Math 152, Fall, 2004, Workshop 8

Honors Section

1. This problem is based on the fact that bounded monotonic sequences converge.

   (a) Given that $1 < x < y$, find a similar relation between $\dfrac{1}{x}$ and $\dfrac{1}{y}$. Then show that if $1 < x < y < 4$ then $1 < 4 - \dfrac{1}{x} < 4 - \dfrac{1}{y} < 4$. (Note: you must show that each of these three inequalities is true).

   (b) Compute five terms $(a_1, \ldots, a_5)$ of the recursively defined sequence

   $$a1 = 1, \quad a_{n+1} = 4 - \frac{1}{a_n}.$$

   (c) Show that $1 < a_1 < a_2 < 4$; then, using (a), show that the sequence $\{a_n\}$ is increasing and bounded. Does it converge?

   (d) Find the limit of the sequence in (b) (call the limit Q, and take the $n \to \infty$ limit on both sides of the recursion to obtain an equation for Q.)

2. (a) Using the integral test, show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges

   (b) Let s denote the sum of this series: $s = \displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. If you attempted to estimate s by computing a partial sum of the series—that is, by adding up the first few terms—how many terms would you have to include to guarantee that the error (the difference between s and the sum you calculate) would be less than 0.05? Explain why you are certain of this accuracy. (Use the inequality

   $$\sum_{n=N+1}^{\infty} f(n) \leq \int_N^{\infty} f(x)dx.$$

   You may ignore any calculator (round-off) error arising in your addition.)

3. Consider the following three series:

$$\text{(i)} \sum_{n=1}^{\infty} \frac{4 - \sin n}{n^2 + 1}, \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{4 - \sin n}{n + 1}, \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{4 - \sin n}{2^n + 1}.$$

(a) Use the comparison test to determine whether or not each of these series converges.

(b) For each series which converges, give an approximation of its sum, together with an error estimate, as follows. First calculate the sum s_5 of the first 5 terms. Then find a number r such that you know that the error $R_5 = s - s_5$ satisfies $|R_5| \leq r$. (Hint: If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $|R_N| = |s - s_N| = |\sum_{n=N+1}^{\infty} a_n| \leq \sum_{n=N=1}^{\infty} b_n$. If $a_n = f(n)$ and $f(x)$ satisfies the hypothesis in the integral test, then $|R_N| \leq \int_N^{\infty} f(x)dx$. This is called estimating the error. In one case you will have to first estimate using another series, then estimate that error using an integral.)