Two constructions of grading-restricted vertex (super)algebras

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Abstract

We give two constructions of grading-restricted vertex (super)algebras. We first give a new construction of a class of grading-restricted vertex (super)algebras originally obtained by Meurman and Primc using a different method. This construction is based on a new definition of vertex operators and a new method. Our second construction is a generalization of the author's construction of the moonshine module vertex operator algebra and a related vertex operator superalgebra. This construction needs properties of intertwining operators formulated and proved by the author.

1 Introduction

Vertex operator (super)algebras are algebraic structures formed by meromorphic fields in two-dimensional (super)conformal field theories. Based on this connection, a program to construct two-dimensional (super)conformal field theories using the representation theory of vertex operator (super)algebras has been successfully carried out during the past 25 years, though there are still many open problems to be solved. For a nontechnical brief description of this program, see the author's blog article [H8]. The construction of vertex operator algebras is a prerequisite for this program.

A vertex operator (super)algebra is a grading-restricted vertex (super)algebra equipped with a conformal element. In this paper, we are interested only in the construction of vertex operators, not the construction of conformal elements. Thus we shall restrict our attention to grading-restricted vertex (super)algebras instead of vertex operator (super)algebras.

The existing constructions of grading-restricted vertex (super)algebras can mostly be divided into three types. The first type includes the constructions of grading-restricted vertex (super)algebras associated to the Heisenberg algebras, Clifford algebras and lattices (see [FLM] and [FFR]). These constructions are based on explicit formulas for vertex operators. The second type includes the constructions of the grading-restricted vertex (super)algebras associated to the Virasoro algebra, affine Lie algebras and superconformal algebras (see [FZ], [KW] and [A]). These constructions are based on a definition of vertex operators using generating fields and the residue of x_1 of the Jacobi identity for vertex (super)algebras. The third type is the so-called orbifold construction, including the constructions of the moonshine module vertex operator algebra and a related vertex operator superalgebra (see [FLM],

[DGM] [H1] and [H2]). These constructions are based on suitable intertwining operators among fixed-point vertex operator subalgebras and modules for the subalgebras.

In this paper, we first give a new construction of a class of grading-restricted vertex (super)algebras. Our construction is based on a new definition of vertex operators using the rational functions obtained from the products of generating fields. This definition is motivated by the associativity for vertex operators. Our construction is also based on a new method. A uniqueness result shows that the class of grading-restricted vertex algebras given by this construction is the same as the class given by Meurman and Primc [MP] using a different construction of the second type.

We also give in this paper a generalization of the author's construction in [H1] of the moonshine module vertex operator algebra. Given a grading-restricted vertex algebra equipped with a compatible $\mathfrak{sl}(2, \mathbb{C})$ action (called quasi-vertex operator algebra in [FHL]) and a non-degenerate symmetric invariant bilinear form and a module equipped with similar additional structures satisfying certain conditions, we construct a quasi-vertex operator algebra or superalgebra structure on the direct sum of the algebra and the module using suitable intertwining operators. Examples of such quasi-vertex operator algebras or superalgebras include, as mentioned above, the moonshine module vertex operator algebra and a related vertex operator superalgebra. The vertex operator superalgebras associated to the Moore-Read states [MR] in the study of quantum Hall states can also be obtained using this construction.

In addition to being useful for obtaining examples of grading-restricted vertex (super)algebras, quasi-vertex operator (super)algebras and vertex operator (super)algebras, these two constructions provide new methods in the study of these algebras and their representations. For example, the author's cohomology theory for grading-restricted vertex algebras [H6] and [H7] are constructed and developed based on the rational functions of the products and iterates of vertex operators and operators corresponding to cochains. Our first construction will be useful in the study of this cohomology theory for those algebras that can be constructed in this way. The (logarithmic) tensor category theory for module categories for vertex operator algebras (or more generally for quasi-vertex operator algebras or Möbius vertex algebras) is based on (logarithmic) intertwining operators (see the expositions [HL] and [HLZ] and the references in these papers for details on this theory). Our second construction uses intertwining operators and is indeed closely related to the tensor category theory. The second construction can be generalized to more complicated orbifold constructions and these generalizations will be studied in future publications.

In the present paper, instead of the formal variable approach, we use the complex analysis approach. Although we do use residues, our method depends mainly on the method of analytic extensions and the properties of rational functions of the special type appearing in the theory of vertex operator algebras. The advantage of our analytic approach is that every definition or proof has a geometric meaning. The geometric meaning is not logically needed in this paper and in many papers studying vertex operator algebras. But it often provides ideas and motivations for many constructions and proofs. One important example is the cohomology theory for grading-restricted vertex algebras [H6] [H7] mentioned above and the author's ongoing study of the representation theory of vertex operator algebras using this cohomology theory. The geometric insights play a crucial role in obtaining new results and developing new methods in this theory.

The present paper is organized as follows: In the next section, we recall the definitions of grading-restricted vertex (super)algebra, module and intertwining operator and some other useful notions. In Sections 3 and 4, we give our first and second constructions, respectively.

2 Grading-restricted vertex (super)algebras

In this section, we recall the definitions of grading-restricted vertex (super)algebra, module and intertwining operator. We also recall some other notions, including the notions of quasi-vertex operator (super)algebra, conformal element, vertex operator (super)algebra, fusion rule and nondegenerate symmetric invariant bilinear form. Except for conformal elements and vertex operator algebras, these are all needed in our constructions.

For a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, as usual, we denote its graded dual space $\coprod_{n\in\mathbb{Z}} V^*_{(n)}$ by V' and its algebraic completion $\prod_{n\in\mathbb{Z}} V_{(n)}$ by $\overline{V}$. On V and V', we use the topology given by the dual pair (V, V'). In the case that $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$, we use the topology on $\overline{V} = (V')^*$ given by the dual pair $(V', \overline{V})$. We give a topology to the space $\mathrm{Hom}(V^{\otimes n}, \overline{V})$ by identifying it with a subspace of the dual space of $V^{\otimes n} \otimes V'$ and then using the topology induced from the dual pair of this subspace and $V^{\otimes n} \otimes V'$. More specifically, a sequence (or more generally a net) $\{f_n\}$ in $\mathrm{Hom}(V\otimes\cdots\otimes V, \overline{V})$ is convergent to $f \in \mathrm{Hom}(V\otimes\cdots\otimes V, \overline{V})$ if for $v_1, \dots, v_n \in V$ and $v' \in V'$, $\langle v', f_n(v_1\otimes\cdots\otimes v_n)\rangle$ is convergent to $\langle v', f(v_1 \otimes \cdots \otimes v_n)\rangle$. In particular, analytic maps from a region in $\mathbb{C}$ to $\mathrm{Hom}(V^{\otimes n}, \overline{V})$ make sense. For a $\mathbb{C}$-graded vector space, we use the same notations and topologies.

We give the definition of grading-restricted vertex (super)algebra first.

Definition 2.1. A *grading-restricted vertex superalgebra* is a $\frac{\mathbb{Z}}{2}$-graded vector space $V = \coprod_{n\in\frac{\mathbb{Z}}{2}} V_{(n)} = V^0 \oplus V^1$, where $V^0 = \coprod_{n\in\mathbb{Z}} V_{(n)}$ and $V^1 = \coprod_{n\in\mathbb{Z}+\frac{1}{2}} V_{(n)}$, equipped with an analytic map

$$\begin{array}{rcl} Y_V : \mathbb{C}^\times & \to & \mathrm{Hom}(V \otimes V, \overline{V}), \\ z & \mapsto & Y_V(\cdot, z)\cdot : u \otimes v \mapsto Y_V(u, z)v \end{array}$$

called *vertex operator map* and a *vacuum* $\mathbf{1} \in V_{(0)}$ satisfying the following axioms:

1. Axioms for the grading: (a) *Grading-restriction condition*: When n is sufficiently negative, $V_{(n)} = 0$ and $\dim V_{(n)} < \infty$ for $n \in \frac{\mathbb{Z}}{2}$. (b) *$L(0)$-bracket formula*: Let $L_V(0) : V \to V$ be defined by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, z)] = \frac{d}{dz}Y_V(v, z) + Y_V(L_V(0)v, z)$$

for $v \in V$.

2. Axioms for the vacuum: (a) *Identity property*: Let 1_V be the identity operator on V. Then $Y_V(\mathbf{1}, z) = 1_V$. (b) *Creation property*: For $u \in V$, $\lim_{z \to 0} Y_V(u, z)\mathbf{1}$ exists and is equal to u.

3. $L(-1)$*-derivative property*: Let $L_V(-1) : V \to V$ be the operator given by

$$L_V(-1)v = \lim_{z\to 0} \frac{d}{dz} Y_V(v, z)\mathbf{1}$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dz} Y_V(v, z) = Y_V(L_V(-1)v, z) = [L_V(-1), Y_V(v, z)].$$

4. *Duality*: For $u_1 \in V^{|u_1|}$, $u_2 \in V^{|u_2|}$, and $v \in V$ where $|u_1|, |u_2|$ are 0 or 1, $v' \in V'$, the series

$$\begin{gathered} \langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle, \\ (-1)^{|u_1||u_2|}\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle, \\ \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle \end{gathered}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

In the case that $V^1 = 0$, the grading-restricted vertex superalgebra just defined is called a *grading-restricted vertex algebra*.

We denote the grading-restricted vertex (super)algebra just defined by $(V, Y_V, \mathbf{1})$ or by V. Note that in the definition above, we use the duality instead of the Jacobi identity or weak commutativity as the main axiom.

Although we are mainly interested in grading-restricted vertex (super)algebras in this paper, in our second construction, we need an operator $L_V(1)$ acting on the algebra V such that together with $L_V(0)$ and $L_V(-1)$, it gives an action of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ on V and satisfies the usual bracket formula between the basis of $\mathfrak{sl}(2, \mathbb{C})$ and vertex operators. Also our main examples all have conformal elements. Because of these reason, we also recall the definitions of quasi-vertex operator (super)algebra, conformal element and vertex operator (super)algebra from [FHL].

Definition 2.2. A *quasi-vertex operator (super)algebra* is a grading-restricted vertex (super)algebra $(V, Y_V, \mathbf{1})$ together with an operator $L_V(1)$ of weight 1 on V satisfying

$$\begin{aligned} [L_V(-1), L_V(1)] &= -2L_V(0), \\ [L_V(1), Y_V(v, z)] &= Y_V(L_V(1)v, z) + 2zY_V(L_V(0)v, z) + z^2Y_V(L_V(-1)v, z) \end{aligned}$$

for $v \in V$.

We denote the quasi-vertex operator (super)algebra just defined by $(V, Y_V, \mathbf{1}, L_V(1))$ or simply by V.

