \\
&\quad\quad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \left(\frac{d}{dx} \int a(x)^+ dx + \alpha(0) x^{-1} \right) e_\alpha x^\alpha \\
&\quad\quad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \alpha(-1) e_\alpha x^\alpha \\
&\quad = e^{\int \alpha(x)^- dx} \left(\frac{d}{dx} \int a(x)^- dx \right) e^{\int \alpha(x)^+ dx} e_\alpha x^\alpha - e^{\int \alpha(x)^- dx} \alpha(-1) e^{\int \alpha(x)^+ dx} e_\alpha x^\alpha \\
&\quad\quad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \left(\frac{d}{dx} \int a(x)^+ dx \right) e_\alpha x^\alpha + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \alpha(0) x^{-1} e_\alpha x^\alpha \\
&\quad\quad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} \alpha(-1) e_\alpha x^\alpha \\
&\quad = \left(\frac{d}{dx} e^{\int \alpha(x)^- dx} \right) e^{\int \alpha(x)^+ dx} e_\alpha x^\alpha + e^{\int \alpha(x)^- dx} \left(\frac{d}{dx} e^{\int \alpha(x)^+ dx} \right) e_\alpha x^\alpha \\
&\quad\quad + e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} e_\alpha \alpha(0) x^\alpha x^{-1} \\
&\quad = \frac{d}{dx} e^{\int \alpha(x)^- dx} e^{\int \alpha(x)^+ dx} e_\alpha x^\alpha \\
&\quad = \frac{d}{dx} Y_{V_L}(e^\alpha, x).
\end{aligned} \tag{2.7.106}$$

Properties 3 and 4 are clear. Property 5 follows immediately from Proposition 2.7.1. □

From Proposition 2.7.2 and Theorem 2.5.7, we obatin:

Theorem 2.7.3. *The vector space V_L equipped with the the vertex operator map Y_{V_L} defined by*

$$\begin{aligned}\langle v', &Y_{V_L}(\alpha_1(-n_1)\cdots\alpha_k(-n_k)e^{\alpha}, z)v\rangle\\ &= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{-n_1}\cdots\xi_k^{-n_k}\xi_{k+1}^{-1}R(\langle v', a_1(\xi_1+z)\cdots a_k(\xi_k+z)Y_{V_L}(e^{\alpha}, \xi_{k+1}+z)v\rangle).\end{aligned} \tag{2.7.107}$$

and the vacuum $\mathbf{1}_{V_L}$ is a grading-restricted vertex algebra. Moreover, this is the unique grading-restricted vertex algebra structure on V_L with the vacuum $\mathbf{1}_{V_L}$ such that $Y(a(-1)\mathbf{1}, x) = a(x)$ for $a \in \mathfrak{h}$ and $Y(e^{\alpha}, x) = Y_{V_L}(e^{\alpha}, x)$ for $\alpha \in L$.

As an $\hat{\mathfrak{h}}$-module, V_L has a $\hat{\mathfrak{h}}$-submodule $S(\hat{\mathfrak{h}}_-)\otimes \mathbb{C}e^0$. Note that $\mathbf{1}_{V_L} \in S(\hat{\mathfrak{h}}_-)\otimes \mathbb{C}e^0$.

Proposition 2.7.4. *The $\hat{\mathfrak{h}}$-submodule $S(\hat{\mathfrak{h}}_-)\otimes \mathbb{C}e^0$ of V_L is closed under the restriction of the vertex operator map Y_{V_L} to $S(\hat{\mathfrak{h}}_-)\otimes \mathbb{C}e^0$, that is, For $u, v \in S(\hat{\mathfrak{h}}_-)\otimes \mathbb{C}e^0$, $Y_{V_L}(u, x)v \in (S(\hat{\mathfrak{h}}_-)\otimes \mathbb{C}e^0)[[x, x^{-1}$. In particular, $S(\hat{\mathfrak{h}}_-)\otimes \mathbb{C}e^0$ is a grading-restricted vertex subalgebra of V_L. Moreover, $S(\hat{\mathfrak{h}}_-)\otimes \mathbb{C}e^0$ as a grading-restricted vertex algebra is isomorphic to $S(\hat{\mathfrak{h}}_-)$.*

Exercise 2.7.5. Prove Proposition 2.7.4.

Since $S(\hat{\mathfrak{h}}_-)$ has a conformal vector ω, $S(\hat{\mathfrak{h}}_-)\otimes \mathbb{C}e^0$ also has a conformal vector $\omega \otimes e^0$. We use ω_{V_L} to denote $\omega \otimes e^0$.

Theorem 2.7.6. *The element ω_{V_L} is a conformal vector of V_L. In particular, V_L is a vertex operator algebra.*

Exercise 2.7.7. Prove Theorem 2.7.6.

We now give the operator production expansion of some vertex operators. By Theorem 2.8.4, V_L has a structure of a grading-restricted vertex algebra. We use Y_{V_L} to denote its vertex operator map. Then from the definition of the vertex operator map in the preceding section, we see that $Y_{V_L}(a(-1)\mathbf{1}, x) = a(x)$ for $a \in \mathfrak{h}$ and $Y_{V_L}(e^{\alpha}, x)$ is exactly the vertex operator associated with $e^{\alpha} = 1 \otimes e^{\alpha} \in V_L$. So the operator product expansion of $a(z_1)$ and $Y_{V_L}(e^{\alpha}, z_2)$ is

$$\begin{aligned}a(z_1)Y_{V_L}(e^{\alpha}, z_2) &= Y_{V_L}(a(-1)\mathbf{1}, z_1)Y_{V_L}(e^{\alpha}, z_2)\\ &= Y_{V_L}(Y_{V_L}(a(-1)\mathbf{1}, z_1 - z_2)e^{\alpha}, z_2)\end{aligned} \tag{2.7.108}$$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. But

$$\begin{aligned}Y_{V_L}(a(-1)\mathbf{1}, &z_1 - z_2)e^{\alpha} = a(z_1 - z_2)e^{\alpha}\\ &= \sum_{n\in\mathbb{Z}} a(n)e^{\alpha}(z_1 - z_2)^{-n-1}\\ &= (a, \alpha)e^{\alpha}(z_1 - z_2)^{-1} + \sum_{n\in -\mathbb{Z}_+} a(n)e^{\alpha}(z_1 - z_2)^{-n-1}.\end{aligned} \tag{2.7.109}$$

Substituting the right-hand side of (2.7.109) into the right-hand side of (2.7.108), we obtain the explicit form

$$a(z_1)Y_{V_L}(e^\alpha, x) = (a, \alpha)Y_{V_L}(e^\alpha, z_2)(z_1 - z_2)^{-1} + \sum_{n\in -\mathbb{Z}_+} Y_{V_L}(a(n)e^\alpha, z_2)(z_1 - z_2)^{-n-1} \quad (2.7.110)$$

of the operator product expansion of $a(z_1)$ and $Y_{V_L}(e^\alpha, x)$ Since the only singular term in $z_1 - z_2$ in the right-hand side of (2.7.110) is $(a, \alpha)Y_{V_L}(e^\alpha, z_2)(z_1 - z_2)^{-1}$, we obtain

$$a(z_1)Y_{V_L}(e^\alpha, x) \sim (a, \alpha)Y_{V_L}(e^\alpha, z_2)(z_1 - z_2)^{-1}.$$

This last formula can also be obtained using the commutator formula (2.7.79) using the same method as in Example **??**.

2.8 Examples: Affine vertex operator algebras

In this section, we give constructions of the grading-restricted vertex algebras and vertex operator algebras associated to affine Lie algebras. See [K] for the theory of Kac-Moody algebras, including in particular, the theory of affine Lie algebras.

2.8.1 The grading-restricted vertex algebra $V(\ell, 0)$

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with a symmetric invariant bilinear form $(\cdot, \cdot)$. The affine Lie algebra $\hat{\mathfrak{g}}$ is given by the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ equipped with the Lie bracket operation

$$\begin{aligned} [a \otimes t^m, b \otimes t^n] &= [a, b] \otimes t^{m+n} + (a, b)m\delta_{m+n,0}\mathbf{k}, \\ [a \otimes t^m, \mathbf{k}] &= 0, \end{aligned}$$

for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. It is a central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Let $\hat{\mathfrak{g}}_\pm = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}]$ and $\hat{\mathfrak{g}}_0 = \mathfrak{g} \oplus \mathbb{C}\mathbf{k}$. Then

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+.$$

Note that $\hat{\mathfrak{g}}_\pm$, $\hat{\mathfrak{g}}_0$, $\mathfrak{g}$ and $\mathbf{k}$ are all subalgebra of $\hat{\mathfrak{g}}$. For a module for a subalgebra of $\hat{\mathfrak{g}}$ (in particular, a $\hat{\mathfrak{g}}$-module), we use $a(n)$ to denote the action of $a \otimes t^n$ on the module for $a \in \mathfrak{g}$, $n \in \mathbb{Z}$ such that $a \otimes t^n$ is in the subalgebra.

