Additional review problems

for the Final Exam, 350 Honors Section

Fall, 2019

You should go through the material covered in the classes and review homework problems, examples in the book and the two midterm exams. Here are some additional review problems:

1. State and prove the following theorems and corollaries in the book:

   (a) The existence of a basis of a vector space generated by a finite subset (Theorem 1.9).

   (b) Replacement Theorem (Theorem 1.10).

   (c) All basis of a vector space generated by a finite subset have the same number of finitely many elements (Corollary 1 in Section 1.6).

   (d) Corollary 2 in Section 1.6.

   (e) Dimension Theorem (Theorem 2.3).

   (f) ϕ_β is an isomorphism (Theorem 2.21).

   (g) Φ_β^γ is an isomorphism (Theorem 2.20).

   (h) $\det(AB) = \det(A)\det(B)$ (Theorem 4.7).

   (i) $\det(A^t) = \det(A)$ (Theorem 4.8).

   (j) A linear operator on a finite-dimensional vector space is diagonalizable if and only if there exists an oredered basis of V consisting of eigenvectors of T. (Theorem 5.1).

   (k) Theorem 5.20.

   (l) Theorem 5.21.

   (m) The Cayley-Hamilton Theorem.

   (n) Cauchy-Schwarz Inequality (Part (c) of Theorem 6.2).

   (o) Triangle Inequality (Part (d) of Theorem 6.2).

   (p) Theorem 6.4.

   (q) Theorem 6.7

2. Prove or disprove the following statements:

(a) Let W be the subset of $\mathbb{F}^3$ consisting of all vectors (x_1, x_2, x_3) such that $3x_1x_2 = 4x_2x_3$ (that is, $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 3x_1x_2 = 4x_2x_3\}$). Then W is a subspace of $\mathbb{F}^3$.

(b) Let W be the subset of $P_2(\mathbb{F})$ consisting of all polynomials $a_2x^2 + a_1x + a_0$ such that $5a_2 - 3a_0 = 2a_1$ (that is, $W = \{a_2x^2 + a_1x + a_0 \in P_2(\mathbb{F}) \mid 5a_2 - 3a_0 = 2a_1\}$). Then W is a subspace of $P_2(\mathbb{F})$.

(c) Let W be the subset of $P_n(\mathbb{F})$ consisting of all polynomials of the form $ax^{n-1} + (3ab)x^{n-2} + b$ for $a, b \in \mathbb{F}$ such that either $a \neq 0$ or $a = b = 0$ (that is, $W = \{ax^{n-1} + (3ab)x^{n-2} + b \mid a, b \in \mathbb{F}, a \neq 0 or a = b = 0\}$). Then W is a subspace of $P_n(\mathbb{F})$.

(d) Let W be the subset of $M_{n\times n}(\mathbb{F})$ consisting of $n \times n$ matrices A such that $\operatorname{tr} A = 0$. Then W is a subspace of $M_{n\times n}(\mathbb{F})$.

(e) Let W be the subset of $M_{n\times n}(\mathbb{F})$ consisting of $n \times n$ matrices A such that $\det A = 0$. Then W is a subspace of $M_{n\times n}(\mathbb{F})$.

3. Let S_1 and S_2 be subsets of a vector space V.

(a) Prove that if $S_1 \subset S_2$, then $\operatorname{span}(S_1) \subset \operatorname{span}(S_2)$.

(b) Prove that if $S_1 \subset S_2$ and $\operatorname{span}(S_1) = V$, then $\operatorname{span}(S_2) = V$.

(c) Prove that if $S_1 \subset S_2$, S_2 is linearly independent and $\operatorname{span}(S_1) = \operatorname{span}(S_2)$, then $S_1 = S_2$.

(d) Prove that $\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.

4. Let $\{u, v, w\}$ be a set of linearly independent vectors of a vector space V.

a) Determine whether the vectors $3u - v + 2w$, $2u + 4v$, $u + v - w$ are linear independent?

b) If $\{u, v, w\}$ is a basis of V, is $\{3u - v + 2w, 2u + 4v, u + v - w\}$ also a basis of V. Prove your answer.

5. Let $\{v_1, \ldots, v_n\}$ be a set of linearly independent vectors of a vector space V. Let A be an $n \times n$ invertible matrix and $u_i = \sum_{j=1}^{n} a_{ij}v_j$ for $i = 1, \ldots, n$.

a) Prove that the vectors $\{u_1, \ldots, u_n\}$ are linear independent.

b) Prove that if $\{v_1, \ldots, v_n\}$ is a basis of V, then $\{u_1, \ldots, u_n\}$ is also a basis of V.

