Prescribing Scalar Curvature on $\mathbb{S}^n$ and Related Problems, Part II: Existence and Compactness

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Dedicated to Professor L. Nirenberg with admiration on the occasion of his 70th birthday

Abstract

This is a sequel to [30], which studies the prescribing scalar curvature problem on $\mathbb{S}^n$. First we present some existence and compactness results for $n = 4$. The existence result extends that of Bahri and Coron [4], Benayed, Chen, Chtioui, and Hammami [6], and Zhang [39]. The compactness results are new and optimal. In addition, we give a counting formula of all solutions. This counting formula, together with the compactness results, completely describes when and where blowups occur. It follows from our results that solutions to the problem may have multiple blowup points. This phenomena is new and very different from the lower-dimensional cases $n = 2, 3$.

Next we study the problem for $n \geqq 3$. Some existence and compactness results have been given in [30] when the order of flatness at critical points of the prescribed scalar curvature functions $K(x)$ is $\beta \in (n-2, n)$. The key point there is that for the class of K mentioned above we have completed L^∞ apriori estimates for solutions of the prescribing scalar curvature problem. Here we demonstrate that when the order of flatness at critical points of $K(x)$ is $\beta = n - 2$, the L^∞ estimates for solutions fail in general. In fact, two or more blowup points occur.

On the other hand, we provide some existence and compactness results when the order of flatness at critical points of $K(x)$ is $\beta \in [n-2, n)$. With this result, we can easily deduce that C^∞ scalar curvature functions are dense in $C^{1,\alpha}$ $(0 < \alpha < 1)$ norm among positive functions, although this is generally not true in the C^2 norm.

We also give a simpler proof to a Sobolev-Aubin-type inequality established in [16].

Some of the results in this paper as well as that of [30] have been announced in [29].© 1996 John Wiley & Sons, Inc.

0. Introduction

Let $(\mathbb{S}^n, g_0)$ be the standard n-sphere. The following question was raised by L. Nirenberg: Which function $K(x)$ on $\mathbb{S}^2$ is the Gauss curvature of a metric g on $\mathbb{S}^2$ conformally equivalent to g_0? Naturally one can extend this question to higher dimensions $\mathbb{S}^n$ $(n > 2)$.

For $n \geqq 3$, we write $g = v^{\frac{4}{n-2}} g_0$; the problem is then equivalent to finding a function v on $\mathbb{S}^n$ that satisfies the following equation:

$$(0.1) \qquad -\Delta_{g_0} v + c(n) R_0 v = c(n) K(x) v^{\frac{n+2}{n-2}}, \quad v > 0 \text{ on } \mathbb{S}^n,$$

where $c(n) = \frac{n-2}{4(n-1)}$, $R_0 = n(n-1)$ is the scalar curvature of g_0, and Δ_{g_0} denotes the Laplace-Beltrami operator associated with the metric g_0.

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For $n = 2$, we write $g = e^{2v}g_0$; the problem is then equivalent to finding a function v on $\mathbb{S}^2$ that satisfies the following equation:

$$-\Delta_{g_0} v + 1 = K(x)e^{2v}. \tag{0.2}$$

A necessary condition for solving (0.1) or (0.2) is that K be positive somewhere. For $n = 2$, this follows from integrating (0.2) on $\mathbb{S}^2$. For $n \geqq 3$, this follows from multiplying (0.1) by v and integrating by parts on $\mathbb{S}^n$. It turns out that there is at least one other obstruction to solving the problem, the Kazdan-Warner condition (see [27]). In particular, if $\mathbb{S}^n$ is embedded as usual in $\mathbb{R}^{n+1}$ and $K(x) \in C^1(\mathbb{S}^n)$ is strictly monotonic in one direction, then the equation cannot be solved. The Kazdan-Warner condition is obtained by exploiting the centered dilation conformal transformations of $\mathbb{S}^n$. In the same spirit, further obstructions are given in [9] by exploiting the full conformal transformation group of $\mathbb{S}^n$. (See [18] and [25] for more discussions of the Kazdan-Warner-type conditions.)

Recall that the centered dilation conformal transformations of $\mathbb{S}^n$ are defined as follows: For $P \in \mathbb{S}^n$, $0 < t < \infty$, we define a centered dilation conformal transformation $\varphi_{P,t} : \mathbb{S}^n \to \mathbb{S}^n$ by $y \mapsto ty$, where $y \in \mathbb{R}^n$ is the stereographic projection coordinates of points on $\mathbb{S}^n$ while the stereographic projection is performed with P as the north pole to the equatorial plane of $\mathbb{S}^n$.

Much work has been devoted to the existence of solutions of (0.1) and (0.2). For the relation between this work and previous work, see the introduction and references in [30].

In this paper we first present some existence and compactness results for the problem on $\mathbb{S}^4$. The existence result extends that of Bahri and Coron [4], Benayed, Chen, Chtioui, and Hammami [6], and Zhang [39]. We also have a complete understanding of the compactness of solutions to the original equations and subcritical approximation equations that give rise to a degree-counting of all solutions. From our results we know when and where blowups occur. In fact, multiple point blowups may occur. The phenomena of multiple point blowups in dimension $n \geqq 4$ is new and very different from that of lower-dimensional cases.

After the study of the problem on $\mathbb{S}^4$, we study the problem on $\mathbb{S}^n$ for all $n \geqq 3$. Notice that the problem is much less understood for higher dimensions compared to lower dimensions. For higher dimensions, one result is due to Escobar and Schoen [21] concerning curvature functions with group symmetry; another is due to Chang and Yang [16] concerning curvature functions close to constants. We also recall one of the results we obtained in [30], which can be viewed as a natural link between theorem II in [15], theorem 1 in [4], and theorem 2.1 in [21].

THEOREM L. ([30], a special case) *For $n \geqq 3$, we suppose that $K \in C^1(\mathbb{S}^n)$ is some positive function for which the following is true: For any critical point q_0 of K, there exists some real number $\beta = \beta(q_0) \in (n-2, n)$ such that in some geodesic normal coordinate system centered at q_0, $K(y) = K(0) + \sum_{j=1}^n a_j|y_j|^\beta + R(y)$, where $a_j = a_j(q_0) \neq 0, \sum_{j=1}^n a_j \neq 0$, and $R(y)$ is $C^{[\beta]-1,1}$ near 0 and satisfies* $\lim_{|y|\to 0} \sum_{0 \leqq |\alpha| \leqq [\beta]} |\partial^\alpha R(y)| |y|^{-\beta+|\alpha|} = 0$.

Assume further that

$$\sum_{\substack{\nabla_{g_0} K(q_0)=0 \\ \sum_{j=1}^n a_j(q_0)<0}} (-1)^{i(q_0)} \neq (-1)^n ,$$

where $i(q_0) = \#\{a_j(q_0) : a_j(q_0) < 0,\ 1 \leqq j \leqq n\}$. *Then* (0.1) *has at least one solution.*

The key point in establishing the above is to obtain L^∞ apriori estimates for solutions of (0.1). More precisely, it is shown in [30] that under the hypotheses of the above theorem, if we let $K_\mu = \mu K + (1-\mu)R_0$, and let v be any solution of (0.1) corresponding to K_μ for some $0 < \mu \leqq 1$, then $\max_{\mathbb{S}^n} v \leqq C$. It is also shown that the Leray-Schauder degree of all solutions of (0.1) is equal to $-1 + (-1)^n \sum_{\nabla_{g_0} K(q_0)=0, \sum_{j=1}^n a_j(q_0)<0} (-1)^{i(q_0)}$. Here the proper flatness hypotheses near critical points of K ($n-2 < \beta < n$) have been used.

A natural question is what happens when β is equal to $n-2$. The subtlety in this case has been illustrated by Bianchi and Egnell, who constructed in [8] some smooth axisymmetric positive function K with the order of flatness at north and south poles equal to $n-2$ and for which there is no axisymmetric solution to (0.1). We will show that, in general, the L^∞ estimates for solutions of (0.1) fail when the order of flatness at critical points of K is allowed to be equal to $n-2$. This is achieved by first establishing some existence results for those axisymmetric K that are close to R_0 in the L^∞ norm (see Theorem 0.18) and then argue by contradiction. Namely, if the L^∞ estimates hold, we will be able to produce an axisymmetric solution to (0.1) by using a degree argument and Theorem 0.18. In fact, we know that what has happened in this case is that two isolated simple blowup points occur to the corresponding solutions of (0.1) simultaneously at the north and south poles. This phenomenon of multiple point blowups shows that higher-dimensional cases ($n \geqq 4$) are substantially different and more difficult than lower-dimensional cases.

By assuming further some smallness hypothesis on the coefficients of $|y|^{n-2}$, we still obtain the L^∞ estimates as in [30] and hence some existence results. It follows from this existence result and section 6 of [30] that C^∞ scalar curvature functions are dense in the $C^{1,\alpha}(0 < \alpha < 1)$ norm among all positive functions. This density result is generally false in the C^2 norm.

We first note some notation and definitions found in [30]. The notion of an isolated simple blowup point was introduced by Schoen in [36] and [37].

Let $\Omega \subset \mathbb{R}^n$ ($n \geqq 3$) be a bounded domain, $\tau_i \geqq 0$ satisfy $\lim_{i\to\infty} \tau_i = 0$, $p_i = \frac{n+2}{n-2} - \tau_i$, and $\{K_i\} \in C^1(\Omega)$ satisfy, for some constant $A_1 > 0$,

$$1/A_1 \leqq K_i(x) \leqq A_1 \quad \text{for all } x \in \Omega . \tag{0.3}$$

Consider

$$-\Delta u_i = c(n) K_i(x) u_i^{p_i}, \quad u_i > 0 \text{ in } \Omega . \tag{0.4}$$

DEFINITION 0.1. Suppose that $\{K_i\}$ satisfies (0.3) and $\{u_i\}$ satisfies (0.4). A point $\bar{y} \in \Omega$ is called a *blowup point* of $\{u_i\}$ if there exists a sequence y_i tending to $\bar{y}$ such that $u_i(y_i) \to \infty$.

DEFINITION 0.2. Suppose that $\{K_i\}$ satisfies (0.3) and $\{u_i\}$ satisfies (0.4). A point $\bar{y} \in \Omega$ is called an *isolated blowup point* of $\{u_i\}$ if there exists $0 < \bar{r} < \operatorname{dist}(\bar{y}, \partial\Omega)$, $\overline{C} > 0$, and a sequence y_i tending to $\bar{y}$ such that y_i is a local maximum of u_i, $u_i(y_i) \to \infty$, and $u_i(y) \leqq \overline{C}|y - y_i|^{-\frac{2}{p_i - 1}}$ for all $y \in B_{\bar{r}}(y_i)$.

Let $y_i \to \bar{y}$ be an isolated blowup point of $\{u_i\}$; we define

$$\bar{u}_i(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(y_i)} u, \quad \overline{w}_i(r) = r^{\frac{2}{p_i - 1}} \bar{u}_i(r), \qquad r > 0.$$

DEFINITION 0.3. $\bar{y} \in \Omega$ is called an *isolated simple blowup point* if $\bar{y}$ is an isolated blowup point such that for some $\rho > 0$ (independent of i) $\overline{w}_i$ has precisely one critical point in $(0, \rho)$.

DEFINITION 0.4. For any real number $\beta \geqq 1$, we say that a sequence of functions $\{K_i\}$ satisfies condition $(*)_\beta$ for some sequences of constants $\{L_1(\beta, i)\}$ and $\{L_2(\beta, i)\}$ in some region Ω_i if $\{K_i\} \in C^{[\beta]-1,1}(\Omega_i)$ satisfies

$$\|\nabla K_i\|_{C^0(\Omega_i)} \leqq L_1(\beta, i)$$

and, if $\beta \geqq 2$, then

$$|\partial^\alpha K_i(y)| \leqq L_2(\beta, i)|\nabla K_i(y)|^{\frac{\beta - |\alpha|}{\beta - 1}} \quad \text{for all } 2 \leqq |\alpha| \leqq [\beta],\ y \in \Omega_i,\ \nabla K_i(y) \neq 0.$$

Remark 0.5. Let $\{K_i\}$ be bounded in $C^\ell(B_1)$ ($\ell \geqq 2$ is an integer) and have the Taylor expansion

$$K_i(y) = K_i(0) + Q_i^{(\ell)}(y) + R_i(y), \quad y \in B_1,$$

with $Q_i^{(\ell)}$ being some homogeneous polynomial of degree ℓ satisfying

$$|\nabla Q_i^{(\ell)}(y)| \geqq A_6|y|^{\ell - 1}, \quad y \in B_1,$$

for some positive constant A_6 independent of i. Furthermore, let $R_i(y)$ satisfy $\sum_{0 \leqq |\alpha| \leqq \ell} |\partial^\alpha R_i(y)||y|^{-\ell + |\alpha|} \to 0$ uniformly for i as $|y| \to 0$. Then $\{K_i\}$ satisfies $(*)_\ell$ for $L_1(\ell)$ and $L_2(\ell)$ near the origin. Here $L_1(\ell)$ and $L_2(\ell)$ are some constants independent of i.

On a Riemannian manifold (M^n, g), $L_g\psi = \Delta_g\psi - c(n)R_g\psi$ is called the conformal Laplacian, where R_g is the scalar curvature of g. The conformal Laplacian has the following invariance properties under the conformal change of metrics.

For $\hat{g} = u^{\frac{4}{n-2}}g, u > 0$, we have

$$L_{\hat{g}}\psi = u^{-\frac{n+2}{n-2}}L_g(\psi u) \quad \text{for all } \psi \in C^\infty(M). \tag{0.5}$$

Another well-known fact is that if $\partial M = \varnothing$, then for all $\psi \in C^\infty(M)$ we have

$$\int_M \{|\nabla_{\hat{g}}\psi|^2 + c(n)R_{\hat{g}}\psi^2\}dV_{\hat{g}} = \int_M \{|\nabla_g(\psi u)|^2 + c(n)R_g(\psi u)^2\}dV_g . \tag{0.6}$$

Equation (0.6) can be derived easily from equation (0.5). See [7] for the proof of (0.5).

Let P be the south pole and make a stereographic projection to the equatorial plane of $\mathbb{S}^n$. Let $x = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{S}^n$, and let $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ denote the stereographic projection coordinates of x. It is easy to see that

$$\begin{cases} x_i = \frac{2y_i}{1+|y|^2}, & 1 \leqq i \leqq n;\ x_{n+1} = \frac{|y|^2-1}{|y|^2+1}, \\ y_i = \frac{x_i}{1-x_{n+1}}, & 1 \leqq i \leqq n. \end{cases}$$

It follows that in the stereographic projection coordinates

$$g_0 = \sum_{i=1}^{n+1} dx_i^2 = \left(\frac{2}{1+|y|^2}\right)^2 dy^2 = \left\{\left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}}\right\}^{\frac{4}{n-2}} dy^2 .$$

For $P \in \mathbb{S}^n$, let $G_P(q)$ be Green's function of L_{g_0} on $\mathbb{S}^n$. It is well-known that G_P satisfies

$$\begin{cases} G_P(q) > 0, \quad L_{g_0}G_P(q) = 0, & \text{for all } q \in \mathbb{S}^n \setminus \{P\}, \\ -⨍_{\mathbb{S}^n} G_P L_{g_0}\psi dV_{g_0} = \psi(P), & \text{for all } \psi \in C^\infty(\mathbb{S}^n). \end{cases}$$

In this paper, $⨍_{\mathbb{S}^n}$ will denote $|\mathbb{S}^n|^{-1}\int_{\mathbb{S}^n}$. The existence and uniqueness of G_P are well-known (see [2]).

Using (0.5) with $g = dy^2$, $\hat{g} = g_0$, it is elementary to see that in the stereographic projection coordinates as introduced above,

$$G_P(y) = \frac{2^{2-n}|\mathbb{S}^n|}{(n-2)|\mathbb{S}^{n-1}|}\left(\frac{1+|y|^2}{|y|^2}\right)^{\frac{n-2}{2}} . \tag{0.7}$$

It is also easy to see that

$$\min_{q\in\mathbb{S}^n} G_P(q) = \frac{2^{2-n}|\mathbb{S}^n|}{(n-2)|\mathbb{S}^{n-1}|} .$$

For $K \in C^2(\mathbb{S}^n)$, we introduce the following notation:

$$\begin{aligned}
\mathscr{K} &= \{q \in \mathbb{S}^n : \nabla_{g_0} K(q) = 0\}, \\
\mathscr{K}^+ &= \{q \in \mathbb{S}^n : \nabla_{g_0} K(q) = 0,\ \Delta K(q) > 0\}, \\
\mathscr{K}^- &= \{q \in \mathbb{S}^n : \nabla_{g_0} K(q) = 0,\ \Delta K(q) < 0\}, \\
\mathscr{M}_K &= \{v \in C^2(\mathbb{S}^n) : v \text{ satisfies (0.1) or (0.2)}\}.
\end{aligned}$$

We first present some compactness results and existence results for $n = 4$. For $K \in C^2(\mathbb{S}^4)$, we associate any k ($k \geqq 1$) distinct points $q^{(1)}, \dots, q^{(k)} \in \mathscr{K} \setminus \mathscr{K}^+$ with a $k \times k$ symmetric matrix $M = \left(M(q^{(1)}, \dots, q^{(k)})\right)$ defined by

$$M_{ij} = \begin{cases} -\dfrac{\Delta_{g_0} K(q^{(i)})}{K(q^{(i)})^2}, & i = j, \\ -\dfrac{48|\mathbb{S}^3|}{|\mathbb{S}^4|} \dfrac{G_{q^{(i)}}(q^{(j)})}{\sqrt{K(q^{(i)}) K(q^{(j)})}}, & i \neq j. \end{cases} \tag{0.8}$$

Let $\mu(M)$ denote the least eigenvalue of M. When $k = 1$, $\mu(M) = M = -\frac{\Delta_{g_0} K(q^{(1)})}{K(q^{(1)})^2}$.

Remark 0.6. Bahri and Coron discovered through the theory of critical points at infinity that some matrices like (0.8) play important roles in establishing existence results for critical exponent equations. See [3] and [4].

Set

$$\begin{aligned}
\mathscr{A} = \{K \in C^2(\mathbb{S}^4) &: K > 0 \text{ on } \mathbb{S}^4,\ \Delta_{g_0} K \neq 0 \text{ on } \mathscr{K}, \\
&\mu(M(q^{(1)}, \dots, q^{(k)})) \neq 0,\ \forall\, q^{(1)}, \dots, q^{(k)} \in \mathscr{K}^-,\ k \geqq 2\}.
\end{aligned}$$

Observe that for any $K \in \mathscr{A}$, there exists some constant $\delta > 0$ depending only on $\min_{\mathbb{S}^4} K$, $\|K\|_{C^2(\mathbb{S}^4)}$ such that for all $q^{(1)}, \dots, q^{(k)}$ with $\min_{j \neq l} |q^{(j)} - q^{(l)}| \leqq \delta$ we have $\mu(M(q^{(1)}, \dots, q^{(k)})) \leqq -1$. It follows that $\mathscr{A}$ is open in $C^2(\mathbb{S}^4)$. It is obvious that $\mathscr{A}$ is dense in $C^2(\mathbb{S}^4)^+ = \{K \in C^2(\mathbb{S}^4) : K > 0 \text{ on } \mathbb{S}^4\}$ with respect to the C^2 norm.

We define Index : $\mathscr{A} \to \mathbb{Z}$ by the following properties:

(i) For any Morse function $K \in \mathscr{A}$ with $\mathscr{K}^- = \{q^{(1)}, \dots, q^{(m)}\}$, we define

$$\operatorname{Index}(K) = -1 + \sum_{k=1}^{m} \sum_{\substack{\mu(M(q^{(i_1)}, \dots, q^{(i_k)})) > 0 \\ 1 \leqq i_1 < \cdots < i_k \leqq m}} (-1)^{k-1+\sum_{j=1}^{k} i(q^{(i_j)})},$$

where $i(q^{(i_j)})$ denotes the Morse index of K at $q^{(i_j)}$.

