Commun. Math. Phys. 200, 421 – 444 (1999)

Communications in
Mathematical Physics

Harnack Type Inequality: the Method of Moving Planes

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Received: 30 September 1997 / Accepted: 21 July 1998

Abstract: A Harnack type inequality is established for solutions to some semilinear elliptic equations in dimension two. The result is motivated by our approach to the study of some semilinear elliptic equations on compact Riemannian manifolds, which originated from some Chern–Simons Higgs model and have been studied recently by various authors.

0. Introduction

Let (M, g) be a compact Riemann surface without boundary, V be a positive function on M, W be a function with $\int_M W dv_g = 1$. Throughout the paper dv_g denotes the volume element of g, Δ_g denotes the Laplace Beltrami operator with respect to g. For $\lambda \in \mathbb{R}$, we seek a solution of

$$-\Delta_g u = \lambda\left(\frac{Ve^u}{\int_M Ve^u dv_g} - W\right) \quad \text{on } M. \qquad (E_u)_\lambda$$

Clearly $\int_M W dv_g = 1$ is a necessary condition for $(E_u)_\lambda$ to have a solution. If we set $\xi = u - \log \int_M Ve^u dv_g$ for a solution of $(E_u)_\lambda$, then ξ satisfies

$$-\Delta_g \xi = \lambda(Ve^\xi - W) \quad \text{on } M, \qquad (E_\xi)_\lambda$$

and

$$\int_M Ve^\xi dv_g = 1. \qquad (1)$$

* Partially supported by the Alfred P. Sloan Foundation Research Fellowship, NSF grant DMS-9706887, and a Rutgers University Research Council grant.

Equation $(E_u)_\lambda$ has been studied by Kazdan and Warner [29] in connection with the prescribed Gauss curvature problem, while, it also arises from some Chern–Simons Higgs model as discussed in Taubes [40, 41], Hong, Kim and Pac [27], Jackiw and Weinberg [28], Spruck and Yang [38], Caffarelli and Yang [11], Tarantello [39], Struwe and Tarantello [35], Ding, Jost, Li and Wang [22, 23], and the references therein. Related problems are studied by Carleson and Chang in [14]. Such equations on bounded domains of $\mathbb{R}^2$ with Dirichlet boundary conditions play an important role in the context of statistical mechanics of point vortices in the mean field limit as discussed in Caglioti, Lions, Marchioro and Pulvirenti [12, 13] and Kiessling [30]. In particular, it is proved in [35], when (M, g) is a flat torus with fundamental cell domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, $V \equiv 1$ and $W \equiv 1/vol(M)$, that Eq. $(E_u)_\lambda$ has at least one nontrivial solution for $8\pi < \lambda < 4\pi^2$. On the other hand Eq. $(E_u)_{8\pi}$, with $W \equiv 1/vol(M)$, is studied in [22] where sufficient conditions are given for the existence of solutions. Such conditions obviously hold when (M, g) is a flat two dimensional torus, $V \equiv 1$ and $W \equiv 1/vol(M)$. The author was recently informed by G. Tarantello that she and M. Nolasco have independently established the existence results in the special case that (M, g) is a flat two-dimensional torus, $V \equiv 1$ and $W \equiv 1/vol(M)$.

In view of our earlier work [31], we propose a different approach to study the existence of solutions of $(E_u)_\lambda$. Clearly $(E_u)_\lambda$ is invariant when replacing u by $u + constant$. Assuming V and W are Lipschitz functions, it is well known that when λ lies in compact subsets of $(-\infty, 8\pi)$, all solutions u of $(E_u)_\lambda$, after a normalization $\int_M u dv_g = 0$, stay bounded in $C^{2,\alpha}(M)$ for $0 < \alpha < 1$. For λ in compact subsets of $\cup_{m=1}^\infty \big(8\pi m, 8\pi(m+1)\big)$, the same conclusion holds due to the results of Brezis and Merle [8] and Li and Shafrir [32]. For $0 < \alpha < 1$, let

$$X_\alpha = \{u \in C^{2,\alpha}(M) \mid \int_M u dv_g = 0\}.$$

X_α, equipped with the $C^{2,\alpha}(M)$ norm, is a Banach space. We introduce an operator $K_\lambda : X_\alpha \to X_\alpha$ by

$$K_\lambda(u) = \lambda(-\Delta_g)^{-1}\left(\frac{Ve^u}{\int_M Ve^u dv_g} - W\right).$$

It follows from standard elliptic theories that K_λ is a well defined compact operator. Equation $(E_u)_\lambda$ is equivalent to $(I - K_\lambda)u = 0$ in X_α. For any bounded open set $O \subset X_\alpha$, the Leray-Schauder degree $\deg(I - K_\lambda, O, 0)$ is well defined provided 0 does not belong to $(I - K_\lambda)(\partial O)$. For the definition of the Leray-Schauder degree and its various properties, see, for example, Nirenberg [34]. Let

$$B_a = \{u \in X_\alpha \mid \|u\|_{X_\alpha} < a\}$$

denote the ball in X_α. Due to the above mentioned a priori estimates of solutions of $(E_u)_\lambda$ for λ in compact subsets of $\mathbb{R} \setminus \cup_{m=1}^\infty \{8\pi m\}$, there exists some continuous function a_λ defined in $\mathbb{R} \setminus \cup_{m=1}^\infty \{8\pi m\}$ such that for all $\lambda \in \mathbb{R} \setminus \cup_{m=1}^\infty \{8\pi m\}$ and $a > a_\lambda$,

$$d_\lambda := \deg(I - K_\lambda, B_a, 0) \tag{2}$$

is well defined and, in view of the homotopy invariance of the Leray-Schauder degree, is independent of a as long as $a > a_\lambda$. Moreover d_λ is a constant in each interval $(8\pi m, 8\pi(m+1))$. The piecewise constant function d_λ is determined by the Euler number of M. We know that d_λ is equal to 1 for $\lambda < 8\pi$. However, due to the possible loss of compactness of solutions of $(E_u)_\lambda$ when λ crosses $8\pi m$, we do not know yet the values of d_λ in the other intervals. Knowing the values of d_λ should lead to new existence results for $(E_u)_\lambda$ since $d_\lambda \not\equiv 0$ implies that $(E_u)_\lambda$ has at least one solution. The situation here is similar to that in [31] where

$$-\Delta u + 2u = \frac{1}{6}K(x)u^3, \quad u > 0, \qquad \text{on } \mathbb{S}^4 \tag{3}$$

is studied. Let $\mathcal{A}$ be the open and dense subset of $C^2(\mathbb{S}^4)^+$, the set of positive twice differentiable functions, defined in [31]. For any Morse function $K \in \mathcal{A}$, let $K_\lambda = (1-\lambda) + \lambda K$ for $0 \leq \lambda \leq 1$. It is not difficult to see from [31] that there exist $0 < \lambda_1 < \cdots < \lambda_l < 1$ such that for all $\lambda \in (0,1] \setminus \{\lambda_1, \cdots, \lambda_l\}$ the total Leray-Schauder degree d_λ of all possible solutions of (3) with $K = K_\lambda$ is well defined and is a constant function of λ in each interval $(\lambda_m, \lambda_{m+1})$. Since $d_1 \not\equiv 0$ implies that (3) has at least one solution, we wish to have a formula of d_1 in terms of K. The formula of d_λ for small λ is known due to the work of Chang and Yang [16]. So, one way to derive a formula of d_1 is to calculate the jump-values of d_λ at λ_m for $1 \leq m \leq l$. These jump-values can be calculated by using the strong pointwise estimates in [31] of blowup solutions u_λ, solutions of (3) with $K = K_\lambda$, as $\lambda \to \lambda_m$. Once these jump-values are known, we have a formula of d_1 in terms of K. This provides an alternative derivation of the formula of d_1 obtained in [31].

We propose to take a similar approach to study $(E_u)_\lambda$, namely, to look for a formula of d_λ in terms of the Euler number of M. Since we know $d_\lambda = 1$ for $\lambda < 8\pi$, we only need to calculate the jump-values of d_λ at $8\pi m$, $m \geq 1$. In view of the results in [31], we tend to believe that a good enough pointwise estimate of blowup solutions $\{u_\lambda\}$ as $\lambda \to 8\pi m$ is the most crucial step in evaluating the jump-value of d_λ at $8\pi m$. Once we know the jump-values for m less than some m_0, we obtain a formula of d_λ in $(-\infty, 8\pi m_0) \setminus \cup_{m=1}^{m_0-1}\{8\pi m\}$. The main purpose of this paper is to start making good pointwise estimates for blowup solutions $\{u_\lambda\}$ as $\lambda \to 8\pi m$. The main analytical result of this paper is a new local estimate given in Theorem 0.3.

We first state a well known fact in the subcritical case $\lambda < 8\pi$.

Theorem 0.1 (well known**).** *Let (M, g) be a compact Riemann surface, V be a positive continuous function on M, $W \in L^\infty(M)$ with $\int_M W dv_g = 1$. Then for all $\epsilon > 0$, $-\epsilon^{-1} \leq \lambda \leq 8\pi - \epsilon$, and all C^2 solutions u of $(E_u)_\lambda$ with $\int_M u dv_g = 0$, we have*

$$\|u\|_{L^\infty(M)} \leq C,$$

where C depends only on $M, g, \epsilon, \|V\|_{L^\infty(M)}, \|W\|_{L^\infty(M)}$, the modulo of continuity of V, and the positive lower bound of V.

