Two-dimensional conformal field theory

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Chapter 1

Axioms for two-dimensional conformal field theories

In this chapter, we give Segal's definition of conformal field theory (see [S1], [S2] and [S3]) using the language of PROPS and algebras over PROPs. A similar definition was also proposed by Kontsevich around the same time (see the review of the paper [S1] in Mathematical Reviews). We also give Segal's definitions of modular functors and weakly conformal field theories (see [S2] and [S3]).

1.1 Definition of conformal field theory

Consider the following geometric category $\mathcal{B}$: Objects of $\mathcal{B}$ are finite sets (including the empty set) of copies of the unit circle S^1. Given two objects, morphisms from one object to another are conformal equivalence classes of Riemann surfaces (including degenerate ones, e.g., circles, and possibly disconnected) with oriented, ordered and parametrized boundary components such that the copies of S^1 in the domain and codomain parametrize the negatively oriented and positively oriented boundary components, respectively. See Figure 1.1. Note that the morphism is not the Riemann surface in the picture itself but instead the conformal equivalence class of the Riemann surface.

For simplicity, we shall call a Riemann surface with the additional data in the description of morphisms above a rigged Riemann surface. For an object containing n copies of the unit circle, the identity on it is the degenerate surface given by the n unit circles with the trivial riggings of the boundary components. Given two composable morphisms, that is, two rigged Riemann surfaces Σ_1 and Σ_2 such that the codomain of Σ_1 is the same as the domain of Σ_2, we can compose them by identifying the positively oriented boundary components of Σ_1 with the negatively oriented boundary components of Σ_2. Then the composition or sewing operation satisfies the associativity and any morphism composed with an identity is equal to itself. Thus we indeed have a category.

We recall the following notion of PROP (see [MacL1] and the exposition [Mak]):

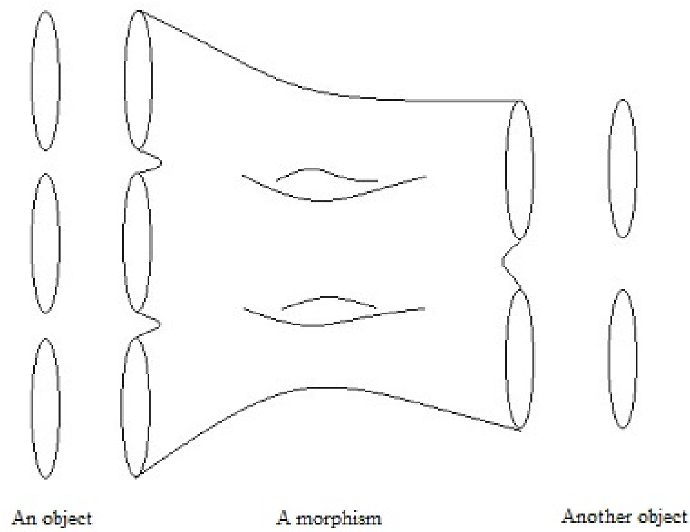

Figure 1.1: Two objects and a morphism between them

Definition 1.1.1. A *PROP* is a symmetric strict monoidal category where every object is of the form $A^{\otimes n}$ for a single object A and $n \in \mathbb{N}$. The object A is called the *generaor of the PROP*.

This category has a symmetric strict monoidal category structure where the tensor product bifunctor is defined by disjoint unions of objects and morphisms. Since the unit circle is an object of $\mathcal{B}$ and objects of $\mathcal{B}$ are all tensor powers of the unit circle, we have:

Proposition 1.1.2. *The category $\mathcal{B}$ has a structure of PROP.*

We shall use $[\Sigma]$ to denote the morphism in the category $\mathcal{B}$ containing a rigged Riemann surface Σ.

We consider the category $\mathcal{H}$ of Hilbert spaces over $\mathbb{C}$ with trace-class maps as morphisms. Here trace class maps are continuous maps for which we can take traces. There is a tensor product bifunctor $\otimes$ such that $\mathcal{H}$ becomes a symmetric tensor category. The category $\mathcal{H}$ induces a projective category $\mathcal{P}(\mathcal{H})$ whose objects are projective spaces of Hilbert spaces and whose morphisms are those maps induces from trace class maps between Hilbert spaces. Then $\mathcal{P}(\mathcal{H})$ is also a symmetric tensor category.