Fix $\ell \in \mathbb{C}$. Let $\mathbb{C}_\ell$ be a copy of $\mathbb{C}$, with the structure of a module for $\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+$ by defining $a(n)1 = 0$ for all $a \in \mathfrak{g}$ and $n \geq 0$, and $\mathbf{k}1 = \ell$. Let

$$V(\ell, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+)} \mathbb{C}_\ell$$

be the corresponding induced $\hat{\mathfrak{g}}$-module. Let $\mathbf{1}_{V(\ell,0)} = 1 \otimes 1 \in V(\ell, 0)$. Recall that $\mathbf{k}\mathbf{1}_{V(\ell,0)} = \ell\mathbf{1}_{V(\ell,0)}$ and if $n \geq 0$, $a(n)\mathbf{1}_{V(\ell,0)} = 0$.

By the Poincare-Birkhoff-Witt Theorem, The induced $\hat{\mathfrak{g}}$-module $V(\ell, 0)$ is linearly isomorphic to $U(\hat{g}_-)$. Since $U(\hat{g}_-)$ is spanned by elements of the form $(a_1 \otimes t^{n_1}) \cdots (a_k \otimes t^{n_k}$ for $a_1, \ldots, a_k \in \mathfrak{g}$ and $n_1, \ldots, n_k \in -\mathbb{Z}_+$ which correspond to $a_1(n_1) \cdots a_k(n_k)\mathbf{1}_{V(\ell,0)} \in V(\ell, 0)$ under this linear isomorphism. In particular, we see that $V(\ell, 0)$ is spanned by elements of the form $a_1(n_1) \cdots a_k(n_k)\mathbf{1}_{V(\ell,0)}$ for $a_1, \ldots, a_k \in \mathfrak{g}$ and $n_1, \ldots, n_k \in -\mathbb{Z}_+$.

The $\hat{\mathfrak{g}}$-module structure on $V(\ell, 0)$ can also be obtained explicitly as follows: For $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we define the action of $a(n)$ on $V(\ell, 0)$ by

$$a(n)(a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}) = a(n)a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}$$

when $n < 0$ for $a_1, \ldots, a_k \in \mathfrak{g}$ and $n_1, \ldots, n_k \in \mathbb{Z}_+$,

$$\begin{aligned} &a(n)(a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}) \\ &\quad = \sum_{i=1}^{k} a_1(-n_1) \cdots a_{i-1}(-n_{i-1})[a(n), a_i(-n_i)]a_{i+1} \cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)} \\ &\quad = \sum_{i=1}^{k} a_1(-n_1) \cdots a_{i-1}(-n_{i-1})([a, a_i](n - n_i) + n(a, a_i)\delta_{n-n_i,0}\ell)a_{i+1} \cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)} \end{aligned}$$

when $n \geq 0$ and

$$\mathbf{k}(a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}) = \ell a_1(-n_1) \cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}.$$

Then it is easy to verify that $V(\ell, 0)$ with this action of $\hat{\mathfrak{g}}$ is a $\hat{\mathfrak{g}}$-module.

Let $a(x) = \sum_{n \in \mathbb{Z}} a(n)x^{-n-1}$ for $a \in \mathfrak{g}$.

Proposition 2.8.1. *For $a, b \in \mathfrak{h}$, we have*

$$\begin{aligned} &[a(x_1), b(x_2)] \\ &\quad = [a, b](x_2)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) - \ell(a, b)\frac{\partial}{\partial x_1}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) \\ &\quad = [a, b](x_2)\left((x_1 - x_2)^{-1} - (-x_2 + x_1)^{-1}\right) + \ell(a, b)\left((x_1 - x_2)^{-2} - (-x_2 + x_1)^{-2}\right). \end{aligned} \tag{2.8.111}$$

Proof. Using the commutator formula for the actions of $\hat{\mathfrak{g}}$, we have

$$\begin{aligned} &[a(x_1), b(x_2)] \\ &\quad = \sum_{m,n \in \mathbb{Z}} [a(m), b(n)]x_1^{-m-1}x_2^{-n-1} \\ &\quad = \sum_{m,n \in \mathbb{Z}} ([a, b](m + n) + m(a, b)\delta_{m+n,0}\ell)\, x_1^{-m-1}x_2^{-n-1} \\ &\quad = \sum_{m,n \in \mathbb{Z}} [a, b](m + n)x_1^{-m-1}x_2^{-n-1} + \sum_{m \in \mathbb{Z}} m(a, b)\ell x_1^{-m-1}x_2^{m-1} \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{k\in\mathbb{Z}}[a,b](k)x_2^{-k-1}\right)\left(\sum_{m\in\mathbb{Z}}x_1^{-m-1}x_2^m\right) - (a,b)\ell\frac{\partial}{\partial x_1}\sum_{m\in\mathbb{Z}}x_1^{-m}x_2^{m-1}\\
&= [a,b](x_2)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) - \ell(a,b)\frac{\partial}{\partial x_1}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\\
&= [a,b](x_2)\left((x_1-x_2)^{-1}-(-x_2+x_1)^{-1}\right) + \ell(a,b)\left((x_1-x_2)^{-2}-(-x_2+x_1)^{-2}\right).
\end{aligned}
$$

□

We define the weight of the element $a_1(n_1)\cdots a_k(n_k)\mathbf{1}_{V(\ell,0)}$ to be $-n_1+\cdots-n_k$. Then we obtain a $\mathbb{Z}$-grading

$$V(\ell,0)=\coprod_{n\in\mathbb{Z}}V_{(n)}(\ell,0),$$

where for $n\in\mathbb{Z}$, $V_{(n)}(\ell,0)$ is the subspace of $V(\ell,0)$ spanned by elements of weight n. In fact, $V_{(n)}(\ell,0)=0$ for $n<0$ and $\dim V_{(n)}(\ell,0)<\infty$ for $n\in\mathbb{Z}$. So

Proposition 2.8.2. *For $n<0$, $V_{(n)}(\ell,0)=0$, $V_{(0)}(\ell,0)=\mathbb{C}\mathbf{1}_{V(\ell,0)}$, and $\dim V_{(n)}(\ell,0)<\infty$.*

Proof. By defintion, $V_{(n)}(\ell,0)=0$ for $n<0$, $\mathbf{1}_{V(\ell,0)}\in V_{(0)}(\ell,0)$ and it is also clear that $V_{(0)}(\ell,0)$ is one-dimensional.

Let $n>0$, and let $\{a_1,\ldots,a_r\}$ be a basis for $\mathfrak{g}$. Then $\{a_i(-m)\mid i,m\in\mathbb{Z},1\le i\le r,m>0\}$ is a basis for $\hat{\mathfrak{g}}_-$ and in particular, $V_{(n)}(\ell,0)$ is spanned by

$$\{a_{i_1}(-n_1)\cdots a_{i_k}(-n_k)\mathbf{1}_{V(\ell,0)}\mid 1\le i_1\cdots\le i_l\le r,n_1,\ldots,n_k>0,n_1+\cdots+n_k=n\},$$

which is a finite set since the positive integer n has only finitely many partitions. □

Let $L_{(V\ell,0)}(0)$ be the grading operator on $V(\ell,0)$. We also define $L_{V(\ell,0)}(-1)$ on $V(\ell,0)$ as follows: Define a linear map $D_{\hat{\mathfrak{g}}}:\hat{\mathfrak{g}}\to\hat{\mathfrak{g}}$ by $D_{\hat{\mathfrak{g}}}\mathbf{k}=0$ and $D_{\hat{\mathfrak{g}}}(a\otimes t^n)=-n(a\otimes t^{n-1})$. Then this defines a derivation on $\hat{\mathfrak{g}}$. This derivation can be extended to a derivation $D_{U(\hat{\mathfrak{g}})}$ on $U(\hat{\mathfrak{g}})$. From the definition of $D_{U(\hat{\mathfrak{g}})}$, we see that the image of $U(\hat{\mathfrak{g}}_-)$ under $D_{U(\hat{\mathfrak{g}})}$ is in $U(\hat{\mathfrak{g}}_-)$. So the restriction of $D_{U(\hat{\mathfrak{g}})}$ to $U(\hat{\mathfrak{g}}_-)$ is a derivation on $U(\hat{\mathfrak{g}}_-)$. Using the isomorphism between $V(\ell,0)$ and $U(\hat{\mathfrak{g}}_-)$, this allows us to define $L_{V(\ell,0)}(-1)$ as a linear map on V by

$$L_{V(\ell,0)}(-1)a_1(n_1)\cdots a_k(n_k)\mathbf{1}=(D_{U(\hat{\mathfrak{g}}_-)}(a_1\otimes t^{n_1})\cdots(a_k\otimes t^{n_k}))\mathbf{1}_{V(\ell,0)}$$

for $a_1,\ldots,a_k\in\mathfrak{g}$ and $n_1,\ldots,n_k\in\mathbb{Z}_+$. More explicitly, we have

$$L_{V(\ell,0)}(-1)\mathbf{1}_{V(\ell,0)}=0$$

and

$$
\begin{aligned}
&L_{V(\ell,0)}(-1)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}\\
&\quad=\sum_{i=1}^{k}n_ia_1(-n_1)\cdots a_{i-1}(-n_{i-1})a_i(-n_i-1)a_{i+1}(-n_{i+1})\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}
\end{aligned}
$$