6. Let V and W be n-dimensional vector spaces, let $T : V \to W$ be a linear transformation from V to W and let $\{v_1, \ldots, v_k\}$ be a subset of V.

(a) If $\{v_1, \ldots, v_k\}$ is linearly independent and T is an isomorphism, prove that $\{T(v_1), \ldots, T(v_k)\}$ is a linearly independent subset of W.

(b) If $\{v_1, \ldots, v_k\}$ is linearly dependent, prove that $\{T(v_1), \ldots, T(v_k)\}$ is a linearly dependent subset of W.

7. Let V be vector space of all 2 by 2 upper triangular matrices. Let $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$.

   (a) Prove that $\beta = \{A_1, A_2, A_3\}$ is a basis of V.

   (b) Let $B = \begin{pmatrix} 3 & 5 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. Find $[B]_\beta$ and $[C]_\beta$.

   (c) Find $[2B - 3C]_\beta$.

8. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the map defined by

$$T(f(x)) = \int_0^x f(t)dt + f'(x).$$

   (a) Show that T a linear transformation.

   (b) Find a basis of the range $R(T)$ of T and justify that your set is indeed a basis.

   (c) What are the nullity and rank of T? Justify your answers.

   (d) Is T one-to-one? Is T Onto? Justify your answers.

   (e) Let $\beta = \{1, x, x^2\}$ (the standard ordered basis of $P_2(\mathbb{R})$) and $\gamma = \{1, x, x^2, x^3\}$ (the standard ordered basis of $P_3(\mathbb{R})$). Calculate $[T]_\beta^\gamma$.

9. Let $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ be the linear transformation defined by

$$T(a_1x + a_0) = (-a_1 + a_0)x + (-a_1 + 2a_0).$$

   Let $\beta = \{1, x\}$ be the standard ordered basis of $P_1(\mathbb{R})$, and also consider the ordered basis $\beta' = \{1, x + 1\}$ of $P_1(\mathbb{R})$.

   (a) Find the matrix $[T]_\beta$ of T. (Note that $[T]_\beta$ can also be written as $[T]_\beta^\beta$.) Justify your answer.

   (b) Find the matrix $[T]_{\beta'}^\beta$ of T. Justify your answer.

   (c) Find the matrix $[T]_{\beta'}(= [T]_{\beta'}^{\beta'})$ of T. Justify your answer.

   (d) Find the change of coordinate matrix $Q = [1_{P_1(\mathbb{R})}]_{\beta'}^\beta$.

10. Find the dual basis of the basis $\{(1, 2), (3, 5)\}$ of $\mathbb{R}^2$.

11. Find the dual basis of the standard basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{F})$.

12. Find the dual basis of the standard basis $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ of $M_{2\times 2}(\mathbb{F})$.

13. Let $V = \mathbb{F}^3$ and let β be the ordered basis $\{v_1, v_2, v_3\}$ for V, where $v_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$. Let $\beta^* = \{f_1, f_2, f_3\}$ be the corresponding dual basis for the vector space V^* consisting of the linear functionals from V to $\mathbb{F}$. Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{F}^2$. Find explicit formulas for $f_1 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $f_2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $f_3 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in terms of a, b and c.

14. Let $A = \begin{pmatrix} -2 & -1 \\ 2 & 2 \end{pmatrix}$.

    (a) Determine whether A is diagonalizable over the field $\mathbb{Q}$ of rational numbers. If it is diagonalizable, write down the diagonal matrix.

    (b) Determine whether A is diagonalizable over the field $\mathbb{R}$ of real numbers. If it is diagonalizable, write down the diagonal matrix.

15. For the following linear operator T on a vector space V over the field $\mathbb{R}$, find the eigenvalues of T and the corresponding eigenvectors, determine whether T is diagonalizable over $\mathbb{R}$ and if T is diagonalizable, write down an ordered basis β such that $[T]_\beta$ is a diagonal matrix:

    (a) $V = \mathbb{R}^2$ and T is defined by $T(a, b) = (a + b, b)$.

    (b) $V = P_1(\mathbb{R})$ and T is defined by $T(ax + b) = (-6a + 2b)x + (-6a + b)$.