We need the following natural notion of an algebra over a PROP in a symmetric monoidal category (see for example [Mak] in the case that the symmetric monoidal category is the category of modules over a commutative unital ring):

Definition 1.1.3. Let $\mathcal{P}$ be a PROP and $\mathcal{C}$ a symmetric monoidal category. An algebra over $\mathcal{P}$ in $\mathcal{C}$ is a symmetric monoidal functor $\psi : \mathcal{P} \to \mathcal{C}$. Let A be the generator of $\mathcal{P}$. The image $\psi(A)$ is called the *object* (or *space* when $\mathcal{C}$ is a category of vector spaces) of the algebra.

If the space of an algebra over a PROP $\mathcal{P}$ given by a functor ψ is H, we shall denote the algebra by (H, ψ). We might often omit ψ to simply call H an algebra over $\mathcal{P}$.

Let Σ be a rigged Riemann surface such that $[\Sigma]$ is a morphism in $\mathcal{B}$ from m copies of S^1 to n copies of S^1. Then by identifying the i-th incoming boundary component of Σ parametrized S^1 with the j-th outgoing boundary component of Σ parametrized by S^1 in the codomain of $[\Sigma]$, we obtain a rigged Riemann surface $[\Sigma_{\widehat{i,j}}]$. (Note that the two copies of S^1 identified might or might not be on a same connected component of Σ.) See Figure 1.2 in the case $i = 4$ and $j = 3$.

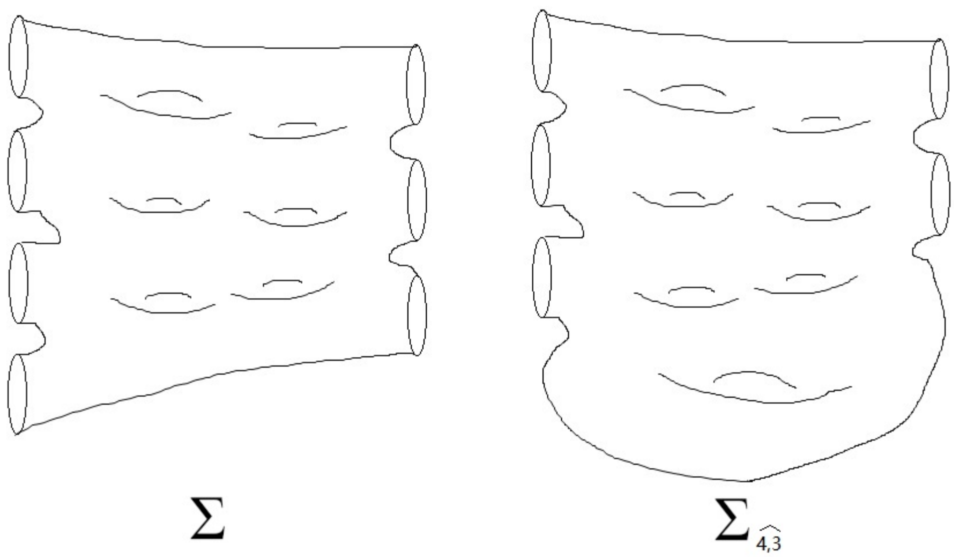

Figure 1.2: Sewing the 4-th incoming boundary component of a riggeed Riemann surface Σ with the 3-rd outgoing boundary component of Σ to obtain $\Sigma_{\Sigma_{\widehat{4,3}}}$

Let Σ be as above. By changing the i-th incoming boundary component of the Σ to the $n+1$-st outgoing boundary component the same Riemann surface, we obtain a rigged Riemann surface $\Sigma_{i \to n+1}$ with $m-1$ ordered incoming boundary conponenents and $n+1$ ordered outgoing boundary componenents.

We now give a definition of two-dimensional conformal field theory due to Segal (see [S1]–[S3] and the review of the paper [S1] in Mathematical Reviews for a similar proposal by Kontsevich):

Definition 1.1.4. A *two-dimensional conformal field theory* (or simply *conformal field theory* when it is clear that it is two dimensional) is a Hilbert space H and a functor ϕ from $\mathcal{B}$ to $\mathcal{P}(\mathcal{H})$ satisfying the following conditions for morphisms fo the form $[\Sigma]$ from m ordered copies of S^1 to n ordered copies of S^1:

1. The trace between the i-th tensor factor of the domain and the j-th tensor factor of the codomain of $\phi([\Sigma])$ exists and is equal to $\phi([\Sigma_{\widehat{i,j}}])$.