Proposition 2.8.3. *The series $a(x)$ for $a \in \mathfrak{g}$, $\mathbf{1}_{V(\ell,0)}$ and the operators $L_{V(\ell,0)}(0)$ and $L_{V(\ell,0)}(-1)$ have the following properties:*

1. *For $a \in \mathfrak{g}$, $[L_{V(\ell,0)}(0), a(x)] = x\frac{d}{dx}a(x) + a(x)$.*

2. *$L_{V(\ell,0)}(-1)\mathbf{1}_{V(\ell,0)} = 0$, $[L_{V(\ell,0)}(-1), a(x)] = \frac{d}{dx}a(x)$ for $a \in \mathfrak{g}$.*

3. *For $a \in \mathfrak{g}$, $a(x)\mathbf{1}_{V(\ell,0)} \in V(\ell,0)[[x]]$. Moreover, $\lim_{x\to 0} a(x)\mathbf{1}_{V(\ell,0)} = a(-1)\mathbf{1}_{V(\ell,0)}$.*

4. *The vector space $V(\ell,0)$ is spanned by elements of the form*
$$a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}$$
for $a_1, \dots, a_k \in \mathfrak{g}$ and $n_1, \dots, n_k \in \mathbb{Z}_+$.

5. *For $a, b \in \mathfrak{h}$,*
$$(x_1 - x_2)^2 a(x_1)b(x_2) = (x_1 - x_2)^2 b(x_2)a(x_1).$$

Proof. For $n \in \mathbb{Z}$, from the definition of $L_{V(\ell,0)}(0)$ and the commutator formula for $\hat{\mathfrak{g}}$, it is clear that $a(n)$ is an operator of weight $-n$, that is,
$$[L_{V(\ell,0)}(0), a(n)] = -na(n).$$

Then
$$\begin{aligned}
[L_{V(\ell,0)}(0), a(x)] &= [L_{V(\ell,0)}(0), \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}] \\
&= \sum_{n\in\mathbb{Z}} [L_{V(\ell,0)}(0), a(n)]x^{-n-1} \\
&= \sum_{n\in\mathbb{Z}} (-n)a(n)x^{-n-1} \\
&= \sum_{n\in\mathbb{Z}} (-n-1)a(n)x^{-n-1} + \sum_{n\in\mathbb{Z}} a(n)x^{-n-1} \\
&= x\frac{d}{dx}a(x) + a(x).
\end{aligned}$$

For $n \in \mathbb{Z}$, $a_1, \dots, a_k \in \mathfrak{g}$ and $n_1, \dots, n_k \in \mathbb{Z}_+$, using the fact that $L_{V(\ell,0)}(-1)$ is obtained from the restriction $D_{U(\hat{\mathfrak{g}}_-)}$ to $U(\hat{\mathfrak{g}}_-)$ of the derivation $D_{U(\hat{\mathfrak{g}})}$ on $U(\hat{\mathfrak{g}})$, we have

$$\begin{aligned}
&[L_{V(\ell,0)}(-1), a(n)]a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)} \\
&\quad = L_{V(\ell,0)}(-1)a(n)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)} - a(n)L_{V(\ell,0)}(-1)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)} \\
&\quad = (D_{U(\hat{\mathfrak{g}}_-)}(a\otimes t^n)(a_1\otimes t^{n_1})\cdots(a_k\otimes t^{n_k}))\mathbf{1}_{V(\ell,0)} \\
&\qquad - a(n)L_{V(\ell,0)}(-1)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)} \\
&\quad = (D_{U(\hat{\mathfrak{g}})}(a\otimes t^n)(a_1\otimes t^{n_1})\cdots(a_k\otimes t^{n_k}))\mathbf{1}_{V(\ell,0)}
\end{aligned}$$

$$
\begin{aligned}
&\quad - a(n)L_{V(\ell,0)}(-1)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}\\
&= -na(n-1)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)} + a(n)(D_{U(\hat{\mathfrak{g}})}(a_1\otimes t^{n_1})\cdots(a_k\otimes t^{n_k}))\mathbf{1}_{V(\ell,0)}\\
&\quad - a(n)L_{V(\ell,0)}(-1)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}\\
&= -na(n-1)a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}.
\end{aligned}
$$

Then $[L_{V(\ell,0)}(-1), a(n)] = -na(n-1)$. Thus

$$
\begin{aligned}
[L_{V(\ell,0)}(-1), a(x)] &= \left[L_{V(\ell,0)}(-1), \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}\right]\\
&= \sum_{n\in\mathbb{Z}}[L_{V(\ell,0)}(-1), a(n)]x^{-n-1}\\
&= \sum_{n\in\mathbb{Z}}(-n)a(n-1)x^{-n-1}\\
&= \frac{d}{dx}a(x).
\end{aligned}
$$

Since $a(n)\mathbf{1}_{V(\ell,0)} = 0$ for $a \in \hat{\mathfrak{g}}$ and $n \in -\mathbb{N}$,

$$
a(x)\mathbf{1}_{V(\ell,0)} = \sum_{n\in\mathbb{Z}} a(n)\mathbf{1}_{V(\ell,0)}x^{-n-1} = \sum_{n\in\mathbb{Z}_+} a(n)\mathbf{1}_{V(\ell,0)}x^{-n-1}.
$$

In particular, $\lim_{x\to 0} a(x)\mathbf{1}_{V(\ell,0)} = a(-1)\mathbf{1}_{V(\ell,0)}$.

Using (2.8.111), we have

$$
\begin{aligned}
(x_1-x_2)^2[a(x_1), b(x_2] &= (x_1-x_2)^2[a,b](x_2)\left((x_1-x_2)^{-1} - (-x_2+x_1)^{-1}\right)\\
&\quad + (x_1-x_2)^2\ell(a,b)\left((x_1-x_2)^{-2} - (-x_2+x_1)^{-2}\right)\\
&= [a,b](x_2)((x_1-x_2) - (-x_2+x_1)) + \ell(a,b)(1-1)\\
&= 0.
\end{aligned}
$$

□

From Proposition 2.8.3 and Theorem 2.5.7, we obatin:

Theorem 2.8.4. *The vector space $V(\ell,0)$ equipped with the the vertex operator map $Y_{V(\ell,0)}$ defined by*

$$
\begin{aligned}
&\langle v', Y_{V(\ell,0)}(a_1(-n_1)\cdots a_k(-n_k)\mathbf{1}_{V(\ell,0)}, z)v\rangle\\
&\quad = \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{-n_1}\cdots\xi_k^{-n_k}\xi_{k+1}^{-1}R(\langle v', a_1(\xi_1+z)\cdots a_k(\xi_k+z)v\rangle).
\end{aligned}
\tag{2.8.112}
$$

and the vacuum $\mathbf{1}_{V(\ell,0)}$ is a grading-restricted vertex algebra. Moreover, this is the unique grading-restricted vertex algebra structure on $V(\ell,0)$ with the vacuum $\mathbf{1}_{V(\ell,0)}$ such that $Y(a(-1)\mathbf{1}, x) = a(x)$ for $a \in \mathfrak{g}$.

2.8.2 Conformal elememts of $V(\ell, 0)$ for $\ell \neq -h^\vee$

We now give a conformal element of $V(\ell, 0)$. Then $V(\ell, 0)$ in fact has a structure of a vertex operator algebra. But the conformal element exists only when some assmptions are satisfied. The first assumption is that the invariant bilinear form on $\hat{\mathfrak{g}}$ is positive definite (for example, in the case that $\hat{\mathfrak{g}}$ is semisimple and the form is obtained from the Killing form).