(ii) Index : $\mathscr{A} \to \mathbb{Z}$ is continuous with respect to the $C^2(\mathbb{S}^4)$ norm of $\mathscr{A}$ and hence is locally constant.

Remark 0.7. The existence and uniqueness of the Index mapping defined above follows from Theorem 0.8 and the proof of Theorem 0.9.

THEOREM 0.8.
(a) *For any $K \in \mathscr{A}$, there exists $\delta = \delta(K) > 0$, $C = C(K) > 0$, such that for all $\widetilde{K} \in C^2(\mathbb{S}^4)$, $\|\widetilde{K} - K\|_{C^2(\mathbb{S}^4)} < \delta$, $v \in \mathscr{M}_{\widetilde{K}}$, we have*

$$C(K)^{-1} < v < C(K) \text{ on } \mathbb{S}^4, \quad \|v\|_{C^3(\mathbb{S}^4)} < C(K).$$

(b) *For any $K \in C^2(\mathbb{S}^4)^+ \setminus \mathscr{A} = \partial\mathscr{A}$, there exists $K_i \to K$ in $C^2(\mathbb{S}^4)$ and $v_i \in \mathscr{M}_{K_i}$ such that*

$$\lim_{i\to\infty}(\max_{\mathbb{S}^4} v_i) = \infty, \qquad \lim_{i\to\infty}(\min_{\mathbb{S}^4} v_i) = 0. \tag{0.9}$$

THEOREM 0.9. *Suppose $K \in \mathscr{A}$. Then for all $0 < \alpha < 1$, there exists some constant C depending only on $\min_{\mathbb{S}^4} K$, $\|K\|_{C^2(\mathbb{S}^4)}$, C^2 modulo of continuity of K, $\min_{\mathscr{K}} |\Delta_{g_0} K|$, and $\min\{\,|\mu\left(M(q^{(1)},\dots,q^{(k)})\right)|\, : q^{(1)},\dots,q^{(k)} \in \mathscr{K}^-,\ k \geqq 2\}$ such that*

$$1/C < v < C, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^4)} < C,$$

for all solutions v of (0.1). *Furthermore, for all $R \geqq C$,*

$$\deg\left(v - \frac{1}{6}(-\Delta_{g_0} + 2)^{-1}(Kv^3), \mathscr{O}_R, 0\right) = \operatorname{Index}(K), \tag{0.10}$$

where $\mathscr{O}_R = \{v \in C^{2,\alpha}(\mathbb{S}^4) : 1/R < v < R,\ \|v\|_{C^{2,\alpha}(\mathbb{S}^4)} < R\}$, and deg *denotes the Leray-Schauder degree in $C^{2,\alpha}(\mathbb{S}^4)$. As a consequence, $\mathscr{M}_K \neq \emptyset$ provided* $\operatorname{Index}(K) \neq 0$.

THEOREM 0.10. *Let $K \in C^2(\mathbb{S}^4)$ be a positive function. There exists some number $\delta^* > 0$ depending only on $\min_{\mathbb{S}^4} K, \|K\|_{C^2(\mathbb{S}^4)}$, and the C^2 modulo of continuity of K with the following property: Let $\{p_i\}$ satisfy $p_i \leqq 3$, $p_i \to 3$, $\{K_i\} \in C^2(\mathbb{S}^4)$ satisfy $K_i \to K$ in $C^2(\mathbb{S}^4)$, $\{v_i\}$ satisfy*

$$-\Delta_{g_0} v_i + 2v_i = \frac{1}{6}K_i v_i^{p_i}, \quad v_i > 0 \text{ on } \mathbb{S}^4,$$

and

$$\lim_{i\to\infty}\max_{\mathbb{S}^4} v_i = \infty.$$

Then after passing to some subsequence we have
(i) *$\{v_i\}$ (still denote the subsequence by $\{v_i\}$) has only isolated simple blowup points $q^{(1)},\dots,q^{(k)} \in \mathscr{K} \setminus \mathscr{K}^+$ ($k \geqq 1$) with $|q^{(j)} - q^{(l)}| \geqq \delta^*$, $\forall j \neq l$, and $\mu\left(M(q^{(1)},\dots,q^{(k)})\right) \geqq 0$. Furthermore, $q^{(1)},\dots,q^{(k)} \in \mathscr{K}^-$ if $k \geqq 2$.*
(ii)

$$\lambda_j := K(q^{(j)})^{-1/2} \lim_{i\to\infty} v_i(q_i^{(1)}) v_i(q_i^{(j)})^{-1} \in (0, \infty)$$

and

$$\mu^{(j)} := \lim_{i\to\infty} \tau_i v_i(q_i^{(j)})^2 \in [0,\infty), \quad \forall j : 1 \leqq j \leqq k,$$

where $q_i^{(j)} \to q^{(j)}$ *is the local maximum of* v_i.

(iii) *When* $k = 1$,

$$\mu^{(1)} = -24K(q^{(1)})^{-2}\Delta_{g_0}K(q^{(1)}). \tag{0.11}$$

When $k \geqq 2$,

$$\sum_{\ell=1}^{k} M_{\ell j}(q^{(1)},\dots,q^{(k)})\lambda_\ell = \frac{1}{24}\lambda_j\mu^{(j)}, \quad \forall j : 1 \leqq j \leqq k. \tag{0.12}$$

(iv) $\mu^{(j)} \in (0,\infty)$, $\forall j : 1 \leqq j \leqq k$, *if and only if* $\mu(M(q^{(1)},\dots,q^{(k)})) > 0$.

COROLLARY 0.11. *Let* $K \in \mathscr{A}$ *be a Morse function satisfying* $\#\mathscr{K}^- \leqq 1$ *or for any distinct* $P, Q \in \mathscr{K}^-$,

$$\Delta_{g_0}K(P)\Delta_{g_0}K(Q) < 9K(P)K(Q).$$

Then for some constant C,

$$1/C < v < C, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^4)} < C,$$

for all solutions v *of* (0.1), *and for all* $R \geqq C$,

$$\begin{aligned}\deg&\left(v - \frac{1}{6}(-\Delta_{g_0}+2)^{-1}(Kv^3),\ \mathscr{O}_R,\ 0\right)\\ &= \mathrm{Index}(K) = -1 + \sum_{\substack{\nabla_{g_0}K(q_0)=0\\ \Delta_{g_0}K(q_0)<0}} (-1)^{i(q_0)},\end{aligned}$$

where $i(q_0)$ *denotes the Morse index of* K *at* q_0. *Furthermore, if*

$$\sum_{\substack{\nabla_{g_0}K(q_0)=0\\ \Delta_{g_0}K(q_0)<0}} (-1)^{i(q_0)} \neq 1,$$

equation (0.1) *has at least one solution.*

Using Theorems 0.8 through 0.10, we can completely characterize blowups of a sequence of solutions for (0.1) when $n = 4$. For $K \in C^2(\mathbb{S}^4)^+$, we define

$$\begin{aligned}\mathscr{S}(K) = \{&(q^{(1)},\dots,q^{(k)}) : k \geqq 1;\ q^{(j)} \in \mathscr{K}\setminus\mathscr{K}^+,\ \forall j : 1 \leqq j \leqq k;\\ &q^{(j)} \neq q^{(l)},\ \forall j \neq l;\ \mu(M(q^{(1)},\dots,q^{(k)})) = 0\}.\end{aligned}$$

It is easy to see that $\mathcal{S}(K) = \emptyset$ if and only if $K \in \mathcal{A}$.

THEOREM 0.10′.
(a) *Let* $K \in C^2(\mathbb{S}^4)^+ \setminus \mathcal{A}$, $K_i \to K$ *in* C^2, *and* $v_i \in \mathcal{M}_{K_i}$ *with* $\max_{\mathbb{S}^4} v_i \to \infty$. *Then for some* $(q^{(1)}, \dots, q^{(k)}) \in \mathcal{S}(K)$, $\{v_i\}$, *after passing to a subsequence, blows up at precisely the* k *points.*
(b) *Let* $K \in C^2(\mathbb{S}^4)^+ \setminus \mathcal{A}$ *and* $(q^{(1)}, \dots, q^{(k)}) \in \mathcal{S}(K)$. *Then there exists* $K_i \to K$ *in* C^2, $v_i \in \mathcal{M}_{K_i}$ *such that* $\{v_i\}$ *blows up at precisely the* k *points.*

COROLLARY 0.12. *For any* k $(k \geqq 1)$ *distinct points* $q^{(1)}, \dots, q^{(k)} \in \mathbb{S}^4$, *there exists a sequence of Morse functions* $\{K_i\} \in \mathcal{A}$ *such that for some* $v_i \in \mathcal{M}_{K_i}$, $\{v_i\}$ *blows up at precisely the* k *points.*

The compactness results (Theorem 0.8, Theorem 0.9, and Theorem 0.10′) are new and optimal. The existence problem on $\mathbb{S}^4$ has already been studied in [4], [6], and [39]. In [4], Bahri and Coron stated some existence result on $\mathbb{S}^4$ with a brief description of the idea of a proof. Benayed, Chen, Chtioui, and Hammami proved in [6] the result. The existence part of Theorem 0.9 extends the result in [6] in two aspects. First is that we do not need to assume that K is a Morse function. In fact, K can have infinitely many critical points. Second, even for a Morse function K, our result assumes only that the *least* eigenvalue of $M(q^{(1)}, \dots, q^{(k)})$ is nonzero instead of *all* the eigenvalues. Notice that only the least eigenvalue of $M(q^{(1)}, \dots, q^{(k)})$ plays a role in counting the total degree of solutions of (0.1) and the compactness of $\mathcal{M}_K$. For instance, considering a continuous family of K, the total degree of solutions of (0.1) changes when the least eigenvalue of $M(q^{(1)}, \dots, q^{(k)})$ crosses zero, while the total degree remains the same when other eigenvalues of $M(q^{(1)}, \dots, q^{(k)})$ cross zero. The existence result on $\mathbb{S}^4$ in [39] is contained in the result of [6].

Next we study (0.1) for $n \geqq 3$ and give an extension of theorem 0.5 in [30] that is more general than Theorem L stated earlier. We assume a $K \in C^1(\mathbb{S}^n)$ such that for any critical point q_0 of K there exists some real number $\beta = \beta(q_0) \in [n-2, n)$ for which, in some geodesic normal coordinate system centered at q_0,

$$(0.13) \qquad K(y) = K(0) + Q^{(\beta)}_{(q_0)}(y) + R_{(q_0)}(y) \quad \text{for all } y \text{ close to } 0,$$

where $Q^{(\beta)}_{(q_0)}$ satisfies

$$Q^{(\beta)}_{(q_0)}(\lambda y) = \lambda^\beta Q^{(\beta)}_{(q_0)}(y), \qquad \forall \lambda > 0,\ y \in \mathbb{R}^n,\ Q^{(\beta)}_{(q_0)} \in C^{[\beta]-1,1}(\mathbb{S}^{n-1}),$$

$R_{(q_0)}(y)$ is $C^{[\beta]-1,1}$ near 0 with $\lim_{y\to 0} \sum_{0 \le |\alpha| \le [\beta]} |\partial^\alpha R_{(q_0)}(y)|\,|y|^{-\beta+|\alpha|} = 0$, and

$$(0.14) \qquad |\nabla Q^{(\beta)}_{(q_0)}(y)| \sim |y|^{(\beta-1)} \quad \text{for all } y \text{ close to } 0,$$

$$(0.15)\qquad \begin{aligned} &\left|\int_{\mathbb{R}^n} Q_{(q_0)}^{(\beta)}(y+\xi)\frac{1-|y|^2}{(1+|y|^2)^{n+1}}dy\right|^2 \\ &+\left|\int_{\mathbb{R}^n} Q_{(q_0)}^{(\beta)}(y+\xi)(1+|y|^2)^{-n}dy\right|^2 \neq 0, \end{aligned}$$

for those $\xi \in \mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \nabla Q_{(q_0)}^{(\beta)}(y+\xi)(1+|y|^2)^{-n}\,dy = 0.$$

Set

$$\mathscr{K}_\alpha = \{q_0 \in \mathbb{S}^n : \nabla_{g_0}K(q_0) = 0,\ \beta(q_0) = \alpha\}, \quad n-2 \leqq \alpha < n.$$

We assume for $q_0 \in \mathscr{K}_{n-2}$ that

$$(0.16)\qquad \int_{\mathbb{R}^n} \nabla Q_{(q_0)}^{(n-2)}(y+\xi)(1+|y|^2)^{-n}dy = 0 \quad \text{if and only if } \xi = 0.$$

Let

$$\mathscr{K}_{n-2}^- = \left\{q_0 \in \mathscr{K}_{n-2} : \int_{\mathbb{R}^n} z\cdot\nabla Q_{(q_0)}^{(n-2)}(z)(1+|z|^2)^{-n}dz < 0\right\},$$

and for any distinct $q^{(1)}, q^{(2)} \in \mathscr{K}_{n-2}^-$, $M = M(q^{(1)}, q^{(2)})$ is a symmetric 2×2 matrix given by

$$M_{ij} = \begin{cases} -\frac{48}{(n-2)|\mathbb{S}^{n-1}|K(q^{(j)})^2}\int_{\mathbb{R}^n} y\cdot\nabla Q_{q^{(j)}}^{(n-2)}(y)(1+|y|^2)^{-n}\,dy, & i=j, \\ -\frac{48|\mathbb{S}^{n-1}|}{|\mathbb{S}^n|}\frac{G_{q^{(i)}}(q^{(j)})}{\sqrt{K(q^{(i)})K(q^{(j)})}}, & i\neq j. \end{cases}$$

THEOREM 0.13. *Suppose $K \in C^1(\mathbb{S}^n)$ $(n \geqq 3)$ satisfies* (0.13), (0.14), (0.15), (0.16), *and either $\#\mathscr{K}_{n-2}^- \leqq 1$ or $M_{11}M_{22} < M_{12}^2$ for all distinct $q^{(1)}, q^{(2)} \in \mathscr{K}_{n-2}^-$, $M = M(q^{(1)}, q^{(2)})$.*

Then for all $0 < \alpha < 1$, there exists some constant C such that

$$1/C < v < C, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C,$$

for all solutions v of (0.1),

$$\int_{\mathbb{S}^n} K\circ\varphi_{P,t}(x)x \neq 0 \quad \textit{for all } P\in\mathbb{S}^n,\ t\geqq C,$$

and for all $R \geqq C$, $t \geqq C$,

$$\begin{aligned} &\deg\left(v - \left(-\Delta_{g_0} + c(n)R_0\right)^{-1}\left(c(n)Kv^{\frac{n+2}{n-2}}\right), \mathscr{O}_R, 0\right) \\ &= (-1)^n \deg\left(\int_{\mathbb{S}^n} K\circ\varphi_{P,t}(x)x, B, 0\right). \end{aligned}$$

If we further assume that

$$\deg\left(\int_{\mathbb{S}^n} K\circ\varphi_{P,t}(x)x, B, 0\right) \neq 0$$

for t large, then (0.1) *has at least one solution.*

Remark 0.14. In Theorem 0.13, B denotes the unit ball of $\mathbb{R}^{n+1}$, $\mathbb{S}^n = \partial B$. The map $\int_{\mathbb{S}^n} K\circ\varphi_{P,t}(x)x$ was introduced in [16], and its many properties were studied in section 6 of [30].

COROLLARY 0.15. *For $n \geqq 3$, let $K \in C^1(\mathbb{S}^n)$ be some positive function satisfying* (0.13) *with $Q^{(\beta)}_{(q_0)}(y) = \sum_{j=1}^n a_j|y_j|^\beta$, where $a_j = a_j(q_0) \neq 0$, $\sum_{j=1}^n a_j \neq 0$. Assume either $\#\mathscr{K}^-_{n-2} \leqq 1$ or $M_{11}M_{22} < M_{12}^2$ for all distinct $q^{(1)}, q^{(2)} \in \mathscr{K}^-_{n-2}$, $M = M(q^{(1)}, q^{(2)})$.*

Then for all $0<\alpha<1$, there exists some constant C such that

$$1/C < v < C, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C,$$

for all solutions v of (0.1), *and for all $R \geqq C$,*

$$\begin{aligned}&\deg\left(v - (-\Delta_{g_0} + c(n)R_0)^{-1}(c(n)Kv^{\frac{n+2}{n-2}}), \mathscr{O}_R, 0\right)\\ &= -1 + (-1)^n \sum_{\substack{\nabla_{g_0}K(q_0)=0\\ \sum_{j=1}^n a_j(q_0)<0}} (-1)^{i(q_0)},\end{aligned}$$

where

$$i(q_0) = \#\{a_j(q_0) : a_j(q_0) < 0,\ 1 \leqq j \leqq n\}.$$

If we further assume that

$$\sum_{\substack{\nabla_{g_0}K(q_0)=0\\ \sum_{j=1}^n a_j(q_0)<0}} (-1)^{i(q_0)} \neq (-1)^n,$$

then (0.1) *has at least one solution.*

COROLLARY 0.16. *Given any positive numbers $\varepsilon > 0$, $0 < \alpha < 1$, and any positive $C^{1,\alpha}$ function K on $\mathbb{S}^n$, there exists $\widetilde{K} \in C^\infty(\mathbb{S}^n)$, $\|\widetilde{K} - K\|_{C^{1,\alpha}(\mathbb{S}^n)} < \varepsilon$, such that $\widetilde{K}$ is the scalar curvature function of some smooth metric conformal to g_0.*

Remark 0.17. A C^2 density result is false in general. For example, it follows from the compactness results in [30] that for $n = 3, 4$, (0.1) has no solution for any function K which is C^2 close to $x^{n+1} + 2$. For $n = 2$, the previous statement is still valid following from the compactness results in [24] and [13].

We have proved in [30] that if K satisfies $(*)_{n-2}$ then solutions of (0.1) either stay bounded or have only isolated simple blowups; if K satisfies $(*)_\beta$ on $\mathbb{S}^n$ for some $\beta > n-2$ then solutions of (0.1) either stay bounded or have *precisely one* isolated simple blowup point on $\mathbb{S}^n$. The following questions are natural.

Question: Assume that K is a positive smooth function on $\mathbb{S}^n$ ($n \geqq 5$) satisfying

$$\partial^\alpha K(q_0) = 0, \quad 2 \leqq |\alpha| \leqq n-3, \qquad \text{for all } q_0 \in \mathbb{S}^n,\ \nabla K(q_0) = 0.$$

Is it true that solutions of (0.1) either stay bounded or have only isolated simple blowup points?

Question: Assume that K is a positive smooth function on $\mathbb{S}^n$ ($n \geqq 4$) satisfying

$$\partial^\alpha K(q_0) = 0, \quad 2 \leqq |\alpha| \leqq n-2, \qquad \text{for all } q_0 \in \mathbb{S}^n,\ \nabla K(q_0) = 0.$$

Is it true that solutions of (0.1) either stay bounded or have precisely one isolated simple blowup point on $\mathbb{S}^n$?

Next we look at a special situation when $K \in C^1(\mathbb{S}^n)$ depends only on the latitude. Here we assume $K \in C^1$ for the sake of simplicity. For most of the results in this case this smoothness condition can be weakened. Notice that not all of our existence results in this case are new; there is much overlap with previous work (see, e.g., [8], [20], [19], and the references therein). Our approach is different, and the interesting part is that we can see that more than one blowup point indeed occur.

