Additional review problems

for Exam 2, 350 Honors Section

Fall, 2019

You should go through the material covered in the classes and review homework problems and examples in the book. Here are some additional review problems:

1. State and prove the following theorems and corollaries in the book:

   (a) $\det(AB) = \det(A)\det(B)$ (Theorem 4.7).

   (b) $\det(A^t) = \det(A)$ (Theorem 4.8).

   (c) A linear operator on a finite-dimensional vector space is diagonalizable if and only if there exists an oredered basis of V consisting of eigenvectors of T. (Theorem 5.1).

   (d) Theorem 5.20.

   (e) Theorem 5.21.

   (f) The Cayley-Hamilton Theorem.

2. Find the dual basis of the basis $\{(1,2),(3,5)\}$ of $\mathbb{R}^2$.

3. Let $V = \mathbb{F}^3$ and let β be the ordered basis $\{v_1, v_2, v_3\}$ for V, where $v_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$. Let $\beta^* = \{f_1, f_2, f_3\}$ be the corresponding dual basis for the vector space V^* consisting of the linear functionals from V to $\mathbb{F}$. Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{F}^2$. Find explicit formulas for $f_1\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $f_2\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $f_3\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in terms of a, b and c.

4. Let $A = \begin{pmatrix} -2 & -1 \\ 2 & 2 \end{pmatrix}$.

   (a) Determine whether A is diagonalizable over the field $\mathbb{Q}$ of rational numbers. If it is diagonalizable, write down the diagonal matrix.

   (b) Determine whether A is diagonalizable over the field $\mathbb{R}$ of real numbers. If it is diagonalizable, write down the diagonal matrix.

5. For the following linear operator T on a vector space V over the field $\mathbb{R}$, find the eigenvalues of T and the corresponding eigenvectors, determine whether T is diagonalizable over $\mathbb{R}$ and if T is diagonalizable, write down an ordered basis β such that $[T]_\beta$ is a diagonal matrix:

   (a) $V = \mathbb{R}^2$ and T is defined by $T(a,b) = (a+b,b)$.

   (b) $V = P_1(\mathbb{R})$ and T is defined by $T(ax+b) = (-6a+2b)x + (-6a+b)$.

   (c) $V = P_2(\mathbb{R})$ and T is defined by $T(f(x)) = f(0) + f(1)(x^2+x)$.

6. Consider the linear operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by $T(a,b,c,d) = (b+2c, a+b+d, c-3d, 5c)$.

   (a) Find n such that $\{(1,0,0,0), T(1,0,0,0), \ldots, T^{n-1}(1,0,0,0)\}$ is an ordered basis of the T-cyclic subspace W of $\mathbb{R}^4$ generated by $(1,0,0,0)$.

   (b) Express $T^n(x)$ as a linear combination of $x, T(x), \ldots, T^{n-1}(x)$, where n is the integer found in (a).

   (c) Use (b) to find the characteristic polynomial of T_W.

7. For the linear transformations T below, find all eigenvalues, eigenvectors and generalized eigenvectors. Then for each eigenvector, find a cycle of generalized eigenvectors for T such that the eigenvector is the initial vector.

   (a) $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ given by $T(a+bx) = (a+b) + (-a+3b)x$.

   (b) $T = L_A : \mathbb{R}^2 \to \mathbb{R}^2$ where $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.