If both V and W are Lipschitz functions, then it is well known that any C^2 solution of $(E_u)_\lambda$ is actually in $C^{2,\alpha}(M)$ for all $0 < \alpha < 1$, and $C^{2,\alpha}$ estimates of u follow from the L^∞ estimates. Thus, we have

Corollary 0.1. *In addition to the hypothesis in Theorem 0.1, assume that both V and W are Lipschitz functions. Then for all $\epsilon > 0$, $-\epsilon^{-1} \leq \lambda \leq 8\pi - \epsilon$, $0 < \alpha < 1$, and all C^2 solution u of $(E_u)_\lambda$ with $\int_M u dv_g = 0$, we have*

$$\|u\|_{C^{2,\alpha}(M)} \leq C,$$

where C depends only on $M, g, \epsilon, \alpha, \|V\|_{L^\infty(M)}, \|\nabla V\|_{L^\infty(M)}, \|\nabla W\|_{L^\infty(M)}$, and the positive lower bound of V.

Corollary 0.2. *In addition to the hypothesis in Theorem 0.1, we assume that both V and W are Lipschitz functions. Then*

$$d_\lambda = 1$$

for all $\lambda < 8\pi$. Consequently, $(E_u)_\lambda$ has at least one solution for every $\lambda < 8\pi$.

Remark 0.1. The existence of one solution to $(E_u)_\lambda$ for $\lambda < 8\pi$ can easily be established by variational methods using the following consequence of the Moser-Trudinger inequality: For every $\epsilon > 0$,

$$\log \int_M e^w dv_g \le (\frac{1}{16\pi} + \epsilon) \int_M |\nabla_g w|^2 dv_g + \frac{1}{vol(M)} \int_M w dv_g + C(\epsilon) \quad \forall\, w \in H^1(M).$$

See, for example, [33, 29] and [22] for more details.

For $\lambda \ge 8\pi$, Eq. $(E_u)_\lambda$ is much more delicate. The difficulty lies in possible loss of compactness of solutions of $(E_u)_\lambda$ for $\lambda \ge 8\pi$. A good understanding of possible blowup behavior of solutions of $(E_u)_\lambda$ is important in the study of $(E_u)_\lambda$. Our next theorem gives some understanding of the possible blowup behavior of solutions of $(E_u)_\lambda$, which should be relevant in the study of the existence of solutions of $(E_u)_\lambda$ for $\lambda \ge 8\pi$.

Let $\{V_n\}$ satisfy

$$\liminf_{n\to\infty} \min_M V_n > 0, \qquad \limsup_{n\to\infty} (\max_M V_n + \|\nabla V_n\|_{L^\infty(M)}) < \infty, \tag{4}$$

$\{W_n\}$ satisfy

$$\limsup_{n\to\infty} \|\nabla W_n\|_{L^\infty(M)} < \infty, \ \int_M W_n dv_g = 1, \tag{5}$$

and $\{\xi_n\}$ satisfy

$$-\Delta_g \xi_n = \lambda_n (V_n e^{\xi_n} - W_n) \qquad \text{on } M, \tag{6}$$

and

$$\int_M V_n e^{\xi_n} dv_g = 1. \tag{7}$$

We will use $d(x, y)$ to denote the distance between x and y in M and will use the notation

$$\overline{\xi}_n = \frac{1}{vol(M)} \int_M \xi_n dv_g,$$

to denote the average of ξ_n on M.

Let $G(x, y)$ denote the Green′s function of $-\Delta_g$ on M, namely,

$$\begin{cases} -\Delta_x G(x,y) = \delta_y - \frac{1}{vol(M)}, \text{ in } M, \\ \int_M G(x,y) dv_g(x) = 0. \end{cases}$$

It is well known (see, e.g., [2]) that $G(x, y)$ is uniquely defined, symmetric in x and y, and a solution of (6) satisfies

$$\xi_n(x) - \overline{\xi}_n = \lambda_n \int_M (V_n(y) e^{\xi_n(y)} - W_n(y)) G(x,y) dv_g(y), \qquad \forall\, x \in M. \tag{8}$$

Theorem 0.2. *Let (M, g) be a compact Riemann surface, $\{V_n\}$ and $\{W_n\}$ satisfy (4) and (5), $\lambda_n \to \overline{\lambda} \in (-\infty, \infty)$, and $\{\xi_n\} \subset C^2(M)$ satisfy (6) and (7). Assume*

$$\max_M |\xi_n| \to \infty. \tag{9}$$

Then after passing to a subsequence (still denoted as $\{\xi_n\}$), there exist m distinct points $\{\overline{x}^{(l)}\}_{1 \le l \le m}$ in M and m sequences of points $\overline{x}_n^{(l)} \to \overline{x}^{(l)}$ such that

(a) $\xi_n \to -\infty$ *uniformly on compact subsets of* $M \setminus \{\overline{x}^{(1)}, \cdots, \overline{x}^{(m)}\}$.

(b) For each $1 \le l \le m$, and n large, $\overline{x}_n^{(l)}$ is the unique maximum point of ξ_n in $\{x \in M \mid d(x, \overline{x}^{(l)}) \le \frac{1}{2} \min_{l' \ne l} dist(\overline{x}^{(l')}, \overline{x}^{(l)})\}$, and $\xi_n(\overline{x}_n^{(l)}) \to \infty$.

(c) For each $1 \le l \le m$, let $g = e^{\varphi_n}(dx_1^2 + dx_2^2)$ be an isothermal coordinate system (with $\varphi_n(0) = 0$) centered at $\overline{x}_n^{(l)}$, we have, for some constant C independent of n,

$$|\xi_n(x) - \log \frac{e^{\xi_n(0)}}{(1 + \frac{\lambda_n V_n(0)}{8} e^{\xi_n(0)} |x|^2)^2}| \le C, \quad \forall\, |x| \le \frac{1}{4} \min_{l' \ne l} dist(\overline{x}^{(l)}, \overline{x}^{(l')}) \text{ and } \forall\, n.$$

(d) For some constant C independent of n,

$$\max_{1 \le l \le m} |\xi_n(x_n^{(l)}) + \overline{\xi}_n| \le C.$$

(e) In $C^2_{loc}(M \setminus \{\overline{x}^{(1)}, \cdots, \overline{x}^{(m)}\})$,

$$\xi_n - \overline{\xi}_n \to 8\pi \sum_{l=1}^{m} G(\cdot, \overline{x}^{(l)}) - 8\pi m \int_M W(y) G(\cdot, y) dv_g(y),$$

where $W = \lim_{n \to \infty} W_n$ weak $*$ in $L^\infty(M)$. Consequently,

$$\lambda_n V_n e^{\xi_n} \rightharpoonup 8\pi \sum_{l=1}^{m} \delta_{\overline{x}^{(l)}} \text{ in the sense of measure, and } \overline{\lambda} = 8\pi m,$$

where $\delta_{\overline{x}^{(l)}}$ denotes the Delta mass at $\overline{x}^{(l)}$.

Remark 0.2. Theorem 0.2 still holds when we replace the metric g by a sequence of metrics g_n converging to g in the C^2 norm. This can be seen easily from the proof of Theorem 0.2.

Due to Theorem 0.2 and Theorem 0.1, there exists some continuous function a_λ defined in $\mathbb{R} \setminus \cup_{m=1}^{\infty} \{8\pi m\}$ such that d_λ in (2) is well defined for all $\lambda \in \mathbb{R} \setminus \cup_{m=1}^{\infty} \{8\pi m\}$ and $a > a_\lambda$. Furthermore, in view of Remark 0.2, d_λ is independent of the metric g. Therefore, in view of the homotopy invariance of the Leray-Schauder degree, d_λ is a constant in each of the open intervals, and all these constants are independent of V, W and the metric g. So d_λ is a piecewise constant function of λ determined completely by the Euler number of M. We know from Corollary 0.2 that d_λ is equal to 1 for $\lambda < 8\pi$, but we do not know yet the values of d_λ in other intervals. Knowing the values of d_λ should lead to new existence results for $(E_u)_\lambda$. As mentioned earlier we wish to calculate the jump-value of d_λ at $8\pi m$. In view of the results in [31], Theorem 0.2, providing pointwise estimates of $\{\xi_n\}$, should be useful in evaluating the jump-value of d_λ at $8\pi m$.