Let $x = (x^1, \dots, x^{n+1}) \in \mathbb{S}^n$, $x^{n+1} = \cos\theta$, $0 \leqq \theta \leqq \pi$. Suppose that $K(x) = K(\theta)$. Here we have abused the notation, but the meaning is evident. In the following we say that K is axisymmetric if K depends only on θ. For $0 < \alpha < 1$, let

$$H_r^1(\mathbb{S}^n) = \{u \in H^1(\mathbb{S}^n) : u \text{ depends only on } \theta\},$$
$$C_r^{2,\alpha}(\mathbb{S}^n) = \{u \in C^{2,\alpha}(\mathbb{S}^n) : u \text{ depends only on } \theta\}.$$

Let N denote the north pole of $\mathbb{S}^n$ and let $\varphi_{N,t}$ ($0 < t < \infty$) denote the conformal transformation defined before. Furthermore, let

$$X_r = \left\{u \in H_r^1(\mathbb{S}^n) : \fint_{\mathbb{S}^n} |u|^{\frac{2n}{n-2}} = 1\right\}, \quad \mathcal{S}_r = \left\{u \in X_r : \fint_{\mathbb{S}^n} x^{n+1} |u|^{\frac{2n}{n-2}} = 0\right\}.$$

For a conformal transformation $\varphi : \mathbb{S}^n \to \mathbb{S}^n$, we set

$$T_\varphi u = u \circ \varphi |\det d\varphi|^{\frac{n-2}{2n}}.$$

We define $\pi : \mathscr{S}_r \times (0,\infty) \to X_r$ by

$$u = \pi(w,t) = T^{-1}_{\varphi_{N,t}} w, \quad w \in \mathscr{S}_r,\ 0 < t < \infty .$$

As in lemma 5.4 of [30], one can prove that π is a C^2 diffeomorphism.

For $\varepsilon_1 > 0$, set

$$\begin{aligned} \mathscr{N}_1^r &= \{w \in \mathscr{S}_r : \|w - 1\| < \varepsilon_1\} , \\ \mathscr{N}_2^r &= \{u \in X_r : u = \pi(w,t) \text{ for some } w \in \mathscr{N}_1^r,\ 0 < t < \infty\} , \\ \mathscr{N}_3^r &= \left\{v \in H_r^1(\mathbb{S}^n) \setminus \{0\} : cv \in \mathscr{N}_2^r \text{ for some } c > 0\right\} . \end{aligned}$$

THEOREM 0.18. *There exist some small constants $\varepsilon_1 = \varepsilon_1(n)$, $\varepsilon_2 = \varepsilon_2(n) > 0$ with the property that for any function $K(\theta) \in C_r^1(\mathbb{S}^n)$ satisfying*

$$\|K - R_0\|_{L^\infty(\mathbb{S}^n)} \leqq \varepsilon_2 ,$$

and

$$\left\{ \begin{aligned} K(\theta) &= K(\pi) + a_1(\pi - \theta)^{\beta_1} + R_1(\theta), \\ &= K(0) + a_2\theta^{\beta_2} + R_2(\theta), \end{aligned} \right. \tag{0.17}$$

where $1 < \beta_1, \beta_2 < n$, $a_1, a_2 \neq 0$, $R_1(\theta) = o((\pi - \theta)^{\beta_1})$, $\frac{dR_1}{d\theta}(\theta) = o((\pi - \theta)^{\beta_1 - 1})$ as $\theta \to \pi$, and $R_2(\theta) = o(\theta^{\beta_2})$, $\frac{dR_2}{d\theta}(\theta) = o(\theta^{\beta_2 - 1})$ as $\theta \to 0$, there exists some large positive constant C_1 such that

$$1/C_1 < v < C_1, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C_1 ,$$

for all solutions $v \in \mathscr{N}_3^r$ (if there are any) of (0.1) *and for all $C \geqq C_1$*

$$\begin{aligned} &\deg\left(v + L_{g_0}^{-1}\left(c(n)Kv^{\frac{n+2}{n-2}}\right), \mathscr{N}_3^r \cap \right. \\ &\qquad \left. \left\{v \in C_r^{2,\alpha} : \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C, 1/C < v < C\right\}, 0\right) \\ &= \frac{1}{2}\left(\operatorname{sign}(a_1) + \operatorname{sign}(a_2)\right) . \end{aligned}$$

In particular, (0.1) *has at least one solution $v \in C_r^2(\mathbb{S}^n) \cap \mathscr{N}_3^r$ provided $a_1 a_2 > 0$.*

THEOREM 0.19. *Let $K(\theta) \in C_r^1(\mathbb{S}^n)$ be a nonnegative function satisfying* (0.17) *with $K(0)$, $K(\pi) > 0$, $a_1, a_2 \neq 0$, $R_1(\theta) = o((\pi - \theta)^{\beta_1})$, $\frac{dR_1}{d\theta}(\theta) = o((\pi - \theta)^{\beta_1 - 1})$ as $\theta \to \pi$, and $R_2(\theta) = o(\theta^{\beta_2})$, $\frac{dR_2}{d\theta}(\theta) = o(\theta^{\beta_2 - 1})$ as $\theta \to 0$. Assume any of the following conditions:*

(i) $n - 2 \leqq \beta_1, \beta_2 < n$, $\max(a_1, a_2) > 0$.

(ii) $n - 2 \leqq \beta_1, \beta_2 < n$, $\beta_1 + \beta_2 \neq 2n - 4$.

(iii) $1 < \beta_1, \beta_2 < n$, $a_1, a_2 > 0$.

Then there exists some positive constant C_1 *such that*

$$1/C_1 < v < C_1, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C_1,$$

for all $C_r^2(\mathbb{S}^n)$ *solutions (if there are any) of* (0.1) *and for all* $C \geqq C_1$,

$$\begin{aligned}&\deg\Big(v + L_{g_0}^{-1}\big(c(n)Kv^{\frac{n+2}{n-2}}\big),\\&\qquad \{v \in C_r^{2,\alpha} : \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C,\ 1/C < v < C\},\ 0\Big)\\&= \frac{1}{2}\left(\mathrm{sign}(a_1) + \mathrm{sign}(a_2)\right).\end{aligned}$$

In particular, (0.1) *has at least one* $C_r^2(\mathbb{S}^n)$ *solution under a further assumption* $a_1 a_2 > 0$.

In the following we set $c_1 = \left(2^{n-2}(n-2)\int_0^\infty \frac{r^{2n-3}dr}{(1+r^2)^n}\right)^2$.

THEOREM 0.20. *Let* $K(\theta) \in C_r^1(\mathbb{S}^n)$ *be a nonnegative function satisfying*

$$\begin{cases} K(\theta) = K(\pi) + a_1(\pi - \theta)^{n-2} + R_1(\theta), \\ \qquad\;\; = K(0) + a_2\theta^{n-2} + R_2(\theta), \end{cases}$$

where $K(0), K(\pi) > 0$, $a_1, a_2 < 0$, $K(0)K(\pi) \neq c_1 a_1 a_2$, $R_1(\theta) = o((\pi - \theta)^{n-2})$, $\frac{dR_1}{d\theta}(\theta) = o((\pi-\theta)^{n-3})$ *as* $\theta \to \pi$, *and* $R_2(\theta) = o(\theta^{n-2})$, $\frac{dR_2}{d\theta}(\theta) = o(\theta^{n-3})$ *as* $\theta \to 0$. *Then there exists some positive number* C_2 *such that*

$$1/C_2 < v < C_2, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C_2, \tag{0.18}$$

for all axisymmetric solutions v *of* (0.1). *Furthermore, for all* $C \geqq C_2$ *we have*

$$\begin{aligned}&\deg\Big(v + L_{g_0}^{-1}\big(c(n)Kv^{\frac{n+2}{n-2}}\big), \{v \in C_r^{2,\alpha}(\mathbb{S}^n) : \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C, 1/C < v < C\}, 0\Big)\\&= \begin{cases} -1, & K(0)K(\pi) > c_1 a_1 a_2, \\ 0, & K(0)K(\pi) < c_1 a_1 a_2. \end{cases}\end{aligned} \tag{0.19}$$

Remark 0.21. Since we often need to work with a family of K, we need to know the dependence of C_2 on K in Theorem 0.20. This can be seen easily from the proof. For example, C_2 is under control provided that $K(0), K(\pi), -a_1, -a_2$, and $|K(0)K(\pi) - c_1 a^1 a^2|$ are bounded above and below by positive constants and that K has certain uniform continuity near the poles.

COROLLARY 0.22. *Under the hypotheses of Theorem* 0.20, (0.1) *has at least one* $C_r^2(\mathbb{S}^n)$ *solution provided* $K(0)K(\pi) > c_1 a_1 a_2$. *On the other hand, if we assume* $K(0)K(\pi) < c_1 a_1 a_2$ *and* $\|K - R_0\|_{L^\infty(\mathbb{S}^n)} \leqq \varepsilon_2$ *(*ε_2 *is defined in Theorem* 0.18*), then* (0.1) *has at least two* $C_r^2(\mathbb{S}^n)$ *solutions.*

Remark 0.23. Examples given in [8] show that when $K(0)K(\pi) < c_1 a_1 a_2$, (0.1) may not have any $C_r^2(\mathbb{S}^n)$ solution.

COROLLARY 0.24. *Let* $K_t (0 \leqq t \leqq 2)$ *be a family of nonnegative* $C_r^1(\mathbb{S}^n)$ *function. Writing*

$$\begin{cases} K_t(\theta) = K_t(\pi) + a_1(t)(\pi - \theta)^{n-2} + R_t^1(\theta), \\ K_t(\theta) = K_t(0) + a_2(t)\theta^{n-2} + R_t^2(\theta), \end{cases}$$

where $K_t(\pi), K_t(0), -a_1(t), -a_2(t)$ *are positive continuous functions on the interval* $0 \leqq t \leqq 2$,

$$\begin{aligned} K_t(0)K_t(\pi) &> c_1 a_1(t) a_2(t), \qquad 0 \leqq t < 1, \\ K_t(0)K_t(\pi) &< c_1 a_1(t) a_2(t), \qquad 1 < t \leqq 2, \end{aligned}$$

$R_t^1(\theta) = \circ((\pi - \theta)^{n-2})$, $\frac{dR^1}{d\theta}(\theta) = \circ((\pi - \theta)^{n-3})$ *as* $\theta \to \pi$, *and* $R_t^2(\theta) = \circ(\theta^{n-2})$, $\frac{dR^2}{d\theta}(\theta) = \circ(\theta^{n-3})$ *as* $\theta \to 0$ *uniformly for* $0 \leqq t \leqq 2$.

Then there exists a sequence $t_j \to 1$ *and a* $v_j \in C_r^2(\mathbb{S}^n)$ *that is the solution of* (0.1) *corresponding to* K_{t_j} *such that*

$$\lim_{j\to\infty} \max_{\mathbb{S}^n} v_j = \infty .$$

Furthermore, $\{v_j\}$ *has precisely two isolated simple blowup points, which are the north and south poles.*

This paper is organized as follows: In Section 1 we recall some results used in [30]. In Section 2 we study the problem on $\mathbb{S}^4$ and establish Theorems 0.8, 0.9, and 0.10, as well as Theorem 0.10′. Theorem 0.10 will be proved first, using results in [30]. Then we use Theorem 0.10 and some results in [30] to prove part (a) of Theorem 0.8. To prove Theorem 0.9, we consider a subcritical approximation of (0.1), namely,

$$-\Delta_{g_0} v + c(n)R_0 v = c(n)K(x)v^{3-\tau}, \quad v > 0 \text{ on } \mathbb{S}^4 \tag{0.20}$$

for $\tau > 0$ small. Thanks to part (a) of Theorem 0.8, we can assume without loss of generality that K is a Morse function. For any k distinct points $\overline{P}_1, \ldots, \overline{P}_k \in \mathscr{K}^-$ with $\mu(M(\overline{P}_1, \ldots, \overline{P}_k)) > 0$, we carefully construct some set $\Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k) \subset H^1(\mathbb{S}^4)$ that consists of suitable functions which are highly concentrated near $\overline{P}_1, \ldots, \overline{P}_k$. Using Theorem 0.10 and some results in [30], we first establish Proposition 2.1, which asserts that for $\tau > 0$ very small, solutions of (0.20) either stay bounded or stay in one of the $\Sigma_\tau(\overline{P}_1, \dots, \overline{P}_k)$. On the other hand, we establish Theorem 2.2′, which asserts that for $\tau > 0$ small enough, (0.20) has precisely one solution in $\Sigma_\tau(\overline{P}_1, \dots, \overline{P}_k)$, which is nondegenerate with Morse index $5k - \sum_{j=1}^k i(\overline{P}_j)$.

Now we point out a well-known fact (Proposition 2.7) which asserts that the H^1 total degree of solutions of (0.20) is equal to -1 for all $0 < \tau < 2$. It follows that the H^1 degree contribution of those solutions of (0.20) which remain bounded as τ tends to zero is equal to Index(K). Some well-known results in degree theory imply that the H^1 degree contribution above is equal to the $C^{2,\alpha}$ degree contribution of those bounded solutions of (0.20). Using part (a) of Theorem 0.8, Theorem 0.10 and the homotopy invariance of the Leray-Schauder degree, we obtain (0.10). Theorem 0.9 is therefore established. Part (b) of Theorem 0.8 is proved by using Theorem 0.9, Theorem 0.10, part (a) of Theorem 0.8 and the homotopy invariance of the Leray-Schauder degree. The proof of Theorem 0.10′ is similar to the proof of part (b) of Theorem 0.8, and is omitted. In Section 3 we establish Theorem 0.13 by proving a more general result. In Section 4 we establish results in the axisymmetric case and demonstrate that when the order of flatness at critical points of $K(x)$ is $n-2$, the L^∞ estimates for solutions fail in general. In Section 5 we give a simpler proof of a Sobolev-Aubin type inequality established in [16]. In Section 6 we list some elementary estimates.

1. Quick Review of Some Known Facts

In this section we recall some results used in [30]. Let $\sigma > 0$ and B_σ be a ball of radius σ in $\mathbb{R}^n$ ($n \geqq 3$).

PROPOSITION 1.1. *Let $p \geqq 1$, K be a C^1 function and u be a C^2 solution of*

$$-\Delta u = c(n)K(x)|u|^{p-1}u, \quad x \in B_\sigma .$$

We have

$$\frac{c(n)}{p+1}\sum_i \int_{B_\sigma} x_i \frac{\partial K}{\partial x_i}|u|^{p+1} + \left(\frac{n}{p+1} - \frac{n-2}{2}\right) c(n)\int_{B_\sigma} K|u|^{p+1} - \frac{\sigma c(n)}{p+1}\int_{\partial B_\sigma} K|u|^{p+1} = \int_{\partial B_\sigma} B(\sigma, x, u, \nabla u)$$

where

$$B(\sigma, x, u, \nabla u) = \frac{n-2}{2}u\frac{\partial u}{\partial \nu} - \frac{\sigma}{2}|\nabla u|^2 + \sigma\left(\frac{\partial u}{\partial \nu}\right)^2 .$$

It is elementary to check that the boundary term $B(\sigma, x, u, \nabla u)$ has the following properties:

PROPOSITION 1.2. *Let $A \in \mathbb{R}$ and $\alpha(x)$ be some differentiable function near the origin with $\alpha(0) = 0$. Then for $u(x) = |x|^{2-n} + A + \alpha(x)$, we have*

$$\lim_{\sigma \to 0} \int_{\partial B_\sigma} B(\sigma, x, u(x), \nabla u(x)) = -\frac{(n-2)^2}{2} A |\mathbb{S}^{n-1}|.$$

PROPOSITION 1.3. *Let $\{K_i\}$ satisfy* (0.3), *$\{u_i\}$ satisfy* (0.4), *and $y_i \to \bar{y} \in \Omega$ be an isolated blowup point. Then for any $0 < r < \bar{r}/3$, we have the following Harnack inequality:*

$$\max_{y \in B_{2r} \setminus B_{r/2}} u_i(y) \leqq C \min_{y \in B_{2r} \setminus B_{r/2}} u_i(y),$$

where C is a positive constant depending only on $n, \overline{C}$, and $\sup_i \|K_i\|_{L^\infty(B_{\bar{r}}(y_i))}$, *and $\bar{r}$ and $\overline{C}$ are the constants in Definition* 0.2.

PROPOSITION 1.4. *Suppose $\{K_i\} \in C^1_{\mathrm{loc}}(\Omega)$ is bounded in $C^1_{\mathrm{loc}}(\Omega)$ satisfying* (0.3) *and $\{u_i\}$ satisfies* (0.4). *Let $\bar{y} \in \Omega$ be an isolated blowup point of $\{u_i\}$ and $\{y_i\}$ be the sequence of points as in Definition* 0.2. *Then for any $R_i \to \infty$, $\varepsilon_i \to 0^+$, we have, after passing to a subsequence (still denoted as $\{u_i\}$, $\{y_i\}$, etc.), that*

$$\|u_i(y_i)^{-1} u_i\left(u_i(y_i)^{-\frac{p_i-1}{2}} \cdot + y_i\right) - (1 + k_i|\cdot|^2)^{\frac{2-n}{2}}\|_{C^2(B_{2R_i}(0))} \leqq \varepsilon_i,$$

$$R_i u_i(y_i)^{-\frac{p_i-1}{2}} \to 0 \quad \text{as } i \to \infty$$

where $k_i = c(n)\left(n(n-2)\right)^{-1} K_i(y_i)$.

PROPOSITION 1.5. *Suppose $\{K_i\} \in C^1_{\mathrm{loc}}(B_2)$ satisfies* (0.3) *with $\Omega = B_2$ and*

$$|\nabla K_i(y)| \leqq A_2 \quad \text{for all } y \in B_2 \tag{1.1}$$

for some positive constant A_2. Suppose also that u_i satisfies (0.4) *with $\Omega = B_2$ and that $y_i \to 0$ is an isolated blowup point with, for some positive constant A_3,*

$$|y - y_i|^{\frac{2}{p_i-1}} u_i(y) \leqq A_3 \quad \text{for all } y \in B_2. \tag{1.2}$$

Then there exists some positive constant $C = C(n, A_1, A_2, A_3)$ such that

$$u_i(y) \geqq C^{-1} u_i(y_i)\left(1 + k_i u_i(y_i)^{p_i-1} |y - y_i|^2\right)^{\frac{2-n}{2}} \quad \text{for all } |y - y_i| \leqq 1.$$

In particular, for any $e \in \mathbb{R}^n$, $|e| = 1$, we have

$$u_i(y_i + e) \geqq C^{-1} u_i(y_i)^{-1+\frac{n-2}{2}\tau_i}.$$

PROPOSITION 1.6. *Suppose* $\{K_i\} \subset C^1_{\text{loc}}(B_2)$ *satisfies* (0.3) *with* $\Omega = B_2$ *and* (1.1) *for some positive constant* A_2. *Suppose also that* u_i *satisfies* (0.4) *with* $\Omega = B_2$ *and* $y_i \to 0$ *is an isolated simple blowup point with* (1.2) *for some positive constant* A_3. *Then there exists some positive constant* $C = C(n, A_1, A_2, A_3, \rho)$ *such that*

$$u_i(y) \leqq C u_i(y_i)^{-1} |y - y_i|^{2-n} \quad \text{for all } |y - y_i| \leqq 1$$

where ρ *is the constant in Definition* 0.3.

Furthermore, for some harmonic function $b(y)$ *in* B_1 *we have, after passing to a subsequence, that*

$$u_i(y_i) u_i(y) \to h(y) = a|y|^{2-n} + b(y) \quad \text{in } C^2_{\text{loc}}(B_1 \setminus \{0\}),$$

where

$$a = \lim_{i\to\infty} k_i^{\frac{2-n}{2}} = c(n)^{\frac{2-n}{2}} [n(n-2)]^{\frac{n-2}{2}} \left(\lim_{i\to\infty} K_i(0)\right)^{\frac{2-n}{2}}.$$

2. Proofs of Theorems 0.8, 0.9, and 0.10

Proof of Theorem 0.10: It follows from theorem 4.1 of [30] that $\{v_i\}$ has only isolated simple blowup points $q^{(1)}, \dots, q^{(k)} \in \mathscr{K}$ $(k \geqq 1)$ with $|q^{(j)} - q^{(l)}| \geqq \delta^*$ $(j \neq l)$ for some δ^* depending only on the data stated in Theorem 0.10.