Definition 2.3. Let $(V, Y_V, \mathbf{1})$ be a grading-restricted vertex (super)algebra. A *conformal element* of V is an element $\omega \in V$ satisfying the following axioms:

1. There exists $c \in \mathbb{C}$ such that $Y(\omega, z)\omega$ expanded as a V-valued Laurent series is equal to $L_V(-1)\omega z^{-1} + 2\omega z^{-2} + \frac{c}{2}\mathbf{1}z^{-4}$ plus a V-valued power series in z.
2. $L_V(-1) = \mathrm{Res}_z Y_V(\omega, z)$ and $L_V(0) = \mathrm{Res}_z z Y_V(\omega, z)$ (Res_z being the operation of taking the coefficient of z^{-1} of a Laurent series).

A grading-restricted vertex (super)algebra equipped with a conformal element is called a *vertex operator (super)algebra* (or, more consistently, *grading-restricted conformal vertex (super)algebra*).

We denote the vertex operator (super)algebra just defined by $(V, Y_V, \mathbf{1}, \omega)$ or simply by V.

Remark 2.4. The absolute convergence of

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle$$

and

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

to rational functions in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, in Definition 2.1 are called the *rationality of the products* and the *rationality of the iterates*, respectively. The statement that

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle$$

and

$$(-1)^{|u_1||u_2|}\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle$$

converges in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively, to a common rational function is called the *commutativity*. The statement that

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle$$

and

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

converges in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function is called the *associativity*. In fact, with all the other properties still hold, it is easy to show that the commutativity is equivalent to the *skew-symmetry*: For $u \in V^{|u|}$ and $v \in V^{|v|}$,

$$Y_V(u, z)v = (-1)^{|u||v|}e^{zL_V(-1)}Y_V(v, -z)u.$$

In particular, we can replace the duality in Definition 2.1 by the rationality of the products and iterates, the associativity and the skew-symmetry. We shall need this fact below.

Next we give the definition of module for a grading-restricted vertex (super)algebra.

Definition 2.5. Let V be a grading-restricted vertex superalgebra. A V*-module* is a $\mathbb{C}\times\mathbb{Z}_2$-graded vector space

$$W = \coprod_{n\in\mathbb{C},\alpha\in\mathbb{Z}_2} W^{\alpha}_{(n)} = \coprod_{n\in\mathbb{C}} W_{(n)} = W^0 \oplus W^1$$

(where $W_{(n)} = W^0_{(n)} \oplus W^1_{(n)}$, $W^0 = \coprod_{n\in\mathbb{C}} W^0_{(n)}$ and $W^1 = \coprod_{n\in\mathbb{C}} V^1_{(n)}$) equipped with a *vertex operator map*

$$\begin{aligned} Y_W : \mathbb{C}^\times &\to \mathrm{Hom}(V\otimes W, \overline{W}),\\ z &\mapsto Y_W(\cdot, z)\cdot : u\otimes w \mapsto Y_W(u,z)w \end{aligned}$$

satisfying the following axioms:

1. Axioms for the gradings: (a) *Grading-restriction condition*: When the real part of n is sufficiently negative, $W_{(n)} = 0$ and $\dim W_{(n)} < \infty$ for $n\in\mathbb{C}$. (b) $L(0)$*-bracket formula*: Let $L_W(0) : V \to V$ be defined by $L_W(0)v = nv$ for $v \in W_{(n)}$. Then

$$[L_W(0), Y_W(v,z)] = \frac{d}{dz}Y_W(v,z) + Y_W(L_V(0)v, z)$$

   for $v\in V$. (c) *Grading compatibility*: For $\alpha,\beta\in\mathbb{Z}_2$, $u\in V^\alpha$ and $w\in W^\beta$, $Y_W(u,z)w \in \overline{W^{\alpha+\beta}}$.

2. *Identity property*: Let 1_W be the identity operator on W. Then $Y_W(\mathbf{1}, z) = 1_W$.

3. $L(-1)$*-derivative property*: There exists $L_W(-1) : W \to W$ such that for $u\in V$,

$$\frac{d}{dz}Y_W(u,z) = Y_W(L_V(-1)u, z) = [L_W(-1), Y_W(u,z)].$$

4. *Duality*: For $u_1 \in V^{|u_1|}$, $u_2\in V^{|u_2|}$, and $w\in W$ where $|u_1|, |u_2|$ are 0 or 1, $w'\in W'$, the series

$$\begin{gathered} \langle w', Y_W(u_1,z_1)Y_W(u_2,z_2)w\rangle,\\ (-1)^{|u_1||u_2|}\langle w', Y_W(u_2,z_2)Y_W(u_1,z_1)w\rangle,\\ \langle w', Y_W(Y_V(u_1,z_1-z_2)u_2,z_2)w\rangle \end{gathered}$$

   are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

When V is a grading-restricted vertex algebra, a V*-module* is a V-module W with $W^1 = 0$ when V is viewed as a grading-restricted vertex superalgebra. When V is a vertex operator (super)algebra, a V*-module* is a V-module when V is viewed as a grading-restricted vertex (super)algebra. When V is a quasi-vertex operator (super)algebra, a *V-module* is a V-module W when V is viewed as a grading-restricted vertex (super)algebra together with an operator $L_W(1)$ of weight 1 on W satisfying

$$\begin{aligned}
[L_W(-1), L_W(1)] &= -2L_W(0), \\
[L_W(1), Y_W(v, z)] &= Y_W(L_V(1)v, z) + 2zY_W(L_V(0)v, z) + z^2 Y_W(L_V(-1)v, z)
\end{aligned}$$

for $v \in V$.

We denote the V-module just defined by (W, Y_W) or simply by W. In the case that V is a quasi-vertex operator (super)algebra, we denote the V-module just defined by $(W, Y_W, L_W(1))$ or simply by W.

Remark 2.6. In Definition 2.5, as in Definition 2.1, we can also separate the rationality of products and iterates, the commutativity and the associativity (see Remark 2.4). In fact, for modules, it is easy to see that the commutativity is a consequence of the rationality, the associativity and the skew-symmetry for V. Since the skew-symmetry for V always holds, the duality in Definition 2.5 can be replaced by the rationality and the associativity.

We also need the important notion of intertwining operator. We could define an intertwining operator to be an analytic map from $\mathbb{C}^\times$ to a suitable space of linear maps. But in general it is not single valued and we need to choose a preferred branch. Because of this complication, we shall define an intertwining operator as usual to be a linear map to a space of formal series with complex powers. After the definition, we shall explain how to choose a special branch of the intertwining operator to obtain an analytic map from $\mathbb{C}^\times$ to the corresponding space of linear maps. These maps are what we shall use in this paper.

Definition 2.7. Let V be a grading-restricted vertex superalgebra (a grading-restricted vertex algebra being a special case) and W_1, W_2, W_3 V-modules. An *intertwining operator of type* $\binom{W_3}{W_1 W_2}$ is a linear map

$$\begin{aligned}
\mathcal{Y} : W_1 \otimes W_2 &\to W_3\{x\} \\
w_1 \otimes w_2 &\mapsto \mathcal{Y}(w_1, x)w_2
\end{aligned}$$

(where $W_3\{x\}$ is the space of formal series of the form $\sum_{n\in\mathbb{C}} a_n x^n$ for $a_n \in W_3$ and x is a formal variable) satisfying the following axioms:

1. *$L(0)$-bracket formula*: For $w_1 \in W_1$,

$$L_{W_3}(0)\mathcal{Y}(w_1, x) - \mathcal{Y}(w_1, x)L_{W_2}(0) = \frac{d}{dx}\mathcal{Y}(w_1, x) + Y_W(L_{W_1}(0)w_1, x).$$

2. *$L(-1)$-derivative property*: For $w_1 \in W_1$,

$$\frac{d}{dx}\mathcal{Y}(w_1, x) = Y_W(L_{W_1}(-1)w_1, x) = L_{W_3}(-1)\mathcal{Y}(w_1, x) - \mathcal{Y}(w_1, x)L_{W_2}(-1).$$

3. *Duality with vertex operators*: For $u \in V^{|u|}$, $w_1 \in W^{|w_1|}$, and $w_2 \in W$ where $|u|, |w_1|$ are 0 or 1, $w_3' \in W_3'$, for any single-valued branch $l(z_2)$ of the logarithm of z_2 in the region $z_2 \neq 0$, $0 \leq \arg z_2 \leq 2\pi$, the series

$$\langle w_3', Y_{W_3}(u, z_1)\mathcal{Y}(w_1, x_2)w_2\rangle\Big|_{x_2^n = e^{nl(z_2)}, n \in \mathbb{C}},$$
$$(-1)^{|u||w_1|}\langle w_3', \mathcal{Y}(w_1, x_2)Y_{W_2}(u, z_1)w_2\rangle\Big|_{x_2^n = e^{nl(z_2)}, n \in \mathbb{C}},$$
$$\langle w_3', \mathcal{Y}(Y_{W_1}(u, z_1 - z_2)w_1, x_2)w_2\rangle\Big|_{x_2^n = e^{nl(z_2)}, n \in \mathbb{C}}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common analytic function in z_1 and z_2 and can be analytically extended to a multivalued analytic functions with the only possible poles $z_1 = 0$ and $z_1 = z_2$ and the only possible branch point $z_2 = 0$.

When V is a quasi-vertex operator superalgebra (a quasi-vertex operator superalgebra being a special case) and W_1, W_2, W_3 are V-modules, an *intertwining operator of type* $\binom{W_3}{W_1 W_2}$ is an intertwining operator $\mathcal{Y}$ of the same type in the sense above satisfying in addition the following condition:

4. *$L(1)$-bracket formula*: For $w_1 \in W_1$,

$$\begin{aligned} &L_{W_3}(1)\mathcal{Y}(w_1, z) - \mathcal{Y}(w_1, z)L_{W_2}(1) \\ &\quad = \mathcal{Y}(L_{W_1}(1)w_1, z) + 2z\mathcal{Y}(L_{W_1}(0)w_1, z) + z^2\mathcal{Y}(L_{W_1}(-1)w_1, z). \end{aligned}$$

The dimension of the space of all intertwining operators of type $\binom{W_3}{W_1 W_2}$ is called the *fusion rule of type* $\binom{W_3}{W_1 W_2}$ and is denoted by $N_{W_1 W_2}^{W_3}$.

We shall need later a formula equivalent to the $L(0)$-bracket formula called *$L(0)$-conjugation formula* for an intertwining operator $\mathcal{Y}$: For $a \in \mathbb{C}$,

$$e^{aL_{W_3}(0)}\mathcal{Y}(w_1, x)e^{-aL_{W_2}(0)} = \mathcal{Y}(e^{aL_{W_1}(0)}w_1, e^a x).$$

For $z \in \mathbb{C}^\times$, let $\log z = \log|z| + i\arg z$, where $0 \leq \arg z < 2\pi$. Let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{W_1 W_2}$. Then for $z \in \mathbb{C}^\times$, $w_1 \in W_1$ and $w_2 \in W_2$, we use $\mathcal{Y}(w_1, z)w_2$ to denote $\mathcal{Y}(w_1, x)w_2\Big|_{x^n = e^{n\log z}, n \in \mathbb{C}}$. In particular, we have a map from $\mathbb{C}^\times$ to $\mathrm{Hom}(W_1 \otimes W_2, \overline{W}_3)$ given by $z \mapsto \mathcal{Y}(w_1, z)w_2$. Using this notation, the three expressions in the duality axiom for intertwining operators can be written as

$$\langle w_3', Y_{W_3}(u, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle,$$
$$(-1)^{|u||w_1|}\langle w_3', \mathcal{Y}(w_1, z_2)Y_{W_2}(u, z_1)w_2\rangle,$$
$$\langle w_3', \mathcal{Y}(Y_{W_1}(u, z_1 - z_2)w_1, z_2)w_2\rangle.$$

We shall always use this notation in this paper.