Let $\{\xi_n\}$ be the subsequence in Theorem 0.2 satisfying (a)-(e). In an isothermal coordinate system centered at $\overline{x}_n^{(l)}$, we set

$$v_n(x) = \xi_n(\delta_n^{(l)}x) + 2\log\delta_n^{(l)}, \qquad |x| < a/\delta_n^{(l)},$$

where $\delta_n^{(l)} = e^{-\xi_n(\overline{x}_n^{(l)})/2}$ and a is some suitably small positive constant. It will be shown by a blow up argument that

$$v_n \to v \qquad \text{in } C^2_{loc}(\mathbb{R}^2)$$

with

$$v(x) = \log\{\frac{1}{(1+\frac{\overline{\lambda}(\lim_{n\to\infty}V_n(0))}{8}|x|^2)^2}\}, \qquad \text{in } \mathbb{R}^2. \tag{10}$$

Consequently,

$$\overline{R}_n^{(l)} := \sup\{R>0 \ : \ \|v_n - v\|_{C^2(\overline{B}_{2R}(0))} + \|v_n - v\|_{H^2(\overline{B}_{2R}(0))} < e^{-R}\} \to \infty.$$

This shows that $\xi_n(x)$ is very well approximated by $\log\{\frac{e^{\xi_n(\overline{x}_n^{(l)})}}{(1+\frac{\lambda_n V_n(0)}{8}e^{\xi_n(\overline{x}_n^{(l)})}|x|^2)^2}\}$ in $|x| \le \overline{R}_n^{(l)}\delta_n^{(l)}$. For $\overline{R}_n^{(l)}\delta_n^{(l)} \le |x| \le \frac{1}{2}\min_{l'\neq l} dist(\overline{x}^{(l')}, \overline{x}^{(l)})$, we will give, using (c), some convergence estimate better than (e). For convenience, we use the notation

$$\zeta_n \sim 0$$

to denote a sequence of functions $\{\zeta_n\}$ in $C^2(M)$ satisfying

$$\lim_{n\to\infty} \max\{\frac{|\zeta_n(x)|}{1+\sum_{j=1}^m |\log d(x,\overline{x}_n^{(j)})|} \ : \ x\in M\setminus \cup_{l=1}^l B_{\overline{R}_n^{(l)}\delta_n^{(l)}}(\overline{x}_n^{(l)})\} = 0, \tag{11}$$

$$\lim_{n\to\infty} \max\{\frac{|\nabla\zeta_n(x)|}{\sum_{j=1}^m d(x,\overline{x}_n^{(j)})^{-1}} \ : \ x\in M\setminus \cup_{l=1}^l B_{\overline{R}_n^{(l)}\delta_n^{(l)}}(\overline{x}_n^{(l)})\} = 0, \tag{12}$$

and

$$\lim_{n\to\infty} \max\{\frac{|\nabla^2\zeta_n(x)|}{\sum_{j=1}^m d(x,\overline{x}_n^{(j)})^{-2}} \ : \ x\in M\setminus \cup_{l=1}^l B_{\overline{R}_n^{(l)}\delta_n^{(l)}}(\overline{x}_n^{(l)})\} = 0. \tag{13}$$

We also write $\zeta \sim_0 0$ for (11), $\zeta \sim_1 0$ for (12), and $\zeta \sim_2 0$ for (13).

Corollary 0.3. *Let $\{\xi_n\}$ be the subsequence in Theorem 0.2 satisfying (a)-(e). Then*

$$\varphi_n := \xi_n - \overline{\xi}_n - 8\pi\sum_{l=1}^m G(\cdot,\overline{x}_n^{(l)}) + 8\pi m\int_M W(y)G(\cdot,y)dv_g(y) \sim 0.$$

Theorem 0.2 will be deduced from some local results on the behavior of blowup solutions to equations of the type $-\Delta u = Ve^u$ in domains of $\mathbb{R}^2$. In particular, a new local estimate, Theorem 0.3, is needed in the proof of Theorem 0.2. We first recall some known results.

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, $0\in\Omega$, $\{V_n\}$ be a sequence of Lipschitz continuous functions satisfying

$$0 < a \le V_n(x) \le b < \infty, \qquad \forall\, x\in\overline{\Omega}, \tag{14}$$

and

$$|\nabla V_n(x)| \leq A, \qquad \forall\, x \in \overline{\Omega}, \tag{15}$$

where a, b and A are positive constants.

Consider

$$-\Delta u_n = V_n e^{u_n}, \qquad \text{in } \Omega, \tag{16}$$

and let $\{u_n\}$ be a sequence of C^2 solutions of (16) satisfying

$$\limsup_{n\to\infty} \int_\Omega V_n e^{u_n} < \infty. \tag{17}$$

It follows from Theorem 3 in Brezis and Merle [8] that, under (14)–(17), there are only three alternatives after passing to a subsequence:

1. $\{u_n\}$ uniformly converges on compact subsets of Ω,
2. $\{u_n\}$ tends to $-\infty$ uniformly on compact subsets of Ω,
3. There exist finitely many blowup points $\{x^{(1)}, \cdots, x^{(l)}\}$ of $\{u_n\}$ such that $\{u_n\}$ tends to $-\infty$ uniformly on compact subsets of $\Omega \setminus \{x^{(1)}, \cdots, x^{(l)}\}$, and

$$V_n e^{u_n} \rightharpoonup \sum_{i=1}^{l} \alpha_i \delta_{x^{(i)}} \qquad \text{in the sense of measure,}$$

with $\alpha_i \geq 4\pi$. Here $\delta_{x^{(i)}}$ is the Dirac mass at $x^{(i)}$.

We recall that a point $\overline{y}$ is called a blowup point of $\{u_n\}$ if there exist $y_n \to \overline{y}$ such that $u_n(y_n) \to \infty$ as $n \to \infty$.

It was conjectured in [8] that each α_i can be written as $\alpha_i = 8\pi m_i$ for some positive integer m_i. This was established by Li and Shafrir in [32]. Chen further demonstrated in [20] that any positive integer m_i can occur in such local situations.

Under (14) and (15), the following Harnack type inequality is proved by Brezis, Li and Shafrir in [9] through the method of moving planes: Every solution of (16) satisfies, on any compact subset K of Ω,

$$\sup_K u_n + \inf_\Omega u_n \leq C(a, b, A, K, \Omega). \tag{18}$$

It is raised as an open question in [9] whether the above Harnack type inequality still holds when replacing $\|\nabla V_n\|_{L^\infty(\Omega)}$ by $\|V_n\|_{C^\alpha(\Omega)}$ $(0 < \alpha < 1)$. The answer is affirmative due to some recent work of Chen and Lin [19].

Now we are ready to state our new local estimate which is essentially equivalent to a Harnack type estimate $|\sup u_n + \inf u_n| \leq C$ under additional hypotheses (20) and (21) below. These additional hypotheses are necessary for such an estimate to hold. We will further assume that

$$u_n(0) = \max_{\overline{\Omega}} u_n \to \infty, \tag{19}$$

and

$$V_n e^{u_n} \rightharpoonup \alpha\delta, \qquad \text{in } \overline{\Omega}, \ \text{ in the sense of measure,} \tag{20}$$

where $\alpha > 0$ is a constant and δ is the Dirac mass at the origin.

Theorem 0.3. *In addition to (14)–(16) and (19)–(20), we assume that*

$$\max_{\partial\Omega} u_n - \min_{\partial\Omega} u_n \le A_1 \tag{21}$$

for some positive constant A_1. Then for some constant C independent of n, we have

$$|u_n(x) - \log \frac{e^{u_n(0)}}{(1 + \frac{V_n(0)}{8} e^{u_n(0)}|x|^2)^2}| \le C, \qquad \forall\, x \in \overline{\Omega} \text{ and } \forall\, n. \tag{22}$$

Theorem 0.3 will be proved by the method of moving planes, which has become a very powerful and convenient tool in the study of nonlinear elliptic partial differential equations starting from the pioneering works of A.D. Alexandrov [1], Serrin [37], and Gidas, Ni and Nirenberg [25, 26]. The method has been further developed in a series of papers by Berestycki, Nirenberg and their collaborators [3]–[7], and Caffarelli, Gidas and Spruck [10]. Many more applications of the moving plane method have been given by various authors. The method of moving planes was used to obtain some Harnack type inequalities by Schoen in [36], subsequently by Brezis, Li, and Shafrir in [9], and by Chen and Lin in [18]. Our proof of Theorem 0.3 requires some new ingredients.

1. Compactness and Existence for $\lambda \in (-\infty, 8\pi)$

Throughout this section V is a positive continuous function on M and $W \in L^\infty(M)$ with $\int_M W dv_g = 1$.

Lemma 1.1. *Let $\epsilon > 0$ and ξ satisfy $(E_\xi)_\lambda$ and (1) with $-\epsilon^{-1} \le \lambda \le 8\pi - \epsilon$. Then*

$$\max_M |\xi| \le C$$

for some constant C depending only on M, g, ϵ, $\|W\|_{L^\infty(M)}$, $\|V\|_{L^\infty(M)}$, the positive lower bound of V, and the modulus of continuity of V.

Proof. Suppose the contrary, then there exist $\{V_n\}$ converging to some positive function in $C(M)$, $\{W_n\}$ bounded in $L^\infty(M)$, $\lambda_n \to \overline{\lambda} \in [-\epsilon^{-1}, 8\pi - \epsilon]$, ξ_n, with $\int_M V_n e^{\xi_n} dv_g = 1$, satisfying $(E_\xi)_{\lambda_n}$ with $V = V_n$ and $W = W_n$, but $\max_M |\xi_n| \to \infty$.