Since $q^{(j)}$ is an isolated simple blowup point of v_i, we let $q_i^{(j)} \to q^{(j)} (i \to \infty)$ be the local maximum of v_i. Let $q_i^{(j)}$ be the south pole and make a stereographic projection to the equatorial plane of $\mathbb{S}^n$ with y as the stereographic projection coordinates. Set

$$u_i(y) = \frac{2}{1 + |y|^2} v_i(y),$$

the equation (0.1) is transformed to

$$-\Delta u_i(y) = \frac{1}{6} K_i(y) H_i^{\tau_i}(y) u_i(y)^{p_i}, \quad y \in \mathbb{R}^4,$$

where $H_i(y) = 2/(1 + |y|^2)$.

Let $y_i^{(j)} \to 0$ be the local maximum of u_i as in Definition 0.2. It follows from proposition 2.3 of [30] that

$$\begin{aligned} &u_i(y_i^{(j)}) u_i(y) \to h^{(j)}(y) \\ &\quad := 48 K(q^{(j)})^{-1} |y|^{-2} + b^{(j)}(y) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^4 \setminus \{q^{(1)}, \dots, q^{(k)}\}), \end{aligned}$$

where $b^{(j)}(y)$ is some regular harmonic function in $R^4 \setminus \cup_{l\neq j}\{q^{(l)}\}$. It follows from the maximum principle that $b^{(j)}(y) \equiv 0$ if $k = 1$, and $b^{(j)}(y) > 0$ if $k \geqq 2$.

It follows from [30] that

$$\lim_{i\to\infty} v_i(q_i^{(j)})v_i(q) = 96|\mathbb{S}^3||\mathbb{S}^4|^{-1}K(q^{(j)})^{-1}G_{q^{(j)}}(q) + \widetilde{b}^{(j)}(q)$$

for $q \neq q^{(j)}$ and close to $q^{(j)}$, where $\widetilde{b}^{(j)}(q)$ is some regular function in $\mathbb{S}^4 \setminus \cup_{l\neq j}\{q^{(l)}\}$ satisfying $L_{g_0}\widetilde{b}^{(j)} = 0$ and the convergence is in the sense of $C^2_{\text{loc}}(\mathbb{S}^4 \setminus \{q^{(1)}, \dots, q^{(k)}\})$.

When $k \geqq 2$, it follows from the maximum principle and [30] that for all $1 \leqq j \leqq k$,

$$\lim_{i\to\infty} v_i(q_i^{(j)})v_i(q) = \frac{96|\mathbb{S}^3|}{|\mathbb{S}^4|}\left\{\frac{G_{q^{(j)}}(q)}{K(q^{(j)})} + \sum_{\ell\neq j}\lim_{i\to\infty}\frac{v_i(q_i^{(j)})}{v_i(q_i^{(\ell)})}\frac{G_{q^{(\ell)}}(q)}{K(q^{(\ell)})}\right\},$$

where the convergence is in $C^2_{\text{loc}}(\mathbb{S}^4\setminus\{q^{(1)}, \dots, q^{(k)}\})$. It is not difficult to see (using the fact that the blowup is isolated simple) that $v_i(y_i^{(j)})v_i(q_i^{(j)})^{-1} \to 1$. It follows from (0.7) that for $|y| > 0$ small,

$$h^{(j)}(y) = \frac{48}{K(q^{(j)})|y|^2} + \frac{384|\mathbb{S}^3|}{|\mathbb{S}^4|}\sum_{\ell\neq j}\lim_{i\to\infty}\frac{v_i(q_i^{(j)})}{v_i(q_i^{(\ell)})}\frac{G_{q^{(\ell)}}(q^{(j)})}{K(q^{(\ell)})} + O(|y|).$$

It follows from lemma 2.4, lemma 2.6, lemma 2.7, and proposition 2.1 in [30] and the evenness of $(1 + |z|^2)^{-2}$ that

$$\text{(2.1)}\qquad \begin{cases} |\nabla K_i(y_i^{(j)})| = O\left(u_i(y_i^{(j)})^{-1}\right), \quad \tau_i = O\left(u_i(y_i^{(j)})^{-2}\right), \\ \displaystyle\sum_{j=1}^{4}\left|\int_{B_\sigma} x_j u_i^{p_i+1}(\cdot + y_i^{(j)})\right| = o\left(u_i(y_i^{(j)})^{-1}\right), \\ \displaystyle\sum_{j\neq l}\left|\int_{B_\sigma} x_j x_l u_i^{p_i+1}(\cdot + y_i^{(j)})\right| = o\left(u_i(y_i^{(j)})^{-2}\right), \\ \displaystyle\int_{\partial B_\sigma} u_i^{p_i+1}\left(\cdot + y_i^{(j)}\right) = O\left(u_i\left(y_i^{(j)}\right)^{-p_i-1}\right), \\ \displaystyle\lim_{i\to\infty} u_i(y_i^{(j)})^2\int_{B_\sigma}|x|^2 u_i^{p_i+1}\left(\cdot + y_i^{(j)}\right) = \frac{18432|\mathbb{S}^3|}{K(q^{(j)})^3}. \end{cases}$$

Part (ii) of Theorem 0.10 follows from (2.1) and [30].

It follows from Proposition 1.1 and (2.1) that for any $0 < \sigma < 1$ we have

$$\begin{aligned}
&\int_{\partial B_\sigma} B(\sigma, x, u_i(\cdot + y_i^{(j)}), \nabla u_i(\cdot + y_i^{(j)})) \\
&= \frac{1}{24}\sum_l \int_{B_\sigma} x_l \frac{\partial (K_i H_i^{\tau_i})}{\partial x_l}(\cdot + y_i^{(j)}) u_i^{p_i+1}(\cdot + y_i^{(j)}) \\
&\quad + \frac{\tau_i}{24}\int_{B_\sigma} K_i(\cdot + y_i^{(j)}) u_i(x + y_i^{(j)})^{p_i+1} + o(u_i(y_i^{(j)})^{-2}) \\
&= \frac{1}{24}\sum_l \int_{B_\sigma} x_l \frac{\partial K_i}{\partial x_l}(\cdot + y_i^{(j)}) u_i^{p_i+1}(\cdot + y_i^{(j)}) \\
&\quad + 8K(q^{(j)})^{-1}|\mathbb{S}^3|\tau_i + o(u_i(y_i^{(j)})^{-2}) \\
&= \frac{1}{24}\int_{B_\sigma} x\cdot\nabla K_i(y_i^{(j)}) u_i^{p_i+1}(\cdot + y_i^{(j)}) \\
&\quad + \frac{1}{24}\sum_{l,m}\int_{B_\sigma} x_l x_m \frac{\partial^2 K_i}{\partial x_l \partial x_m}(y_i^{(j)}) u_i^{p_i+1}(\cdot + y_i^{(j)}) \\
&\quad + 8K(q^{(j)})^{-1}|\mathbb{S}^3|\tau_i + o(u_i(y_i^{(j)})^{-2}) \\
&= \frac{1}{24}\Delta_{g_0} K(q^{(j)}) \int_{B_\sigma} |x|^2 u_i(x + y_i^{(j)})^{p_i+1} + 8K(q^{(j)})^{-1}|\mathbb{S}^3|\tau_i + o(u_i(y_i^{(j)})^{-2}).
\end{aligned}$$

Multiplying the above by $u_i(y_i^{(j)})^2$ and sending i to ∞, we have

$$\int_{\partial B_\sigma} B(\sigma, x, h^{(j)}, \nabla h^{(j)}) = \frac{768|\mathbb{S}^3|\Delta_{g_0} K(q^{(j)})}{K(q^{(j)})^3} + \frac{32|\mathbb{S}^3|\mu^{(j)}}{K(q^{(j)})}.$$

Sending σ to 0, it follows from Proposition 1.2 that

$$768|\mathbb{S}^3|K(q^{(j)})^{-3}\Delta_{g_0}K(q^{(j)}) + 32|\mathbb{S}^3|K(q^{(j)})^{-1}\mu^{(j)} = -96K(q^{(j)})^{-1}|\mathbb{S}^3|b^{(j)}(0).$$

It follows that $q^{(j)} \in \mathscr{K}\setminus\mathscr{K}^+, 1 \leqq j \leqq k$, and, when $k \geqq 2$, $q^{(j)} \in \mathscr{K}^-, 1 \leqq j \leqq k$.

When $k = 1$, $b^{(j)}(0) = 0$, we have verified (0.11).

When $k \geqq 2$, for all $1 \leqq j \leqq k$,

$$b^{(j)}(0) = 384|\mathbb{S}^3|\,|\mathbb{S}^4|^{-1}\sum_{\ell\neq j}\frac{\lambda_\ell}{\lambda_j}\frac{G_{q^{(\ell)}}(q^{(j)})}{\sqrt{K(q^{(j)})K(q^{(\ell)})}}.$$

It follows that

$$-48|\mathbb{S}^3||\mathbb{S}^4|^{-1}\sum_{\ell\neq j}\frac{G_{q^{(\ell)}}(q^{(j)})}{\sqrt{K(q^{(j)})K(q^{(\ell)})}}\lambda_\ell - \frac{\Delta_{g_0}K(q^{(j)})}{K(q^{(j)})^2}\lambda_j = \frac{1}{24}\lambda_j\mu^{(j)}.$$

We have established (0.12) and thus verified part (iii) of Theorem 0.10. It follows from linear algebra that there exists some $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_k) \neq 0$, $\overline{\lambda}_\ell \geqq 0\ \forall \ell$, such that

$$\sum_{\ell=1}^{k} M_{\ell j}(q^{(1)}, \ldots, q^{(k)})\overline{\lambda}_\ell = \mu(M)\overline{\lambda}_j, \quad 1 \leqq j \leqq k.$$

Multiplying (0.12) by $\overline{\lambda}_j$ and summing over j, we have

$$\mu(M)\sum_\ell \lambda_\ell \overline{\lambda}_\ell = \sum_{\ell,j} M_{\ell j}\lambda_\ell\overline{\lambda}_j = \frac{1}{24}\sum_j \lambda_j\overline{\lambda}_j\mu^{(j)} \geqq 0.$$

It follows that $\mu(M) \geqq 0$. We have verified part (i) of Theorem 0.10.

Part (iv) of Theorem 0.10 follows from (i)-(iii) and some elementary arguments.

Proof of part (a) of Theorem 0.8: Suppose the contrary is true. Then it is easy to see that there exists $K_i \to K$ in $C^2(\mathbb{S}^4)$ such that $\max_{\mathbb{S}^4} v_i \to \infty$ for some $v_i \in \mathcal{M}_{K_i}$. It follows from Theorem 0.10 that $\{v_i\}$ has only isolated simple blowup points $\{q^{(1)}, \ldots, q^{(k)}\}$. It follows from theorem 4.4 in [30] that $k > 1$. It follows from Theorem 0.10 that $q^{(1)}, \ldots, q^{(k)} \in \mathcal{K}^-$ and for all $1 \leqq j \leqq k$, $\sum_{\ell=1}^k M_{\ell j}\lambda_\ell = 0$, where $\lambda_\ell > 0$ $(1 \leqq \ell \leqq k)$.

Since $\mu(M)$ has at least one nonnegative eigenvector and since eigenvectors with respect to different eigenvalues are orthogonal to each other, we have $\mu(M) = 0$. This fact contradicts the fact that $K \in \mathcal{A}$.

The rest of this section is devoted to the proof of Theorem 0.9, and then part (b) of Theorem 0.8. Due to part (a) of Theorem 0.8, we only need to prove Theorem 0.9 for $K \in \mathcal{A}$ being a Morse function. Once this is achieved, Index(K) is well-defined. Therefore we always assume that $K \in \mathcal{A}$ is a Morse function in the rest of this section.

For $n \geqq 3$, $P \in \mathbb{S}^n$, $t > 0$, set $\delta_{P,t} = T_{\varphi_{P,t}} 1$.

It is well-known that $u = \delta_{P,t}$ satisfies $-L_{g_0}u = c(n)R_0 u^{\frac{n+2}{n-2}}$.

We denote the H^1 inner product and norm by

$$\langle u, v\rangle = -\int_{\mathbb{S}^n}(L_{g_0}u)v, \quad \|u\| = \sqrt{\langle u, u\rangle}.$$

Set for $\tau > 0$ small

$$I_\tau(u) = \frac{1}{2}\int_{\mathbb{S}^4}(|\nabla u|^2 + 2u^2) - \frac{1}{6(4-\tau)}\int_{\mathbb{S}^4} K|u|^{4-\tau}.$$

Let $\overline{P}_1, \ldots, \overline{P}_k \in \mathcal{K}^-$ be critical points of K with $M(\overline{P}_1, \ldots, \overline{P}_k) > 0$. For $\varepsilon_0 > 0$ small, let $\Omega_{\varepsilon_0} = \Omega_{\varepsilon_0}(\overline{P}_1, \ldots, \overline{P}_k) \subset \mathbb{R}_+^k \times \mathbb{R}_+^k \times (\mathbb{S}^4)^k$ be defined as

$$\Omega_{\varepsilon_0} = \big\{(\alpha, t, P) \in \mathbb{R}_+^k \times \mathbb{R}_+^k \times (\mathbb{S}^4)^k : |\alpha_i - (12/K(P_i))^{1/2}| < \varepsilon_0, \\ |P_i - \overline{P}_i| < \varepsilon_0,\ t_i > 1/\varepsilon_0,\ 1 \leqq i \leqq k\big\}.$$

It follows from [3] and [4] that there exists $\varepsilon_0 > 0$ small (depending only on $\min_{\mathbb{S}^4} K$ and $\|K\|_{C^2(\mathbb{S}^4)}$) with the following property: For any $u \in H^1(\mathbb{S}^4)$ satisfying for some $(\tilde{\alpha}, \tilde{t}, \tilde{P}) \in \Omega_{\varepsilon_0/2}$ the inequality $\|u - \sum_{i=1}^k \tilde{\alpha}_i \delta_{\tilde{P}_i, \tilde{t}_i}\| < \varepsilon_0/2$, we have a unique representation

$$u = \sum_{i=1}^{k} \alpha_i \delta_{P_i, t_i} + v$$

with $(\alpha, t, P) \in \Omega_{\varepsilon_0}$ and

$$\langle v, \delta_{P_i,t_i} \rangle = \left\langle v, \frac{\partial \delta_{P_i,t_i}}{\partial P_i^{(l)}} \right\rangle = \left\langle v, \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right\rangle = 0 . \tag{2.2}$$

We work in some orthonormal basis near $\overline{P}_i$. $\frac{\partial}{\partial P_i^{(l)}}$ denotes the corresponding derivatives. We denote the set of $v \in H^1(\mathbb{S}^4)$ satisfying (2.2) by $E_{t,P}$. It follows that in a small tubular neighborhood (independent of τ) of $\{\sum_{i=1}^k \alpha_i \delta_{P_i,t_i} : (\alpha, t, P) \in \Omega_{\varepsilon_0/2}\}$, (α, t, P, v) is a good parameterization.

Set for large constant A

$$\begin{aligned}
\Sigma_\tau &= \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k) \\
&= \{(\alpha, t, P, v) \in \Omega_{\varepsilon_0/2} \times H^1(\mathbb{S}^4) : \\
&\qquad |P_i - \overline{P}_i| < \tau^{1/2} |\log \tau|,\ A^{-1}\tau^{-1/2} < t_i < A\tau^{-1/2},\ v \in E_{t,P},\ \|v\| < \nu_0\} .
\end{aligned}$$

Without confusion we use the same notation for

$$\Sigma_\tau = \left\{ u = \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v : (\alpha, t, P, v) \in \Sigma_\tau \right\} \subset H^1(\mathbb{S}^4) .$$

Proposition 2.1. *For $K \in \mathscr{A}$ a Morse function, $\alpha \in (0,1)$, there exists some positive constants $\nu_0 \ll 1$, $A \gg 1$, $R \gg 1$ depending only on K such that when $\tau > 0$ is sufficiently small,*

$$u \in \mathcal{O}_R \bigcup \{ \cup_{k \geqq 1} \cup_{\overline{P}_1, \ldots, \overline{P}_k \in \mathscr{K}^-, M(\overline{P}_1, \ldots, \overline{P}_k) > 0} \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)\}$$

for all u satisfying $u \in H^1(\mathbb{S}^4)$, $u > 0$ a.e., $I_\tau'(u) = 0$.

Proof: The proposition follows from Theorem 0.10, (2.1), the properties of isolated simple blowup points established in [30], and some elementary arguments. We omit the details.

Theorem 2.2. *Let $K \in \mathscr{A}$ be a Morse function, $\nu_0 > 0$ be suitably small, $A > 0$ be suitably large. Then for $\overline{P}_1, \ldots, \overline{P}_k \in \mathscr{K}^-$, $M(\overline{P}_1, \ldots, \overline{P}_k) > 0$, $k \geqq 1$, and $\tau > 0$ sufficiently small, we have*

$$\begin{aligned}
&\deg_{H^1}\left(u - \frac{1}{6}(-\Delta_{g_0} + 2)^{-1}(K|u|^{2-\tau}u), \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k), 0\right) \\
&= (-1)^{k + \sum_{j=1}^k i(\overline{P}_j)} ,
\end{aligned}$$

where $\deg_{H^1}$ *denotes the Leray-Schauder degree on* $H^1(\mathbb{S}^4)$.

In fact, we can establish a stronger result.

THEOREM 2.2′. *Under the same hypotheses of Theorem 2.2, there exists a unique critical point of* I_τ *in* $\sum_\tau(\overline{P}_1,\dots,\overline{P}_k)$. *The critical point is nondegenerate with Morse index* $5k - \sum_{i=1}^k i(\overline{P}_i)$.

Remark 2.3. For $\tau > 0$ small, $u \in \Sigma_\tau, I'_\tau(u) = 0$ implies $u > 0$ on $\mathbb{S}^4$. See [28] for a proof.

In this paper we will prove only Theorem 2.2, which is enough to establish the results in this paper. The proof of Theorem 2.2′ is similar to that of Theorem 2.2. The difference is that we make the calculation at the level of one more derivative of I_τ. With Theorem 2.2′ and the compactness results we have established in Theorems 0.8 and 0.9, we can immediately establish some more general existence results by recording the information at the level of Morse inequality.

PROPOSITION 2.4. *For* $\tau > 0$ *small,* $(\alpha, t, P, 0) \in \Sigma_\tau = \Sigma_\tau(\overline{P}_1,\dots,\overline{P}_k)$, $\overline{P}_1,\dots,\overline{P}_k \in \mathscr{K}^-$, $k \geqq 1$, *there exists a unique minimizer* $\overline{v} = \overline{v}_\tau(\alpha,t,P) \in E_{t,P}$ *of* $I_\tau(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v)$ *with respect to* $\{v \in E_{t,P} : \|v\| < \nu_0\}$. *Furthermore,*

$$\|\overline{v}\| \leqq C\sum_{i=1}^k |\nabla K(P_i)|\tau^{1/2} + C\tau|\log\tau| \leqq C\tau|\log\tau|,$$

$$\left\langle I'_\tau\left(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v\right), v\right\rangle \neq 0 \qquad \forall(\alpha,t,P,v) \in \Sigma_\tau,\ v \neq \overline{v},$$

and $(\tau,\alpha,t,P) \mapsto \overline{v}_\tau(\alpha,t,P)$ *is a* C^2 *map to* $H^1(\mathbb{S}^4)$.

Proof: For $(\alpha,t,P,v) \in \Sigma_\tau$, it follows from (2.2) and Lemma A.2 of the Appendix that

$$\begin{aligned}
&I_\tau\Big(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v\Big)\\
&= |\mathbb{S}^4|\sum_{i=1}^k \alpha_i^2 + 2\sum_{i<j}\alpha_i\alpha_j\int_{\mathbb{S}^4}(\delta_{P_i,t_i})^3\delta_{P_j,t_j}\\
&\quad - \frac{1}{6(4-\tau)}\int_{\mathbb{S}^4}K\Big(\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}\Big)^{4-\tau} - \frac{1}{6}\int_{\mathbb{S}^4}K\Big(\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}\Big)^{3-\tau}v\\
&\quad - \frac{1}{2}\int_{\mathbb{S}^4}L_{g_0}v\cdot v - \frac{(3-\tau)}{12}\int_{\mathbb{S}^4}K\Big(\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}\Big)^{2-\tau}v^2\\
&\quad + V(\tau,\alpha,t,P,v)
\end{aligned}$$

where $|V(\tau, \alpha, t, P, v)| \leqq C\|v\|^3$ and C depends only on K, ν_0, and A.