Finally we define nondegenerate symmetric invariant bilinear forms on modules for a quasi-vertex operator algebra.

Definition 2.8. Let V be a quasi-vertex operator algebra and W a V-module. A *nondegenerate symmetric invariant bilinear form on* W is a nondegenerate symmetric bilinear form $(\cdot,\cdot)_W : W \otimes W \to \mathbb{C}$ on W such that for $m, n \in \mathbb{C}$, $m \neq n$, $w_1 \in W_{(m)}$ and $w_2 \in W_{(n)}$, $(w_1, w_2)_W = 0$ and for $v \in V$, $w_1, w_2 \in W$,

$$\begin{aligned}(w_1, L_W(1)w_2)_W &= (L_W(-1)w_1, w_2)_W,\\ (w_1, Y_W(v,z)w_2)_W &= (Y_W(e^{zL_V(1)}(-z^{-2})^{L_V(0)}v, z^{-1})w_1, w_2)_W.\end{aligned}$$

In particular, a *nondegenerate symmetric invariant bilinear form on* V is defined to be a nondegenerate symmetric invariant bilinear form on V when V is viewed as a V-module.

3 The first construction

In this section, we give our first construction of grading-restricted vertex (super)algebras. By a uniqueness result, our construction gives the same class of grading-restricted vertex algebras as in [MP]. What is new in this section is the construction, including a new formula for the vertex operator map and a new method. As mentioned in the introduction, we use the complex analysis approach. But since in this section, we work only with rational functions, the results and the proofs in this section still work for grading-restricted vertex (super)algebras over any field of characteristic 0.

Let $V = \coprod_{n\in\frac{\mathbb{Z}}{2}}$ be a $\frac{\mathbb{Z}}{2}$-graded vector space such that $V_{(n)} = 0$ for n sufficiently negative and $\dim V_{(n)} < \infty$ for $n \in \frac{\mathbb{Z}}{2}$. Since $\dim V_{(n)} < \infty$ for $n \in \frac{\mathbb{Z}}{2}$, we have $\overline{V} = (V')^*$. Elements of $V_{(n)}$ is said to have *weight* n. Elements of $V^0 = \coprod_{n\in\mathbb{Z}} V_{(n)}$ are said to be *even* and elements of $V^1 = \coprod_{n\in\mathbb{Z}+\frac{1}{2}} V_{(n)}$ are said to be *odd*. Let $L_V(0) : V \to V$ be the operator defined by the grading on V, that is, by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then for $a \in \mathbb{C}$, the operator $e^{aL_V(0)}$ on V defined by $e^{aL_V(0)}v = e^{an}v$ for $v \in V_{(n)}$ has a natural extension to $\overline{V}$. For $n \in \mathbb{Z}$, we use π_n to denote the projection from V or $\overline{V}$ to $V_{(n)}$.

An operator O on V satisfying $[L_V(0), O] = nO$ is said to have *weight* n. Similarly for operators on the graded dual V' of V.

Lemma 3.1. *Let* ϕ *be an analytic map from* $\mathbb{C}^\times$ *to* $\mathrm{Hom}(V, \overline{V})$*. If there exists* $\mathrm{wt}\ \phi \in \mathbb{Z}$ *such that*

$$[L_V(0), \phi(z)] = z\frac{d}{dz}\phi(z) + (\mathrm{wt}\ \phi)\phi(z),$$

then we have a Laurent expansion $\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-1}$ *where for* $n \in \mathbb{Z}$, $\phi_n \in \mathrm{Hom}(V, V)$ *is homogeneous of weight* $\mathrm{wt}\ \phi - n - 1$*. Moreover, for* $v \in V$, $\phi(z)v$ *as a Laurent series in* z *has only finitely many negative power terms and for* $v' \in V'$, $\langle v', \phi(z)\cdot\rangle$ *as a Laurent series with coefficients in* V' *has only finitely many positive powers of* z.

Proof. From the bracket formula for $L_V(0)$ and $\phi(z)$, we obtain that for $a \in \mathbb{C}$,

$$e^{aL_V(0)}\phi(z)e^{-aL_V(0)} = e^{a(\mathrm{wt}\ \phi)}\phi(e^a z).$$

In particular, taking a such that $e^{-a} = z$, we have $\phi(z) = e^{a(\mathrm{wt}\ \phi)}e^{-aL_V(0)}\phi(1)e^{aL_V(0)}$. Let $\phi_n : V \to V$ be defined by $\phi_n v = \pi_{(\mathrm{wt}\ \phi)-n-1+m}\phi(1)v$ for $v \in V_{(m)}$. Then ϕ_n is of weight $\mathrm{wt}\ \phi - n - 1$ and $\sum_{n\in\mathbb{Z}}\phi_n = \phi(1)$. Moreover, $e^{a(\mathrm{wt}\ \phi)}e^{-aL_V(0)}\phi_n e^{aL_V(0)}v = \phi_n v z^{-n-1}$ for $n \in \mathbb{Z}$ and $v \in V$. Thus

$$\begin{aligned}
\phi(z)v &= e^{a(\mathrm{wt}\ \phi)}e^{-aL_V(0)}\phi(1)e^{aL_V(0)}v \\
&= \sum_{n\in\mathbb{Z}} e^{a(\mathrm{wt}\ \phi)}e^{-aL_V(0)}\phi_n e^{aL_V(0)}v \\
&= \sum_{n\in\mathbb{Z}} \phi_n v z^{-n-1}
\end{aligned}$$

for $v \in V$.

Since $V_{(n)} = 0$ for n sufficiently negative and the weight of ϕ_n is $\mathrm{wt}\ \phi - n - 1$, for $v \in V$, $\phi(z)v$ has only finitely many negative power terms and for $v' \in V'$, $\langle v', \phi(z)\cdot\rangle$ as a Laurent series with coefficients in V' has only finitely many positive powers of z. ∎

Let ϕ^i for $i \in I$ be analytic maps from $\mathbb{C}^\times$ to $\mathrm{Hom}(V, \overline{V})$ and let $\mathbf{1} \in V_{(0)}$. Assume that V, ϕ^i for $i \in I$ and $\mathbf{1} \in V_{(0)}$ satisfy the following conditions:

1. For $i \in I$, there exists $\mathrm{wt}\ \phi^i \in \mathbb{Z}$ such that $[L_V(0), \phi^i(z)] = z\frac{d}{dz}\phi^i(z) + (\mathrm{wt}\ \phi^i)\phi^i(z)$.

2. There exists an operator $L_V(-1)$ on V such that $L_V(-1)\mathbf{1} = 0$ and $[L_V(-1), \phi^i(z)] = \frac{d}{dz}\phi^i(z)$ for $i \in I$.

3. The limits $\lim_{z\to 0}\phi^i(z)\mathbf{1}$ for $i \in I$ exist (the limits can be taken in the topology of $\overline{V}$, but by Lemma 3.1 above, the existence of these limits mean that the expansions of $\phi^i(z)$ have only nonnegative powers). These elements are either in V^0 or V^1. We define $|\phi^i| = 0$ if $\phi^i_{-1}\mathbf{1} = \lim_{z\to 0}\phi^i(z)\mathbf{1} \in V^0$ and $|\phi^i| = 1$ if $\phi^i_{-1}\mathbf{1} = \lim_{z\to 0}\phi^i(z)\mathbf{1} \in V^1$.

4. The vector space V is spanned by elements of the form $\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}$ for $i_1, \dots, i_k \in I$ and $n_1, \dots, n_k \in \mathbb{Z}$.

5. For $v' \in V'$, $v \in V$ and $i_1, \dots, i_k \in I$, the series $\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)v\rangle$ (in fact a Laurent series in $z_1, \dots, z_k$ with complex coefficients by Lemma 3.1) is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ to a rational function $R(\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)v\rangle)$ in $z_1, \dots, z_k$ with the only possible poles at $z_i = 0$ for $i = 1, \dots, k$ and $z_j = z_l$ for $j \neq l$. In addition, the order of the pole $z_j = z_l$ is independent of ϕ^{i_n} for $n \neq j, l$, v and v' and the order of the pole $z_j = 0$ is independent of ϕ^{i_n} for $n \neq j$ and v'.

6. For $v \in V$, $v' \in V'$, $i_1, i_2 \in I$,
$$R(\langle v', \phi^{i_1}(z_1)\phi^{i_2}(z_2)v\rangle) = (-1)^{|\phi^{i_1}||\phi^{i_2}|}R(\langle v', \phi^{i_2}(z_2)\phi^{i_1}(z_1)v\rangle).$$

Proposition 3.2. *The space V, the maps ϕ^i for $i \in I$, $L_V(-1)$ and $\mathbf{1}$ have the following properties:*

7. *For $a \in \mathbb{C}$ and $i \in I$, $e^{aL_V(0)}\phi^i(z)e^{-aL_V(0)} = e^{a(\mathrm{wt}\ \phi^i)}\phi^i(e^a z)$.*

8. $L_V(-1)\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1} = \sum_{j=1}^{k}\phi^{i_1}_{n_1}\cdots\phi^{i_{j-1}}_{n_{j-1}}(-n_j\phi^{i_j}_{n_j-1})\phi^{i_{j+1}}_{n_{j+1}}\cdots\phi^{i_k}_{n_k}\mathbf{1}$.

9. *For $a \in \mathbb{C}$, $z \in \mathbb{C}^\times$ satisfying $|z| > |a|$ and $i \in I$, $e^{aL_V(-1)}\phi^i(z)e^{-aL_V(-1)} = \phi^i(z+a)$.*

10. *The operator $L_V(-1)$ has weight 1 and its adjoint $L_V(-1)'$ as an operator on V' has weight -1 (the weight of an operator on V' is defined in the same way as that of an operator on V). In particular, $e^{zL_V(-1)'}v' \in V'$ for $z \in \mathbb{C}$ and $v' \in V'$.*

11. *For $v \in V$, $v' \in V'$ and $\sigma \in S_k$,*

$$R(\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)v\rangle) = \pm R(\langle v', \phi^{i_{\sigma(1)}}(z_{\sigma(1)})\cdots\phi^{i_{\sigma(k)}}(z_{\sigma(k)})v\rangle),$$

where the sign $\pm$ is uniquely determined by σ and $|\phi^{i_1}|, \ldots, |\phi^{i_k}|$ (here we omit its explicit but complicated formula that can be calculated easily for special cases using Condition 6).