Let y_n be a maximum point of ξ_n. If $\xi_n(y_n) \to \infty$ along a subsequence (still denoted as $\{\xi_n\}$), we work in some isothermal coordinate system $x = (x_1, x_2)$ centered at y_n. Without loss of generality, we may assume $y_n \to \overline{y}$. In a neighborhood of $\overline{y}$, $g = e^{\varphi_n}(dx_1^2 + dx_2^2)$, where $\varphi_n(0) = 1$ and $\{\varphi_n\}$ converges in the neighborhood with respect to C^2 norms, and, in the neighborhood, the equation of ξ_n takes the form

$$-\Delta\xi_n = \lambda_n(V_n e^{\varphi_n} e^{\xi_n} - e^{\varphi_n} W_n),$$

where $\Delta = \partial_{x_1x_1} + \partial_{x_2x_2}$. Consider

$$v_n(x) = \xi_n(\delta_n x) + 2\log\delta_n, \qquad |x| < a\delta_n^{-1},$$

where $\delta_n = e^{-\xi_n(0)/2} \to 0$ and $a > 0$ is some constant. Clearly v_n satisfies

$$\begin{cases} -\Delta v_n(x) = \lambda_n \left(V_n(\delta_n x) e^{\varphi_n(\delta_n x)} e^{v_n(x)} - \delta_n^2 e^{\varphi_n(\delta_n x)} W_n(\delta_n x)\right), & |x| < a\delta_n^{-1}, \\ \int_{|x| \le a\delta_n^{-1}} V_n(\delta_n x) e^{\varphi_n(\delta_n x)} e^{v_n(x)} \le 1, & \\ v_n(x) \le v_n(0) = 0, & |x| < a\delta_n^{-1}. \end{cases}$$

For any $R > 1$, let f_n be the solution of

$$\begin{cases} -\Delta f_n(x) = \lambda_n \left(V_n(\delta_n x) e^{\varphi_n(\delta_n x)} e^{v_n(x)} - \delta_n^2 e^{\varphi_n(\delta_n x)} W_n(\delta_n x) \right), & |x| < R, \\ f_n(x) = 0, & |x| = R. \end{cases}$$

Then $|f_n|$ is bounded from above by some constant $C = C(R)$ in $|x| \le R$, so $C + f_n - v_n$ is a nonnegative harmonic function in $|x| \le R$ with value at the origin not larger than $2C$. The Harnack inequality yields the upper bound of $C + f_n - v_n$ in $|x| \le R/2$, which in turn yields the lower bound of v_n in $|x| \le R/2$. Therefore, after passing to a subsequence, we have, by applying $W^{2,p}$ estimates to v_n, that

$$v_n \to v \quad \text{in } C^1_{loc}(\mathbb{R}^2),$$

where v satisfies, in the distribution sense,

$$\begin{cases} -\Delta v = \overline{\lambda}(\lim_{n\to\infty} V_n(0)) e^v, \text{ in } \mathbb{R}^2, \\ (\lim_{n\to\infty} V_n(0)) \int_{\mathbb{R}^2} e^v \le 1, \\ v(x) \le v(0) = 0, & \text{in } \mathbb{R}^2. \end{cases} \tag{23}$$

In fact, due to standard elliptic estimates, $v \in C^2(\mathbb{R}^2)$.

It is easy to show (see the Appendix) that there is no solution to (23) if $\overline{\lambda} \le 0$, so $\overline{\lambda} > 0$. On the other hand, due to the classification of all solutions of (23) (see, for example, [19, 21] and [15]), we know that v is the function given in (10). It follows that

$$\overline{\lambda}(\lim_{n\to\infty} V_n(0)) \int_{\mathbb{R}^2} e^v = 8\pi.$$

Consequently, $\overline{\lambda} \ge 8\pi$. This is a contradiction. Thus $\{\xi_n\}$ is bounded from above and $\xi_n(\hat{y}_n) = -\min_M \xi_n \to \infty$ for some $\hat{y}_n \in M$. Without loss of generality, we may assume $\hat{y}_n \to \hat{y}$. Let $\Omega \subset M$ be any smooth open connected set containing $\hat{y}$, $\partial\Omega \neq \phi$. Define η_n by

$$\begin{cases} -\Delta_g \eta_n = \lambda_n (V_n e^{\xi_n} - W_n), \text{ in } \Omega, \\ \eta_n = 0, & \text{on } \partial\Omega. \end{cases}$$

In view of the upper bound of ξ_n, we derive from standard elliptic estimates that $\{\eta_n\}$ is uniformly bounded in $\overline{\Omega}$. Let $w_n = \xi_n - \eta_n$, then w_n satisfies

$$-\Delta_g w_n = 0, \qquad w_n \le C, \qquad \text{in } \Omega.$$

Applying the Harnack inequality to $C - w_n$ on compact subsets of Ω, we have, in view of $C - w_n(\hat{y}_n) \to \infty$, that $C - w_n \to \infty$ uniformly on compact subsets of Ω. Namely, $\xi_n \to -\infty$ uniformly on compact subsets of Ω. Since Ω can be chosen arbitrarily, $\xi_n \to -\infty$ uniformly on M which violates $\int_M V_n e^{\xi_n} = 1$. Lemma 1.1 is established. □

Theorem 0.1 can be deduced from Lemma 1.1 as follows.

Proof of Theorem 0.1. Set $\xi = u - \log \int_M V e^u dv_g$. We know from Lemma 1.1 that $|\xi| \le C$ on M. Since $\int_M u dv_g = 0$, u vanishes somewhere in M. It follows that $|\log \int_M V e^u dv_g| \le C$. Consequently, $|u| \le C$. □

2. A New Local Estimate by the Method of Moving Planes

In this section we establish Theorem 0.3 by the method of moving planes.

Let $\overline{G}(x,y)$ be the Green′s function of $-\Delta$ in $\Omega \subset \mathbb{R}^2$ with respect to the zero boundary condition:

$$\begin{cases} -\Delta_x \overline{G}(x,y) = \delta_y, \text{ in } \Omega, \\ \quad\quad \overline{G}(x,y) = 0, \quad x \in \partial\Omega. \end{cases}$$

Consider

$$\tilde{u}_n(x) = \int_\Omega \overline{G}(x,y) V_n(y) e^{u_n(y)} dy.$$

Namely, $\tilde{u}_n$ is the solution of

$$\begin{cases} -\Delta \tilde{u}_n = V_n e^{u_n}, \text{ in } \Omega, \\ \tilde{u}_n = 0, \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

Lemma 2.1. *Under the hypothesis of Theorem 0.3, for all $r > 0$,*

$$\tilde{u}_n(x) \to \alpha \overline{G}(x,0) \qquad \textit{in } C^1(\overline{\Omega} \setminus B_r).$$

Proof. Write

$$\tilde{u}_n(x) = \overline{G}(x,0) \int_\Omega V_n(y) e^{u_n(y)} dy + \int_\Omega [\overline{G}(x,y) - \overline{G}(x,0)] V_n(y) e^{u_n(y)} dy.$$

As $y \to 0$, $\overline{G}(x,y) - \overline{G}(x,0) \to 0$ uniformly for $x \in \overline{\Omega} \setminus B_r$. Consequently, using (20),

$$\tilde{u}_n(x) \to \alpha \overline{G}(x,0) \qquad \text{in } C^0(\overline{\Omega} \setminus B_r).$$

On the other hand, we have

$$\nabla \tilde{u}_n(x) = \int_\Omega \nabla_x \overline{G}(x,y) V_n(y) e^{u_n(y)} dy,$$

and, as $y \to 0$, $\nabla_x \overline{G}(x,y) - \nabla_x \overline{G}(x,0) \to 0$ uniformly for $x \in \overline{\Omega} \setminus B_r$. The C^1 convergence of $\tilde{u}_n(x)$ to $\alpha \nabla_x \overline{G}(x,0)$ follows immediately. □

Lemma 2.2. *Under the hypotheses of Theorem 0.3, for all $r > 0$, there exists some constant $C = C(r, \Omega, a, b, A, A_1, \alpha)$ such that*

$$\max_{\overline{\Omega} \setminus B_r} u_n - \min_{\overline{\Omega} \setminus B_r} u_n \le C.$$

Proof. It follows from Lemma 2.1 and (21) that the oscillation of $u_n - \tilde{u}_n$ on $\partial\Omega$ is bounded. Since $u_n - \tilde{u}_n$ is a harmonic function, it follows from the maximum principle that the oscillation of $u_n - \tilde{u}_n$ in $\overline{\Omega}$ is bounded. Lemma 2.2 follows. □

Due to Lemma 2.2, we only need to establish Theorem 0.3 for a special case: $\Omega = B_1$ is the unit ball in $\mathbb{R}^2$. Without loss of generality, we assume that $V_n(0) = 8$. Set

$$\delta_n = e^{-u_n(0)/2},$$

$$\overline{v}_n(x) = u_n(\delta_n x) + 2\log\delta_n, \qquad \text{for } |x| \le 1/\delta_n,$$

$$\overline{w}_n(x) = \overline{v}_n(x) + 2\log|x|, \qquad \text{for } |x| \le 1/\delta_n.$$

It is clear that $\overline{v}_n$ satisfies

$$\begin{cases} -\Delta\overline{v}_n(x) = V_n(\delta_n x)e^{\overline{v}_n(x)} & \text{for } |x| \le 1/\delta_n, \\ \overline{v}_n(x) \le \overline{v}_n(0) = 0 & \text{for } |x| \le 1/\delta_n. \end{cases}$$

Arguing as in Sect. 1,

$$\overline{v}_n \to \overline{v} \qquad \text{in } C^2_{loc}(\mathbb{R}^2), \tag{24}$$

and therefore

$$\overline{w}_n - \overline{w} \to 0 \qquad \text{in } C^2_{loc}(\mathbb{R}^2), \tag{25}$$

where

$$\overline{v}(x) = \log\left\{\frac{1}{(1+|x|^2)^2}\right\},$$

$$\overline{w}(x) = \log\left\{\frac{|x|^2}{(1+|x|^2)^2}\right\}.$$

For convenience, we work in cylindrical coordinates (t, θ) with

$$\begin{cases} x_1 = e^t\cos\theta, \\ x_2 = e^t\sin\theta. \end{cases} \tag{26}$$

It is easy to check that the transformation given by (26): $(x_1, x_2) \to (t, cos\theta, \sin\theta)$ is a conformal transformation of $\mathbb{R}^2 \setminus \{0\}$ to the cylinder $\mathbb{R} \times \mathbb{S}^1 = \{(t, cos\theta, \sin\theta)\}$.