In this case, there exists an orthonormal basis $\{u^i\}_{i=1}^{\dim \mathfrak{g}}$ for $\mathfrak{g}$ with respect to the form $(\cdot, \cdot)$. Let

$$\Omega = \sum_{i=1}^{\dim \mathfrak{g}} u^i u^i \in U(\mathfrak{g})$$

be the Casimir element of $\mathfrak{g}$. (In Definition C.17, we have introduced a Casimir element associated to a representation of a finite-dimensional Lie algebra.)

We also assume that Ω acts on $\mathfrak{g}$ by a scalar $2h^\vee$, where $h^\vee \in \mathbb{C}$. This assumption is satisfied if $\mathfrak{g}$ is a simple Lie algebra. In fact, $\mathfrak{g}$ with the adjoint action is a faithful module of $\mathfrak{g}$. By Proposition C.19, Ω acting on $\mathfrak{g}$ commutes with $\mathrm{ad}\, a$ for every $a \in \mathfrak{g}$. Since $\mathfrak{g}$ is simple, Ω must act as a scalar, which we denote by $2h^\vee$. (See [Hum], Section 6.) The number $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. We further assume $\ell + h^\vee \neq 0$.

Let

$$\omega_{V(\ell,0)} = \frac{1}{2(\ell + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} u^i(-1) u^i(-1) \mathbf{1}_{V(\ell,0)} = \frac{1}{2(\ell + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} u^i(-1)^2 \mathbf{1}_{V(\ell,0)} \in V_{(2)}(\ell, 0).$$

Theorem 2.8.5. *The element $\omega_{V(\ell,0)}$ is a conformal element of $V(\ell, 0)$ such that the central charge for the corresponding Virasoro operators is $\dfrac{\ell \dim \mathfrak{g}}{\ell + h^\vee}$. In particular, $V(\ell, 0)$ is a vertex operator algebra with the conformal element $\omega_{V(\ell,0)}$ and central charge $\dfrac{\ell \dim \mathfrak{g}}{\ell + h^\vee}$.*

Proof. We need to calculate $Y_{V(\ell,0)}(\omega, x)\omega$. We first calculate

$$Y_{V(\ell,0)}(u^i(-1)^2 \mathbf{1}, x)\omega$$

for $i, j = 1, \ldots, \dim \mathfrak{g}$. By the definition of $Y_{V(\ell,0)}$, we have

$$u^i(-1)^2 \mathbf{1} = \mathrm{Res}_{x_0} x_0^{-1} u^i(x_0) u^i(-1)\mathbf{1} = \mathrm{Res}_{x_0} x_0^{-1} Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_0) u^i(-1)\mathbf{1}.$$

Using this and the first equality in (2.4.30), we have

$$\begin{aligned}
&Y_{V(\ell,0)}(u^i(-1)^2 \mathbf{1}, x_2) \\
&\quad = \mathrm{Res}_{x_0} x_0^{-1} Y_{V(\ell,0)}(Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_0) u^i(-1)\mathbf{1}, x_2) \\
&\quad = \mathrm{Res}_{x_0} x_0^{-1} \mathrm{Res}_{x_1} x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_1) Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_2) \\
&\quad\quad - \mathrm{Res}_{x_0} x_0^{-1} \mathrm{Res}_{x_1} x_0^{-1} \delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_2) Y_{V(\ell,0)}(u^i(-1)\mathbf{1}, x_1)
\end{aligned}$$

$$= \mathrm{Res}_{x_1}(x_1-x_2)^{-1}u^i(x_1)u^i(x_2) - \mathrm{Res}_{x_1}(-x_2+x_1)^{-1}u^i(x_2)u^i(x_1)$$
$$= \left(\sum_{m\in\mathbb{N}} u^i(-m-1)x_2^m\right)u^i(x_2) + u^i(x_2)\left(\sum_{m\in-\mathbb{Z}_+} u^i(-m-1)x_2^m\right). \tag{2.8.113}$$

Using $\omega \in V_{(2)}$, $V_{(n)}(\ell,0) = 0$ for $n < 0$ and $u^i(p)u^i(q) = u^i(q)u^i(p)$ for $p \neq -q$ and $i = 1, \ldots, \dim \mathfrak{g}$, we obtain

$$\begin{aligned}
&Y_{V(\ell,0)}(u^i(-1)^2\mathbf{1}, x_2)\omega \\
&\quad= \left(\sum_{m\in\mathbb{N}} u^i(-m-1)x_2^m\right)u^i(x_2)\omega + u^i(x_2)\left(\sum_{m\in-\mathbb{Z}_+} u^i(-m-1)x_2^m\right)\omega \\
&\quad= \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} u^i(-m-1)u^i(n)\omega x_2^{m-n-1} + \sum_{m\in-\mathbb{Z}_+}\sum_{n\in\mathbb{Z}} u^i(n)u^i(-m-1)\omega x_2^{m-n-1} \\
&\quad= \sum_{0\le m\le n\le 2} u^i(-m-1)u^i(n)\omega x_2^{m-n-1} + \sum_{-3\le m\le n\le m+3\le 2} u^i(n)u^i(-m-1)\omega x_2^{m-n-1} \\
&\qquad + F_i(x_2) \\
&\quad= \sum_{m=0}^{2} u^i(-m-1)u^i(2)\omega x_2^{m-3} + \sum_{m=0}^{1} u^i(-m-1)u^i(1)\omega x_2^{m-2} + u^i(-1)u^i(0)\omega x_2^{-1} \\
&\qquad + \sum_{n=-3}^{0} u^i(n)u^i(2)\omega x_2^{-n-4} + \sum_{n=-2}^{1} u^i(n)u^i(1)\omega x_2^{-n-3} + \sum_{n=-1}^{2} u^i(n)u^i(0)\omega x_2^{-n-2} + F_i(x_2) \\
&\quad= 2u^i(-3)u^i(2)\omega x_2^{-1} + 2u^i(-2)u^i(1)\omega x_2^{-1} + 2u^i(-1)u^i(0)\omega x_2^{-1} + 2u^i(-2)u^i(2)\omega x_2^{-2} \\
&\qquad + 2u^i(-1)u^i(1)\omega x_2^{-2} + u^i(0)^2\omega x_2^{-2} + 2u^i(-1)u^i(2)\omega x_2^{-3} + 2u^i(0)u^i(1)\omega x_2^{-3} \\
&\qquad + 2u^i(0)u^i(2)\omega x_2^{-4} + u^i(1)u^i(1)\omega x_2^{-4} + F_i(x_2),
\end{aligned} \tag{2.8.114}$$

where $F_i(x_2) \in V(\ell,0)[[x_2]]$ for $i = 1, \ldots, \dim \mathfrak{g}$.

Using the commutator formula for the affine Lie algebra operators, the formulas

$$\begin{aligned}
[u^i, u^i] = [u^j, u^j] &= [[u^i, u^j], [u^i, u^j]] = 0, \\
(u^i, u^j) &= \delta_{ij}. \\
(u^i, [u^i, u^j]) &= ([u^i, u^i].u^j) = 0, \\
([u^i, u^j], u^j) &= (u^i, [u^j, u^j]) = 0, \\
([u^i, [u^i, u^j]], u^j) &= -([u^i, u^j], [u^i, u^j]),
\end{aligned}$$

the fact that $\{u^i\}_{i=1}^{\dim \mathfrak{g}}$ is an orthonormal basis and the invariance of the bilinear form $(\cdot,\cdot)$, we have

$$u^i(2)u^j(-1)^2\mathbf{1} = 0, \tag{2.8.115}$$

$$u^i(1)u^j(-1)^2\mathbf{1} = [[u^i,u^j],u^j](-1)\mathbf{1} + 2\ell\delta_{ij}u^j(-1)\mathbf{1}, \tag{2.8.116}$$

$$\begin{aligned} u^i(0)u^j(-1)^2\mathbf{1} &= [u^i,u^j](-1)u^j(-1)\mathbf{1} + u^j(-1)[u^i,u^j](-1)\mathbf{1} \\ &= \sum_{k=1}^{\dim\mathfrak{g}}([u^i,u^j],u^k)u^k(-1)u^j(-1)\mathbf{1} + \sum_{k=1}^{\dim\mathfrak{g}}([u^i,u^j],u^k)u^j(-1)u^k(-1)\mathbf{1} \\ &= \sum_{k=1}^{\dim\mathfrak{g}}(u^i,[u^j,u^k])u^k(-1)u^j(-1)\mathbf{1} + \sum_{k=1}^{\dim\mathfrak{g}}(u^i,[u^j,u^k])u^j(-1)u^k(-1)\mathbf{1}. \end{aligned} \tag{2.8.117}$$