2. The maps $\phi([\Sigma])$ and $\phi([\Sigma_{i\to n+1}])$ are related by the map from $\mathrm{Hom}(P(H)^{\otimes m}), P(H)^{\otimes n}))$ to $\mathrm{Hom}(P(H)^{\otimes m-1}), P(H)^{\otimes (n+1)})$ obtained using the map $H \to H^*$ corresponding to the bilinear form $(\cdot, \cdot)$.

We shall denote a conformal field theory defined in Definition 1.1.4 by (H, ϕ) or simply by H.

Let Σ be as above and $\overline{\Sigma}$ the rigged Riemann surface obtained by taking the complex conjugate complex structure of the one on $[\Sigma]$ and takeing the incoming and outgoing boundary components of Σ to be the outgoing and incoming blundary components of $\overline{\Sigma}$ with the same orders (note that the orientations of the boundary components are reversed). The correspondence $\Sigma \to \overline{\Sigma}$ is a functor from the category of rigged Riemann surfaces to it self.

Definition 1.1.5. A *two-dimensional real conformal field theory* (or a *real conformal field theory* is a (two-dimensional) conformal field theory (H, ϕ) together with an anti-linear involution θ from H to itself such that $\phi([\overline{\Sigma}]) = P(\theta)^{\otimes m} \circ \phi^*([\Sigma]) \circ (P(\theta)^{-1})^{\otimes n}$ where $\phi^*([\Sigma])$ is the adjoint of $\Phi([\Sigma])$ and $P(\theta)$ is the map from $P(H)$ to itself induced from θ.

Remark 1.1.6. In the definitions above, we do not discuss which classes of parametrizations of the boundary componenets of Riemanssurfaces we are considering. The parametrizations that are easiest to work with are analytic parametrizations. But limits of sequences of analytic parametrizations are in general not analytic. So they are not the correct class of parametrizations. Segal in [S2] chose smooth parametrizations. But the space smooth parametrizations do not have all the good properties that we need. The correct classes of parametrizations are the quasi-symmetric parametrizations given by Radnell and Schippers [RS1]-[RS3] and Radnell, Schippers and Staubach [RSS1]-[RSS5].

Remark 1.1.7. The definitions above define a conformal field theory using a Hilbert space H over $\mathbb{C}$ and the projective space of H. In fact, we can replace H in the definition by a complete nuclear space over $\mathbb{C}$ eith a nodegenerate bilinear form. Also we can use the the space H and its tensor power in the definition instead of the projective space of H. But then we have to determinant lines and canonical isomorphisms between them. For simplicity, we omit the discussions on these subtle issues here.

1.2 Definition of modular functor and the corresponding PROP

The definition of conformal field theory in the preceding section do not reveal many important properties that physicists and mathematicians discovered. In particular, they do not give the detailed structures of chiral and anti-chiral parts of conformal field theories, that is, parts of conformal field theories depending on the moduli space parameters analytically and anti-analytically. We will see in the next chapter that meromorphic fields in a conformal field theory form a vertex operator algebra. The representations of this vertex operator algebra form the chiral parts of the theory. Therefore to construct conformal field theories from vertex operator algebras, it is necessary to study first chiral and anti-chiral parts of conformal field theories. Axiomatically, chiral and anti-chiral parts of conformal field theories are weakly conformal field theories introduced by G. Segal in [S2] and [S3] and are generalizations of conformal field theories defined in Section 1.1.

To define weakly conformal field theories, we first need to define modular functors, which was also introduced by Segal in [S2] and [S3]. We need to consider rigged Riemann surfaces with additional labels on their boundary components by a set. Let $\mathcal{A}$ be a set. An *$\mathcal{A}$-labeled and rigged Riemann surfaces* (or simply a *labeled and rigged Riemann surfaces* when the set $\mathcal{A}$ of "labels" is clear) is a rigged Riemann surfaces with the boundary components labeled by elements of $\mathcal{A}$. We call the boundary component labeled by an element of $\mathcal{A}$ an *$\mathcal{A}$-labeled boundary component* (or simply a *labeled boundary component*). We consider a category whose objects are conformal equivalence classes of $\mathcal{A}$-labeled and rigged Riemann surfaces and whose morphisms are given by the sewing operation, that is, if one such equivalent class can be obtained from another using the sewing operation, then the procedure of obtaining the second surface from the first one is a morphism. We also use $[\Sigma]$ to denote the conformal equivalence class of a surface Σ.