Proof. These properties follow immediately from Conditions 1–6. ∎

For $N \in \mathbb{Z}_+$, $(z_i - z_j)^N$ is a polynomial in z_1 and z_2. We shall use $(z_i - z_j)^{N_{ij}}\phi(z_i)\phi(z_j)$ to denote the Laurent series obtained by multiplying the polynomial $(z_i - z_j)^{N_{ij}}$ to the Laurent series $\phi(z_i)\phi(z_j)$. We warn the reader that, unless otherwise stated, $(z_i - z_j)^{N_{ij}}\phi(z_i)\phi(z_j)$ is not the Laurent series obtained by multiplying the complex number $(z_i - z_j)^{N_{ij}}$ to the Laurent series $\phi(z_i)\phi(z_j)$.

Proposition 3.3. *Let $V = \coprod_{n\in\frac{\mathbb{Z}}{2}} V_{(n)}$ be a $\frac{\mathbb{Z}}{2}$-graded vector space, ϕ^i for $i \in I$ analytic maps from $\mathbb{C}^\times$ to $\mathrm{Hom}(V, \overline{V})$, $L_V(-1)$ an operator on V and $\mathbf{1} \in V_{(0)}$. Assume that they satisfy Conditions 1–4. Then Conditions 5 and 6 are equivalent to the following weak commutativity:*

12. *For $i, j \in I$, there exists $N_{ij} \in \mathbb{Z}_+$ such that*

$$(z_1 - z_2)^{N_{ij}}\phi^i(z_1)\phi^j(z_2) = (z_1 - z_2)^{N_{ij}}(-1)^{|\phi^i||\phi^j|}\phi^j(z_2)\phi^i(z_1). \tag{3.1}$$

In particular, when Conditions 1–4 and Property 12 holds, properties 7–11 also hold.

Proof. It is clear that Conditions 5 and 6 imply Property 12.

Now we assume that Property 12 holds. Consider the Laurent series

$$\prod_{1\le p<q\le k}(z_p - z_q)^{N_{i_p i_q}}\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)v\rangle. \tag{3.2}$$

For $1 \le l \le k$, using (3.1), the Laurent series (3.2) is equal to

$$\prod_{1\le p<q\le k}(z_p - z_q)^{N_{i_p i_q}}\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_{l-1}}(z_{l-1})\phi^{i_{l+1}}(z_{l+1})\cdots\phi^{i_k}(z_k)\phi^{i_l}(z_l)v\rangle. \tag{3.3}$$

By Lemma 3.1, (3.3) has only finitely many negative power terms in z_l. So the same is true for (3.2). On the other hand, using (3.1) again, (3.2) is equal to

$$\prod_{1\le p<q\le k}(z_p-z_q)^{N_{i_pi_q}}\langle v', \phi^{i_l}(z_l)\phi^{i_1}(z_1)\cdots\phi^{i_{l-1}}(z_{l-1})\phi^{i_{l+1}}(z_{l+1})\cdots\phi^{i_k}(z_k)v\rangle. \tag{3.4}$$

By Lemma 3.1 again, (3.4) has only finitely many positive power terms in z_l. So the same is true for (3.2). Thus (3.2) must be a Laurent polynomial in z_l. Since this is true for $1\le l\le k$, (3.2) is a Laurent polynomial in $z_1,\dots,z_k$.

For fixed $1\le p<q\le k$, the expansion coefficients of

$$\langle v', \phi(z_1)\cdots\phi(z_k)v\rangle \tag{3.5}$$

as Laurent series in z_l for $l\ne p,q$ are of the form

$$\langle v', \phi^{i_1}_{n_1}\cdots\phi^{i_{p-1}}_{n_{p-1}}\phi^{i_p}(z_p)\phi^{i_{p+1}}_{n_{p+1}}\cdots\phi^{i_{q-1}}_{n_{q-1}}\phi^{i_q}(z_q)\phi^{i_{q+1}}_{n_{q+1}}\cdots\phi^{i_k}_{n_k}v\rangle \tag{3.6}$$

for $n_l\in\mathbb{Z}$, $l\ne p,q$. Clearly (3.6) contains only finitely many negative powers in z_q and finitely many positive powers in z_p. But we have shown that when multiplied by $(z_p-z_q)^{N_{pq}}$, it becomes a Laurent polynomial. Thus (3.6) must be the product of a Laurent polynomial in z_p and z_q and the expansion of $(z_p-z_q)^{-N_{pq}}$ as a Laurent series in nonnegative powers of z_q, or equivalently, in the region $|z_p|>|z_q|>0$. Since p and q are arbitrary, we see that (3.5) is equal to the product of a Laurent polynomial and the expansion of $\prod_{1\le p<q\le k}(z_p-z_q)^{-N_{pq}}$ in the region $|z_1|>\cdots>|z_k|>0$. This is Condition 5. Condition 6 follows immediately from Condition 5 in the case $k=2$ and (3.1). ∎

We now define a vertex operator map. We first give the motivation of this definition. The vertex operator map we want to define is a map

$$\begin{aligned} Y_V:\mathbb{C}^\times &\to \mathrm{Hom}(V\otimes V,\overline{V}),\\ z &\mapsto Y_V(\cdot,z)\cdot : u\otimes v\mapsto Y_V(u,z)v. \end{aligned}$$

We define $Y_V(\phi^i_{-1}\mathbf{1},z)v=\phi^i(z)v$ for $i\in I$ and $v\in V$. The vertex operator map should satisfy the rationality and associativity property. In particular, we should have

$$R(\langle v', Y_V(\phi^{i_1}(\xi_1)\cdots\phi^{i_k}(\xi_k)\mathbf{1},z)v\rangle)=R(\langle v', \phi^{i_1}(\xi_1+z)\cdots\phi^{i_k}(\xi_k+z)v\rangle)$$

for $i_1,\dots,i_k\in I$, $v\in V$ and $v'\in V'$.

Motivated by this associativity formula, we define the vertex operator map as follows: For $v'\in V'$, $v\in V$, $i_1,\dots,i_k\in I$, $m_1,\dots,m_k\in\mathbb{Z}$, we define Y_V by

$$\begin{aligned}&\langle v', Y_V(\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1},z)v\rangle\\ &\quad=\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1}\cdots\xi_k^{m_k}R(\langle v', \phi^{i_1}(\xi_1+z)\cdots\phi^{i_k}(\xi_k+z)v\rangle).\end{aligned} \tag{3.7}$$

Note that for a meromorphic function $f(\xi)$, $\mathrm{Res}_{\xi=0}f(\xi)$ means expanding $f(\xi)$ as a Laurent series in $0<|\xi|<r$ for r sufficiently small so that no other poles are in this disk and then

taking the coefficient of ξ^{-1}. We can also expand $f(\xi)$ as a Laurent series in a different region. In general, the coefficient of ξ^{-1} in this Laurent series might be different from $\mathrm{Res}_{\xi=0}f(\xi)$. Also note that the order to take these residues is important. Different orders in general give vertex operators for different elements.

Since $\overline{V} = (V')^*$, for fixed $\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1}, v \in V$, the formula above indeed gives an element

$$Y_V(\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1}, z)v \in \overline{V}.$$

Since there might be relations among elements of the form $\phi^{i_1}_{m_1}\cdots\phi^{i_k}_{m_k}\mathbf{1}$, we first have to show that the definition above indeed gives a well-defined map from $\mathbb{C}^\times$ to $\mathrm{Hom}(V\otimes V, \overline{V})$. Let ϕ^0 be the map from $\mathbb{C}^\times$ to $\mathrm{Hom}(V, \overline{V})$ given by $\phi^0(z) = 1_V$. Let wt $\phi^0 = 0$. Then Conditions 1 to 6 and Properties 7 to 12 above still hold for ϕ^i, $i \in \tilde{I} = I \cup \{0\}$. Then any relation among such elements can always be written as

$$\sum_{p=1}^{q} \lambda_p \phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1} = 0$$

for some $i_j^p \in \tilde{I}$ and $m_j^p \in \mathbb{Z}$, $p = 1, \dots, q$, $j = 1, \dots, k$.

Lemma 3.4. *If*

$$\sum_{p=1}^{q} \lambda_p \phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1} = 0,$$

then

$$\sum_{p=1}^{q} \lambda_p \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p} R(\langle v', \phi^{i_1^p}(\xi_1+z)\cdots\phi^{i_k^p}(\xi_k+z)v\rangle) = 0$$

for $v \in V$ and $v' \in V'$.

Proof. By Condition 4, we can take v to be of the form $\phi^{j_1}_{n_1}\cdots\phi^{j_l}_{n_l}\mathbf{1}$. Moreover, in this case,

$$\begin{aligned}
&R(\langle v', \phi^{i_1^p}(z_1)\cdots\phi^{i_k^p}(z_k)\phi^{j_1}_{n_1}\cdots\phi^{j_l}_{n_l}\mathbf{1}\rangle)\\
&\quad = \mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_k^{n_l} R(\langle v', \phi^{i_1^p}(z_1)\cdots\phi^{i_k^p}(z_k)\phi^{j_1}(\zeta_1)\cdots\phi^{j_l}(\zeta_l)\mathbf{1}\rangle).
\end{aligned}$$

Then

$$\begin{aligned}
&\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p} R(\langle v', \phi^{i_1^p}(\xi_1+z)\cdots\phi^{i_k^p}(\xi_k+z)v\rangle)\\
&\quad = \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1^p}(\xi_1+z)\cdots\phi^{i_k^p}(\xi_k+z)\phi^{j_1}(\zeta_1)\cdots\phi^{j_l}(\zeta_l)\mathbf{1}\rangle)\\
&\quad = \prod_{r=1}^{k}\prod_{s=1}^{l}(-1)^{|\phi^{i_r}||\phi^{j_s}|}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{j_1}(\zeta_1)\cdots\phi^{j_l}(\zeta_l)\phi^{i_1^p}(\xi_1+z)\cdots\phi^{i_k^p}(\xi_k+z)\mathbf{1}\rangle)
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{r=1}^{k}\prod_{s=1}^{l}(-1)^{|\phi^{i_r}||\phi^{j_s}|}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad\cdot R(\langle e^{zL_V(-1)'}v', \phi^{j_1}(\zeta_1-z)\cdots\phi^{j_l}(\zeta_l-z)\phi^{i_1^p}(\xi_1)\cdots\phi^{i_k^p}(\xi_k)\mathbf{1}\rangle)\\
&= \prod_{r=1}^{k}\prod_{s=1}^{l}(-1)^{|\phi^{i_r}||\phi^{j_s}|}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad\cdot R(\langle e^{zL_V(-1)'}v', \phi^{j_1}(\zeta_1-z)\cdots\phi^{j_l}(\zeta_l-z)\phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1}\rangle).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\sum_{p=1}^{q}\lambda_p\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{m_1^p}\cdots\xi_k^{m_k^p}R(\langle v', \phi^{i_1^p}(\xi_1+z)\cdots\phi^{i_k^p}(\xi_k+z)v\rangle)\\
&\quad= \sum_{p=1}^{q}\prod_{r=1}^{k}\prod_{s=1}^{l}(-1)^{|\phi^{i_r}||\phi^{j_s}|}\lambda_p\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad\cdot R(\langle e^{zL_V(-1)'}v', \phi^{j_1}(\zeta_1-z)\cdots\phi^{j_l}(\zeta_l-z)\phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1}\rangle)\\
&\quad= \prod_{r=1}^{k}\prod_{s=1}^{l}(-1)^{|\phi^{i_r}||\phi^{j_s}|}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_l=0}\zeta_1^{n_1}\cdots\zeta_l^{n_l}\cdot\\
&\qquad\qquad\cdot R\left(\left\langle e^{zL_V(-1)'}v', \phi^{j_1}(\zeta_1-z)\cdots\phi^{j_l}(\zeta_l-z)\left(\sum_{p=1}^{q}\lambda_p\phi^{i_1^p}_{m_1^p}\cdots\phi^{i_k^p}_{m_k^p}\mathbf{1}\right)\right\rangle\right)\\
&\quad= 0,
\end{aligned}
$$

proving the lemma. ∎

From this lemma, we see that the vertex operator map Y_V is well defined. We are now ready to formulate and prove the main result of this section.