Set, for $t < 0$ and $\theta \in [0, 2\pi]$,

$$\tilde{w}_n(t,\theta) = u_n(e^t\cos\theta, e^t\sin\theta) + 2t,$$

and

$$\tilde{w}(s) = \log\left\{\frac{e^{2s}}{(1+e^{2s})^2}\right\} = 2s - 2\log(1+e^{2s}).$$

Under transformation (26),

$$\overline{w}_n(x) = \tilde{w}_n(t + \log\delta_n, \theta), \qquad \overline{w}(x) = \tilde{w}(t).$$

We derive from (25) that in the new variables,

$$\lim_{n\to\infty} \|\tilde{w}_n(s + \log\delta_n, \theta) - \tilde{w}(s)\|_{L^\infty(s\le\alpha, \theta\in[0,2\pi])} = 0, \qquad \forall\, \alpha \in \mathbb{R}. \tag{27}$$

Clearly, under the above conformal transformation of $\mathbb{R}^2\setminus\{0\}$ to $\mathbb{R}\times\mathbb{S}^1$, the equation of u_n is transformed to the following equation on the half cylinder $\mathbb{R}_- \times \mathbb{S}^1$:

$$-(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2})\tilde{w}_n = \tilde{V}_n(t,\theta)e^{\tilde{w}_n} \qquad \text{in } Q,$$

where

$$Q = \{(t,\theta)\ :\ t \le 0 \text{ and } 0 \le \theta \le 2\pi\},$$

and

$$\tilde{V}_n(t,\theta) = V_n(e^t \cos\theta, e^t \sin\theta).$$

Note that $\tilde{w}$ achieves its maximum at $s = 0$, $\tilde{w}'(s) > 0$ for $s < 0$, and $\tilde{w}(-s) = \tilde{w}(s)$ for all s.

Let us first describe the ideas of the proof. For some $R_n \to \infty$, estimate (22) inside the shrinking balls $|x| \le R_n\delta_n$ follows from the usual blow up argument. What we need to estimate is in the region $R_n\delta_n \le |x| \le 1$. We work on $\mathbb{R}_- \times \mathbb{S}^1$, the left half cylinder. It is not difficult to see that the desired estimate (22) in the region $C\delta_n \le |x| \le 1$ is equivalent to

$$|\tilde{w}_n(t,\theta) - \tilde{w}(t - \log\delta_n)| \le C, \ \ \forall\ \log\delta_n + C \le t \le 0 \text{ and } \forall\theta. \tag{28}$$

Here and in the following, C denotes various constants independent of n.

The blow up argument gives a precise estimate to $\tilde{w}_n(t,\theta) - \tilde{w}(t - \log\delta_n)$ for $t \le \log\delta_n + C$. Since $\tilde{w}(t - \log\delta_n)$ is symmetric with respect to $t = \log\delta_n$, estimate (28) is then, in view of (27), equivalent to

$$|\tilde{w}_n(t,\theta) - \tilde{w}_n(2\log\delta_n - t,\theta)| \le C, \ \forall\ \log\delta_n + C \le t \le 0 \text{ and } \forall\theta. \tag{29}$$

To establish (29) we will introduce two functions, $\hat{w}_n$ and w_n^*, which differ from $\tilde{w}_n$ by some uniformly bounded functions. The function $\hat{w}_n$ will be chosen so that the method of moving planes can be applied to $\hat{w}_n$ from the left to obtain

$$\hat{w}_n(t,\theta) \ge \hat{w}_n(2\lambda_n - t,\theta), \ \ \forall\ \lambda_n \le t \le 0 \text{ and } \forall\theta, \tag{30}$$

where λ_n is some number smaller than $\log\delta_n + 2$. On the other hand, w_n^* will be chosen so that the method of moving planes can be applied to w_n^* from the right to obtain

$$w_n^*(t,\theta) \le w_n^*(2\lambda_n^* - t,\theta), \ \ \forall\ \lambda_n^* \le t \le 0 \text{ and } \forall\theta, \tag{31}$$

where λ_n^* is some number larger than $\log\delta_n - C$. We emphasize that in order to apply the moving plane method to w_n^* from the right we need (30) and Lemma 2.1-2.2 so that the plane moving process can get started. These estimates are also needed to ensure that

$$|\lambda_n - \log\delta_n| + |\lambda_n^* - \log\delta_n| \le C. \tag{32}$$

The desired estimate (29) follows from (30), (31), (32) and (27).

We first introduce

$$\hat{w}_n(t,\theta) = \tilde{w}_n(t,\theta) - \frac{A}{a}e^t \qquad \text{in } Q.$$

Clearly $\hat{w}_n$ satisfies

$$-(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2})\hat{w}_n = \hat{V}_n e^{\hat{w}_n} + \frac{A}{a}e^t, \tag{33}$$

where

$$\hat{V}_n(t,\theta) = \tilde{V}_n(t,\theta)e^{Ae^t/a}.$$

It is easy to see that

$$\frac{\partial}{\partial t}\left\{\hat{V}_n(t,\theta)e^{\xi} + \frac{A}{a}e^t\right\} \geq 0 \qquad \forall\,(t,\theta) \in Q, \forall\,\xi \in \mathbb{R}. \tag{34}$$

We recall some estimates obtained for $\hat{w}_n$ in [9] by the method of moving planes. For $\lambda < 0$ and $\lambda \leq t < 0$, we set $t^\lambda = 2\lambda - t$ and

$$\hat{w}_n^\lambda(t,\theta) = \hat{w}_n(t^\lambda,\theta).$$

Clearly $\hat{w}_n^\lambda$ satisfies

$$-(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2})\hat{w}_n^\lambda = \hat{V}_n^\lambda e^{\hat{w}_n^\lambda} + \frac{A}{a}e^{t^\lambda}, \tag{35}$$

where $\hat{V}_n^\lambda = \hat{V}_n(t^\lambda,\theta)$.

It is easy to see that $\hat{w}_n(t,\theta)$ behaves like $2t$ for t very negative and therefore for λ very negative (depending on n), we have

$$\hat{w}_n^\lambda(t,\theta) - \hat{w}_n(t,\theta) < 0 \qquad \text{for } \lambda < t \leq 0, 0 \leq \theta \leq 2\pi.$$

Define

$$\lambda_n = \sup\{\mu < 0 : \hat{w}_n^\lambda(t,\theta) - \hat{w}_n(t,\theta) < 0 \ \text{ for all } \lambda < \mu, \lambda < t \leq 0, 0 \leq \theta \leq 2\pi\}.$$

For every fixed $\alpha \in \mathbb{R}$, we know that $\tilde{w}_n(t,\theta)$ approximates $\tilde{w}(t - \log\delta_n,\theta)$ very well in $t \leq \log\delta_n + \alpha$. Therefore (see [9] for details)

$$\lambda_n \leq \log\delta_n + 2. \tag{36}$$

Using the fact

$$\hat{w}_n^\lambda(t,\theta) - \hat{w}_n(t,\theta) < 0 \qquad \forall\,\lambda < t < 0, \lambda < \lambda_n, 0 \leq \theta \leq 2\pi,$$

it is not difficult to see from (34), (33), (35) and the mean value theorem that

$$-\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2}\right)\left(\hat{w}_n^\lambda(t,\theta) - \hat{w}_n(t,\theta)\right) \leq 0 \qquad \text{for } \lambda \leq t \leq 0, \lambda \leq \lambda_n \text{ and } 0 \leq \theta \leq 2\pi.$$

Since the plane moving process stops at λ_n, we derive, using the Hopf lemma and the strong maximum principle, that

$$\min_{0\leq\theta\leq 2\pi}\{\hat{w}_n(0,\theta) - \hat{w}_n(2\lambda_n,\theta)\} = 0. \tag{37}$$

Next, we introduce

$$w_n^*(t,\theta) = \tilde{w}_n(t,\theta) + \frac{A}{a}e^t \qquad \text{in } Q.$$

Clearly

$$-\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial \theta^2}\right) w_n^* = V_n^* e^{w_n^*} - \frac{A}{a}e^t, \tag{38}$$

where $V_n^*(t,\theta) = \tilde{V}_n(t,\theta)e^{-Ae^t/a}$.

It is easy to see that

$$\frac{\partial}{\partial t}\left\{V_n^*(t,\theta)e^{\xi} - \frac{A}{a}e^t\right\} < 0 \qquad \forall\,(t,\theta)\in Q, \forall\,\xi\in\mathbb{R}. \tag{39}$$

We will apply the method of moving planes to w_n^*, but from the opposite direction. For $\lambda < 0$ and $2\lambda \le t \le \lambda$, we set

$$w_n^{*\lambda}(t,\theta) = w_n^*(t^\lambda,\theta),$$

where, as before, $t^\lambda = 2\lambda - t$.

Clearly

$$-\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial \theta^2}\right) w_n^{*\lambda} = V_n^{*\lambda}(t,\theta) e^{w_n^{*\lambda}} - \frac{A}{a}e^{t^\lambda}, \tag{40}$$

where $V_n^{*\lambda}(t,\theta) = V_n^*(t^\lambda,\theta)$.