    (c) $V = P_2(\mathbb{R})$ and T is defined by $T(f(x)) = f(0) + f(1)(x^2 + x)$.

16. Consider the linear operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by $T(a, b, c, d) = (b + 2c, a + b + d, c - 3d, 5c)$.

    (a) Find n such that $\{(1, 0, 0, 0), T(1, 0, 0, 0), \ldots, T^{n-1}(1, 0, 0, 0)\}$ is an ordered basis of the T-cyclic subspace W of $\mathbb{R}^4$ generated by $(1, 0, 0, 0)$.

    (b) Express $T^n(x)$ as a linear combination of $x, T(x), \ldots, T^{n-1}(x)$, where n is the integer found in (a).

    (c) Use (b) to find the characteristic polynomial of T_W.

17. For the linear transformations T below, find all eigenvalues, eigenvectors and generalized eigenvectors. Then for each eigenvector, find a cycle of generalized eigenvectors for T such that the eigenvector is the initial vector.

    (a) $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ given by $T(a + bx) = (a + b) + (-a + 3b)x$.

    (b) $T = L_A : \mathbb{R}^2 \to \mathbb{R}^2$ where $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

18. Let T be the linear operator on $P_2(\mathbb{R})$ defined by

$$T(ax^2+bx+c) = (-4b-2c)x^2 + (a+4b+c)x + (-a-2b+c).$$

(a) Find the characteristic polynomial of T.
(b) Find all the eigenvalues of T.
(c) Find all the eigenvectors of T.
(d) Find the Jordan canonical form J of T and the corresponding Jordan canonical basis of $P_2(\mathbb{R})$ for T (that is, a basis β of $P_2(\mathbb{R})$ such that $[T]_\beta$ is the Jordan canonical form J).

19. Let T be a linear operator on a 8-dimensional vector space V over $\mathbb{C}$, that is, let T a linear transform from a 8-dimensional vector space V over $\mathbb{C}$ to V itself. Suppose that we have

$$\text{Rank } (T-I) = 6, \ \text{Rank } (T-I)^2 = 6,$$
$$\text{Rank } (T-3I) = 6, \ \text{Rank } (T-3I)^2 = 4, \ \text{Rank } (T-3I)^3 = 3,$$
$$\text{Rank } (T-3I)^4 = 2, \ \text{Rank } (T-3I)^5 = 2,$$

Find the Jordan canonical form of T.

20. Let $A = \begin{pmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.
(b) Find all the eigenvalues of A.
(c) Find all the eigenvectors of A corresponding to each eigenvalue.
(d) Find all the other generalized eigenvectors (not eigenvectors) of A corresponding to each eigenvalue.
(e) Find the Jordan canonical form J of A.

21. Suppose that the characteristic polynomial of a linear operator T splits and the eigenvalues of T are 1, 3 and 5. If the dot diagrams for the restrictions of T to its generalized eigenspaces are

$\lambda_1 = 1$:

$$\begin{array}{ccc} \bullet & \bullet & \bullet \end{array}$$

$\lambda_2 = 3$:

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \\ \bullet & & \end{array}$$

$\lambda_3 = 5$:

$$\begin{array}{c} \bullet \\ \bullet \end{array}$$

Find the multiplicities of eigenvalues of T and the Jordan canonical form of T.

22. Find the Jordan canonical form of the following matrix and linear operator:

    (a) The derivative operator D on $P_2(\mathbb{R})$ defined by $D(f(x)) = f'(x)$.

    (b) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$

23. Consider the vector space $P_2(\mathbb{R})$ and the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

for $f, g \in P_2(\mathbb{R})$.

    (a) Find an orthonormal basis of the inner product space defined above starting from the standard basis $\{1, x, x^2\}$.

    (b) Express the polynomial $1 + x$ as a linear combination of the elements of the orthonormal basis that you found in Part (a).

24. Prove that for any $n \times n$ matrix A, $R(L_{A^*})^{\perp} = N(L_A)$, where $A^* = \bar{A}^t$.

25. Let V be the vector space of all continuous real-valued functions on $[-1, 1]$ with an inner product defined by

$$(f, g) = \int_{-1}^1 f(x)g(x)dx$$

for $f, g \in V$. Let W_e and W_o be the subspace of V consisting even and odd functions, respectively. Prove $W_e^{\perp} = W_o$ and $W_o^{\perp} = W_e$.