Using the defintion of ω,

$$\sum_{i=1}^{\dim\mathfrak{g}}[[u^j,u^i],u^i] = \sum_{i=1}^{\dim\mathfrak{g}}[u^i,[u^i,u^j]] = \Omega u^j = 2h^\vee u^j,$$
$$\sum_{j=1}^{\dim\mathfrak{g}}[[u^i,u^j],u^j] = \sum_{j=1}^{\dim\mathfrak{g}}[u^j,[u^j,u^i]] = \Omega u^i = 2h^\vee u^i$$

and (2.8.115)– (2.8.117), we obtain

$$u^i(2)\omega = 0, \tag{2.8.118}$$

$$u^i(1)\omega = \frac{1}{2(\ell+h^\vee)}\sum_{j=1}^{\dim\mathfrak{g}}([[u^i,u^j],u^j](-1)\mathbf{1} + 2\ell\delta_{ij}u^i(-1)\mathbf{1}) = u^i(-1)\mathbf{1}, \tag{2.8.119}$$

$$\begin{aligned} u^i(0)\omega &= \frac{1}{2(\ell+h^\vee)}\sum_{j=1}^{\dim\mathfrak{g}}\sum_{k=1}^{\dim\mathfrak{g}}\big((u^i,[u^j,u^k])u^k(-1)u^j(-1)\mathbf{1} + (u^i,[u^j,u^k])u^j(-1)u^k(-1)\mathbf{1}\big) \\ &= \frac{1}{2(\ell+h^\vee)}\sum_{j=1}^{\dim\mathfrak{g}}\sum_{k=1}^{\dim\mathfrak{g}}\big((u^i,[u^k,u^j])u^j(-1)u^k(-1)\mathbf{1} + (u^i,[u^j,u^k])u^j(-1)u^k(-1)\mathbf{1}\big) \\ &= 0. \end{aligned} \tag{2.8.120}$$

From (2.8.118)–(2.8.120), we obtain

$$u^i(0)^2\omega = 0, \tag{2.8.121}$$
$$u^i(0)u^i(1)\omega = 0, \tag{2.8.122}$$
$$u^i(1)u^i(1)\omega = \ell\mathbf{1}. \tag{2.8.123}$$

Substituting (2.8.115)–(2.8.123) into the right-hand side of (2.8.114), summing over i and using the formula

$$L_{V(\ell,0)}(-1)u^i(-1)^2\mathbf{1} = u^i(-2)u^i(-1) + u^i(-1)u^i(-2)\mathbf{1} = 2u^i(-2)u^i(-1)\mathbf{1},$$

we obtain

$$\begin{aligned}
&\sum_{i=1}^{\dim \mathfrak{g}} Y_{V(\ell,0)}(u^i(-1)^2\mathbf{1}, x_2)\omega \\
&\quad = \sum_{i=1}^{\dim \mathfrak{g}} 2u^i(-2)u^i(-1)x_2^{-1} + \sum_{i=1}^{\dim \mathfrak{g}} 2u^i(-1)u^i(-1)\mathbf{1}x_2^{-2} + \sum_{i=1}^{\dim \mathfrak{g}} \ell\mathbf{1}x_2^{-4} + \sum_{i=1}^{\dim \mathfrak{g}} F_i(x_2) \\
&\quad = 2(\ell+h^\vee)L_{V(\ell,0)}(-1)\omega x_2^{-1} + 4(\ell+h^\vee)\omega x_2^{-2} + \ell \dim \mathfrak{g}\mathbf{1}x_2^{-4} + \sum_{i=1}^{\dim \mathfrak{g}} F_i(x_2). \qquad (2.8.124)
\end{aligned}$$

Dividing both sides of (2.8.124) by $2(\ell+h^\vee)$ and let $G(x_2) = \frac{1}{2(\ell+h^\vee)}\sum_{i=1}^{\dim \mathfrak{g}} F_i(x_2),$, we obtain

$$Y_{V(\ell,0)}(\omega, x_2)\omega = L_{V(\ell,0)}(-1)\omega x_2^{-1} + 2\omega x_2^{-2} + \frac{c}{2}\mathbf{1}x_2^{-4} + G(x_2),$$

where

$$c = \frac{\ell \dim \mathfrak{g}}{\ell + h^\vee}.$$

Using the commutator formula for $Y_{V(\ell,0)}$, we have

$$\begin{aligned}
&[a(x_1), Y_{V(\ell,0)}(\omega, x_2)] \\
&\quad = [Y_{V(\ell,0)}(a(-1)\mathbf{1}, x_1), Y_{V(\ell,0)}(\omega, x_2)] \\
&\quad = \mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_{V(\ell,0)}(Y_{V(\ell,0)}(a(-1)\mathbf{1}, x_0)\omega, x_2) \\
&\quad = \sum_{n\in\mathbb{Z}} \mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_{V(\ell,0)}(a(n)\omega, x_2)x_0^{-n-1} \qquad (2.8.125)
\end{aligned}$$

for $a \in \mathfrak{g}$. From (2.8.115)–(2.8.117) and the fact that $\{u^i\}_{i\in I}$ is a basis of $\mathfrak{g}$, by writing $a \in \mathfrak{g}$ as a linear combination of u^i for $i \in I$, we obtain

$$a(2)u^j(-1)^2\mathbf{1} = 0, \qquad (2.8.126)$$

$$a(1)u^j(-1)^2\mathbf{1} = [[a, u^j], u^j](-1)\mathbf{1} + 2\ell(a, u^j)u^i(-1)\mathbf{1}, \qquad (2.8.127)$$

$$a(0)u^j(-1)^2\mathbf{1} = \sum_{k=1}^{\dim \mathfrak{g}}(a, [u^j, u^k])u^k(-1)u^j(-1)\mathbf{1} + \sum_{k=1}^{\dim \mathfrak{g}}(a, [u^j, u^k])u^j(-1)u^k(-1)\mathbf{1}. \qquad (2.8.128)$$

From (2.8.126)–(2.8.128), we obtain

$$a(2)\omega = 0, \qquad (2.8.129)$$

$$a(1)\omega = a(-1)\mathbf{1}, \qquad (2.8.130)$$

$$a(0)\omega = 0. \qquad (2.8.131)$$

Using (2.8.129)–(2.8.131) and the formal Taylor's theorem, we see that the right-hand side of (2.8.125) becomes

$$\begin{aligned}
&\mathrm{Res}_{x_0} x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) Y_{V(\ell,0)}(a(-1)\mathbf{1}, x_2)x_0^{-2} \\
&\quad = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{N}} \mathrm{Res}_{x_0} \frac{x_0^k}{k!}\frac{d^k}{dx_2^k} x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) a(n)x_0^{-2}x_2^{-n-1} \\
&\quad = \sum_{n\in\mathbb{Z}} \frac{d}{dx_2} x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) a(n)x_2^{-n-1}
\end{aligned} \tag{2.8.132}$$

Taking the coefficient of $x_1^{-m-1}x_2^{-2}$ for $m \in \mathbb{Z}$ on the left-hand sides of (2.8.125) and the right-hand side of (2.8.125), we obtain

$$[a(m), \mathrm{Res}_{x_2} x_2 Y_{V(\ell,0)}(\omega, x_2)] = ma(m). \tag{2.8.133}$$

Taking the coefficient of $x_1^{-m-1}x_2^{-1}$ for $m \in \mathbb{Z}$ on the left-hand sides of (2.8.125) and the right-hand side of (2.8.125), we obtain

$$[a(m), \mathrm{Res}_{x_2} Y_{V(\ell,0)}(\omega, x_2)] = ma(m-1). \tag{2.8.134}$$

Since $Y_{V(\ell,0)}(\omega, x_2)\mathbf{1} \in V(\ell, 0)[[x_2]]$,

$$\mathrm{Res}_{x_2} x_2 Y_{V(\ell,0)}(\omega, x_2)\mathbf{1} = \mathrm{Res}_{x_2} Y_{V(\ell,0)}(\omega, x_2)\mathbf{1} = 0. \tag{2.8.135}$$

From (2.8.133)–(2.8.135) and the definitions of $L_{V(\ell,0)}(0)$ and $L_{V(\ell,0)}(-1)$, we obtain

$$\begin{aligned}
[a(m), \mathrm{Res}_{x_2} x_2 Y_{V(\ell,0)}(\omega, x_2)] &= [a(m), L_{V(\ell,0)}(0)], \\
[a(m), \mathrm{Res}_{x_2} Y_{V(\ell,0)}(\omega, x_2)] &= [a(m), L_{V(\ell,0)}(-1)], \\
\mathrm{Res}_{x_2} x_2 Y_{V(\ell,0)}(\omega, x_2)\mathbf{1} &= L_{V(\ell,0)}(0)\mathbf{1}, \\
\mathrm{Res}_{x_2} Y_{V(\ell,0)}(\omega, x_2)\mathbf{1} &= L_{V(\ell,0)}(-1)\mathbf{1}.
\end{aligned}$$