In order to get started with the plane moving process, appropriate estimates are needed for w_n^*. For that purpose, we first use the harmonicity of $u_n - \tilde{u}_n$ in B_1, the boundedness of the oscillation of $u_n - \tilde{u}_n$ in B_1, and standard elliptic estimates to obtain

$$|\nabla(u_n - \tilde{u}_n)| \le C \qquad \text{in } B_{1/2}. \tag{41}$$

Taking $-\Lambda_1 > -\Lambda_2 >> 1$, we derive from (41) and Lemma 2.1, for large n (depending on Λ_1 and Λ_2), that

$$\frac{\partial u_n}{\partial t}(t,\theta) \le -\frac{\alpha}{2\pi} + 1, \qquad \forall\ \Lambda_1 \le t \le \Lambda_2,\ 0 \le \theta \le 2\pi.$$

Notice that $\alpha \ge 8\pi$, we have

$$\frac{\partial \tilde{w}_n}{\partial t}(t,\theta) = \frac{\partial u_n}{\partial t}(t,\theta) + 2 \le -1, \qquad \forall\ \Lambda_1 \le t \le \Lambda_2,\ 0 \le \theta \le 2\pi.$$

Consequently,

$$\frac{\partial w_n^*}{\partial t}(t,\theta) \le -1/2, \qquad \forall\ \Lambda_1 \le t \le \Lambda_2,\ 0 \le t \le \theta. \tag{42}$$

Fix Λ_2 first. It follows from (37), (27), (36) and Lemma 2.2 that

$$w_n^*(t,\theta) \le \tilde{w}(2\lambda_n - \log\delta_n) + C(\Lambda_2) \le 2(2\lambda_n - \log\delta_n) + C(\Lambda_2)$$

for $\Lambda_2 \le t \le 0, 0 \le \theta \le 2\pi$. Therefore, for all $\Lambda_0 < \Lambda_2$, $2\Lambda_0 \le t \le 2\Lambda_0 - \Lambda_2$, we have

$$w_n^{*\Lambda_0}(t,\theta) = w_n^*(t^{\Lambda_0},\theta) \le 2(2\lambda_n - \log\delta_n) + C(\Lambda_2). \tag{43}$$

Using the definition of λ_n, we have

$$w_n^*(t,\theta) \geq \hat{w}_n(t,\theta) - C \geq \hat{w}_n^{\lambda_n}(t,\theta) - C \qquad \forall\, \lambda_n \leq t \leq 0, 0 \leq \theta \leq 2\pi,$$

where C is some constant independent of n, Λ_2, Λ_1 and Λ_0. Namely, for all $\lambda_n \leq t \leq 0, 0 \leq \theta \leq 2\pi$ we have

$$w_n^*(t,\theta) \geq \hat{w}_n(2\lambda_n - t,\theta) - C.$$

Therefore for all $\lambda_n \leq t \leq \Lambda_0$, $0 \leq \theta \leq 2\pi$, we have, in view of (36) and (27),

$$\begin{aligned} w_n^*(t,\theta) &\geq \hat{w}_n(2\lambda_n - t,\theta) - C \\ &\geq \tilde{w}(2\lambda_n - \log\delta_n - t,\theta) - C \\ &\geq 2(2\lambda_n - \log\delta_n - t) - C. \end{aligned} \tag{44}$$

We see from (43) and (44) that there exists some $\overline{\Lambda}_0 < \Lambda_2$ such that for all $\Lambda_0 < \overline{\Lambda}_0$, and all $\lambda_n \leq 2\Lambda_0 \leq t \leq 2\Lambda_0 - \Lambda_2$ and $0 \leq \theta \leq 2\pi$, we have

$$w_n^{*\Lambda_0}(t,\theta) < w_n^*(t,\theta). \tag{45}$$

Fix one such $\Lambda_0 < \overline{\Lambda}_0$.

Using (42) with $\Lambda_1 = 2\Lambda_0$, we have, for n large,

$$w_n^{*\Lambda_0}(t,\theta) < w_n^*(t,\theta), \qquad \forall\, 2\Lambda_0 - \Lambda_2 \leq t < \Lambda_0, 0 \leq \theta \leq 2\pi. \tag{46}$$

Define

$$\lambda_n^* = \inf\{\mu \leq \Lambda_0 : w_n^{*\lambda}(t,\theta) - w_n^*(t,\theta) < 0\, \forall\, \mu \leq \lambda \leq \Lambda_0, 2\lambda \leq t < \lambda, 0 \leq \theta \leq 2\pi\}.$$

Due to (45) and (46), λ_n^* is well defined for large n. It is easy to see from (27), for large n, that

$$\lambda_n^* \geq \log\delta_n - 2. \tag{47}$$

Using the fact

$$w_n^{*\lambda}(t,\theta) - w_n^*(t,\theta) < 0, \qquad \forall\, 2\lambda < t < \lambda, \lambda_n^* \leq \lambda \leq \Lambda_0, 0 \leq \theta \leq 2\pi,$$

we derive from (38), (40), (39) and the mean value theorem that

$$\begin{gathered} -\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial\theta^2}\right)\big(w_n^{*\lambda}(t,\theta) - w_n^*(t,\theta)\big) \leq 0 \\ \forall\, 2\lambda < t < \lambda, \lambda_n^* \leq \lambda \leq \Lambda_0, 0 \leq \theta \leq 2\pi. \end{gathered}$$

Since the plane moving process stops at λ_n^*, we have, by using the strong maximum principle and the Hopf lemma, that

$$\max_{0\leq\theta\leq 2\pi}\left\{w_n^{*\lambda_n^*}(2\lambda_n^*,\theta) - w_n^*(2\lambda_n^*,\theta)\right\} = 0.$$

Namely,

$$\max_{0\leq\theta\leq 2\pi}\{w_n^*(0,\theta) - w_n^*(2\lambda_n^*,\theta)\} = 0. \tag{48}$$

It follows from the definition of w_n^* and (21) that

$$\min_{\partial B_1} u_n - C \le w_n^*(0,\theta) \le \min_{\partial B_1} u_n + C \qquad \forall 0 \le \theta \le 2\pi. \tag{49}$$

Using (27) and the definition of λ_n, we also know that

$$\begin{aligned}\min_{0\le\theta\le 2\pi} w_n^*(2\lambda_n^*,\theta) &\ge \min_{0\le\theta\le 2\pi} \hat{w}_n(2\lambda_n^*,\theta) - C \\ &\ge \begin{cases} \min_{0\le\theta\le 2\pi} \hat{w}_n(2\lambda_n - 2\lambda_n^*) - C \text{ if } 2\lambda_n^* \ge \lambda_n \\ \min_{0\le\theta\le 2\pi} \hat{w}_n(2\lambda_n^*) - C \text{ if } 2\lambda_n^* < \lambda_n \end{cases} \\ &\ge \begin{cases} 2(2\lambda_n - 2\lambda_n^* - \log\delta_n) - C \text{ if } 2\lambda_n^* \ge \lambda_n \\ 2(2\lambda_n^* - \log\delta_n) - C \text{ if } 2\lambda_n^* < \lambda_n \end{cases}.\end{aligned} \tag{50}$$

Combining (48), (49) and (50), we have

$$\max_{\partial B_1} u_n \ge \begin{cases} 2(2\lambda_n - 2\lambda_n^* - \log\delta_n) - C \text{ if } 2\lambda_n^* \ge \lambda_n \\ 2(2\lambda_n^* - \log\delta_n) - C \text{ if } 2\lambda_n^* < \lambda_n \end{cases}. \tag{51}$$

On the other hand, we know from (37) and (27) that

$$\min_{\partial B_1} u_n \le 2(2\lambda_n - \log\delta_n) + C. \tag{52}$$

It follows from (21), (51), (52) that either

$$-\lambda_n^* \le C, \tag{53}$$

or

$$\lambda_n^* \le \lambda_n + C. \tag{54}$$

We rule out (53) as follows. Suppose (53) happens, then, since $\lambda_n^* < \overline{\Lambda}_0$, we derive from (45), for all $2\lambda_n^* \le t \le 2\lambda_n^* - \Lambda_2$ and $0 \le \theta \le 2\pi$, that

$$w_n^{*\lambda_n^*}(t,\theta) < w_n^*(t,\theta).$$

Now, in view of (42), we have, for n large, $2\lambda_n^* - \Lambda_2 \le t \le \Lambda_2$, and $0 \le \theta \le 2\pi$, that

$$\frac{\partial w_n^*}{\partial t}(t,\theta) \le -1/2 < 0.$$

These imply, for some $\epsilon > 0$ and $\lambda \in [\lambda_n^* - \epsilon, \lambda_n^*]$, that

$$w_n^{*\lambda}(t,\theta) < w_n^*(t,\theta)$$

for all $2\lambda \le t < \lambda$ and $0 \le \theta \le 2\pi$. This violates the definition of λ_n^*, so (53) can not happen. Therefore we always have (54) and, in view of (47) and (36), that

$$|\lambda_n - \log\delta_n| + |\lambda_n^* - \log\delta_n| \le C. \tag{55}$$

Recall that

$$w_n^{*\lambda_n^*}(t,\theta) \le w_n^*(t,\theta), \qquad \forall\, 2\lambda_n^* \le t \le \lambda_n^*, 0 \le \theta \le 2\pi,$$

and

$$\hat{w}_n^{\lambda_n}(t,\theta) \le \hat{w}_n(t,\theta), \qquad \forall\, \lambda_n \le t \le 0, 0 \le \theta \le 2\pi.$$