Definition 1.2.1. Let $\mathcal{A}$ be a set. A *modular functor with the labeling set $\mathcal{A}$* (or simply a *modular functor* when the set $\mathcal{A}$ is clear) is a functor E from the category of $\mathcal{A}$-labeled and rigged Riemann surfaces to the category of finite-dimensional vector spaces over $\mathbb{C}$ satisfying the following conditions:

1. $E([\Sigma])$ depends on Σ holomorphically.

2. $E([\Sigma_1 \sqcup \Sigma_2])$ is naturally isomorphic to $E([\Sigma_1]) \otimes E([\Sigma_2])$.

3. If Σ is obtained from another surface Σ_a by sewing two boundary components with opposite orientations (meaning one incoming and one outgoing) but with the same label $a \in \mathcal{A}$ of Σ_a, then $E([\Sigma])$ is naturally isomorphic to $\oplus_{b\in\mathcal{A}} E([\Sigma_b])$ where for $b \neq a$, Σ_b is the surface obtained from Σ_a by changing the label a to b on the boundary components to be sewn.

4. $E([S_a])$ is canonically isomorphic to $\mathbb{C}$, where S_a is the degenerate $\mathcal{A}$-labeled and rigged Riemann surface given by the unit circle with one incoming boundary component and one outgoing boundary component (both are the same as the degenerate surface itself, the obvious parametrizations and the labels by $a \in \mathcal{A}$.

The simplest nontrivial example of a modular functor is given by the determinant line bundles.

From a modular functor E, we can construct a symmetric strict monoidal category $\mathcal{B}_E$ extending the category $\mathcal{B}$ as follows: We call a copy of S^1 together with an element of $\mathcal{A}$ a labeled S^1. Objects of $\mathcal{B}_E$ are finite sets (including the empty set) of labeled S^1. A morphism of $\mathcal{B}_E$ is a pair consisting of an equivalence class $[\Sigma]$ of $\mathcal{A}$-labeled and rigged Riemann surfaces (including the degenerate ones) and an element λ of the vector space $E([\Sigma])$, such that each labeled boundary component of Σ matches with an labeled S^1 in the domain or codomain depending on whether the boundary component is incoming or outgoing, respectively. The identity on an object is given by the obvious degenerate $\mathcal{A}$-labeled and rigged Riemann surface S (given by the same number of copies of S^1 as the object) with the obvious labeling by elements of $\mathcal{A}$ determined by the object and the element in $E(S)$ corresponding to $1 \in \mathbb{C}$ (since $E(S)$ is canonically isomorphic to $\mathbb{C}$) . Let $([\Sigma_1], \lambda_1)$ and $([\Sigma_2], \lambda_2)$ be two composable morphisms. Let Σ be the $\mathcal{A}$-labeled and rigged Riemann surface obtained by sewing Σ_1 and Σ_2. The sewing procedure to obtain Σ from Σ_1 and Σ_2 is a morphism of the category of $\mathcal{A}$-labeled and rigged Riemann surfaces above. Since E is a functor, applying E to this morphism, we have a linear map from $E([\Sigma_1 \sqcup \Sigma_2])$ to $E([\Sigma])$. Since $E([\Sigma_1 \sqcup \Sigma_2])$ is naturally isomorphic to $E([\Sigma_1]) \otimes E([\Sigma_2])$, we also have a linear map from $E([\Sigma_1]) \otimes E([\Sigma_2])$ to $E([\Sigma])$. We define the composition of $([\Sigma_1], \lambda_1)$ and $([\Sigma_2], \lambda_2)$ is $([\Sigma], \lambda)$, where Σ, as discussed above, is the $\mathcal{A}$-labeled and rigged Riemann surface obtained by sewing Σ_1 and Σ_2 and λ is the image of $\lambda_1 \otimes \lambda_2 \in E([\Sigma_1]) \otimes E([\Sigma_2])$ under the linear map from $E([\Sigma_1 \sqcup \Sigma_2])$ to $E([\Sigma])$. Then the composition clearly satisfies the associativity and any morphism composed with an identity is equal to itself. In particular, we obtain a category $\det_{\mathcal{B}}^{c} \otimes \overline{\det}_{\mathcal{B}}^{\bar{c}}$.