Theorem 3.5. *Let $V=\coprod_{n\in\frac{\mathbb{Z}}{2}}V_{(n)}$ be a $\frac{\mathbb{Z}}{2}$-graded vector space, ϕ^i for $i\in I$ maps from $\mathbb{C}^\times$ to $\mathrm{Hom}(V,\overline{V})$, $L_V(-1)$ an operator on V and $\mathbf{1}\in V_{(0)}$. Assume that they satisfy Conditions 1–6. Then the triple $(V, Y_V, \mathbf{1})$ is a grading-restricted vertex algebra generated by $\phi^i_{-1}\mathbf{1}$ for $i\in I$. Moreover, this is the unique grading-restricted vertex algebra structure on V with the vacuum $\mathbf{1}$ such that $Y(\phi^1_{-1}\mathbf{1},z)=\phi^i(z)$ for $i\in I$.*

Proof. The vertex operator map Y_V is clearly analytic. The grading-restriction axiom is by assumption satisfied. The $L(-1)$-bracket formula follows from Condition 1 and the definition of Y_V. The identity property and the creation property also follow from of the definition of Y_V.

Let $L_V(0)'$ be the adjoint operator of $L_V(0)$. For $v'\in V'$, $v\in V$, $i_1,\dots,i_k\in I$ and $n_1,\dots,n_k\in\mathbb{Z}$, $a\in\mathbb{C}^\times$

$$
\langle v', a^{L_V(0)}Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1},z)a^{-L_V(0)}v\rangle
$$

$$
\begin{aligned}
&= \langle a^{L_V(0)'}v', Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z)a^{-L_V(0)}v\rangle\\
&= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{n_1}\cdots\xi_k^{n_k}R(\langle a^{L_V(0)'}v', \phi^{i_1}(\xi_1+z)\cdots\phi^{i_k}(\xi_k+z)a^{-L_V(0)}v\rangle)\\
&= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{n_1}\cdots\xi_k^{n_k}R(\langle v', a^{L_V(0)}\phi^{i_1}(\xi_1+z)\cdots\phi^{i_k}(\xi_k+z)a^{-L_V(0)}v\rangle)\\
&= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_k=0}\xi_1^{n_1}\cdots\xi_k^{n_k}a^{\mathrm{wt}\ \phi^{i_1}+\cdots\mathrm{wt}\ \phi^{i_k}}R(\langle v', \phi^{i_1}(a\xi_1+az)\cdots\phi^{i_k}(a\xi_k+az)v\rangle)\\
&= \mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}a^{\mathrm{wt}\ \phi^{i_1}+\cdots\mathrm{wt}\ \phi^{i_k}-k-n_1-\cdots-n_k}\cdot\\
&\qquad\qquad\qquad\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1+az)\cdots\phi^{i_k}(\zeta_k+az)v\rangle)\\
&= \langle v', Y_V(a^{L_V(0)}\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, az)v\rangle).
\end{aligned}
$$

This formula implies the $L(0)$-bracket formula.

From Condition 2 and the definition of Y_V, we obtain

$$
\frac{d}{dz}Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z) = [L_V(-1), Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z)].
$$

From Property 8 and the definition of Y_V, we obtain

$$
\frac{d}{dz}Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z) = Y_V(L_V(-1)\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z).
$$

Applying both sides of this formula to $\mathbf{1}$, taking the limit $z\to 0$ and then using the creation property, we obtain

$$
L_V(-1)\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1} = \lim_{z\to 0}\frac{d}{dz}Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z)\mathbf{1}.
$$

The $L(-1)$-derivative property is proved.

Let $\{e_n\}_{n\in\mathbb{Z}}$ be a homogeneous basis of V and $\{e'_n\}_{n\in\mathbb{Z}}$ its dual basis in V'. Then we have

$$
\begin{aligned}
&\langle v', Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1)Y_V(\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle\\
&\quad= \sum_{n\in\mathbb{Z}}\langle v', Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1)e_n\rangle\langle e'_n, Y_V(\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle\\
&\quad= \sum_{n\in\mathbb{Z}}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad \cdot R(\langle v', \phi^{i_1}(\zeta_1+z_1)\cdots\phi^{i_k}(\zeta_k+z_1)e_n\rangle)R(\langle e'_n, \phi^{j_1}(\xi_1+z_2)\cdots\phi^{j_l}(\xi_l+z_2)v\rangle)\\
&\quad= \mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad \cdot\sum_{n\in\mathbb{Z}}R(\langle v', \phi^{i_1}(\zeta_1+z_1)\cdots\phi^{i_k}(\zeta_k+z_1)e_n\rangle)R(\langle e'_n, \phi^{j_1}(\xi_1+z_2)\cdots\phi^{j_l}(\xi_l+z_2)v\rangle).
\end{aligned}
\tag{3.8}
$$

By Condition 5, when $|z_1| > \cdots > |z_{k+l}| > 0$,

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}}R(\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)e_n\rangle)R(\langle e'_n, \phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle)\\
&\qquad= \sum_{n\in\mathbb{Z}}\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)e_n\rangle\langle e'_n, \phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle\\
&\qquad= \langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle
\end{aligned}
\tag{3.9}
$$

is absolutely convergent to the rational function

$$R(\langle v', \phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}(z_{k+1})\cdots\phi^{j_l}(z_{k+l})v\rangle) \tag{3.10}$$

in $z_1, \dots, z_{k+l}$. On the other hand, since the only possible poles of (3.10) are $z_i - z_j = 0$ for $i \neq j$ and $z_i = 0$, there is a unique expansion of such a rational function in the region $|z_1|, \dots, |z_k| > |z_{k+1}|, \dots, |z_{k+l}| > 0$, $z_i \neq z_j$ for $i \neq j$, $i, j = 1, \dots, k$ and $i, j = k+1, \dots, k+l$ such that each term is a product of two rational functions, one in $z_1, \dots, z_k$ and the other in $z_{k+1}, \dots, z_{k+l}$. Since the left-hand side of (3.9) is a series of the same form and is absolutely convergent in the region $|z_1| > \cdots > |z_{k+l}| > 0$ to (3.10), it must be absolutely convergent in the larger region $|z_1|, \dots, |z_k| > |z_{k+1}|, \dots, |z_{k+l}| > 0$, $z_i \neq z_j$ for $i \neq j$, $i, j = 1, \dots, k$ and $i, j = k+1, \dots, k+l$ to (3.10).

Substituting $\zeta_i + z_1$ for z_i for $i = 1, \dots, k$ and $\xi_j + z_2$ for z_{k+j} for $j = 1, \dots, l$, we see that

$$\sum_{n\in\mathbb{Z}} R(\langle v', \phi^{i_1}(\zeta_1 + z_1)\cdots\phi^{i_k}(\zeta_k + z_1)e_n\rangle)R(\langle e'_n, \phi^{j_1}(\xi_1 + z_2)\cdots\phi^{j_l}(\xi_l + z_2)v\rangle)$$

is absolutely convergent to

$$R(\langle v', \phi^{i_1}(\zeta_1 + z_1)\cdots\phi^{i_k}(\zeta_k + z_1)\phi^{j_1}(\xi_1 + z_2)\cdots\phi^{j_l}(\xi_l + z_2)v\rangle)$$

when $|\zeta_1 + z_1|, \dots, |\zeta_k + z_1| > |\xi_1 + z_2|, \dots, |\xi_l + z_2| > 0$, $\zeta_i \neq \zeta_j$ for $i, j = 1, \dots, k$ and $\xi_i \neq \xi_j$ for $i, j = 1, \dots, l$. When $|z_1| > |z_2| > 0$, we can always find sufficiently small neighborhood of 0 such that when $\zeta_1, \dots, \zeta_k, \xi_1, \dots, \xi_l$ are in this neighborhood, $|\zeta_1 + z_1|, \dots, |\zeta_k + z_1| > |\xi_1 + z_2|, \dots, |\xi_l + z_2| > 0$ holds. Thus we see that when $|z_1| > |z_2| > 0$, the right-hand side of (3.8) is absolutely convergent to

$$\begin{aligned}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot&\\ \cdot R(\langle v', \phi^{i_1}(\zeta_1 + z_1)\cdots\phi^{i_k}(\zeta_k + z_1)\phi^{j_1}(\xi_1 + z_2)\cdots\phi^{j_l}(\xi_l + z_2)v\rangle).&\end{aligned} \tag{3.11}$$

This is a rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. In particular, the left-hand side of (3.8), that is,

$$\langle v', Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1)Y_V(\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle, \tag{3.12}$$

is absolutely convergent in the region $|z_1| > |z_2| > 0$ to this rational function.

We have proved the rationality of the product of two vertex operators. We are ready to prove the commutativity. The calculation above also shows that

$$\langle v', Y_V(\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1)v\rangle \tag{3.13}$$

is absolutely convergent to the rational function

$$\begin{aligned}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\cdot&\\ \cdot R(\langle v', \phi^{j_1}(\xi_1 + z_2)\cdots\phi^{j_l}(\xi_l + z_2)\phi^{i_1}(\zeta_1 + z_1)\cdots\phi^{i_k}(\zeta_k + z_1)v\rangle),&\end{aligned} \tag{3.14}$$

in the regions $|z_2| > |z_1| > 0$, respectively. By Property 11, the rational functions (3.11) and (3.14) multiplied by

$$\prod_{r=1}^{k}\prod_{s=1}^{l}(-1)^{|\phi^{i_r}||\phi^{j_s}|} = (-1)^{|\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}||\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}|}$$

are equal. Thus (3.12) and (3.13) multiplied by the sign $(-1)^{|\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}||\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}|}$ are absolutely convergent in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively, to a common rational function with the only possible poles at $z_1 = z_2$, $z_1 = 0$ and $z_2 = 0$.