Namely,

$$w_n^*(t,\theta) \le w_n^*(2\lambda_n^* - t,\theta), \qquad \forall\, \lambda_n^* \le t \le 0, 0 \le \theta \le 2\pi,$$

and

$$\hat{w}_n(t,\theta) \ge \hat{w}_n(2\lambda_n - t,\theta), \qquad \forall\, \lambda_n \le t \le 0, 0 \le \theta \le 2\pi.$$

Since

$$|\hat{w}_n(t,\theta) - \tilde{w}_n(t,\theta)| + |w_n^*(t,\theta) - \tilde{w}_n(t,\theta)| \le C,$$

for all $t \le 0$ and θ, we have

$$\begin{cases} \tilde{w}_n(t,\theta) \le \tilde{w}_n(2\lambda_n^* - t,\theta) + C, \ \forall\, \lambda_n^* \le t \le 0, 0 \le \theta \le 2\pi, \\ \tilde{w}_n(t,\theta) \ge \tilde{w}_n(2\lambda_n - t,\theta) - C, \forall\, \lambda_n \le t \le 0, 0 \le \theta \le 2\pi. \end{cases} \tag{56}$$

Due to (55), we have

$$2\lambda_n^* - t \le \log \delta_n + C \qquad \forall\, \lambda_n^* \le t \le 0,$$

and

$$2\lambda_n - t \le \log \delta_n + C \qquad \forall\, \lambda_n \le t \le 0.$$

So we can use (27) to estimate the right hand sides of (56) and obtain, using again (55), that

$$2(\log \delta_n - t) - C \le \tilde{w}_n(t,\theta) \le 2(\log \delta_n - t) + C, \quad \forall\ \log \delta_n \le t \le 0, \forall\, \theta.$$

In terms of u_n, this means

$$|u_n(x) + u_n(0) + 4\log|x|| \le C, \qquad \forall\, \delta_n \le |x| \le 1. \tag{57}$$

The standard blow up argument (see (24)) yields, for some $R_n \to \infty$,

$$\max_{|x| \le R_n \delta_n} |u_n(x) - \log \frac{\delta_n^{-2}}{(1 + \delta_n^{-2}|x|^2)^2}| \to 0 \qquad \text{as } n \to \infty.$$

On the other hand, (57) is equivalent to

$$|u_n(x) - \log \frac{\delta_n^{-2}}{(1 + \delta_n^{-2}|x|^2)^2}| \le C, \forall\, \delta_n \le |x| \le 1.$$

Theorem 0.3 follows from the above two estimates.

3. Proof of Theorem 0.2

In this section we establish Theorem 0.2 by using Theorem 0.3.

Proof of Theorem 0.2. We know from Theorem 0.1 that $\overline{\lambda} \in [8\pi, \infty)$. For any point $\overline{y} \in M$, let $x = (x_1, x_2)$ be some isothermal coordinate system centered at $\overline{y}$. The metric g takes the form $e^{\varphi}(dx_1^2 + dx_2^2)$ in $B_r(0) := \{x \mid x_1^2 + x_2^2 < r\}$ with $\varphi(0) = 0$. Then ξ_n satisfies

$$-\Delta \xi_n = \lambda_n e^{\varphi}(V_n e^{\xi_n} - W_n), \qquad \text{in } x_1^2 + x_2^2 < r,$$

where $\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2}$. Define ζ_n by

$$\begin{cases} -\Delta \zeta_n = \lambda_n e^{\varphi} W_n + \Delta\varphi, & \text{in } B_r(0), \\ \zeta_n = 0, & \text{on } \partial B_r(0), \end{cases}$$

and set $\eta_n = \xi_n + \zeta_n + \varphi$. Then η_n satisfies

$$-\Delta \eta_n = \lambda_n e^{-\zeta_n} V_n e^{\eta_n}, \text{ in } B_r(0).$$

It is clear that $\{\zeta_n\}$ is uniformly bounded in $\overline{B}_r(0)$. We see from (7) that $\{\int_M e^{\xi_n} dv_g\}$ is bounded from above, so $\lambda_n \int_{B_r(0)} e^{-\zeta_n} V_n e^{\eta_n} \le C$. Therefore it follows from Theorem 3 of [8] that, after passing to a subsequence, there are only three possibilities:

(i) $\{\eta_n\}$ uniformly converges in $C^2(\overline{B}_{r/2}(0))$,

(ii) $\{\eta_n\}$ tends to $-\infty$ uniformly on $\overline{B}_{r/2}(0)$,

(iii) There exist finitely many blowup points $\{x^{(1)}, \cdots, x^{(l)}\}$ of $\{\eta_n\}$ such that $\{\eta_n\}$ tends to $-\infty$ uniformly on compact subsets of $\overline{B}_{r/2}(0) \setminus \{x^{(1)}, \cdots, x^{(l)}\}$.

Clearly, in view of the boundedness of $\{\zeta_n\}$, there are only the above three possibilities for $\{\xi_n\}$ as well. Since M is connected, we know that, after passing to a subsequence, there are only three possibilities for $\{\xi_n\}$ on M:

1° $\{\xi_n\}$ uniformly converges in $C^2(M)$,

2° $\{\xi_n\}$ tends to $-\infty$ uniformly on M,

3° There exist finitely many blowup points $\{\overline{x}^{(1)}, \cdots, \overline{x}^{(m)}\}$ of $\{\xi_n\}$ such that $\{\xi_n\}$ tends to $-\infty$ uniformly on compact subsets of $M \setminus \{\overline{x}^{(1)}, \cdots, \overline{x}^{(m)}\}$.

Since we know from (7) that $\{\int_M e^{\xi_n} dv_g\}$ has a positive lower bound, so 2° can not occur. 1° can not occur either because of (9). We are left with 3°. Applying the result in [32], we know that $\lambda_n V_n e^{\xi_n} \rightharpoonup \sum_{l=1}^m 8\pi N_l \delta_{\overline{x}^{(l)}}$ for some positive integers N_l. Consequently, in view of (7), $\overline{\lambda} = 8\pi \sum_{l=1}^m N_l$. We then derive from (8) that, in $C^0_{loc}(M \setminus \{\overline{x}^{(1)}, \cdots, \overline{x}^{(m)}\})$,

$$\xi_n - \overline{\xi}_n \to 8\pi \sum_{l=1}^m N_l G(\cdot, \overline{x}^{(l)}) - \overline{\lambda} \int_M W(y) G(\cdot, y) dv_g(y). \tag{58}$$

Due to (58), $\{\xi_n\}$ has bounded oscillations in compact subsets of $M \setminus \{\overline{x}^{(1)}, \cdots, \overline{x}^{(m)}\}$. Let $0 < a_l < \frac{1}{2} \min_{l' \neq l} d(\overline{x}^{(l)}, \overline{x}^{(l')})$ be some small constant, $\overline{x}_n^{(l)}$ be a maximum point of ξ_n in $\{y \in M \mid d(y, \overline{x}^{(l)}) < a_l\}$, and $x = (x_1, x_2)$ be some isothermal coordinate system centered at $\overline{x}_n^{(l)}$. The metric g takes the form $e^{\varphi_n}(dx_1^2 + dx_2^2)$ in $B_{a_l}(0) := \{x \mid x_1^2 + x_2^2 < a_l\}$ with $\varphi_n(0) = 0$. Define ζ_n and η_n in $B_r(0)$ as at the beginning of this section with $r = a_l$, then, by applying Theorem 0.3 to η_n, we have

$$|\eta_n(x) - \log \frac{e^{\eta_n(0)}}{(1 + \frac{\lambda_n e^{-\zeta_n(0)} V_n(0)}{8} e^{\eta_n(0)} |x|^2)^2}| \le C, \qquad \forall\, |x| \le a_l,$$

namely,

$$|\xi_n(x) - \log \frac{e^{\xi_n(0)}}{(1 + \frac{\lambda_n V_n(0)}{8} e^{\xi_n(0)} |x|^2)^2}| \le C, \qquad \forall\, |x| \le a_l. \tag{59}$$

It follows easily that

$$\lambda_n V_n e^{\xi_n} \rightharpoonup 8\pi \sum_{l=1}^{m} \delta_{\overline{x}^{(l)}} \quad \text{in the sense of measure.}$$

In the isothermal coordinate system centered at $\overline{x}_n^{(l)}$, we define

$$v_n(x) = \xi_n(\delta_n x) + 2 \log \delta_n, \qquad |x| < a_l \delta_n^{-1},$$

where $\delta_n = e^{-\xi_n(0)/2} \to 0$. Set

$$\overline{R}_n^{(l)} := \sup\{R > 0 \;:\; \|v_n - v\|_{C^2(\overline{B}_{2R}(0))} + \|v_n - v\|_{H^2(\overline{B}_{2R}(0))} < e^{-R}\},$$

where

$$v(x) = \log\{\frac{1}{(1 + \frac{\overline{\lambda} \lim_{n\to\infty} V_n(0)}{8} |x|^2)^2}\}, \qquad \text{in } \mathbb{R}^2.$$

Arguing by contradiction using the standard blow up argument as in Sect. 1, we can show that $\overline{R}_n^{(l)} \to \infty$ as $n \to \infty$. Clearly,

$$\lambda_n \int_{d(y,\overline{x}_n^{(l)}) < \overline{R}_n^{(l)} e^{-\xi_n(\overline{x}_n^{(l)})/2}} V_n e^{\xi_n} \to 8\pi,$$

and ξ_n, for large n, has a unique critical point in $\{y \in M \mid d(y, \overline{x}_n^{(l)}) < \overline{R}_n^{(l)} e^{-\xi_n(\overline{x}_n^{(l)})/2}\}$ due to the fact that v has a unique nondegenerate critical point at the origin. It is easy to see from (59) that

$$\int_{\overline{R}_n^{(l)} e^{-\xi_n(\overline{x}_n^{(l)})/2} < d(y, \overline{x}_n^{(l)}) < a_l} V_n e^{\xi_n} \to 0.$$

Consequently, $N_l = 1$ for all l and $\overline{\lambda} = 8\pi m$. (e) then follows from (58). We easily derive (d) and (c) from (59) and (58).