The category $\mathcal{B}_E$ is also a symmetric strict monoidal category: The monoid or tensor product bifunctor is defined by disjoint unions of objects, the disjoint union of the rigged surface part of the morphisms and the tensor products of the elements in the corresponding finite-dimensional vector spaces.

Proposition 1.2.2. *The category $\mathcal{B}_E$ has a structure of PROP.*

1.3 Definition of weakly conformal field theory

Let Σ be an $\mathcal{A}$-labeled and rigged Riemann surface such that $[\Sigma]$ is a morphism in $\mathcal{B}$ from m copies of S^1 to n copies of S^1. Assuming that the labels at the i-th incoming boundary component and the j-th outgoing boundary component of Σ are the same. Then by identifying the i-th incoming boundary component of Σ with the j-th outgoing boundary component of Σ in the codomain of $[\Sigma]$ using the parametrizations of these boundary components, we obtain a $\mathcal{A}$-labeled and rigged Riemann surface $[\Sigma_{\widehat{i,j}}]$.

Since from Σ to $\Sigma_{\widehat{i,j}}$ is an sewing procedure, it is a morphism in the category of $\mathcal{A}$-labeled and rigged Riemann surfaces. By the definition of modular functor, we have a linear map $\rho^E_{[\Sigma];i,j} : E([\Sigma]) \to E([\Sigma_{\widehat{i,j}}])$.

Let Σ be as above. By changing the i-th incoming boundary component of the Σ to the $n+1$-st outgoing boundary component the same Riemann surface, we obtain a $\mathcal{A}$-labeled and rigged Riemann surface $\Sigma_{i \to n+1}$ with $m-1$ ordered incoming boundary conponenents and $n+1$ ordered outgoing boundary componenents. Since $E([\Sigma])$ is independent of the orientation of the boundary components of $[\Sigma]$, $E([\Sigma])$ is in fact the same as $E([\Sigma_{i \to n+1}])$. We shall denote the identity map from $E([\Sigma])$ to $E([\Sigma_{i \to n+1}])$ by $\rho_{[\Sigma];i \to n+1}$.

Definition 1.3.1. Let E be a modular functor labeled by $\mathcal{A}$. A *weakly conformal field theory over E* is a set $\{H^a \mid a \in \mathcal{A}\}$ of Hilbert spaces and a functor ϕ from $E_{\mathcal{B}}$ to $\mathcal{H}$ satisfying the following conditions for morphisms in $E_{\mathcal{B}}$ of the form $([\Sigma], \lambda)$, where $[\Sigma]$ is a morphism in $\mathcal{B}$ from m ordered copies of S^1 to n ordered copies of S^1 and $\lambda \in E([\Sigma])$:

1. If the i-th tensor factor of the domain and the j-th tensor factor of the codomain of $\phi([\Sigma], \lambda)$ are labeled by the same element of A, then the trace between the i-th tensor factor of the domain and the j-th tensor factor of the codomain of $\phi([\Sigma], \lambda)$ exists and is equal to $\phi([\Sigma_{\widehat{i,j}}], \rho^E_{[\Sigma];i,j}(\lambda))$.

2. The maps $\Phi([\Sigma], \lambda)$ and $\Phi([\Sigma_{i \to n+1}], \rho_{[\Sigma];i \to n+1}(\lambda))$ are related by the map from

$$\mathrm{Hom}(H^{a_1} \otimes \cdots \otimes H^{a_m}, H^{b_1} \otimes \cdots \otimes H^{b_n})$$

to

$$\mathrm{Hom}(H^{a_1} \otimes \otimes \widehat{H^{a_i}} \otimes \cdots \otimes H^{a_m}, H^{b_1} \otimes \cdots \otimes H^{b_n} \otimes H^{a^i})$$

obtained using the map $H \to H^*$ corresponding to the bilinear form $(\cdot, \cdot)$, where we use $\widehat{H^{a_i}}$ to denote the tensor factor H^{a_i} is missing.