We now prove the associativity. For $i_1, \dots, i_k, j_1, \dots, j_l \in I$, $m_1, \dots, m_l \in \mathbb{Z}$, $v \in V$ and $v' \in V'$, using the expansion of $\phi^{i_1}(\xi_1), \dots, \phi^{i_k}(\xi_k)$ and the definition of Y_V, we have

$$\begin{aligned}
&\langle v', Y_V(\phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z)v\rangle\\
&\quad= \sum_{p_1,\dots,p_k\in\mathbb{Z}} \langle v', Y_V(\phi^{i_1}_{p_1}\cdots\phi^{i_k}_{p_k}\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z)v\rangle z_1^{-p_1-1}\cdots z_k^{-p_k-1}\\
&\quad= \sum_{p_1,\dots,p_k\in\mathbb{Z}} \mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{p_1}\cdots\zeta_k^{p_k}\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad\cdot R(\langle v', \phi^{i_1}(\zeta_1+z)\cdots\phi^{i_k}(\zeta_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle) z_1^{-p_1-1}\cdots z_k^{-p_k-1}.
\end{aligned} \tag{3.15}$$

We now expand

$$R(\langle v', \phi^{i_1}(\zeta_1+z)\cdots\phi^{i_k}(\zeta_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle)$$

as a Laurent series $\sum_{l\in\mathbb{Z}} f_l(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)\zeta_k^{-l-1}$ in ζ_k in the region $|z|, |\zeta_1|, \dots, |\zeta_{k-1}| > |\zeta_k| > |\xi_1|, \dots, |\xi_l|$, where $f_l(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)$ are rational functions in $\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l$ and z. Then in the region that the Laurent series expansion holds, we have

$$\begin{aligned}
&\sum_{p_k\in\mathbb{Z}} \mathrm{Res}_{\zeta_k=0}\zeta_k^{p_k}\left(\sum_{l\in\mathbb{Z}} f_l(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z)\zeta_k^{-l-1}\right) z_k^{-p_k-1}\\
&\quad= \sum_{p_k\in\mathbb{Z}} f_{p_k}(\zeta_1, \dots, \zeta_{k-1}, \xi_1, \dots, \xi_l, z) z_k^{-p_k-1}\\
&\quad= R(\langle v', \phi^{i_1}(\zeta_1+z)\cdots\phi^{i_{k-1}}(\zeta_{k-1}+z)\phi^{i_k}(z_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle).
\end{aligned} \tag{3.16}$$

Repeating this step for the variables $\zeta_{k-1}, \dots, \zeta_1$, we see that the right-hand side of (3.15) is equal to the expansion of

$$\mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l} R(\langle v', \phi^{i_1}(z_1+z)\cdots\phi^{i_k}(z_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle) \tag{3.17}$$

as a Laurent series in $z_1 \dots, z_k$ in the region $|z| > |z_1| > \cdots > |z_k| > 0$. Thus the left-hand side of (3.15) is absolutely convergent to (3.17) in the region for this Laurent series expansion. In particular, in the region $|z| > |z_1| > \cdots > |z_k| > 0$,

$$\begin{aligned}
&\langle v', Y_V(\phi^{i_1}(z_1)\cdots\phi^{i_k}(z_k)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z)v\rangle\\
&\quad= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad\cdot R(\langle v', \phi^{i_1}(z_1+z)\cdots\phi^{i_k}(z_k+z)\phi^{j_1}(\xi_1+z)\cdots\phi^{j_l}(\xi_l+z)v\rangle). \qquad (3.18)
\end{aligned}$$

Now we have

$$\begin{aligned}
&\langle v', Y_V(Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle\\
&\quad= \sum_{n\in\mathbb{Z}}\langle v', Y_V(e_n, z_2)v\rangle\langle e'_n, Y_V(\phi^{i_1}_{n_1}\cdots\phi^{i_k}_{n_k}\mathbf{1}, z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle\\
&\quad= \sum_{n\in\mathbb{Z}}\langle v', Y_V(e_n, z_2)v\rangle\mathrm{Res}_{\zeta_1=0}\cdots\mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1}\cdots\zeta_k^{n_k}\cdot\\
&\qquad\qquad\cdot R(\langle e'_n, \phi^{i_1}(\zeta_1+z_1-z_2)\cdots\phi^{i_k}(\zeta_k+z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle). \qquad (3.19)
\end{aligned}$$

But by (3.18), in the region $|z_2| > |\zeta_1+z_1-z_2| > \cdots > |\zeta_k+z_1-z_2| > 0$, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}\langle v', Y_V(e_n, z_2)v\rangle\langle e'_n, \phi^{i_1}(\zeta_1+z_1-z_2)\cdots\phi^{i_k}(\zeta_k+z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle\\
&\quad= \langle v', Y_V(\phi^{i_1}(\zeta_1+z_1-z_2)\cdots\phi^{i_k}(\zeta_k+z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}, z_2)v\rangle\\
&\quad= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad\cdot R(\langle v', \phi^{i_1}(\zeta_1+z_1)\cdots\phi^{i_k}(\zeta_k+z_1)\phi^{j_1}(\xi_1+z_2)\cdots\phi^{j_l}(\xi_l+z_2)v\rangle). \qquad (3.20)
\end{aligned}$$

The right-hand side of (3.20) is a rational function in $\zeta_1, \dots, \zeta_k$, z_1 and z_2 with the only possible poles $\zeta_i - \zeta_j = 0$, for $i \neq j$, $\zeta_i + z_1 = 0$, $\zeta_i + z_1 - z_2 = 0$ and $z_2 = 0$. There is a unique expansion of such a rational function in the region $|z_2| > |\zeta_1+z_1-z_2|, \dots, |\zeta_k+z_1-z_2| > 0$, $\zeta_i \neq \zeta_j$ for $i \neq j$, $i, j = 1, \dots, k$, such that each term is a product of two rational functions, one in z_2 and the other in $\zeta_1, \dots, \zeta_k$ and z_1. Since

$$\sum_{n\in\mathbb{Z}}\langle v', Y_V(e_n, z_2)v\rangle R(\langle e'_n, \phi^{i_1}(\zeta_1+z_1-z_2)\cdots\phi^{i_k}(\zeta_k+z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle)$$

is a series of the same form and is equal to the left-hand side of (3.20) in the region $|z_2| > |\zeta_1+z_1-z_2| > \cdots > |\zeta_k+z_1-z_2| > 0$, it must be absolutely convergent to the right-hand side of (3.20) in the larger region $|z_2| > |\zeta_1+z_1-z_2|, \dots, |\zeta_k+z_1-z_2| > 0$. Thus we obtain

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}\langle v', Y_V(e_n, z_2)v\rangle R(\langle e'_n, \phi^{i_1}(\zeta_1+z_1-z_2)\cdots\phi^{i_k}(\zeta_k+z_1-z_2)\phi^{j_1}_{m_1}\cdots\phi^{j_l}_{m_l}\mathbf{1}\rangle)\\
&\quad= \mathrm{Res}_{\xi_1=0}\cdots\mathrm{Res}_{\xi_l=0}\xi_1^{m_1}\cdots\xi_l^{m_l}\cdot\\
&\qquad\qquad\cdot R(\langle v', \phi^{i_1}(\zeta_1+z_1)\cdots\phi^{i_k}(\zeta_k+z_1)\phi^{j_1}(\xi_1+z_2)\cdots\phi^{j_l}(\xi_l+z_2)v\rangle) \qquad (3.21)
\end{aligned}$$

in the region $|z_2| > |\zeta_1 + z_1 - z_2|, \ldots, |\zeta_k + z_1 - z_2| > 0$. Thus when $|z_2| > |z_1 - z_2| > 0$, the right-hand side of (3.19) is absolutely convergent to

$$\begin{aligned}\mathrm{Res}_{\zeta_1=0} \cdots \mathrm{Res}_{\zeta_k=0}\zeta_1^{n_1} \cdots \zeta_k^{n_k}\mathrm{Res}_{\xi_1=0} \cdots \mathrm{Res}_{\xi_l=0}\xi_1^{m_1} \cdots \xi_l^{m_l}\cdot \\ \cdot R(\langle v', \phi^{i_1}(\zeta_1 + z_1) \cdots \phi^{i_k}(\zeta_k + z_1)\phi^{j_1}(\xi_1 + z_2) \cdots \phi^{j_l}(\xi_l + z_2)v\rangle),\end{aligned} \tag{3.22}$$

which is proved above to be equal to the left hand side of (3.8) in the region $|z_1| > |z_2| > 0$. The associativity is proved.

To prove the uniqueness, we need only show that any grading-restricted vertex superalgebra structure on V with the vacuum $\mathbf{1}$ must have the vertex operator map defined by (3.7). But this is clear from the motivation that we discussed before the definition (3.7) of the vertex operator map Y_V. ∎

We call the grading-restricted vertex algebra given in Theorem 3.5 the *grading-restricted vertex algebra generated by* ϕ^i, $i \in I$. The maps ϕ^i, $i \in I$, are called the *generating fields* of the grading-restricted vertex algebra V.

A consequence follows immediately from Proposition 3.3 and Theorem 3.5 is the following result:

Corollary 3.6. *Let* $V = \coprod_{n\in\frac{\mathbb{Z}}{2}} V_{(n)}$ *be a* $\frac{\mathbb{Z}}{2}$*-graded vector space,* ϕ^i *for* $i \in I$ *maps from* $\mathbb{C}^\times$ *to* $\mathrm{Hom}(V, \overline{V})$*,* $L_V(-1)$ *an operator on* V *and* $\mathbf{1} \in V_{(0)}$*. Assume that they satisfy Conditions 1–4 and Property 12. Then the triple* $(V, Y_V, \mathbf{1})$ *is a grading-restricted vertex algebra generated by* $\phi^i_{-1}\mathbf{1}$ *for* $i \in I$*. Moreover, this is the unique grading-restricted vertex algebra structure on* V *with the vacuum* $\mathbf{1}$ *such that* $Y(\phi^i_{-1}\mathbf{1}, z) = \phi^i(z)$ *for* $i \in I$*.* ∎

Remark 3.7. From the uniqueness part of Corollary 3.6, we see that the class of grading-restricted vertex algebras given in Corollary 3.6 is the same as the class given by Meurman and Primc in [MP] using a different construction. In particular, the two expressions for the vertex operator maps in the present paper and in [MP] must be equal. But the constructions are completely different. First, the definitions of vertex operator maps are different. As in all the other constructions of the second type (see the introduction of the present paper), the vertex operator map in [MP] is defined using the residue of x_1 in the Jacobi identity for vertex operator algebras. Our construction uses a completely different definition motivated by the associativity. Second, the proofs are necessarily different. As in most of the other constructions of the second type, the proof in [MP] is based on a result stating that if a generating field is local with two other fields (or satisfies certain identities together with two other fields), then this field is also local with the fields generated by the other two fields (or satisfies certain identities together with the fields generated by the other two fields). Our proof does not use and does not need to use such a result.

4 The second construction

In this section, we give our second construction. It gives us a quasi-vertex operator algebra structure on the direct sum of a quasi-vertex operator algebra and a module satisfying suitable conditions.