For $\varphi, v \in E_{t,P}$, set

$$f_\tau(v) = -\frac{1}{6}\int_{\mathbb{S}^4} K\Big(\sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i}\Big)^{3-\tau} v,$$

$$Q_\tau(\varphi, v) = -\frac{1}{2}\int_{\mathbb{S}^4} L_{g_0}\varphi \cdot v - \frac{(3-\tau)}{12}\int_{\mathbb{S}^4} K\Big(\sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i}\Big)^{2-\tau}\varphi v,$$

$$Q_0(\varphi, v) = -\frac{1}{2}\int_{\mathbb{S}^4} L_{g_0}\varphi \cdot v - \frac{1}{4}\int_{\mathbb{S}^4} \sum_{i=1}^{k} \delta_{P_i,t_i}^2 \varphi v.$$

It is proved in [3] and [4] that there exists some $\delta_0 > 0$ (independent of τ) such that

$$Q_0(v, v) \geqq \delta_0 \|v\|^2, \quad \forall v \in E_{t,P}.$$

We choose $\varepsilon_0 > 0$ sufficiently small from the beginning. Using some elementary estimates as in the Appendix, we have, for $\tau > 0$ small,

$$Q_\tau(v, v) \geqq \delta_0/2\|v\|^2, \quad \forall(\alpha, t, P, v) \in \Sigma_\tau. \tag{2.3}$$

It follows from Lemma A.1, (A.15), (2.2), and (A.18) that

$$\begin{aligned} f_\tau(v) &= -\frac{1}{6}\int_{\mathbb{S}^4} K\left(\sum_{i=1}^{k} \alpha_i^{3-\tau}\delta_{P_i,t_i}^{3-\tau}\right) v + O\left(\sum_{i\neq j}\int_{\mathbb{S}^4} \delta_{P_i,t_i}^{2-\tau}\delta_{P_j,t_j}|v|\right) \\ &= -\frac{1}{6}\int_{\mathbb{S}^4} (K - K(P_i))\sum_{i=1}^{k} \alpha_i^{3-\tau}\delta_{P_i,t_i}^{3} v + O\left(\sum_{i=1}^{k}\int_{\mathbb{S}^4} |\delta_{P_i,t_i}^{3-\tau} - \delta_{P_i,t_i}^{3}||v|\right) \\ &\quad + O(\tau\|v\|) \\ &= O\left(\sum_{i=1}^{k} |\nabla K(P_i)| \int_{\mathbb{S}^4} |\cdot - P_i| \delta_{P_i,t_i}^{3}|v|\right) + O\left(\sum_{i=1}^{k}\int_{\mathbb{S}^4} |\cdot - P_i|^2 \delta_{P_i,t_i}^{3}|v|\right) \\ &\quad + O(\tau|\log\tau|\|v\|). \end{aligned}$$

Using (A.20), we have, for all $(\alpha, t, P, v) \in \Sigma$, that

$$\|f_\tau(v)\| \leqq C\left\{\tau^{1/2}\sum_{i=1}^{k} |\nabla K(P_i)| + \tau|\log\tau|\right\}\|v\| \leqq C\tau|\log\tau|\|v\|.$$

The existence, uniqueness, and C^2 dependence of the minimizer $\bar{v} = \bar{v}_\tau(\alpha, t, P)$ as stated in Proposition 2.4 follows from standard arguments in functional analysis.

Setting $\beta = (\beta_1, \dots, \beta_k)$, $\beta_i = \alpha_i - (12/K(P_i))^{1/2}$, $\forall i$. It follows from (A.9), (A.11), Lemma A.2, Lemma A.1, and (A.15) that

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha_i} I_\tau\left(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\right) \\
&= -\sum_{j=1}^k \alpha_j \int_{\mathbb{S}^4} (L_{g_0}\delta_{P_j,t_j})\delta_{P_i,t_i} \\
&\quad - \frac{1}{6}\int_{\mathbb{S}^4} K\left|\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + v\right|^{2-\tau}\left(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + v\right)\delta_{P_i,t_i} \\
&= 2|\mathbb{S}^4|\alpha_i - \frac{1}{6}\int_{\mathbb{S}^4} K\left|\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}\right|^{3-\tau}\delta_{P_i,t_i} - \frac{1}{2}\int_{\mathbb{S}^4} K\left|\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}\right|^{2-\tau} v\delta_{P_i,t_i} \\
&\quad + O(\tau) + O(\|v\|^2) \\
&= 2|\mathbb{S}^4|\alpha_i - \frac{1}{6}\int_{\mathbb{S}^4} K\left(\sum_{j=1}^k \alpha_j^{3-\tau}\delta_{P_j,t_j}^{3-\tau}\right)\delta_{P_i,t_i} - \frac{1}{2}\int_{\mathbb{S}^4} K(\alpha_i^{2-\tau}\delta_{P_i,t_i}^{2-\tau})v\delta_{P_i,t_i} \\
&\quad + O\left(\sum_{j\neq i}\|\delta_{P_j,t_j}^{2-\tau}\delta_{P_i,t_i}\|_{L^{4/3}(\mathbb{S}^4)}\right) + O(\|v\|^2) + O(\tau).
\end{aligned}
$$

Using (A.2), (A.15), (A.6), (2.2), (A.20), and (A.19), we have

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha_i} I_\tau\left(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\right) \\
&= 2|\mathbb{S}^4|\alpha_i - \frac{1}{6}\int_{\mathbb{S}^4} K\alpha_i^3\delta_{P_i,t_i}^{4-\tau} - \frac{1}{2}\int_{\mathbb{S}^4} K\alpha_i^2\delta_{P_i,t_i}^{3-\tau}v + O(\tau) + O(\|v\|^2) \\
&= 2|\mathbb{S}^4|\alpha_i - \frac{1}{6}\int_{\mathbb{S}^4} K(P_i)\alpha_i^3\delta_{P_i,t_i}^{4-\tau} - \frac{1}{2}\int_{\mathbb{S}^4} K(P_i)\alpha_i^2\delta_{P_i,t_i}^{3-\tau}v + O(\tau) + O(\|v\|^2) \\
&= -4\beta_i\int_{\mathbb{S}^4}\delta_{P_i,t_i}^{4-\tau} - \frac{1}{2}\int_{\mathbb{S}^4} K(P_i)\alpha_i^2(\delta_{P_i,t_i}^{3-\tau} - \delta_{P_i,t_i}^3)v \\
&\quad + O(|\beta|^2) + O(\tau|\log\tau|) + O(\|v\|^2) \\
&= -4|\mathbb{S}^4|\beta_i + O(|\beta|^2) + O(\tau|\log\tau|) + O(\|v\|^2).
\end{aligned}
$$

Hence

$$
\frac{\partial}{\partial\alpha_i} I_\tau\left(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\right) = -4|\mathbb{S}^4|\beta_i + V_{\alpha_i}(\tau,\alpha,t,P,v), \tag{2.4}
$$

where V_{α_i} satisfies, for $(\alpha,t,P,v) \in \Sigma_\tau$, $V_{\alpha_i}(\tau,\alpha,t,P,v) = O(|\beta|^2 + \tau|\log\tau| + \|v\|^2)$.

It follows that

$$\frac{\partial}{\partial\alpha_i} I_\tau(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + \overline{v}) = -4|\mathbb{S}^4|\beta_i + V_{\alpha_i}(\tau,\alpha,t,P,\overline{v})$$

$$= -4|\mathbb{S}^4|\beta_i + O(|\beta|^2 + \tau|\log\tau|). \tag{2.5}$$

Using Lemma A.2 we have

$$\frac{\partial}{\partial t_i} I_\tau\left(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\right)$$
$$= 2\sum_j \int_{\mathbb{S}^4} \alpha_i\alpha_j \delta_{P_j,t_j} \frac{\partial \delta^3_{P_i,t_i}}{\partial t_i} - \frac{1}{6}\int_{\mathbb{S}^4} K\left(\sum_j \alpha_j\delta_{P_j,t_j}\right)^{3-\tau} \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial t_i}$$
$$- \frac{(3-\tau)}{6}\int_{\mathbb{S}^4} K\left(\sum_j \alpha_j \delta_{P_j,t_j}\right)^{2-\tau} v\alpha_i \frac{\partial\delta_{P_i,t_i}}{\partial t_i} + O\left(\|v\|^2 \left\|\frac{\partial \delta_{P_i,t_i}}{\partial t_i}\right\|\right)$$

LEMMA 2.5.
$$\left|\int_{\mathbb{S}^4} K\delta^{2-\tau}_{P_i,t_i} \frac{\partial\delta_{P_i,t_i}}{\partial t_i} v\right| \leqq C\tau\|v\|.$$

Proof: Using (2.2) we have

$$\int_{\mathbb{S}^4} \delta^2_{P_i,t_i}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} v = \frac{1}{3}\int_{\mathbb{S}^4} v \frac{\partial \delta^3_{P_i,t_i}}{\partial t_i}$$
$$= \frac{1}{6}\int_{\mathbb{S}^4} v\frac{\partial}{\partial t_i}(-L_{g_0}\delta_{P_i,t_i})$$
$$= \frac{1}{6}\frac{\partial}{\partial t_i}\langle v, \delta_{P_i,t_i}\rangle$$
$$= \frac{1}{6}\langle v, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}\rangle$$
$$= 0$$

It follows from (A.6), (A.10), and (A.20) that

$$\left|\int_{\mathbb{S}^4} K\delta^{2-\tau}_{P_i,t_i}\frac{\partial\delta_{P_i,t_i}}{\partial t_i} v\right|$$
$$= \left|\int_{\mathbb{S}^4} [K - K(P_i)]\delta^{2-\tau}_{P_i,t_i}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} v + K(P_i)\int_{\mathbb{S}^4}(\delta^{2-\tau}_{P_i,t_i} - \delta^2_{P_i,t_i})\frac{\partial\delta_{P_i,t_i}}{\partial t_i} v\right|$$
$$\leqq C\left\||\cdot - P_i|\delta^{2-\tau}_{P_i,t_i}\right\|_{L^2}\left\|\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\right\|\|v\| + C\left\|\delta^{2-\tau}_{P_i,t_i} - \delta^2_{P_i,t_i}\right\|_{L^2}\left\|\frac{\partial \delta_{P_i,t_i}}{\partial t_i}\right\|\|v\|$$
$$\leqq C\tau\|v\|.$$

LEMMA 2.6.

$$\left|\int_{\mathbb{S}^4} K\Big(\sum_j \alpha_j \delta_{P_j,t_j}\Big)^{2-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} v\right| \leqq C\tau\|v\|.$$

Proof: It follows from Lemma A.2, Lemma 2.5, and (A.16) that

$$\begin{aligned}
&\left|\int_{\mathbb{S}^4} K\left(\sum_j \alpha_j \delta_{P_j,t_j}\right)^{2-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} v\right| \\
&\leqq \left|\int_{\mathbb{S}^4} K\alpha_i^{2-\tau}\delta_{P_i,t_i}^{2-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} v\right| + C\sum_{j\neq i}\int_{\mathbb{S}^4} \delta_{P_i,t_i}^{1-\tau}\delta_{P_j,t_j}\left|\frac{\partial \delta_{P_i,t_i}}{\partial t_i}\right| |v| \\
&\quad + C\sum_{j\neq i}\int_{\mathbb{S}^4} \delta_{P_j,t_j}^{2-\tau}\left|\frac{\partial \delta_{P_i,t_i}}{\partial t_i}\right| |v| \\
&\leqq C\tau\|v\| + C\sum_{j\neq i}\left\|\delta_{P_i,t_i}^{1-\tau}\delta_{P_j,t_j}\frac{\partial \delta_{P_i,t_i}}{\partial t_i}\right\|_{L^{4/3}}\|v\| + C\sum_{j\neq i}\left\|\delta_{P_j,t_j}^{2-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial t_i}\right\|_{L^{4/3}}\|v\| \\
&\leqq C\tau\|v\|.
\end{aligned}$$

It follows from Lemma 2.6, (A.10), Lemma A.1, (A.17), (A.18), and (A.15) that

$$\begin{aligned}
&\frac{\partial}{\partial t_i} I_\tau\left(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\right) \\
&= 2\sum_j \alpha_i\alpha_j \frac{\partial}{\partial t_i}\int_{\mathbb{S}^4} \delta_{P_j,t_j}\delta_{P_i,t_i}^3 \\
&\quad - \frac{1}{6}\int_{\mathbb{S}^4} K\left\{\alpha_i^{3-\tau}\delta_{P_i,t_i}^{3-\tau} + (3-\tau)\alpha_i^{2-\tau}\delta_{P_i,t_i}^{2-\tau}\left(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\right)\right. \\
&\qquad\qquad \left. + \left(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\right)^{3-\tau}\right\}\alpha_i\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&\quad + O(\tau\|v\|) + O(\sqrt{\tau}\|v\|^2) + o(\tau^{3/2}) \\
&= 2\sum_j \alpha_i\alpha_j \frac{\partial}{\partial t_i}\int_{\mathbb{S}^4} \delta_{P_j,t_j}\delta_{P_i,t_i}^3 - \frac{1}{6}\int_{\mathbb{S}^4} K\alpha_i^4\delta_{P_i,t_i}^{3-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&\quad - \frac{1}{2}\int_{\mathbb{S}^4} \alpha_i^3 K\left(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\right)\delta_{P_i,t_i}^{2-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&\quad - \frac{1}{6}\int_{\mathbb{S}^4} \alpha_i K\left(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\right)^{3-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&\quad + O(\tau\|v\|) + O(\sqrt{\tau}\|v\|^2) + o(\tau^{3/2}).
\end{aligned}$$

It follows from Lemma A.1 that

$$\left(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\right)^{3-\tau} = \sum_{j\neq i}(\alpha_j\delta_{P_j,t_j})^{3-\tau} + O\Big(\sum_{\substack{j\neq i\\ l\neq i\\ j\neq l}}\delta_{P_j,t_j}^{2-\tau}\delta_{P_l,t_l}\Big).$$

Using (A.24) and the above, we have

$$\begin{aligned}
&\frac{\partial}{\partial t_i}I_\tau\left(\sum_{j=1}^k\alpha_j\delta_{P_j,t_j}+v\right)\\
&= 2\sum_j\alpha_i\alpha_j\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta_{P_j,t_j}\delta_{P_i,t_i}^3 - \frac{1}{6}\int_{\mathbb{S}^4}K\alpha_i^4\delta_{P_i,t_i}^{3-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&\quad - \frac{1}{6}\alpha_i^3\sum_{j\neq i}\int_{\mathbb{S}^4}K\alpha_j\delta_{P_j,t_j}\frac{\partial\delta_{P_i,t_i}^{3-\tau}}{\partial t_i} - \frac{1}{6}\sum_{j\neq i}\int_{\mathbb{S}^4}\alpha_iK\alpha_j^3\delta_{P_j,t_j}^{3-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&\quad + O(\tau\|v\|) + O(\sqrt{\tau}\|v\|^2) + \circ(\tau^{3/2}).
\end{aligned}$$

Using (A.25),

$$\begin{aligned}
&\int_{\mathbb{S}^4}K\delta_{P_j,t_j}\frac{\partial\delta_{P_i,t_i}^{3-\tau}}{\partial t_i}\\
&= \frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}K\delta_{P_j,t_j}\delta_{P_i,t_i}^{3-\tau}\\
&= K(P_i)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta_{P_j,t_j}\delta_{P_i,t_i}^{3-\tau} + \frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}[K-K(P_i)]\delta_{P_j,t_j}\delta_{P_i,t_i}^{3-\tau}\\
&= K(P_i)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta_{P_j,t_j}\delta_{P_i,t_i}^{3-\tau} + \circ(\tau^{3/2}).
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{S}^4}K\delta_{P_j,t_j}^{3-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&= \frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}K\delta_{P_j,t_j}^{3-\tau}\delta_{P_i,t_i}\\
&= K(P_j)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta_{P_j,t_j}^{3-\tau}\delta_{P_i,t_i} + \frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}[K-K(P_j)]\delta_{P_j,t_j}^{3-\tau}\delta_{P_i,t_i}\\
&= K(P_j)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta_{P_j,t_j}^{3-\tau}\delta_{P_i,t_i} + \circ(\tau^{3/2}).
\end{aligned}$$

Using (A.4), (A.5), (A.3), and (A.7), we have

$$
\begin{aligned}
&\frac{\partial}{\partial t_i} I_\tau\left(\sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v\right) \\
&= 2\sum_j \alpha_i\alpha_j \frac{\partial}{\partial t_i}\int_{\mathbb{S}^4} \delta_{P_j,t_j}\delta^3_{P_i,t_i} - \frac{1}{6}\int_{\mathbb{S}^4} K\alpha_i^4 \delta^{3-\tau}_{P_i,t_i}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&\quad - \frac{1}{6}\alpha_i^3 K(P_i)\sum_{j\neq i}\alpha_j \frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta_{P_j,t_j}\delta^3_{P_i,t_i} \\
&\quad - \frac{1}{6}\alpha_i \sum_{j\neq i} K(P_j)\alpha_j^3 \frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta^3_{P_j,t_j}\delta_{P_i,t_i} \\
&\quad + O(\tau\|v\|) + O(\sqrt{\tau}\|v\|^2) + \circ(\tau^{3/2}) \\
&= 2\sum_j \alpha_i\alpha_j \frac{\partial}{\partial t_i}\int_{\mathbb{S}^4} \delta_{P_j,t_j}\delta^3_{P_i,t_i} - \frac{1}{6}\int_{\mathbb{S}^4} K(P_i)\alpha_i^4 \delta^{3-\tau}_{P_i,t_i}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&\quad - \frac{1}{6}\int_{\mathbb{S}^4} \nabla K(P_i)\cdot(\cdot - P_i)\alpha_i^4 \delta^{3-\tau}_{P_i,t_i}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&\quad - \frac{1}{6}\int_{\mathbb{S}^4} \frac{1}{2}\nabla_{\alpha\beta}K(P_i)(\cdot - P_i)_\alpha(\cdot - P_i)_\beta \alpha_i^4 \delta^{3-\tau}_{P_i,t_i}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&\quad + \int_{\mathbb{S}^4} \circ(|\cdot - P_i|^2)\delta^{3-\tau}_{P_i,t_i}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&\quad - \frac{1}{6}\sum_{j\neq i}\left\{\alpha_i^3\alpha_j K(P_i) + \alpha_i\alpha_j^3 K(P_j)\right\}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta_{P_j,t_j}\delta^3_{P_i,t_i} \\
&\quad + O(\tau\|v\|) + O(\sqrt{\tau}\|v\|^2) + \circ(\tau^{3/2}) \\
&= \frac{1}{6}\sum_{j\neq i}\left\{12\alpha_i\alpha_j - \alpha_i^3\alpha_j K(P_i) - \alpha_i\alpha_j^3 K(P_j)\right\}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta_{P_j,t_j}\delta^3_{P_i,t_i} \\
&\quad - \frac{1}{6(4-\tau)}\alpha_i^4 K(P_i)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta^{4-\tau}_{P_i,t_i} \\
&\quad - \frac{1}{48(4-\tau)}\alpha_i^4 \Delta_{g_0}K(P_i)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}|\cdot - P_i|^2\delta^{4-\tau}_{P_i,t_i} \\
&\quad + O(\tau\|v\|) + O(\sqrt{\tau}\|v\|^2) + \circ(\tau^{3/2}) \\
&= -24\sum_{j\neq i}\frac{1}{\sqrt{K(P_i)K(P_j)}}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta_{P_j,t_j}\delta^3_{P_i,t_i} - \frac{6}{K(P_i)}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}\delta^{4-\tau}_{P_i,t_i} \\
&\quad - \frac{3}{4}\frac{\Delta K(P_i)}{K(P_i)^2}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^4}|\cdot - P_i|^2\delta^{4-\tau}_{P_i,t_i} \\
&\quad + O(\tau\|v\|) + O(\sqrt{\tau}\|v\|^2) + \circ(\tau^{3/2}) + O(|\beta|\tau^{3/2})
\end{aligned}
$$

It follows from (A.7), (A.3), and (A.8) that

$$\begin{aligned}
&\frac{\partial}{\partial t_i} I_\tau(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v) \\
&= \Gamma_3 K(P_i)^{-1} \frac{\tau}{t_i} + \Gamma_4 K(P_i)^{-2} \Delta_{g_0} K(P_i) \frac{1}{t_i^3} \\
&\quad + \Gamma_5 \sum_{j \neq i} K(P_i)^{-1/2} K(P_j)^{-1/2} G_{P_i}(P_j) \frac{1}{t_i^2 t_j} + V_{t_i}(\tau, \alpha, t, P, v),
\end{aligned} \tag{2.6}$$

where $V_{t_i}(\tau,\alpha,t,P,v) = O(|\beta|\tau^{3/2}) + O(\tau\|v\|) + O(\sqrt{\tau}\|v\|^2) + \circ(\tau^{3/2})$, $\Gamma_3 = 8|\mathbb{S}^3|$, $\Gamma_4 = 16|\mathbb{S}^3|$, and $\Gamma_5 = 768|\mathbb{S}^3|^2/|\mathbb{S}^4|$.