From these formulas, we obtain

$$\begin{aligned}
L_{V(\ell,0)}(0) &= \mathrm{Res}_{x_2} x_2 Y_{V(\ell,0)}(\omega, x_2), \\
L_{V(\ell,0)}(-1) &= \mathrm{Res}_{x_2} Y_{V(\ell,0)}(\omega, x_2).
\end{aligned}$$

Thus ω is a conformal element and $V(\ell, 0)$ is a vertex operator algebra. □

2.8.3 The vertex operator algebra $L(\ell, 0)$ (for Wess-Zumino-Witten models)

Even for simple $\mathfrak{g}$ and $\ell \in \mathbb{Z}_+$, the vertex operator algebra $V(\ell, 0)$ is in fact not the vertex algebra for the Wess-Zumino-Witten model associated. We need to take an irreducible quotient of $V(\ell, 0)$.

Let $I(\ell, 0)$ be the maximal proper submodule of the $\hat{\mathfrak{g}}$-module $V(\ell, 0)$. In fact, it is easy to see that $I(\ell, 0)$ exists: Consider all $\hat{\hat{\mathfrak{g}}}$-submodules of $V(\ell, 0)$ that do not contain $\mathbf{1}$. In particular, homogeneous elements of these $\hat{\hat{\mathfrak{g}}}$-submodules have weights greater than 0. Take $I(\ell, 0)$ to be the sum of all such $\hat{\hat{\mathfrak{g}}}$-submodules. Then $\mathbf{1} \notin I(\ell, 0)$ since $\mathbf{1}$ has weight 0. $I(\ell, 0)$ is maximal. Let $L(\ell, 0) = V(\ell, 0)/I(\ell, 0)$. Then as a $\hat{\hat{\mathfrak{g}}}$-module, $L(\ell, 0)$ is irreducible, that is, there is no $\hat{\hat{\mathfrak{g}}}$-submodule of $L(\ell, 0)$ that is not 0 or $L(\ell, 0)$ itself.

We take the vacuum of $L(\ell, 0)$ to be the equivalent class of the vacuum of $V(\ell, 0)$. We define the vertex operator map $Y_{L(\ell,0)} : L(\ell, 0) \otimes L(\ell, 0) \to L(\ell, 0)((x))$ by

$$Y_{L(\ell,0)}(u + I(\ell, 0), x)(v + I(\ell, 0) = Y_{V(\ell,0)}(u, x)v + I(\ell, 0).$$

The vacuum $\mathbf{1}_{L(\ell,0)}$ is defined to $\mathbf{1}_{V(\ell,0)} + I(\ell, 0)$ and in the case $\ell + h^\vee \neq 0$, the conformal element $\omega_{L(\ell,0)}$ is defined to be $\omega_{V(\ell,0)} + I(\ell, 0)$.

Theorem 2.8.6. *The graded vector space $L(\ell, 0)$ equipped with $Y_{L(\ell,0)}$ and $\mathbf{1}$ is a grading-restricted vertex algebra. When $\ell + h^\vee \neq 0$, $L(\ell, 0)$ equipped with $Y_{L(\ell,0)}$, $\mathbf{1}_{L(\ell,0)}$ and $\omega_{L(\ell,0)}$ is a vertex operator algebra.*

Proof. We need only verify that $Y_{L(\ell,0)}$ is well defined; all the axioms can be verified using the properties of $V(\ell, 0)$. To prove that $Y_{L(\ell,0)}$ is well defined, we need only show that $Y_{V(\ell,0)}(u, x)v \in I(\ell, 0)((x))$ when one of u and v is in $I(\ell, 0)$. Since $I(\ell, 0)$ is a $\hat{\mathfrak{g}}$-submodule of $V(\ell, 0)$, we have $a(x)v \in I(\ell, 0)((x))$ for $a \in \mathfrak{g}$ and $v \in I(\ell, 0)$. Then by the definition of the vertex operator map $Y_{V(\ell,0)}$ (see (2.5.49), we have

$$\begin{aligned}\langle v', &Y_{V(\ell,0)}(a_1(m_1) \cdots a_k(m_k)\mathbf{1}, z)v\rangle \\ &= \mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_k=0} \xi_1^{m_1} \cdots \xi_k^{m_k} R(\langle v', a_1(\xi_1 + z) \cdots a_k(\xi_k + z)v\rangle) \qquad (2.8.136)\end{aligned}$$

for $a_1, \dots, a_k \in \mathfrak{g}$, $m_1, \dots, m_k \in \mathbb{Z}$, $v \in V(\ell, 0)$ and $v' \in V(\ell, 0)'$. To prove that $Y_{V(\ell,0)}(u, x)v \in I(\ell, 0)((x))$ for $u \in V(\ell, 0)$ and $v \in I(\ell, 0)$, we need only prove $Y_{V(\ell,0)}(a_1(m_1) \cdots a_k(m_k)\mathbf{1}, z)v \in I(\ell, 0)((x))$ for $a_1, \dots, a_k \in \mathfrak{g}$, $m_1, \dots, m_k \in \mathbb{Z}$ and $v \in I(\ell, 0)$. Let $I(\ell, 0)^0$ be the annihilator of $I(\ell, 0)$, that is, the subspace of $V(\ell, 0)^0$ containing all linear functionals v' on $V(\ell, 0)$ such that $\langle v', v\rangle = 0$ for all $v \in I(\ell, 0)$. Then $Y_{V(\ell,0)}(a_1(m_1) \cdots a_k(m_k)\mathbf{1}, z)v \in I(\ell, 0)((x))$ if and only if $\langle v', Y_{V(\ell,0)}(a_1(m_1) \cdots a_k(m_k)\mathbf{1}, z)v\rangle = 0$ for all $v' \in I(\ell, 0)^0$. From (2.8.136) and $a(x)v \in I(\ell, 0)((x))$ for $a \in \mathfrak{g}$ and $v \in I(\ell, 0)$, we see that the right-hand side of (2.8.136) is 0 for all $v' \in I(\ell, 0)^0$. Thus the left-hand side of (2.8.136) is also 0 for all $v' \in I(\ell, 0)^0$.

We also need to prove $Y_{V(\ell,0)}(u, x)v \in I(\ell, 0)((x))$ for $u \in I(\ell, 0)$ and $v \in V(\ell, 0)$. Since $L(-1)$ can be expresed as a linear combination of products of $a(n)$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$ and $I(\ell, 0)$ is a $\mathfrak{g}$-submodule of $V(\ell, 0)$, we have $L(-1)I(\ell, 0)v \in I(\ell, 0)$ for $v \in I(\ell, 0)$. Then by the skew-symmetry (2.4.28),

$$Y_{V(\ell,0)}(u, x)v = e^{xL(-1)}Y_{V(\ell,0)}(v, -x)u \in I(\ell, 0)((x)).$$

□

The vertex operator algebra underlying the Wess-Zumino-Witten model associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$ and a level $\ell \in \mathbb{Z}_+$ is exactly $L(\ell, 0)$. In this case, there is an explicit formula $I(\ell, 0) = U(\hat{\mathfrak{g}})e_\theta(-1)^{\ell+1}\mathbf{1}$, where θ is the highest root of $\mathfrak{g}$ and e_θ is a root vector in $\mathfrak{g}_\theta$ (see [K] and [LL]).

2.9 Examples: Virasoro vertex operator algebras

In this section, we give the constructions of the grading-restricted vertex algebras and vertex operator algebras associated to the Virasoro algebra.

2.9.1 The vertex operator algebras $V(c, 0)$

We have seen the Virasoro operators above. Here we introduced the Virasoro algebra abstractly. The Virasoro algebra is the graded vector space $\mathrm{Vir} = \coprod_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}\mathbf{c}$ with the bracket operation given by

$$\begin{aligned}[L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\mathbf{c}, \\ [\mathbf{c}, L_n] &= 0\end{aligned}$$

for $m, n \in \mathbb{Z}$. The center element $\mathbf{c}$ usually acts as a fixed number c on a Vir-module. This number is called the central charge of the module. In the preceding section, we have shown that $V(\ell, 0)$ is a Vir-module with the central charge $c = \frac{\ell \dim \mathfrak{g}}{\ell + h^\vee}$. The Virasoro algebra Vir has a triangle decomposition: $\mathrm{Vir} = \mathrm{Vir}_+ \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_-$, where $\mathrm{Vir}_\pm = \coprod_{n\in\pm\mathbb{Z}_+} \mathbb{C}L_n$ and $\mathrm{Vir}_0 = \mathbb{C}L_0 \oplus \mathbb{C}\mathbf{c}$ are subalgebras of Vir.