The above discussion also yields the uniqueness of the maximum point $\overline{x}_n^{(l)}$ since otherwise another maximum point $\hat{x}_n^{(l)}$ would lead to $\lambda_n \int_{d(y,\hat{x}_n^{(l)}) < \overline{R}_n^{(l)} e^{-\xi_n(\hat{x}_n^{(l)})/2}} V_n e^{\xi_n} \to 8\pi$, and due to the definition of $\overline{R}_n^{(l)}$, the two small balls $\{y \in M \mid d(y, \overline{x}_n^{(l)}) < \overline{R}_n^{(l)} e^{-\xi_n(\overline{x}_n^{(l)})/2}\}$ and $\{y \in M \mid d(y, \hat{x}_n^{(l)}) < \overline{R}_n^{(l)} e^{-\xi_n(\hat{x}_n^{(l)})/2}\}$ have no intersections. This would violate $N_l = 1$. Theorem 0.2 is thus established. □

In the rest of this section we derive Corollary 0.3 from Theorem 0.2.

Proof of Corollary 0.3. Using (8), we write φ_n as

$$\varphi_n = \varphi_n^{(1)} + \varphi_n^{(2)}$$

with

$$\varphi_n^{(1)} := \lambda_n \int_M V_n(y) e^{\xi_n(y)} G(\cdot, y) dv_g(y) - 8\pi \sum_{l=1}^m G(\cdot, \overline{x}_n^{(l)})$$

and

$$\varphi_n^{(2)} := 8\pi m \int_M W(y) G(\cdot, y) dv_g(y) - \lambda_n \int_M W_n(y) G(\cdot, y) dv_g(y).$$

Since $\lambda_n W_n \to 8\pi m W$ in $C^\alpha(M)$ for $0 < \alpha < 1$, we derive from Schauder estimates that $\varphi_n^{(2)} \to 0$ in $C^{2,\alpha}(M)$, so

$$\varphi_n^{(2)} \sim 0. \tag{60}$$

Without loss of generality, we may assume $d(x, \overline{x}_n^{(1)}) = \min_{1 \le l \le m} d(x, \overline{x}_n^{(l)})$. Write

$$\varphi_n^{(1)} = \varphi_n^{(11)} + \varphi_n^{(12)} + \varphi_n^{(13)} + \varphi_n^{(14)}$$

with

$$\varphi_n^{(11)} = \lambda_n \sum_{l=1}^m \int_{B_{\sqrt{\overline{R}_n^{(l)}}\delta_n^{(l)}}(\overline{x}_n^{(l)})} V_n(y) e^{\xi_n(y)} [G(\cdot, y) - G(\cdot, \overline{x}_n^{(l)})] dv_g(y),$$

$$\varphi_n^{(12)} = \sum_{l=1}^m \left(\int_{B_{\sqrt{\overline{R}_n^{(l)}}\delta_n^{(l)}}(\overline{x}_n^{(l)})} \lambda_n V_n(y) e^{\xi_n(y)} dv_g(y) - 8\pi \right) G(\cdot, \overline{x}_n^{(l)}),$$

$$\varphi_n^{(13)} = \lambda_n \int_{d(y,x) \le d(x, \overline{x}_n^{(1)})/4} V_n(y) e^{\xi_n(y)} G(\cdot, y) dv_g(y),$$

$$\varphi_n^{(14)} = \lambda_n \int_{d(y,x) \ge d(x,\overline{x}_n^{(1)})/4, y \in M \setminus \cup_{l=1}^m B_{\sqrt{\overline{R}_n^{(l)}}\delta_n^{(l)}}(\overline{x}_n^{(l)})} V_n(y) e^{\xi_n(y)} G(\cdot, y) dv_g(y).$$

For $x \in M \setminus \cup_{l=1}^m B_{\overline{R}_n^{(l)}\delta_n^{(l)}}(\overline{x}_n^{(l)})$, we derive from (c)-(d) in Theorem 0.2 that

$$\begin{aligned} |\varphi^{(13)}(x)| &\le C \frac{e^{\xi_n(\overline{x}_n^{(1)})}}{(1 + e^{\xi_n(\overline{x}_n^{(1)})} d(x, \overline{x}_n^{(1)})^2)^2} \int_{d(y,x) \le d(x,\overline{x}_n^{(1)})/4} |G(x,y)| dv_g(y) \\ &\le C(1 + |\log d(x, \overline{x}_n^{(1)})|) \frac{e^{\xi_n(\overline{x}_n^{(1)})} d(x, \overline{x}_n^{(1)})^2}{(1 + e^{\xi_n(\overline{x}_n^{(1)})} d(x, \overline{x}_n^{(1)})^2)^2}, \end{aligned}$$

which implies, in view of

$$e^{\xi_n(\overline{x}_n^{(1)})} d(x, \overline{x}_n^{(1)})^2 \ge e^{\xi_n(\overline{x}_n^{(1)})} (\overline{R}_n^{(1)} \delta_n^{(1)}))^2 = (\overline{R}_n^{(1)})^2 \to \infty,$$

that

$$\varphi^{(13)} \sim_0 0. \tag{61}$$

The usual blow up argument as in Sect. 1 yields

$$\varphi^{(12)} \sim_0 0. \tag{62}$$

Apparently, for $x \in M \setminus \cup_{l=1}^m B_{overline R_n^{(l)}\delta_n^{(l)}}(\overline{x}_n^{(l)})$,

$$|\varphi^{(11)}(x)| \leq C \sum_{l=1}^m \int_{B_{\sqrt{\overline{R}_n^{(l)}}\delta_n^{(l)}}(\overline{x}_n^{(l)})} e^{\xi_n(y)} |\log \frac{d(x,y)}{d(x,\overline{x}_n^{(i)})}| dv_g(y)$$
$$\leq C \sum_{l=1}^m \log \frac{\overline{R}_n^{(l)} + \sqrt{\overline{R}_n^{(l)}}}{\overline{R}_n^{(l)} - \sqrt{\overline{R}_n^{(l)}}}.$$

Consequently,

$$\varphi^{(11)} \sim_0 0. \tag{63}$$

Using (a) and (c), we have

$$|\varphi^{(14)}(x)| \leq C(1 + |\log d(x, \overline{x}_n^{(1)})|) \int_{M \setminus \cup_{l=1}^m B_{\overline{R}_n^{(l)}\delta_n^{(l)}}(\overline{x}_n^{(1)})} e^{\xi_n} dv_g$$
$$= \circ(1)(1 + |\log d(x, \overline{x}_n^{(l)})|).$$

Namely,

$$\varphi^{(14)} \sim_0 0. \tag{64}$$

Combining (61)–(64), we have

$$\varphi^{(1)} \sim_0 0.$$

Differentiating $\varphi^{(1)}$ under the integral sign and making estimates as above, we can easily show (details are left to readers) that

$$\varphi^{(1)} \sim_1 0 \quad \text{and} \quad \varphi^{(1)} \sim_2 0.$$

Therefore

$$\varphi^{(1)} \sim 0. \tag{65}$$

Corollary 0.3 follows from (60) and (65). □

4. Appendix

For readers' convenience, we provide a proof of the following well known fact.

Lemma 4.1. *There is no C^2 solution to*

$$\begin{cases} \Delta v = e^v, & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^v < \infty. \end{cases}$$

Proof. Suppose the contrary, v is a C^2 solution. Set

$$\overline{v}(r) = \frac{1}{2\pi r} \int_{\partial B_r} v$$

for $r > 0$. We derive from Jensen′s inequality that

$$\frac{1}{2\pi r} \int_{\partial B_r} e^v \geq e^{\overline{v}(r)}. \tag{66}$$

It follows that $\overline{v}$ satisfies

$$\Delta\overline{v} \geq e^{\overline{v}(r)}, \qquad \text{in } \mathbb{R}^2,$$

namely,

$$\frac{1}{r}(r\overline{v}'(r))' \geq e^{\overline{v}(r)}.$$

We derive from the above that

$$r\overline{v}'(r) \geq \int_0^r se^{\overline{v}(s)} \geq 0 \qquad \text{for all } r \geq 0.$$

Consequently,

$$r\overline{v}'(r) \geq \int_0^r se^{\overline{v}(0)} = e^{\overline{v}(0)}\frac{r^2}{2} \qquad \text{for all } r \geq 0.$$

In turn we have

$$\overline{v}(r) \geq \overline{v}(0) + e^{\overline{v}(0)}\frac{r^2}{4} \qquad \text{for all } r \geq 0. \tag{67}$$

It follows from (66) and (67) that

$$\int_{\mathbb{R}^2} e^v = \infty.$$

Contradiction. □

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Communicated by J. L. Lebowitz