Let $(V, Y_V, \mathbf{1}, L_V(1))$ be a simple quasi-vertex operator algebra. Assume that V-modules are all completely reducible and $\mathbb{R}$-graded. Also assume that intertwining operators among V-modules satisfy the associativity property, that is, for V-modules $W_1, \dots, W_5$ and intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_4}{W_1 W_5}$ and $\binom{W_5}{W_2 W_3}$, respectively, there exist a V-module W_6 and intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ of types $\binom{W_4}{W_6 W_3}$ and $\binom{W_6}{W_1 W_2}$, respectively, such that for $w_4' \in W_4'$, $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$,

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle = \langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle \tag{4.1}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$. See [H5] for conditions on V such that this associativity holds.

For simplicity, when $|z_1| > |z_2| > 0$, we use $\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)$ to also denote the element of $\mathrm{Hom}(W_3, \overline{W}_4)$ obtained from the sum of the series $\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$ for $w_4' \in W_4'$ and $w_3 \in W_3$. Similarly, when $|z_2| > |z_1 - z_2| > 0$, we have an element $\mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2) \in \mathrm{Hom}(W_3, \overline{W}_4)$. Then when $|z_1| > |z_2| > |z_1 - z_2| > 0$, (4.1) for all $w_4' \in W_4'$ and $w_3 \in W_3$ can be written as

$$\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2) = \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2).$$

We shall also say that

$$\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)$$

defined on $|z_1| > |z_2| > 0$ and

$$\mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)$$

defined on $|z_2| > |z_1 - z_2| > 0$ are *analytic extensions of each other*. We shall use also the similar notations and terminology below for other similar expressions and equalities.

Let $(W, Y_W, L_W(1))$ be an irreducible V-module. Assume that W is graded either by $\mathbb{Z}$ or by $\mathbb{Z} + \frac{1}{2}$. Also assume that for any irreducible V-module $(M, Y_M, L_M(1))$, the fusion rule $N_{WW}^M = 1$ when M is equivalent to V and $N_{WW}^M = 0$ when M is not equivalent to V. Assume in addition that there exist nondegenerate symmetric invariant bilinear forms $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$ on V and W, respectively.

Let $V_e = V \oplus W$. We shall identify V and W with the corresponding subspaces of V_e so that V and W become subspaces of V_e. We define a vertex operator map

$$\begin{aligned} Y_{V_e} : \mathbb{C}^\times &\to \mathrm{Hom}(V_e \otimes V_e, \overline{V}_e) \\ z &\mapsto Y_{V_e}(\cdot, z)\cdot \end{aligned}$$

by

$$\begin{aligned}
Y_{V_e}(u,z)v &= Y_V(u,z)v, \quad u,v\in V,\\
Y_{V_e}(u,z)w &= Y_W(u,z)w, \quad u\in V, w\in W,\\
Y_{V_e}(w,z)v &= Y_{WV}^W(w,z)v, \quad v\in V, w\in W,\\
Y_{V_e}(w_1,z)w_2 &= Y_{WW}^V(w_1,z)w_2, \quad w_1,w_2\in W,
\end{aligned}$$

where Y_{WV}^W and Y_{WW}^V are intertwining operators of types $\binom{W}{WV}$ and $\binom{V}{WW}$, respectively, constructed in [FHL]. We recall their definitions:

$$Y_{WV}^W(w,z)v = e^{zL_V(-1)}Y_W(v,-z)w$$

for $v\in V$ and $w\in W$ and

$$(v, Y_{WW}^V(w_1,z)w_2)_V = (Y_{WV}^W(e^{zL_W(1)}e^{\pi i L_W(0)}z^{-2L_W(0)}w_1, z^{-1})v, w_2)_W, \tag{4.2}$$

for $v\in V$ and $w_1,w_2\in W$, where $i=\sqrt{-1}$. Note that in (4.2), since W is graded by $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$, $z^{-2L_W(0)}w_1$ is well defined and involves only integer powers of z. From (4.2), we also have

$$(w_2, Y_{WV}^W(w_1,z)v)_W = (Y_{WW}^V(z^{-2L_W(0)}e^{-\pi i L_W(0)}e^{-z^{-1}L_W(1)}w_1, z^{-1})w_2, v)_V. \tag{4.3}$$

Since Y_{WV}^W and Y_{WW}^V are intertwining operators, they satisfy the $L(0)$-conjugation formula.

Let $\mathbf{1}_{V_e} = \mathbf{1}_V$. Let $L_{V_e}(0), L_{V_e}(-1), L_{V_e}(1) : V_e \to V_e$ be the operators that act as $L_V(0), L_V(-1)$, $L_V(1)$, respectively, on V and as $L_W(0), L_W(-1), L_W(1)$, respectively, on W.

Here is our second construction theorem:

Theorem 4.1. *The triple $(V_e, Y_{V_e}, \mathbf{1}_{V_e}, L_{V_e}(1))$ is a quasi-vertex operator algebra if W is graded by $\mathbb{Z}$ and is a quasi-vertex operator superalgebra if W is graded by $\frac{1}{2}+\mathbb{Z}$. In the case that V has a conformal element ω_V, V_e also has a conformal element $\omega_{V_e}=\omega_V$.*

Proof. We first prove the skew-symmetry. For $u,v\in V$ and $w\in W$, the skew-symmetry

$$Y_{V_e}(u,z)v = e^{zL_{V_e}(-1)}Y_{V_e}(v,-z)u$$

and

$$Y_{V_e}(u,z)w = e^{zL_{V_e}(-1)}Y_{V_e}(w,-z)u$$

follow from either the skew-symmetry for the vertex operator algebra V or the definition of $Y_{WV}^W(w,z)u$.

We now prove the skew-symmetry

$$Y_{V_e}(w_1,z)w_2 = \epsilon_W e^{zL_{V_e}(-1)}Y_{V_e}(w_2,-z)w_1 \tag{4.4}$$

for $w_1, w_2 \in W$, where ϵ_W is 1 if W is graded by $\mathbb{Z}$ and is -1 if W is graded by $\mathbb{Z} + \frac{1}{2}$. By definition, for $u \in V$ and $w_1, w_2 \in W$,

$$\begin{aligned}
&(u, Y_{V_e}(w_1, z)w_2)_V \\
&\quad = (u, Y_{WW}^V(w_1, z)w_2)_V \\
&\quad = (Y_{WV}^W(e^{zL_W(1)}e^{\pi i L_W(0)}z^{-2L_W(0)}w_1, z^{-1})u, w_2)_W \\
&\quad = (e^{z^{-1}L_W(-1)}Y_W(u, -z^{-1})e^{zL_W(1)}e^{\pi i L_W(0)}z^{-2L_W(0)}w_1, w_2)_W. \qquad (4.5)
\end{aligned}$$

Using the fact that the adjoint operator of $L_W(n)$ is $L_W(-n)$ for $n = -1, 0, 1$, the invariance of the bilinear form on V and the definitions of Y_{WV}^W and Y_{WW}^V, (4.3), the right-hand side of (4.5) is equal to

$$\begin{aligned}
&(e^{\pi i L_W(0)}z^{-2L_W(0)}w_1, e^{zL_W(-1)}Y_W(e^{-z^{-1}L_V(1)}(-z^2)^{L_V(0)}u, -z)e^{z^{-1}L_W(1)}w_2)_W \\
&\quad = (e^{\pi i L_W(0)}z^{-2L_W(0)}w_1, Y_{WV}^W(e^{z^{-1}L_W(1)}w_2, z)e^{-z^{-1}L_V(1)}(-z^2)^{L_V(0)}u)_W \\
&\quad = ((-z^2)^{L_V(0)}e^{-z^{-1}L_V(-1)}Y_{WW}^V(z^{-2L_W(0)}e^{-\pi i L_W(0)}w_2, z^{-1})e^{\pi i L_W(0)}z^{-2L_W(0)}w_1, u)_V. \qquad (4.6)
\end{aligned}$$

Since the weight of $L_V(-1)$ is 1, we have

$$\begin{aligned}
(-z^2)^{L_V(0)}e^{-z^{-1}L_V(-1)} &= e^{zL_V(-1)}(-z^2)^{L_V(0)} \\
&= e^{zL_V(-1)}z^{2L_V(0)}e^{-\pi i L_V(0)}.
\end{aligned}$$

Using this formulas, the $L(0)$-conjugation formula for Y_{WW}^V and the fact that $e^{-2\pi i L_W(0)} = \epsilon_W$, we see that the right-hand side of (4.6) is equal to

$$\begin{aligned}
&(\epsilon_W e^{zL_V(-1)}Y_{WW}^V(w_2, -z)w_1, u)_V \\
&\quad = (\epsilon_W e^{zL_{V_e}(-1)}Y_{V_e}(e^{-2\pi i L_W(0)}w_2, -z)w_1, u)_V \\
&\quad = (u, \epsilon_W e^{zL_{V_e}(-1)}Y_{V_e}(e^{-2\pi i L_W(0)}w_2, -z)w_1)_V, \qquad (4.7)
\end{aligned}$$

proving the skew-symmetry (4.4) and thus the skew-symmetry for Y_{V_e} holds.

Next we prove the associativity. The associativity properties for the vertex operator maps Y_V and Y_W give the associativity

$$Y_{V_e}(u, z_1)Y_{V_e}(v, z_2) = Y_{V_e}(Y_{V_e}(u, z_1 - z_2)v, z_2)$$

for $u, v \in V$ when $|z_1| > |z_2| > |z_1 - z_2| > 0$. From [FHL], we obtain the following associativity properties

$$\begin{aligned}
Y_{V_e}(u, z_1)Y_{V_e}(w_1, z_2)w_2 &= Y_{V_e}(Y_{V_e}(u, z_1 - z_2)w_1, z_2)w_2, \\
Y_{V_e}(w_1, z_1)Y_{V_e}(u, z_2)w_2 &= Y_{V_e}(Y_{V_e}(w_1, z_1 - z_2)u, z_2)w_2, \\
Y_{V_e}(w_1, z_1)Y_{V_e}(w_2, z_2)u &= Y_{V_e}(Y_{V_e}(w_1, z_1 - z_2)w_2, z_2)u
\end{aligned}$$

for $u \in V$ and $w_1, w_2 \in W$ when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

We still need to prove the associativity

$$Y_{V_e}(w_1, z_1)Y_{V_e}(w_2, z_2)w_3 = Y_{V_e}(Y_{V_e}(w_1, z_1 - z_2)w_2, z_2)w_3$$

for $w_1, w_2, w_3 \in W$ when $|z_1| > |z_2| > |z_1 - z_2| > 0$. By definition,

$$Y_{V_e}(w_1, z_1)Y_{V_e}(w_2, z_2)w_3 = Y^W_{WV}(w_1, z_1)Y^V_{WW}(w_2, z_2)w_3.$$

By assumption, the associativity property for intertwining operators for the quasi-vertex operator algebra V holds. In particular, there exist a V-module M and intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W}{MW}$ and $\binom{M}{WW}$, respectively, such that

$$Y^W_{WV}(w_1, z_1)Y^V_{WW}(w_2, z_2)w_3 = \mathcal{Y}_1(\mathcal{Y}_2(w_1, z_1 - z_2)w_2, z_2)w_3$$

when $|z_1| > |z_2| > |z_1-z_2| > 0$. By assumption, M is equivalent to a direct sum of irreducible V-modules and the fusion rule $N^M_{WW} = 1$ when M is equivalent to V and $N^M_{WW} = 0$ when M is not equivalent to V. Then we see that M must be equivalent to V and $\mathcal{Y}_2$ is proportional to Y^V_{WW} after we identify M with V. On the other hand, it is a general fact that the fusion rule N^W_{VW} is 1 (see [FHL]). In particular, $\mathcal{Y}_1$ is proportional to $Y^W_{VW}(w_1, z_1)$. Thus there exists $\lambda \in \mathbb{C}$ such that

$$Y^W_{WV}(w_1, z_1)Y^V_{WW}(w_2, z_2)w_3 = \lambda Y^W_{VW}(Y^V_{WW}(w_1, z_1 - z_2)w_2, z_2)w_3 \tag{4.8}$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$. From the genus-zero Moore-Seiberg equations (see [MS] and [H3]), one can show that $\lambda = 1$ and thus the associativity property is proved. Since the proof in this case is easy, to make our proof self contained, here we give a direct proof that $\lambda = 1$.