Using Lemma A.2, Lemma A.1, and (2.2), we have

$$\begin{aligned}
&\frac{\partial}{\partial P_i} I_\tau(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v) \\
&= -\int_{\mathbb{S}^4} L_{g_0}\left(\sum_j \alpha_j \delta_{P_j,t_j} + v\right) \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&\quad - \frac{1}{6}\int_{\mathbb{S}^4} K \left|\sum_j \alpha_j \delta_{P_j,t_j} + v\right|^{2-\tau} \left(\sum_j \alpha_j \delta_{P_j,t_j} + v\right) \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&= -\sum_{j\neq i} \alpha_i \alpha_j \int_{\mathbb{S}^4} (L_{g_0} \delta_{P_j,t_j}) \frac{\partial \delta_{P_i,t_i}}{\partial P_i} - \frac{1}{6}\int_{\mathbb{S}^4} K \left(\sum_j \alpha_j \delta_{P_j,t_j}\right)^{3-\tau} \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&+ \frac{(3-\tau)}{6}\int_{\mathbb{S}^4} K \left(\sum_j \alpha_j \delta_{P_j,t_j}\right)^{2-\tau} v \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} + O\left(\|v\|^2 \left\|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right\|\right).
\end{aligned}$$

It follows from Lemma A.2 that

$$\begin{aligned}
\left(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j}\right)^{2-\tau} &= \left(\alpha_i \delta_{P_i,t_i} + \sum_{j\neq i} \alpha_j \delta_{P_j,t_j}\right)^{2-\tau} \\
&= (\alpha_i \delta_{P_i,t_i})^{2-\tau} + O\left(\sum_{j\neq i} \delta_{P_i,t_i}^{1-\tau} \delta_{P_j,t_j} + \sum_{j\neq i} \delta_{P_j,t_j}^{2-\tau}\right).
\end{aligned}$$

Using (A.18), (A.9), (A.6), (A.23), the Sobolev embedding theorem, and the above we have

$$\begin{aligned}
&\int_{\mathbb{S}^4} K\Big(\sum_j \alpha_j \delta_{P_j,t_j}\Big)^{2-\tau} v\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&= \int_{\mathbb{S}^4} K(\alpha_i\delta_{P_i,t_i})^{2-\tau}\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} v + O\left(\sum_{j\neq i}\int_{\mathbb{S}^4} \delta_{P_i,t_i}^{1-\tau}\delta_{P_j,t_j}\left|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right| |v|\right) \\
&\quad + O\left(\sum_{j\neq i}\int_{\mathbb{S}^4} \delta_{P_j,t_j}^{2-\tau}\left|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right| |v|\right) \\
&= \int_{\mathbb{S}^4} K(P_i)(\alpha_i\delta_{P_i,t_i})^{2-\tau}\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} v + O\left(\int_{\mathbb{S}^4} |\cdot - P_i|\delta_{P_i,t_i}^{2-\tau}\left|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right| |v|\right) \\
&\quad + O\left(\sum_{j\neq i}\int_{\mathbb{S}^4} \delta_{P_i,t_i}^{1-\tau}\delta_{P_j,t_j}\left|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right| |v|\right) + O\left(\sum_{j\neq i}\int_{\mathbb{S}^4} \delta_{P_j,t_j}^{2-\tau}\left|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right| |v|\right) \\
&= O\left(\int_{\mathbb{S}^4} |\delta_{P_i,t_i}^{2-\tau} - \delta_{P_i,t_i}^{2}|\left|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right| |v|\right) + O(\|v\|) \\
&= O(\|v\|).
\end{aligned}$$

Using Lemma A.1, (A.24), (A.9), (A.23), and (A.15), we have

$$\begin{aligned}
&\frac{\partial}{\partial P_i} I_\tau\left(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + v\right) \\
&\quad = 2\sum_{j\neq i}\alpha_i\alpha_j \frac{\partial}{\partial P_i}\int_{\mathbb{S}^4}\delta_{P_j,t_j}^3\delta_{P_i,t_i} - \frac{1}{6}\alpha_i\int_{\mathbb{S}^4} K(\alpha_i\delta_{P_i,t_i})^{3-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&\quad - \frac{(3-\tau)}{6}\alpha_i\int_{\mathbb{S}^4} K\left(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\right)(\alpha_i\delta_{P_i,t_i})^{2-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&\quad - \frac{1}{6}\alpha_i\int_{\mathbb{S}^4} K\left(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\right)^{3-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&\quad + O\left(\sum_{j\neq i}\int_{\mathbb{S}^4}\delta_{P_i,t_i}^{1-\tau}\delta_{P_j,t_j}^2\left|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right|\right) \\
&\quad + O(\|v\|) + O(\tau^{-1/2}\|v\|^2) \\
&\quad = -\frac{1}{6}\alpha_i^4\int_{\mathbb{S}^4} K\delta_{P_i,t_i}^{3-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial P_i} + O(\sqrt{\tau}) + O(\|v\|) + O(\tau^{-1/2}\|v\|^2) \\
&\quad = -\frac{1}{6}\alpha_i^4\nabla K(P_i)\int_{\mathbb{S}^4}(\cdot - P_i)\delta_{P_i,t_i}^{3-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial P_i} + O\left(\int_{\mathbb{S}^4}|\cdot - P_i|^2\left|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right|\right) \\
&\quad + O(\sqrt{\tau}) + O(\|v\|) + O(\tau^{-1/2}\|v\|^2)
\end{aligned}$$

$$= -\Gamma_6 \nabla K(P_i) + O(\sqrt{\tau}) + O(\|v\|) + O(\tau^{-1/2}\|v\|^2),$$

namely,

$$\frac{\partial}{\partial P_i} I_\tau \left(\sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right) = -\Gamma_6(i,\tau,\alpha,t,P,v)\nabla K(P_i) + V_{P_i}(\tau,\alpha,t,P,v), \tag{2.7}$$

where $\Gamma_6(i,\tau,\alpha,t,P,v) \geqq \nu_1 > 0$ with ν_1 independent of τ and

$$V_{P_i}(\tau,\alpha,t,P,v) = O(\sqrt{\tau}) + O(\|v\|) + O(\tau^{-1/2}\|v\|^2).$$

At $u = \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v \in \Sigma_\tau$,

$$T_u H^1(\mathbb{S}^4) = E_{t,P} \bigoplus \operatorname{span}\left\{ \delta_{P_i,t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \right\}.$$

We write $I'_\tau(u) \in T_u H^1(\mathbb{S}^4)$ as

$$I'_\tau(u) = \xi + \eta,$$

where $\xi \in E_{t,P}$, $\eta \in \operatorname{span}\{\delta_{P_i,t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\}$.

For all $\varphi \in E_{t,P}$, it follows, as in the proof of Proposition 2.4, that

$$\langle \xi, \varphi \rangle = I'_\tau(u)\varphi = f_\tau(\varphi) + 2Q_\tau(\varphi, v) + \langle V_v(\tau,\alpha,t,P,v), \varphi \rangle,$$

where V_v is some function satisfying $\|V_v(\tau,\alpha,t,P,v)\| \leqq C\|v\|^2$.

Taking $\varphi = v$, we have, using (2.3), that

$$\|\xi\| \geqq \delta_0 \|v\| - \|f_\tau\| - O(\|v\|^2) \geqq \frac{\delta_0}{2}\|v\| - \|f_\tau\|.$$

Set

$$\widetilde{\Sigma}_\tau = \left\{ u = \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v \in \Sigma_\tau \ : \|v\| < \tau|\log\tau|^3,\ |\beta| < \tau|\log\tau|^2 \right\}.$$

It follows from Proposition 2.4 and (2.5) that $I'_\tau(u) \neq 0$, $\forall u \in \Sigma_\tau \setminus \widetilde{\Sigma}_\tau$. For $u = \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v \in \widetilde{\Sigma}_\tau$, it follows from (2.4) that

$$\langle \eta, \delta_{P_i,t_i} \rangle = \frac{\partial}{\partial \alpha_i} I_\tau \left(\sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v \right) = -4|\mathbb{S}^4|\beta_i + V_{\alpha_i}(\tau,\alpha,t,P,v)$$

with V_{α_i} satisfying $|V_{\alpha_i}(\tau,\alpha,t,P,v)| \leqq C(|\beta|^2 + \tau|\log\tau|)$.

It follows from (2.6) that

$$\left\langle \eta, \frac{\partial\delta_{P_i,t_i}}{\partial t_i}\right\rangle = \frac{1}{\alpha_i}\frac{\partial}{\partial t_i}I_\tau\left(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v\right)$$
$$= \frac{1}{\alpha_i}\left\{\frac{\Gamma_3\tau}{K(P_i)t_i} + \frac{\Gamma_4\Delta_{g_0}K(P_i)}{K(P_i)^2t_i^3} + \sum_{j\neq i}\frac{\Gamma_5 G_{P_i}(P_j)}{\sqrt{K(P_i)K(P_j)}t_i^2t_j} + V_{t_i}(\tau,\alpha,t,P,v)\right\},$$

with V_{t_i} satisfying $|V_{t_i}(\tau,\alpha,t,P,v)| = \mathrm{o}(\tau^{3/2})$.

It follows from (2.7) that

$$\left\langle \eta, \frac{\partial\delta_{P_i,t_i}}{\partial P_i}\right\rangle = \frac{1}{\alpha_i}\frac{\partial}{\partial P_i}I_\tau\left(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v\right)$$
$$= \frac{1}{\alpha_i}\{-\Gamma_6(i,\tau,\alpha,t,P,v)\nabla K(P_i) + V_{P_i}(\tau,\alpha,t,P,v)\},$$

with V_{P_i} satisfying $|V_{P_i}(\tau,\alpha,t,P,v)| \leqq C\sqrt{\tau}$.

It is well-known that $I'_\tau(u) = \xi + \eta$ is of the form Id +compact in $H^1(\mathbb{S}^4)$.

We define

$$X_\theta = \xi_\theta + \eta_\theta, \quad 0 \leqq \theta \leqq 1,$$

by the following: For all $\varphi \in E_{t,P}$, $0 \leqq \theta \leqq 1$,

$$\langle\xi_\theta,\varphi\rangle = \theta f_\tau(\varphi) + (1-\theta)\langle v,\varphi\rangle + 2\theta Q_\tau(\varphi,v) + \theta\langle V_v(\tau,\alpha,t,P,v),\varphi\rangle,$$
$$\langle\eta_\theta,\delta_{P_i,t_i}\rangle = -4|\mathbb{S}^4|\{\alpha_i - \theta(12/K(P_i))^{1/2} - (1-\theta)(12/K(\overline{P}_i))^{1/2}\} + \theta V_{\alpha_i}(\tau,\alpha,t,P,v),$$
$$\left\langle \eta_\theta, \frac{\partial\delta_{P_i,t_i}}{\partial t_i}\right\rangle = \left\{\frac{\theta}{\alpha_i} + (1-\theta)\right\}\left\{\frac{\Gamma_3\tau}{K(P_i(\theta))t_i} + \frac{\Gamma_4\Delta_{g_0}K(P_i(\theta))}{K(P_i(\theta))^2t_i^3} + \sum_{j\neq i}\frac{\Gamma_5 G_{P_i(\theta)}(P_j(\theta))}{\sqrt{K(P_i(\theta))K(P_j(\theta))}t_i^2t_j}\right\} + \frac{\theta}{\alpha_i}V_{t_i}(\tau,\alpha,t,P,v),$$

where $P_i(\theta)$ is the shortest geodesic trajectory on $\mathbb{S}^4$ with $P_i(1) = P_i, P_i(0) = \overline{P}_i$.

$$\langle\eta_\theta, \frac{\partial\delta_{P_i,t_i}}{\partial P_i}\rangle = -\left[(1-\theta) + \frac{\theta}{\alpha_i}\Gamma_6\right]\nabla K(P_i) + \frac{\theta}{\alpha_i}V_{P_i}(\tau,\alpha,t,P,v).$$

It is easy to see that X_θ is well-defined in $\widetilde{\Sigma}_\tau$. In addition, from the Sobolev compact embedding theorem, the explicit forms of $V_v, V_{\alpha_i}, V_{t_i}, V_{P_i}$, $A^{-2} < t_i^2/\tau <$

A^2, and the estimates we have obtained, X_θ is of the form Id + compact. Furthermore, it is not difficult to see that X_θ $(0 \leqq \theta \leqq 1)$ is an admissible homotopy with $X_\theta|_{\partial\widetilde{\Sigma}_\tau} \neq 0$.

It follows that

$$\deg_{H^1}(X_1, \widetilde{\Sigma}_\tau, 0) = \deg_{H^1}(X_0, \widetilde{\Sigma}_\tau, 0). \tag{2.8}$$

Clearly,

$$X_0 = \xi_0 + \eta_0,$$

where $\xi_0 \in E_{t,P}$, $\eta_0 \in \operatorname{span}\left\{\delta_{P_i,t_i}, \frac{\partial\delta_{P_i,t_i}}{\partial t_i}, \frac{\partial\delta_{P_i,t_i}}{\partial P_i}\right\}$ satisfy

$$\langle\xi_0, \varphi\rangle = \langle v, \varphi\rangle, \quad \forall\varphi \in E_{t,P},$$
$$\langle\eta_0, \delta_{P_i,t_i}\rangle = -4|\mathbb{S}^4|\{\alpha_i - (12/K(\overline{P}_i))^{1/2}\},$$
$$\left\langle\eta_0, \frac{\partial\delta_{P_i,t_i}}{\partial t_i}\right\rangle = \frac{\Gamma_3\tau}{K(\overline{P}_i)t_i} + \frac{\Gamma_4\Delta_{g_0}K(\overline{P}_i)}{K(\overline{P}_i)^2 t_i^3} + \sum_{j\neq i}\frac{\Gamma_5 G_{\overline{P}_i}(\overline{P}_j)}{\sqrt{K(\overline{P}_i)K(\overline{P}_j)}t_i^2 t_j},$$
$$\left\langle\eta_0, \frac{\partial\delta_{P_i,t_i}}{\partial P_i}\right\rangle = -\nabla K(P_i).$$

It is easy to see that $X_0(u) = 0$, $u = \sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v \in \widetilde{\Sigma}_\tau$, if and only if

$$\alpha_i = (12/K(\overline{P}_i))^{1/2}, \qquad P_i = \overline{P}_i, \qquad v = 0,$$
$$\frac{1}{2}K(\overline{P}_i)^{-1}\frac{\tau}{t_i} - \sum_{j=1}^k M_{ij}(\overline{P}_1, \ldots, \overline{P}_k)\frac{1}{t_i^2 t_j} = 0.$$

Setting

$$F(s_1, \ldots, s_k) = -\frac{\tau}{2}\sum_{j=1}^k K(\overline{P}_j)^{-1}\log s_j + \frac{1}{2}\sum_{i,j} M_{ij}(\overline{P}_1, \ldots, \overline{P}_k)s_i s_j,$$
$$\overline{F}(t_1, \ldots, t_k) = F(s_1, \ldots, s_k), \quad s_i = t_i^{-1}.$$

It is easy to see that

$$\frac{\partial\overline{F}}{\partial t_i}(t_1, \ldots, t_k) = \frac{1}{2}K(\overline{P}_i)^{-1}\frac{\tau}{t_i} - \sum_{j=1}^k M_{ij}(\overline{P}_1, \ldots, \overline{P}_k)\frac{1}{t_i^2 t_j}.$$

Therefore

$$\left\langle\eta_0, \frac{\partial\delta_{P_i,t_i}}{\partial t_i}\right\rangle = 16|\mathbb{S}^3|\frac{\partial\overline{F}}{\partial t_i}(t_1, \ldots, t_k).$$

Clearly $\nabla\overline{F}(t_1, \ldots, t_k) = 0$ if and only if $\nabla F(s_1, \ldots, s_k) = 0$. It is also easy to see that $F(s_1, \ldots, s_k)$ is a strictly convex function having a unique critical point in the first quadrant, which has a positive definite Hessian. It follows that $\overline{F}(t_1, \dots, t_k)$ has a unique critical point in the first quadrant with Morse index zero.

Since X_0 has precisely one nondegenerate zero in $\widetilde{\Sigma}_\tau$, it is elementary to see that

$$\deg_{H^1}(X_0, \widetilde{\Sigma}_\tau, 0) = (-1)^{k+\sum_{i=1}^k i(\overline{P}_i)}. \tag{2.9}$$

Theorem 2.2 follows from (2.8) and (2.9).

PROPOSITION 2.7. *Let $K \in C^0(\mathbb{S}^4)$ be a positive function, $0 < \tau_0 \leqq \tau \leqq 4/(n-2) - \tau_0$. There exists some constant C depending only on $\tau_0, \min_{\mathbb{S}^4} K$ and the modulo of continuity of K such that*

$$\{u \in H^1(\mathbb{S}^4) : u > 0 \text{ a.e., } I'_\tau(u) = 0\} \subset \mathscr{V}_C, \tag{2.10}$$

where $\mathscr{V}_C = \{ u \in H^1(\mathbb{S}^4) : \|u^-\| < 1/C,\ 1/C < \|u\| < C\}$. Furthermore,

$$\deg_{H^1}\left(u - \frac{1}{6}(-\Delta_{g_0} + 2)^{-1}(K|u|^{2-\tau}u), \mathscr{V}_C, 0\right) = -1. \tag{2.11}$$

Proof: Consider $K_t = tK + (1-t)K^*, K^*(x) = x^5 + 2$. It follows from the Kazdan-Warner condition that there is no solution to (0.1) with $K = K^*$. Estimate (2.10) follows from the compactness results of [22]. Therefore we only need to establish (2.11) for K^* and $\tau > 0$ very small. This follows from Theorem 0.10, Proposition 2.1 and Theorem 2.2.

For $\delta > 0$ suitably small, let $\mathscr{O}_{R,\delta} = \{u \in H^1(\mathbb{S}^4) : \inf_{w \in \mathscr{O}_R} \|u - w\| < \delta\}$.