Fix $c \in \mathbb{C}$. Consider the one dimensional vector space $\mathbb{C}w_{c,0}$ with a basis $w_{c,0}$. For any $n \in \mathbb{Z}$, we shall use $L(n)$ to denote the action of L_n on a Vir-module or a module for a subalgebra of Vir containing the element L_n. We give $\mathbb{C}w_{c,0}$ a structure of $\mathrm{Vir}_+ \oplus \mathrm{Vir}_0$-module by defining $L(n)w_{c,0} = 0$ for $n \geq 0$ and$\mathbf{c}w_{c,0} = cw_{c,0}$. Note that $L(0)w_{c,0} = 0w_{c,0}$, which means that $w_{c,0}$ is an eigenvector of L_0 with eigenvalue 0. This is the reason why we use the notation $w_{c,0}$. We have the induced Vir-module $M(c, 0) = U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}_+ \oplus \mathrm{Vir}_0)} \mathbb{C}w_{c,0}$, which by the Poincaré-Birkhoff-Witt theorem is linearly isomorphic to $U(\mathrm{Vir}_-) \otimes w_{c,0}$. We shall identify $M(c, 0)$ with $U(\mathrm{Vir}_-) \otimes w_{c,0}$ so that the Vir-module structure on $M(c, 0)$ give a Vir-module structure on $U(\mathrm{Vir}_-) \otimes w_{c,0}$. Then $M(c, h)$ is spanned by elements of the form $L(-n_1) \cdots L(-n_k)w_{c,0}$ for $n_1, \ldots, n_k \in \mathbb{Z}_+$ and the action of Vir on $M(c, h)$ is given explicitly by

$$L(-n)(L(-n_1) \cdots L(-n_k)w_{c,0}) = L(-n)L(-n_1) \cdots L(-n_k)w_{c,0}$$

for $n \in \mathbb{Z}_+$ and

$$\begin{aligned}&L(n)(L(-n_1) \cdots L(-n_k)w_{c,0}) \\ &\quad= \sum_{i=1}^{k} L(-n_1) \cdots L(-n_{i-1})[L(n), L(-n_i)]L(-n_{i+1}) \cdots L(-n_k)w_{c,0} \\ &\quad\quad + L(-n_1) \cdots L(-n_k)L(n)w_{c,0} \\ &\quad= \sum_{i=1}^{k}(n + n_i)L(-n_1) \cdots L(-n_{i-1})L(n - n_i)L(-n_{i+1}) \cdots L(-n_k)w_{c,0} \\ &\quad\quad + \sum_{i=1}^{k} \frac{c}{12}(n^3 - n)\delta_{n-n_i,0}L(-n_1) \cdots L(-n_{i-1})L(-n_{i+1}) \cdots L(-n_k)w_{c,0}\end{aligned}$$

$$+ \delta_{n,0} h L(-n_1) \cdots L(-n_k) w_{c,0}$$

for $n \in \mathbb{N}$. Moreover, $M(c,0)$ has a basis consisting of elements of the form $L(-n_1) \cdots L(-n_k) w_{c,0}$ for $n_1 \geq \cdots \geq n_k \geq 1$.

Note that $L(-1)w_{c,0} \neq 0$ in $M(c,0)$. To construct a vertex operator algebra from $M(c,0)$, we need to take a quotient of $M(c,0)$ to make sure that $L(-1)$ acting on the vacumm is 0. Let $V(c,0) = M(c,0)/U(\mathrm{Vir})L(-1)w_{c,0}$. Then $V(c,0)$ is a nonzero Vir-module since $U(\mathrm{Vir})L(-1)w_{c,0} \neq M(c,0)$. Let $\mathbf{1}_c = w_{c,0} + U(\mathrm{Vir})L(-1)w_{c,0}$. Then using the Virasoro braket relations, we see that $V(c,0)$ is spanned by elements of the form $L(-n_1) \cdots L(-n_k)\mathbf{1}$ for $n_1, \ldots, n_k \in \mathbb{Z}_+ + 1$. We now give $V(c,0)$ a vertex operator algebra structure.

Define the weight of $L(-n_1) \cdots L(-n_k)\mathbf{1}$ to be $n_1 + \cdots + n_k$. Then $V(c,0) = \coprod_{n \in \mathbb{N}} V_{(n)}(c,0)$, where $V_{(n)}(c,0)$ is the subspace of $V(c,0)$ consisting of elements of weight n. The vacuum for $V(c,0)$ is $\mathbf{1}_c$ and the conformal vector is $\omega_c = L(-2)\mathbf{1}_c$. Let $T(x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2}$. This is the generating field for $V(c,0)$.

Proposition 2.9.1. *We have*

$$[T(x_1), T(x_2)] = \left(\frac{d}{dx_2} T(x_2)\right) x_2^{-1} \delta\left(\frac{x_1}{x_2}\right) - 2T(x_2) \frac{\partial}{\partial x_1} x_2^{-1} \delta\left(\frac{x_1}{x_2}\right) - \frac{c}{12} \frac{\partial^3}{\partial x_1^3} x_2^{-1} \delta\left(\frac{x_1}{x_2}\right).$$

Proof. By the definition of $T(x)$ and the Virasoro commutator relations, we have

$$\begin{aligned}
&[T(x_1), T(x_2)] \\
&\quad = \sum_{m,n \in \mathbb{Z}} [L(m), L(n)] x_1^{-m-2} x_2^{-n-2} \\
&\quad = \sum_{m,n \in \mathbb{Z}} (m-n) L(m+n) x_1^{-m-2} x_2^{-n-2} + \frac{c}{12} \sum_{m,n \in \mathbb{Z}} (m^3 - m) \delta_{m+n,0} x_1^{-m-2} x_2^{-n-2} \\
&\quad = \sum_{m,k \in \mathbb{Z}} (2m-k) L(k) x_1^{-m-2} x_2^{-k+m-2} + \frac{c}{12} \sum_{m \in \mathbb{Z}} (m^3 - m) x_1^{-m-2} x_2^{m-2} \\
&\quad = \sum_{m,k \in \mathbb{Z}} (-k-2) L(k) x_2^{-k-3} x_2^{-1} \left(\frac{x_1}{x_2}\right)^{-m-2} - 2 \sum_{m,k \in \mathbb{Z}} L(k) x_2^{-k-2} x_2^{-1} (-m-1) x_1^{-m-2} x_2^{m+1} \\
&\qquad - \frac{c}{12} \sum_{m \in \mathbb{Z}} (-m+1)(-m)(-m-1) x_1^{-m-2} x_2^{m-2} \\
&\quad = \left(\frac{d}{dx_2} T(x_2)\right) x_2^{-1} \delta\left(\frac{x_1}{x_2}\right) - 2T(x_2) \frac{\partial}{\partial x_1} x_2^{-1} \delta\left(\frac{x_1}{x_2}\right) - \frac{c}{12} \frac{\partial^3}{\partial x_1^3} x_2^{-1} \delta\left(\frac{x_1}{x_2}\right)
\end{aligned}$$

□

Proposition 2.9.2. *The space $V(c,0)$, the series $T(x)$, the vacuum $\mathbf{1}_c$ and the Virasoro operators $L(n)$ for $n \in \mathbb{Z}$ have the following properties:*

1. *For $a \in \mathfrak{h}$, $[L_(0), T(x)] = x \frac{d}{dx} T(x) + T(x)$.*

2. $L(-1)\mathbf{1}_c = 0$, $[L(-1), T(x)] = \frac{d}{dx}T(x)$.

3. $T(x)\mathbf{1}_c \in V(c,0)[[x]]$. *Moreover,* $\lim_{x\to 0} T(x)\mathbf{1}_c = L(-2)\mathbf{1}_c = \omega_c$.

4. *The vector space* $V(c,0)$ *is spanned by elements of the form*

$$L(-n_1)\cdots L(-n_k)\mathbf{1}_c$$

for $n_1, \ldots, n_k \in \mathbb{Z}_+$.