Note that we have proved the skew-symmetry

$$Y^V_{WW}(w_1, z)w_2 = \epsilon_W e^{zL(-1)}Y^V_{WW}(w_2, -z)w_1 \tag{4.9}$$

for $w_1, w_2 \in W$, where ϵ_W is $+1$ if W is graded by $\mathbb{Z}$ and is -1 if W is graded by $\mathbb{Z} + \frac{1}{2}$. We shall use $\sim$ to mean that two expressions are analytic extensions of each other. Then for $w_1, w_2, w_3 \in W$, using the associativity (4.8) and the definition of Y^W_{WV}, we have

$$\begin{aligned}
&Y^W_{WV}(w_1, z_1)Y^V_{WW}(w_2, z_2)w_3\\
&\quad\sim \lambda Y_W(Y^V_{WW}(w_1, z_1 - z_2)w_2, z_2)w_3\\
&\quad\sim \lambda e^{z_2L(-1)}Y^W_{WV}(w_3, -z_2)Y^V_{WW}(w_1, z_1 - z_2)w_2\\
&\quad\sim \lambda^2 e^{z_2L(-1)}Y_W(Y^V_{WW}(w_3, -z_1)w_1, z_1 - z_2)w_2.
\end{aligned} \tag{4.10}$$

On the other hand, using (4.8) and (4.9), we have

$$\begin{aligned}
&Y^W_{WV}(w_1, z_1)Y^V_{WW}(w_2, z_2)w_3\\
&\quad\sim \epsilon_W Y^W_{WV}(w_1, z_1)e^{z_2L(-1)}Y^V_{WW}(w_3, -z_2)w_2\\
&\quad\sim \epsilon_W e^{z_2L(-1)}Y^W_{WV}(w_1, z_1 - z_2)Y^V_{WW}(w_3, -z_2)w_2\\
&\quad\sim \epsilon_W \lambda e^{z_2L(-1)}Y_W(Y^V_{WW}(w_1, z_1)w_3, -z_2)w_2\\
&\quad\sim \epsilon_W^2 \lambda e^{z_2L(-1)}Y_W(e^{z_1L(-1)}Y^V_{WW}(w_3, -z_1)w_1, -z_2)w_2\\
&\quad\sim \lambda e^{z_2L(-1)}Y_W(Y^V_{WW}(w_3, -z_1)w_1, z_1 - z_2)w_2.
\end{aligned} \tag{4.11}$$

Since both the right-hand sides of (4.10) and (4.11) are defined on the same region and they are analytic extensions of each other, they must be equal on the region that they are defined. Thus $\lambda = 1$.

Note that the skew-symmetry (4.4) has a sign ϵ_W. Since Y_{V_e} satisfies the associativity and skew-symmetry, from [H3], it also satisfies the commutativity for quasi-vertex operator algebras when W is graded by $\mathbb{Z}$ and satisfies the commutativity for quasi-vertex operator superalgebras when W is graded by $\mathbb{Z} + \frac{1}{2}$. Since Y_{V_e} involves only integer powers of the variable, the rationality for products and iterates also hold (see also [H3]). The other axioms for grading-restricted vertex operator (super)algebras can be easily verified. Thus we have proved that $(V_{V_e}, Y_{V_e}, \mathbf{1}_{V_e}, L_{V_e}(1))$ is a quasi-vertex operator algebra when W is graded by $\mathbb{Z}$ and is a quasi-vertex operator superalgebra when W is graded by $\mathbb{Z} + \frac{1}{2}$. ∎

Example 4.2. As is mentioned above, this second construction is in fact a generalization of the construction of the moonshine module vertex operator algebra and the corresponding vertex operator superalgebra in [H1]. In fact, take V to be the fixed point vertex operator subalgebra V_Λ^+ of the Leech lattice vertex operator algebra V_Λ under the automorphism induced from the automorphism $\alpha \mapsto -\alpha$ of the Leech lattice Λ. Take W to be the fixed point V_Λ^+-submodule $(V_\Lambda^T)^+$ of the irreducible twisted V_Λ-module V_Λ^T. Then the conditions to use Theorem 4.1 are satisfied. By Theorem 4.1 and the fact that V_Λ^+ has a conformal element, the moonshine module $V^\natural = V_\Lambda^+ \oplus (V_\Lambda^T)^+$ has a structure of vertex operator algebra for which the vertex operator map, the vacuum and the conformal element are given above. If we take V to be the same and W to be the eigenspace $(V_\Lambda^T)^-$ for the action of the automorphism above on V_Λ^T with the eigenvalue -1. Then $V_\Lambda^+ \oplus (V_\Lambda^T)^-$ has a structure of vertex operator superalgebra given above. See [FLM], [H1] and [H2] for the background material that can be used to verify the conditions to use Theorem 4.1.

Example 4.3. Another application is a construction of the vertex operator superalgebra structure on the direct sum $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ of the vertex operator algebra $L(\frac{1}{2}, 0)$ for the minimal model of central charge $\frac{1}{2}$ and the $L(\frac{1}{2}, 0)$-module $L(\frac{1}{2}, \frac{1}{2})$ of lowest weight $\frac{1}{2}$. The conditions to use Theorem 4.1 are satisfied by the results obtained in [W] and [H4] or [H5]. Thus $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ has a structure of vertex operator superalgebra given above. After tensoring the vertex operator algebra $L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$ with the vertex operator algebra associated to the lattice of rank 1 generated by α satisfying $(\alpha, \alpha) = m \in 2\mathbb{Z}_+$, we obtain the vertex operator superalgebra for the Moore-Read state with the filling factors $\nu = \frac{1}{m}$ in the conformal-field-theoretic study of fractional quantum Hall states. We refer the reader to [H9] for details on these examples.

References

[A] D. Adamović, Representations of the $N = 2$ vertex operator superalgebra, *Internat. Math. Res. Notices* **1999** (1999), 62–79.

[DGM] L. Dolan, P. Goddard and P. Montague, Conformal field theory of twisted vertex operators, *Nucl. Phys.* **B338** (1990), 529–601.

[FFR] A. Feingold, I. Frenkel and J. Ries, Spinor Construction of Vertex Operator Algebras, Triality, and $E_8^{(1)}$, *Contemporary Math.*, Vol. 121, Amer. Math. Soc., Providence, RI, 1991.

[FHL] I. Frenkel, Y. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs American Math. Soc.* **104**, 1993.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, *Pure and Applied Math.,* Vol. 134, Academic Press, 1988.

[FZ] I. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123–168.

[H1] Y.-Z. Huang, A nonmeromorphic extension of the moonshine module vertex operator algebra, in: *Moonshine, the Monster and related topics, Proc. Joint Summer Research Conference, Mount Holyoke, 1994*, ed. C. Dong and G. Mason, Contemporary Math., Vol. 193, Amer. Math. Soc., Providence, 1996, 123–148.

[H2] Y.-Z. Huang, Introduction to vertex operator algebras, III, in: *Moonshine and vertex operator algebras*, RIMS Kokyuroku 904, RIMS, Kyoto, Japan, 1995, 51–77.

[H3] Y.-Z. Huang, Generalized rationality and a Jacobi identity for intertwining operator algebras, *Selecta Math.* **6** (2000), 225–267.

[H4] Y.-Z. Huang, Virasoro vertex operator algebras, (nonmeromorphic) operator product expansion and the tensor product theory, *J. Alg.* **182** (1996), 201–234.

[H5] Y.-Z. Huang, Differential equations and intertwining operators, *Comm. Contemp. Math.* **7** (2005), 375–400.

[H6] Y.-Z. Huang, A cohomology theory of grading-restricted vertex algebras, *Comm. Math. Phys.* **327** (2014), 279–307.

[H7] Y.-Z. Huang, First and second cohomologies of grading-restricted vertex algebras, *Comm. Math. Phys.* **327** (2014), 261–278.

[H8] Y.-Z. Huang, A program to construct and study conformal field theories, article in the blog *Conformal Field Theory*, URL address: https://qcft.wordpress.com/2014/09/16/a-program-to-construct-and-study-conformal-field-theories/

[H9] Y.-Z. Huang, Fractional quantum Hall states, vertex operator superalgebras and topological orders, to appear.

[HL] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebra and vertex tensor categories, in: *Lie Theory and Geometry*, in honor of Bertram Kostant, ed. R. Brylinski, J.-L. Brylinski, V. Guillemin, V. Kac, Birkhäuser, Boston, 1994, 349–383.

[HLZ] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: Introduction and strongly graded algebras and their generalized modules, in: *Conformal Field Theories and Tensor Categories, Proceedings of a Workshop Held at Beijing International Center for Mathematics Research*, ed. C. Bai, J. Fuchs, Y.-Z. Huang, L. Kong, I. Runkel and C. Schweigert, Mathematical Lectures from Beijing University, Vol. 2, Springer, New York, 2014, 169–248.

[KW] V. Kac and W. Wang, Vertex operator superalgebras and their representations, in: *Mathematical aspects of conformal and topological field theories and quantum groups (South Hadley, MA, 1992)*, Contemp. Math., Vol. 175, Amer. Math. Soc., Providence, 1994, 161–191.

[MP] A. Meurman and M. Primc, Annihilating fields of standard modules of $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$ and combinatorial identities, preprint, 1994; *Mem. Amer. Math. Soc.* **652**, 1999.

[MR] G. Moore and N. Read, Non-Abelions in the fractional quantum Hall effect, *Nucl. Phys.* **B360** (1991), 362–396.

[MS] G. Moore and N. Seiberg, Classical and quantum conformal field theory *Comm. Math. Phys.* **123** (1989), 177–254.

[W] W. Wang, Rationality of Virasoro vertex operator algebras, *Duke Math. J.* **71** (1993), 197–211.

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