It follows from theorem B.2 of [30] that

$$\begin{aligned} &\deg_{H^1}\left(u - \frac{1}{6}(-\Delta_{g_0} + 2)^{-1}(K|u|^{2-\tau}u), \mathscr{O}_{R,\delta}, 0\right) \\ &\quad = \deg\left(u - \frac{1}{6}(-\Delta_{g_0} + 2)^{-1}(K|u|^{2-\tau}u), \mathscr{O}_R, 0\right). \end{aligned}$$

Using (2.11), Proposition 2.1, Theorem 2.2, and the homotopy invariance of the Leray-Schauder degree, we have

$$\begin{aligned} &\deg\left(u - \frac{1}{6}(-\Delta_{g_0} + 2)^{-1}(Ku^3), \mathscr{O}_R, 0\right) \\ &\quad = \deg\left(u - \frac{1}{6}(-\Delta_{g_0} + 2)^{-1}(K|u|^{2-\tau}u), \mathscr{O}_R, 0\right) = \text{Index}(K). \end{aligned}$$

Theorem 0.9 follows from the above.

Proof of part (b) of Theorem 0.8: It is not difficult to see that Morse functions in $C^2(\mathbb{S}^4)^+ \setminus \mathscr{A} = \partial\mathscr{A}$ are dense in $C^2(\mathbb{S}^4)^+ \setminus \mathscr{A} = \partial\mathscr{A}$. Therefore we assume without loss of generality that $K \in \partial\mathscr{A}$ is a Morse function. Let $\mathscr{K} \setminus \mathscr{K}^+ = \{q^{(1)}, \dots, q^{(m)}\}$, it follows from the definition that there exists $1 \leqq i_1 < \cdots < i_k \leqq m$ ($k \geqq 1$) such that $\mu(M(K; q^{(i_1)}, \dots, q^{(i_k)})) = 0$. By making small C^2 perturbations of K, we can assume without loss of generality that there is only one such $(i_1, \dots, i_k)$. We can easily produce a smooth, one-parameter family of Morse functions $\{K_t\}$ ($-1 \leqq t \leqq 1$) with the following properties:

(i) For $-1 \leqq t \leqq 1$, $\{K_t\}$ have the same critical points with the same Morse index as K, $K_0 = K$, and $\{K_t\}$ are identically the same as K except in some small balls around $q^{(i_1)}, \dots, q^{(i_k)}$.
(ii) $K_t \in \mathscr{A}$ for $t \neq 0$.
(iii) For any $1 \leqq j_1 < \cdots < j_l \leqq m, (j_1, \dots, j_l) \neq (i_1, \dots, i_k)$, $\mu(M(K_t; q^{(j_1)}, \dots, q^{(j_l)}))$ have the same sign for $-1 < t < 1$.
(iv) $\mu(M(K_t; q^{(i_1)}, \dots, q^{(i_k)})) < 0$ for $-1 < t < 0$, but $\mu(M(K_t; q^{(i_1)}, \dots, q^{(i_k)})) > 0$ for $0 < t < 1$.

The above can be achieved easily. The idea is to perturb the function K near $q^{(i_1)}, \dots, q^{(i_k)}$ to change the Hessian of K at $q^{(i_1)}, \dots, q^{(i_k)}$.

Using (0.10), we see that

$$\text{Index}(K_1) = \text{Index}(K_{-1}) + (-1)^{k-1+\sum_{j=1}^{k} i(q^{(i_j)})} \neq \text{Index}(K_{-1}). \tag{2.12}$$

It follows from Theorem 0.10, (2.12), and the homotopy invariance of the Leray-Schauder degree that there exist t_i and $v_i \in \mathscr{M}_{K_{t_i}}$ such that

$$\lim_{i \to \infty} \|v_i\|_{C^{2,\alpha}(\mathbb{S}^4)} = \infty \quad \text{or} \quad \lim_{i \to \infty} (\min_{\mathbb{S}^4} v_i) = 0.$$

It follows from the above, the Harnack inequality, and standard elliptic estimates that (0.9) holds. Using part (a) of Theorem 0.9 and (ii), we know that $t_i \to 0$. In fact, we know that $\{v_i\}$ blows up exactly at the k points $q^{(i_1)}, \dots, q^{(i_k)}$.

Proof of Theorem 0.10′: The proof of Theorem 0.10 is similar to the proof of part (b) of Theorem 0.8 and is left to the reader.

3. Proof of Theorem 0.13

We consider a situation more general than that in Theorem 0.13. Let $K \in C^1(\mathbb{S}^n)$ be some positive function satisfying that for any critical point $q_0 \in \mathbb{S}^n$ of K, there exists some real number $\beta = \beta(q_0) \in [n-2, n)$ such that (0.13), (0.14), and (0.15) hold in some geodesic normal coordinate system centered at q_0.

Let $\mathscr{K}_{n-2}^-$ denote the set of critical points q_0 of K with $\beta(q_0) = n-2$ and

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} \nabla Q_{(q_0)}^{(n-2)}(y+\eta_0)(1+|y|^2)^{-n} dy = 0, \\ \displaystyle\int_{\mathbb{R}^n} y \cdot \nabla Q_{(q_0)}^{(n-2)}(y+\eta_0)(1+|y|^2)^{-n} dy < 0. \end{cases} \tag{3.1}$$

can be solved simultaneously for some $\eta_0 \in \mathbb{R}^n$.

When $\#\mathscr{K}^-_{n-2} \geqq 2$, for distinct $q^{(1)}, \dots, q^{(k)} \in \mathscr{K}^-_{n-2}$, $\eta^{(j)} \in \mathbb{R}^n$ $(1 \leqq j \leqq m)$ satisfying (3.1) with $q_0 = q^{(j)}, \eta_0 = \eta^{(j)}$, we define a $k \times k$ symmetric metric $M = M(q^{(1)}, \dots, q^{(k)}, \eta^{(1)}, \dots, \eta^{(k)})$ by

$$M_{ij} = \begin{cases} -\frac{48}{(n-2)|\mathbb{S}^{n-1}|K(q^{(j)})^2} \int_{\mathbb{R}^n} y \cdot \nabla Q^{(n-2)}_{q^{(j)}}(y + \eta^{(j)})(1+|y|^2)^{-n}\, dy, & i = j, \\ -\frac{48|\mathbb{S}^{n-1}|}{|\mathbb{S}^n|} \frac{G_{q^{(i)}}(q^{(j)})}{\sqrt{K(q^{(i)})K(q^{(j)})}}, & i \neq j. \end{cases}$$

THEOREM 3.1. *Suppose that $K \in C^1(\mathbb{S}^n)$ is some positive function satisfying that for any critical point $q_0 \in \mathbb{S}^n$ of K, there exists some real number $\beta = \beta(q_0) \in [n-2, n)$ such that* (0.13), (0.14), *and* (0.15) *hold in some geodesic normal coordinate system centered at q_0. Suppose further that either $\#\mathscr{K}^-_{n-2} \leqq 1$ or for any two points $q^{(1)}, q^{(2)} \in \mathscr{K}^-_{n-2}$ and any $\eta^{(j)} \in \mathbb{R}^n$ solving* (3.1) *with $q_0 = q^{(j)}, \eta_0 = \eta^{(j)}$, we have $M_{11}M_{22} < M_{12}^2$.*

Then for all $0 < \alpha < 1$, there exists some constant C such that

$$1/C < v < C, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C,$$

for all solutions v of (0.1),

$$\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x)x \neq 0 \quad \text{for all } P \in \mathbb{S}^n,\ t \geqq C,$$

and for all $R \geqq C$, $t \geqq C$,

$$\begin{aligned} &\deg\left(v - (-\Delta_{g_0} + c(n)R_0)^{-1}\left(c(n)Kv^{\frac{n+2}{n-2}}\right), \mathscr{O}_R, 0\right) \\ &\quad = (-1)^n \deg\left(\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x)x, B, 0\right). \end{aligned}$$

If we further assume that

$$\deg\left(\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x)x, B, 0\right) \neq 0$$

for t large, then (0.1) *has at least one solution.*

PROPOSITION 3.2. *Suppose that $K \in C^1(\mathbb{S}^n)$, $\min_{q\in\mathbb{S}^n} K(q) \geqq 1/A_1$ for some positive constant A_1, and for any $q_0 \in \mathbb{S}^n$, $\nabla_{g_0}K(q_0) = 0$, there exists some real number $\beta = \beta(q_0) \in [n-2, n)$ such that* (0.13), (0.14), *and* (0.15) *hold in some geodesic normal coordinate system centered at q_0.*

Let $\{v_i\}$ be a sequence of solutions of (0.1) *that blows up at $\{q^{(1)}, \dots, q^{(k)}\}$, $k \geqq 2$. Then we have $\{q^{(1)}, \dots, q^{(k)}\} \in \mathscr{K}^-_{n-2}$, and for some $\eta^{(j)} \in \mathbb{R}^n$ satisfying* (3.1) *with $q_0 = q^{(j)}, \eta_0 = \eta^{(j)}$ $(1 \leqq j \leqq k)$, the equation*

$$\sum_{\ell=1}^{k} M_{j\ell}\lambda_\ell = 0$$

has at least one solution $\lambda_1, \dots, \lambda_k > 0$.

Proof: It follows from theorem 4.1 in [30] that, after passing to a subsequence, $\{v_i\}$ has only isolated simple blowup points. If $\{v_i\}$ blows up at $\{q^{(1)}, \dots, q^{(k)}\}$, $k \geqq 2$, then it follows from the proof of theorem 4.2 in [30] that $\beta(q^{(j)}) = n-2$, $1 \leqq j \leqq k$.

Since $q^{(j)}$ is an isolated simple blowup point of v_i, we let $q_i^{(j)} \to q^{(j)}$ $(i \to \infty)$ be the local maximum of v_i. Let $q_i^{(j)}$ be the south pole and make a stereographic projection to the equatorial plane of $\mathbb{S}^n$ with y being the stereographic projection coordinates. Set

$$u_i(y) = \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}} v_i(y),$$

the equation (0.1) is transformed to

$$-\Delta u_i(y) = c(n)K(y)u_i(y)^{\frac{n+2}{n-2}}, \quad y \in \mathbb{R}^n. \tag{3.2}$$

It follows that (see (0.7))

$$\begin{aligned}&\lim_{i\to\infty} v_i(q_i^{(j)})v_i(q)\\ &\quad = (n-2)|\mathbb{S}^{n-1}||\mathbb{S}^n|^{-1}c(n)^{\frac{2-n}{2}}[n(n-2)]^{\frac{n-2}{2}}K(q^{(j)})^{\frac{2-n}{2}}G_{q^{(j)}}(q) + \tilde{b}(q)\end{aligned}$$

for $q \neq q^{(j)}$ and close to $q^{(j)}$ where $\tilde{b}(q)$ is some regular function near $q^{(j)}$ satisfying $L_{g_0}\tilde{b}(q) = 0$ near $q^{(j)}$ and the convergence is in the sense of C^2_{loc}.

After passing to a subsequence, it follows from the maximum principle that for $q \neq q^{(j)}$, $\forall 1 \leqq j \leqq k$,

$$\begin{aligned}\lim_{i\to\infty} v_i(q_i^{(j)})v_i(q) = (n-2)\frac{|\mathbb{S}^{n-1}|}{|\mathbb{S}^n|}c(n)^{\frac{2-n}{2}}[n(n-2)]^{\frac{n-2}{2}}\Big\{&K(q^{(j)})^{\frac{2-n}{2}}G_{q^{(j)}}(q)\\ &+ \sum_{\ell\neq j}\lim_{i\to\infty} v_i(q_i^{(j)})v_i(q_i^{(\ell)})^{-1}K(q^{(\ell)})^{\frac{2-n}{2}}G_{q^{(\ell)}}(q)\Big\}.\end{aligned}$$

Let $y_i^{(j)} \to 0$ denote the local maximum of u_i; it is not difficult to see (using the fact that the blowup is isolated simple) that $v_i(y_i^{(j)})v_i(q_i^{(j)})^{-1} \to 1$. It follows that for $|y| > 0$ small,

$$\begin{aligned}&\lim_{i\to\infty} u_i(y_i^{(j)})u_i(y)\\ &\quad = c(n)^{\frac{2-n}{2}}[n(n-2)]^{\frac{n-2}{2}}K(q^{(j)})^{\frac{2-n}{2}}|y|^{2-n} + 2^{n-2}(n-2)|\mathbb{S}^{n-1}||\mathbb{S}^n|^{-1}c(n)^{\frac{2-n}{2}}\\ &\qquad [n(n-2)]^{\frac{n-2}{2}}\sum_{\ell\neq j}\lim_{i\to\infty} v_i(q_i^{(j)})v_i(q_i^{(\ell)})^{-1}K(q^{(\ell)})^{\frac{2-n}{2}}G_{q^{(\ell)}}(q^{(j)}) + O(|y|)\\ &\quad =: h^{(j)}(y).\end{aligned}$$

Applying Proposition 1.1, we have, for $\sigma > 0$ small, that

$$\frac{c(n)(n-2)}{2n}\int_{B_\sigma} x\cdot\nabla K(x+y_i^{(j)})u_i(x+y_i^{(j)})^{\frac{2n}{n-2}} - O\left(u_i(y_i^{(j)})^{-\frac{2n}{n-2}}\right) = \int_{\partial B_\sigma} B\left(\sigma, x, u_i(\cdot+y_i^{(j)}), \nabla u_i(\cdot+y_i^{(j)})\right).$$

Multiplying the above by $u_i(y_i^{(j)})^2$, sending i to ∞, and arguing as before, we have

$$\int_{\partial B_\sigma} B(\sigma, x, h^{(j)}, \nabla h^{(j)}) = \frac{c(n)(n-2)2^{n-2}}{2n}\int_{\mathbb{R}^n}\frac{z\cdot\nabla Q_{q^{(j)}}^{(n-2)}(z+\xi^{(j)})dz}{(1+k^{(j)}|z|^2)^n} + \circ_\sigma(1),$$

where $\xi^{(j)} = \lim_{i\to\infty} u_i(y_i^{(j)})^{\frac{2}{n-2}}y_i^{(j)} \in \mathbb{R}^n$ (lemma 2.6 in [30] is used here) and $k^{(j)} = c(n)[n(n-2)]^{-1}K(q^{(j)})$.

It follows from Proposition 1.2 that

$$\lim_{\sigma\to 0}\int_{\partial B_\sigma} B(\sigma, x, h^{(j)}, \nabla h^{(j)}) = -2^{n-3}(n-2)^3c(n)^{2-n}[n(n-2)]^{n-2}|\mathbb{S}^{n-1}|^2\,|\mathbb{S}^n|^{-1} \sum_{\ell\neq j} K(q^{(j)})^{\frac{2-n}{2}}K(q^{(\ell)})^{\frac{2-n}{2}}G_{q^{(\ell)}}(q^{(j)})\lim_{i\to\infty} v_i(q_i^{(j)})v_i(q_i^{(\ell)})^{-1}.$$

It follows that

$$\int_{\mathbb{R}^n}\frac{z\cdot\nabla Q_{q^{(j)}}^{(n-2)}(z+\xi^{(j)})dz}{(1+k^{(j)}|z|^2)^n} = -\frac{[n(n-2)]^{n-1}(n-2)|\mathbb{S}^{n-1}|^2}{c(n)^{n-1}|\mathbb{S}^n|} \sum_{\ell\neq j} K(q^{(j)})^{\frac{2-n}{2}}K(q^{(\ell)})^{\frac{2-n}{2}}G_{q^{(\ell)}}(q^{(j)})\lim_{i\to\infty}\frac{v_i(q_i^{(j)})}{v_i(q_i^{(\ell)})}.$$

Let

$$\lambda_j = K(q^{(j)})^{(3-n)/2}\lim_{i\to\infty} v_i(q_i^{(1)})v_i(q_i^{(j)})^{-1}.$$

It follows from Proposition 1.6, Proposition 1.5, and lemma 2.3 in [30] that $0 < \lambda_j < \infty$.

Making a change of variable, we obtain that

$$K(q^{(j)})^{-2}\lambda_j\int_{\mathbb{R}^n} y\cdot\nabla Q_{q^{(j)}}^{(n-2)}(y+\eta^{(j)})(1+|y|^2)^{-n}dy = -(n-2)|\mathbb{S}^{n-1}|^2\,|\mathbb{S}^n|^{-1}\sum_{\ell\neq j}K(q^{(j)})^{-1/2}K(q^{(\ell)})^{-1/2}G_{q^{(\ell)}}(q^{(j)})\lambda_\ell, \tag{3.3}$$

where $\eta^{(j)} = \sqrt{k^{(j)}}\xi^{(j)}$.

Next we derive the equation satisfied by $\eta^{(j)}$. Multiplying (3.2) by ∇u_i and integrating on $B_\sigma(y_i^{(j)})$, we have

$$-\int_{B_\sigma(y_i^{(j)})} \nabla u_i \Delta u_i = c(n)\int_{B_\sigma(y_i^{(j)})} \nabla u_i u_i^{\frac{n+2}{n-2}} K .$$

Integrating by parts we have

$$\left|\int_{B_\sigma(y_i^{(j)})} \nabla u_i \Delta u_i\right| \leqq C\int_{\partial B_\sigma(y_i^{(j)})} |\nabla u_i|^2 \leqq C u_i(y_i^{(j)})^{-2} .$$

It follows that

$$\int_{B_\sigma} \nabla K(x+y_i^{(j)}) u_i(x+y_i^{(j)})^{\frac{2n}{n-2}} dx = O(u_i(y_i^{(j)})^{-2}) .$$

Multiplying the above by $u_i(y_i^{(j)})^{\frac{2}{n-2}(n-3)}$ we have

$$\begin{aligned}&\int_{B_\sigma} \nabla Q_{q^{(j)}}^{(n-2)}(u_i(y_i^{(j)})^{\frac{2}{n-2}}x + u_i(y_i^{(j)})^{\frac{2}{n-2}}y_i^{(j)}) u_i(x+y_i^{(j)})^{\frac{2n}{n-2}}\, dx \\ &\quad = \circ_\sigma(1)\int_{B_\sigma} \left|u_i(y_i^{(j)})^{\frac{2}{n-2}}x + u_i(y_i^{(j)})^{\frac{2}{n-2}}y_i^{(j)}\right|^{n-3} u_i(x+y_i^{(j)})^{\frac{2n}{n-2}} dx + \circ(1) .\end{aligned}$$

Sending i to ∞ we have

$$\int_{\mathbb{R}^n} \nabla Q_{q^{(j)}}^{(n-2)}(z+\xi^{(j)})(1+k^{(j)}|z|^2)^{-n} dz = \circ_\sigma(1) .$$

Sending σ to 0 we have

$$\int_{\mathbb{R}^n} \nabla Q_{q^{(j)}}^{(n-2)}(z+\xi^{(j)})(1+k^{(j)}|z|^2)^{-n}\, dz = 0 .$$

Making a change of variable, we have

$$\int_{\mathbb{R}^n} \nabla Q_{q^{(j)}}^{(n-2)}(y+\eta^{(j)})(1+|y|^2)^{-n} dy = 0 . \tag{3.4}$$

Proposition 3.2 follows from (3.3) and (3.4).

Proof of Theorem 3.1: Let h be a C^∞ function satisfying

$$h(r) = \begin{cases} 1, & 0 \leqq r \leqq 1/2 , \\ 0, & r \geqq 1 , \end{cases}$$

with $h'(r) \leqq 0$ for all $r \geqq 0$.