5. *For* $a, b \in \mathfrak{h}$,

$$(x_1 - x_2)^4 T(x_1)T(x_2) = (x_1 - x_2)^4 T(x_2)T(x_1).$$

Proof. Properties 1–4 are either obvious or can be verified easily. Property 5 follows from Proposition 2.9.1. □

We are ready to define a vertex operator map

$$\begin{aligned} Y_{V(c,0)} : V(c,0) \otimes V(c,0) &\to V(c,0)[[x, x^{-1}]] \\ u \otimes v &\mapsto Y_{V(c,0)}(u, x)v. \end{aligned}$$

Note that $V(c,0)$ has a basis

$$\{L(-n_1)\cdots L(-n_k)\mathbf{1}_c \mid n_1, \ldots, n_k \in \mathbb{Z}_+ + 1, n_1 \geq \cdots \geq n_k\}.$$

So we need only define the vertex operator $Y_{V(c,0)}(L(-n_1)\cdots L(-n_k)\mathbf{1}_c, x)$.

By Proposition 2.5.2, we know that for $v \in V(c,0)$, $v' \in V(c,0)'$, the series $\langle v', T(z_1)\cdots T(z_k)v\rangle$ is absolutely convergent on the region $|z_1| > \cdots > |z_k| > 0$ to a rational function $R(\langle v', T(z_1)\cdots T(z_k)v\rangle)$ in the variables $z_1, \ldots, z_k$ with the only poles at $z_i = 0$ and $z_j = z_l$ for $j \neq l$. Using these rational functions, we define $Y_{V(c,0)}(L(-n_1)\cdots L(-n_k)\mathbf{1}_c, z)$ for any $z \in \mathbb{C}^\times$ by

$$\begin{aligned} \langle v', &Y_{V(c,0)}(L(-n_1)\cdots L(-n_k)\mathbf{1}_c, z)v\rangle \\ &= \mathrm{Res}_{\xi_1=0}\cdots \mathrm{Res}_{\xi_k=0}\xi_1^{-n_1}\cdots \xi_k^{-n_k} R(\langle v', T(\xi_1 + z)\cdots T(\xi_k + z)v\rangle). \end{aligned} \tag{2.9.137}$$

Theorem 2.9.3. *The* $\mathbb{Z}$*-graded vector space* $V(c,0)$ *equipped with the the vertex operator map* $Y_{V(c,0)}$ *defined by* (2.9.137) *for* $n_1, \ldots, n_k \in \mathbb{Z}_+ + 1$, $v \in V(c,0)$ *and* $v' \in V(c,0)'$, *the vacuum* $\mathbf{1}_c$ *and the conformal element* $\omega_c = L(-2)\mathbf{1}_c$ *is a vertex operator algebra. Moreover, this is the unique vertex operator algebra structure on* $V(c,0)$ *with the vacuum* $\mathbf{1}_c$ *and conformal element* ω_c *such that* $Y(L(-2)\mathbf{1}_c, z) = T(x)$.

Proof. From Proposition 2.9.2, we see that the conditions to apply Theorem 2.5.7 in are satisfied. Then by Theorem 2.5.7, we see that $(V(c,0), Y_{V(0,c)}, \mathbf{1}_c)$ is a grading-restricted vertex algebra and it is the unique grading-restricted vertex algebra structure on $V(c,0)$ with the vacuum $\mathbf{1}_c$ such that $Y(L(-2)\mathbf{1}_c, z) = T(x)$. It is clear that the axioms for the conformal element ω_c holds. In particular, $(V(c,0), Y_{V(0,c)}, \mathbf{1}_c, \omega_c)$ is a vertex operator algebra with the uniqueness stated in the theorem. □

2.9.2 The Virasoro vertex operator algebras $L(c_{p,q}, 0)$ (for minimal models)

When $c = c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$ for coprime $p, q \in \mathbb{Z}_+ + 1$, the associated conformal field theories are the minimal models. The vertex operator algebras for the minimal models are not $V(c_{p,q}, 0)$. As in the case of Wess-Zumino-Novikov-Witten modules, we have to take an irreducible quotient of $V(c_{p,q}, 0)$, which are also an irreducible quotient of $M(c_{p,q}, 0)$.

To obtain an irreducible quotient of $M(c_{p,q}, 0)$, we need to give a maximal proper Vir-submodule of $M(c_{p,q}, 0)$. In fact, there is only one maximal proper Vir-submodule of $M(c_{p,q}, 0)$. Consider the set of all proper Vir-submodules of $M(c_{p,q}, 0)$ that do not contain $w_{c_{p,q},0}$. This set is nonempty because we have at least the Vir-submodule $U(\mathrm{Vir})L(-1)w_{c_{p,q},0}$ generated by $L(-1)w_{c_{p,q},0}$. Take the sum of all proper Vir-submodules in this set. Then this sum also does not contain $w_{c_{p,q},0}$ and is thus the unique maximal proper Vir-submodule $I(c_{p,q}, 0)$ of $M(c_{p,q}, 0)$.

Let $L(c_{p,q}, 0) = M(c_{p,q}, 0)/I(c_{p,q}, 0)$. Then as a Vir-module, $L(c_{p,q}, 0)$ is irreducible. Since $I(c_{p,q}, 0)$ is maximal, it must contain $U(\mathrm{Vir})L(-1)w_{c_{p,q},0}$. Hnece $L(c_{p,q}, 0)$ must also be a quotient of $V(c_{p,q}, 0)$. In particular, there exists a proper Vir-submodule $J(c_{p,q}, 0)$ of $V(c_{p,q}, 0)$ such that $L(c_{p,q}, 0) = V(c_{p,q}, 0)/J(c_{p,q}, 0)$. Let $\mathbf{1}_{L(c_{p,q},0)} = \mathbf{1}_{c_{p,q}} + J(c_{p,q}, 0) \in L(c_{p,q}, 0)$ and $\omega_{L(c_{p,q},0)} = L(-2)\mathbf{1}_{c_{p,q}} + J(c_{p,q}, 0) \in L(c_{p,q}, 0)$. Since $\mathbf{1}_{c_{p,q}}$ generates $V(c_{p,q}, 0)$, $J(c_{p,q}, 0)$ cannot contain $\mathbf{1}_{c_{p,q}}$. So $\mathbf{1}_{L(c_{p,q},0)} \neq 0$. We also need to show that $L(-2)w_{c_{p,q},0}$ is not in $I(c_{p,q}, 0)$. If $L(-2)w_{c_{p,q},0} \in J(c_{p,q}, 0)$, then

$$L(2)L(-2)\mathbf{1}_{c_{p,q}} = L(-2)L(2)\mathbf{1}_{c_{p,q}} + \frac{c_{p,q}}{6}\mathbf{1}_{c_{p,q}} = \frac{c_{p,q}}{6}\mathbf{1}_{c_{p,q}}.$$

Since $c_{p,q} \neq 0$, $\mathbf{1}_{c_{p,q}}$ must also in $J(c_{p,q}, 0)$ and thus $J(c_{p,q}, 0) = V(c_{p,q}, 0)$. Contradiction. Thus $L(-2)\mathbf{1}_{c_{p,q}}$ is not in $J(c_{p,q}, 0)$ and consequently, $\omega_{L(c_{p,q},0)} \neq 0$. For $u, v \in V(c_{p,q}, 0)$, we define

$$Y_{L(c_{p,q},0)}((u + J(c_{p,q}, 0)), x)(v + J(c_{p,q}, 0)) = Y_{V(c_{p,q},0)}(u, x)v + J(c_{p,q}, 0).$$

This is well defined since for $u \in V(c_{p,q}, 0)$ and $v \in J(c_{p,q}, 0)$ or $u \in J(c_{p,q}, 0)$ and $v \in V(c_{p,q}, 0)$, $Y_{V(c_{p,q},0)}(u, x)v \in J(c_{p,q}, 0)$. In particular, we have

$$Y_{L(c_{p,q},0)}(\omega_{L(c_{p,q},0)}, x)(v + J(c_{p,q}, 0)) = T(x)v + J(c_{p,q}, 0)[[x, x^{-1}]] \tag{2.9.138}$$

for $v \in V(c_{p,q}, 0)$. Then we the following result:

Theorem 2.9.4. *The $\mathbb{Z}$-graded vector space $L(c_{p,q}, 0)$ equipped with the the vertex operator map $Y_{L(c_{p,q},0)}$, the vacuum $\mathbf{1}_{L(c_{p,q},0)}$ and the conformal element $\omega_{L(c_{p,q},0)}$ is a vertex operator algebra. Moreover, this is the unique vertex operator algebra structure on $L(c_{p,q}, 0)$ with the vacuum $\mathbf{1}_{L(c_{p,q},0)}$ and conformal element $\omega_{L(c_{p,q},0)}$ such that* (2.9.138) *holds.*