For $0 < \bar{\varepsilon} < 1$, we choose $\bar{\delta} = \bar{\delta}(\bar{\varepsilon}) > 0$ very small and define, for $\bar{\varepsilon} \leqq s \leqq 1$,

$$K^s(y) = K(0) + \left(1 - (1-s)h(|y|/\bar{\delta})\right)Q^{(\beta)}(y) + R(y),$$

in $\bar{\delta}$-geodesic balls of critical points of K and $K^s = K$ elsewhere. Clearly $K = K^1$. It is elementary to see that if $\bar{\delta}$ is small enough, K^s ($\bar{\varepsilon} \leqq s \leqq 1$) have the same critical points as K and we can, under the hypotheses of Theorem 3.1, apply Proposition 3.2and the Kazdan-Warner identity to this family to conclude that all solutions v of (0.1) with K replaced by K^s ($\bar{\varepsilon} \leqq s \leqq 1$) satisfies $C(\bar{\varepsilon})^{-1} < v < C(\bar{\varepsilon})$ on $\mathbb{S}^n$ and therefore $\|v\|_{C^{2,\alpha}} < C(\bar{\varepsilon})$. Setting $\widetilde{K} = K^{\bar{\varepsilon}}$, it follows from the homotopy invariance of the Leray-Schauder degree that for all $R \geqq C(\bar{\varepsilon})$,

$$\text{(3.5)} \quad \deg\left(v + L_{g_0}^{-1}\left(c(n)Kv^{\frac{n+2}{n-2}}\right), \mathcal{O}_R, 0\right) = \deg\left(v + L_{g_0}^{-1}\left(c(n)\widetilde{K}v^{\frac{n+2}{n-2}}\right), \mathcal{O}_R, 0\right).$$

Set

$$X = \left\{u \in H^1(\mathbb{S}^n) : \fint_{\mathbb{S}^n} |u|^{\frac{2n}{n-2}} = 1\right\}, \qquad \mathcal{S}_0 = \left\{u \in X : \fint_{\mathbb{S}^n} x|u|^{\frac{2n}{n-2}} = 0\right\}.$$

Let B denote the open unit ball in $\mathbb{R}^{n+1}$, $\partial B = \mathbb{S}^n$, and define $\pi : \mathcal{S}_0 \times B \to X$ by $u = \pi(w, \xi) = T_{\varphi_{P,t}}^{-1} w$, where $w \in \mathcal{S}_0$, $\xi = sP (0 \leqq s < 1)$, $P \in \mathbb{S}^n$, $s = \frac{t-1}{t} (1 \leqq t < \infty)$.

It is easy to see that

$$T_{\varphi_{P,t}}^{-1} w = T_{\varphi_{P,t}^{-1}} w; \; \varphi_{P,t}^{-1} = \varphi_{-P,t} = \varphi_{P,t^{-1}}; \; \varphi_{P,1} = \text{Id}, \quad \forall P \in \mathbb{S}^n.$$

It follows that

$$u = \pi(w, \xi) = T_{\varphi_{-P,t}} w, \qquad w = T_{\varphi_{P,t}} u.$$

It follows from lemma 5.4 in [30] that $\pi : \mathcal{S}_0 \times B \to X$ is a C^2 diffeomorphism. Now define on X the following functionals:

$$E_{R_0}(u) = \frac{\fint_{\mathbb{S}^n} (|\nabla u|^2 + c(n)R_0 u^2)}{\left(\fint_{\mathbb{S}^n} R_0 |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}, \qquad E_K(u) = \frac{\fint_{\mathbb{S}^n} (|\nabla u|^2 + c(n)R_0 u^2)}{\left(\fint_{\mathbb{S}^n} K |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}.$$

Let $P \in \mathbb{S}^n$, $t \geqq 1$, and let $\varphi = \varphi_{P,t}$ be the conformal transformation we have defined. It is well-known that

$$\begin{cases} \int_{\mathbb{S}^n} |\nabla_{g_0} T_\varphi u|^2 + c(n)R_0 |T_\varphi u|^2 = \int_{\mathbb{S}^n} |\nabla_{g_0} u|^2 + c(n)R_0 u^2, \\ \int_{\mathbb{S}^n} |T_\varphi u|^{\frac{2n}{n-2}} = \int_{\mathbb{S}^n} |u|^{\frac{2n}{n-2}}. \end{cases}$$

It is elementary to derive that

$$\begin{aligned} T_1 X &= \text{span}\{\text{spherical harmonics of degree} \geqq 1\}, \\ T_1 \mathcal{S}_0 &= \text{span}\{\text{spherical harmonics of degree} \geqq 2\}, \end{aligned}$$

where T_1X denotes the tangent space of X at $u \equiv 1$ and $T_1\mathscr{S}_0$ denotes the tangent space of $\mathscr{S}_0$ at $u \equiv 1$.

It is also elementary to see that for $\widetilde{w} \in T_1\mathscr{S}_0$, $\widetilde{w}$ close to 0, there exist $\mu(\widetilde{w}) \in \mathbb{R}$, $\eta = \eta(\widetilde{w}) \in \mathbb{R}^{n+1}$ that are C^2 functions such that

$$⨍_{\mathbb{S}^n} |1 + \widetilde{w} + \mu + \eta \cdot x|^{\frac{2n}{n-2}} = 1, \qquad ⨍_{\mathbb{S}^n} |1 + \widetilde{w} + \mu + \eta \cdot x|^{\frac{2n}{n-2}} x = 0.$$

Furthermore, $\mu(0) = 0$, $\eta(0) = 0$, $D\mu(0) = 0$, $D\eta(0) = 0$.

Let us use $\widetilde{w} \in T_1\mathscr{S}_0$ as local coordinates of $w \in \mathscr{S}_0$ near $w \equiv 1$. $\widetilde{w} = 0$ corresponds to $w \equiv 1$.

Let

$$E_0(\widetilde{w}) = E_{R_0}(w) = R_0^{\frac{2-n}{2}} ⨍_{\mathbb{S}^n} |\nabla w|^2 + c(n)R_0 w^2,$$

where $\widetilde{w} \in T_1\mathscr{S}_0$ and $w = 1 + \widetilde{w} + \mu(\widetilde{w}) + \eta(\widetilde{w}) \cdot x$ as above.

It follows from a straightforward computation that

$$E_0(\widetilde{w}) = c(n)R_0^{2/n} + R_0^{\frac{2-n}{n}} ⨍_{\mathbb{S}^n} \left(|\nabla \widetilde{w}|^2 - n\widetilde{w}^2\right) + \circ(\|\widetilde{w}\|^2), \qquad \widetilde{w} \in T_1\mathscr{S}_0. \tag{3.6}$$

The quadratic form in (3.6) is clearly positive definite in $T_1\mathscr{S}_0$.

The following propositions are established in [30].

PROPOSITION 3.3. *There exist $\varepsilon_3 = \varepsilon_3(n) > 0$, $\varepsilon_4 = \varepsilon_4(n) > 0$, such that, if $\|K - R_0\|_{L^\infty(\mathbb{S}^n)} \leqq \varepsilon \leqq \varepsilon_3$,*

$$\min_{\substack{w \in \mathscr{S}_0 \\ \|w-1\| < \varepsilon_4}} E_K(w)$$

has a unique minimizer w_K. Furthermore, $D^2E_K|_{\mathscr{S}_0}(w_K)$ is positive definite and

$$w_K > 0 \quad on\ \mathbb{S}^n,$$

$$\|w_K - 1\| \leqq C(n) \inf_{c \in \mathbb{R}} \|K - c\|_{L^{\frac{2n}{n+2}}(\mathbb{S}^n)},$$

$$\|w_K - 1\|_{C^1} \leqq \circ_\varepsilon(1),$$

where $\circ_\varepsilon(1)$ denotes some quantity depending only on n, which tends to 0 as ε tends to 0.

Let

$$\begin{cases} \mathscr{N}_1 = \{w \in \mathscr{S}_0 : \|w - 1\| < \varepsilon_4(n)\}, \\ \mathscr{N}_2(\widetilde{t}) = \Big\{u \in X : u = \pi(w, \xi) \text{ for some } w \in \mathscr{N}_1 \text{ and } \xi = sP, \\ \qquad P \in \mathbb{S}^n,\ s = \dfrac{t-1}{t}\ (1 \leqq t < \widetilde{t})\Big\}, \\ \mathscr{N}_2 = \mathscr{N}_2(\infty), \\ \mathscr{N}_3(\widetilde{t}) = \{v \in H^1(\mathbb{S}^n) \setminus \{0\} : cv \in \mathscr{N}_2(\widetilde{t}) \text{ for some } c > 0\}, \\ \mathscr{N}_3 = \mathscr{N}_3(\infty). \end{cases}$$

Remark 3.4. Since $\varepsilon_4 = \varepsilon_4(n) > 0$ is chosen to be small, it is not difficult to see that for $\|K - R_0\|_{L^\infty} \leqq 1$, any nonzero solution $v \in \mathcal{N}_3$ of $-L_{g_0}v = c(n)K|v|^{4/n-2}v$ on $\mathbb{S}^n$ has to be positive and $\int_{\mathbb{S}^n}|v|^{2n/(n-2)} \geqq 1/C(n)$ for some positive constant $C(n)$ depending only on n.

PROPOSITION 3.5. *There exists some constant $\varepsilon_5 = \varepsilon_5(n) \in (0, \varepsilon_3)$ such that for any $T_1 > 0$ and any nonincreasing positive continuous function $\omega(t)$ $(1 \leqq t < \infty)$ satisfying $\lim_{t\to\infty}\omega(t) = 0$, if a nonconstant function $K \in C^0(\mathbb{S}^n)$ satisfies, for $t \geqq T_1$, that*

$$\|K - R_0\|_{L^\infty(\mathbb{S}^n)} \leqq \varepsilon_5,$$

$$\|K \circ \varphi_{P,t} - K(P)\|_{L^2(\mathbb{S}^n)}\|K \circ \varphi_{P,t} - K(P)\|_{L^{\frac{2n}{n+2}}(\mathbb{S}^n)} \leqq \omega(t)\left|\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x)x\right|$$

for all $P \in \mathbb{S}^n$ and

$$\deg\left(\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x)x, B, 0\right) \neq 0, \quad t \geqq T_1.$$

then (0.1) *has at least one solution. Furthermore, if we assume that $K \in C^\alpha(\mathbb{S}^n)$ $(0 < \alpha < 1)$, then there exists some positive constant C_2, T_2 depending only on n, α, T_1, and ω such that, for all $t \geqq T_2$, $R \geqq C_2$,*

$$\deg\left(v + L_{g_0}^{-1}c(n)Kv^{\frac{n+2}{n-2}}, \mathcal{N}_3(t) \cap \mathcal{O}_R, 0\right) = (-1)^n \deg\left(\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x)x, B, 0\right).$$

Set $K_\mu = \mu\widetilde{K} + (1-\mu)R_0$ for $0 \leqq \mu \leqq 1$.

Claim: There exists some constant $\varepsilon_7 > 0$ such that, for $0 \leqq \mu \leqq \varepsilon_7$,

$$\|K_\mu - R_0\|_{L^\infty(\mathbb{S}^n)} < \varepsilon_5(n).$$

In addition, if we write any solution v of (0.1) with $K = K_\mu$ $(0 \leqq \mu \leqq \varepsilon_7)$ in the form $\pi^{-1}(cv) = (w, \xi)$, $(w, \xi) \in \mathcal{S}_0 \times B$, $cv \in X$, then we have $w \in \mathcal{N}_1$.

For the proof of the above claim, use the proof of a similar claim in section 7 of [30] and just substitute Proposition 3.2 from this paper where theorem 4.2 is cited there.

Once $\bar{\varepsilon} > 0$ is chosen small enough, we can apply Proposition 3.2, theorem 4.4 in [30], and the Harnack inequality to conclude that there exists some constant $R > 1$ such that for all $\varepsilon_7 \leqq \mu \leqq 1$,

$$1/R < v_\mu < R,$$

where v_μ is any solution of (0.1) with $K = K_\mu$.

It follows from the homotopy invariance of the Leray-Schauder degree and corollary 6.1 in [30] that

$$\begin{aligned}\deg\left(v + L_{g_0}^{-1}(c(n)\widetilde{K}v^{\frac{n+2}{n-2}}), \mathcal{O}_R, 0\right) &= \deg\left(v + L_{g_0}^{-1}(c(n)K_{\varepsilon_7}v^{\frac{n+2}{n-2}}), \mathcal{O}_R, 0\right)\\ &= (-1)^n \deg\left(\int_{\mathbb{S}^n} K_{\varepsilon_7} \circ \varphi_{P,t}(x)x, \mathbb{S}^n, 0\right)\\ &= (-1)^n \deg\left(\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x)x, \mathbb{S}^n, 0\right).\end{aligned}$$

Theorem 3.1 follows immediately from the above, (3.5), and Proposition 3.5.

Proof of Theorem 0.13: This theorem follows from Theorem 3.1.

Proof of Corollary 0.15: This corollary follows from Theorem 0.13 and corollary 6.2 of [30].

4. Axisymmetric Case and More Than One Blowup Point

Let $y = (y^1, \dots, y^n) \in \mathbb{R}^n$, $r = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$. In this section all functions are radially symmetric, namely, depending only on r. Let $B_1 \subset \mathbb{R}^n$ be the unit ball, we consider

$$\begin{cases} -\Delta u = c(n)K(r)u^p \quad \text{in } B_1, \\ u > 0, \quad u \text{ is radially symmetric}, \\ p = \dfrac{n+2}{n-2} - \tau, \quad 0 \leqq \tau < \dfrac{2}{n-2}. \end{cases} \tag{4.1}$$

PROPOSITION 4.1. *Suppose that $K \in C_r^0(B_1)$ $(n \geqq 3)$ satisfies, for some positive constant A_1, that*

$$1/A_1 \leqq K(r) \leqq A_1 \quad \forall r : 0 \leqq r \leqq 1. \tag{4.2}$$

Let u satisfy (4.1). Then for any $0 < \varepsilon < \frac{1}{4}$, we have

$$u(r) \leqq C, \quad \forall r : \varepsilon \leqq r \leqq 1 - \varepsilon,$$

where C is some positive constant depending only on n, ε, A_1, and the modulo of continuity of K in B_1.

PROPOSITION 4.2. *Suppose that $K \in C_r^0(B_1)$ $(n \geqq 3)$ satisfies (4.2) for some positive constant A_1. Then there exists some constant C_2 depending only on n, A_1, and the modulo of continuity of K in B_1 such that, for any solution u of (4.1), we have*

$$u(0) = \max_{B_{1/2}} u \tag{4.3}$$

and

$$r^{\frac{2}{p-1}} u(r) \leqq C_2, \quad \forall r \leqq 1/2. \tag{4.4}$$

Proof of Propositions 4.1 and 4.2: Since u is superharmonic, (4.3) follows from the maximum principle. Estimate (4.4) follows from some quite standard blowup arguments, uniqueness results of [11] and [22], and the radial symmetry of u. See, for example, the proof of proposition 2.1 in [28] for an idea of the proof.

PROPOSITION 4.3. *Let $K \in C^0_r(B_1)$ be a nonnegative function and u satisfy (4.1). Then $u'(r) \leqq 0$ for all $0 \leqq r \leqq 1$.*

Proof: Once we write Δ as $r^{1-n}(r^{n-1}u')'$, the proposition follows immediately.

Let $0 \leqq \tau_i \leqq \frac{2}{n-2}$ satisfy $\lim_{i\to\infty} \tau_i = 0$, $p_i = \frac{n+2}{n-2} - \tau_i$. We consider

$$\begin{cases} -\Delta u_i = c(n) K_i(r) u_i^{p_i} & \text{in } B_1, \\ u_i > 0, \quad u_i \text{ is radially symmetric}. \end{cases} \tag{4.5}$$

Remark 4.4. It follows from Proposition 4.2 that if $\{K_i\}$ is a sequence of functions in $C^1_r(B_1)$ with uniform C^1 modulo of continuity that satisfy (4.2) for some positive constant A_1 and $\{u_i\}$ is a sequence of solutions of (4.5) with $\lim_{i\to\infty} \max_{B_{1/2}} u_i = \infty$, then, by setting $y_i = 0$, $y_i \to 0$ is an isolated blowup point of $\{u_i\}$ (see Definition 0.2).

It is not difficult to see that

$$T_1 X_r = \left\{ \varphi \in H^1_r(\mathbb{S}^n) : \fint_{\mathbb{S}^n} \varphi = 0 \right\},$$

$$T_1 \mathcal{S}_r = \left\{ \varphi \in H^1_r(\mathbb{S}^n) : \fint_{\mathbb{S}^n} \varphi = 0, \fint_{\mathbb{S}^n} x^{n+1} \varphi = 0 \right\}.$$

Therefore we have

$$T_1 X_r = T_1 \mathcal{S}_r \bigoplus \operatorname{span}\{x^{n+1}\}.$$

Set

$$E_K(u) = \frac{\fint_{\mathbb{S}^n} (|\nabla u|^2 + c(n) R_0 u^2)}{\left(\fint_{\mathbb{S}^n} K |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}.$$

PROPOSITION 4.5. *For $\widetilde{w} \in T_1 \mathcal{S}_r$, $\widetilde{w}$ close to 0, there exist C^2 functions $\mu(\widetilde{w}) \in \mathbb{R}$ and $\eta = \eta(\widetilde{w}) \in \mathbb{R}$ such that*

$$\begin{cases} \fint_{\mathbb{S}^n} |1 + \widetilde{w} + \mu + \eta x^{n+1}|^{\frac{2n}{n-2}} = 1, \\ \fint_{\mathbb{S}^n} |1 + \widetilde{w} + \mu + \eta x^{n+1}|^{\frac{2n}{n-2}} x^{n+1} = 0. \end{cases}$$

Furthermore, $\mu(0) = 0$, $\eta(0) = 0$, $D\mu(0) = 0$, $D\eta(0) = 0$.

The proof of Proposition 4.5 is elementary.

Let

$$E_0(\tilde{w}) = E_{R_0}(w) = R_0^{\frac{2-n}{2}} \fint_{\mathbb{S}^n} |\nabla w|^2 + c(n) R_0 w^2 ,$$

where $\tilde{w} \in T_1\mathscr{S}_r$ and $w = 1 + \tilde{w} + \mu(\tilde{w}) + \eta(\tilde{w})x^{n+1}$ as in Proposition 4.5. It follows from a straightforward computation that

$$E_0(\tilde{w}) = c(n) R_0^{2/n} + R_0^{\frac{2-n}{n}} \fint_{\mathbb{S}^n} (|\nabla \tilde{w}|^2 - n\tilde{w}^2) + o(\|\tilde{w}\|^2), \quad \tilde{w} \in T_1\mathscr{S}_r . \tag{4.6}$$

The quadratic form in (4.6) is clearly positive definite in $T_1\mathscr{S}_r$.

Proof of Theorem 0.18: With the above results, we only need to follow the arguments in section 6 of [30]. The details are left to the reader.

Proof of Theorem 0.19: Consider $K_\mu = \mu K + (1-\mu)R_0$ $(0 \leqq \mu \leqq 1)$. Choose $\varepsilon > 0$ small so that $\|K_\varepsilon - R_0\|_{L^\infty} \leqq \varepsilon_2$, where ε_2 is the constant in Theorem 0.18. Under hypothesis (i) or (ii), it follows from Propositions 4.2 and 4.3 and the results in section 4 of [30] that there exists C_1 (depending on n, K, ε) such that

$$1/C_1 < v < C_1, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C_1 ,$$

for all $C_r^2(\mathbb{S}^n)$ positive solutions of (0.1) with $K = K_\mu$, $\varepsilon \leqq \mu \leqq 1$. Under hypothesis (iii), the above estimate is still valid. This can be seen from the computation in [30]. In addition, we know $\theta K'(\theta) \geqq 0$ for θ small and $(\pi - \theta)K'(\theta) \leqq 0$ for θ close to π. Therefore, in the geodesic normal coordinate system centered at the north pole or south pole, we have $y \cdot \nabla K(y) \geqq 0$. This information is enough to establish that the blowup has to be isolated simple and cannot have more than one isolated simple blowup point on $\mathbb{S}^n$. Applying the Kazdan-Warner identity as in [30], we eventually conclude that a blowup can never occur.

With the above estimates we can establish Theorem 0.19 by using the homotopy invariance of the Leray-Schauder degree and Theorem 0.18.

Proof of Theorem 0.20: Estimate (0.18) follows from Propositions 4.2 and 4.3, Proposition 3.2, and some standard elliptic estimates. In the following we establish (0.19).

Case I. $K(0)K(\pi) > c_1 a_1 a_2$.

In this case, we can actually assume

$$K(0) = K(\pi) = 1, \qquad |a_1| + |a_2| \ll 1 . \tag{4.7}$$

This can be achieved by constructing a nice family of nonnegative functions $K_t (0 \leqq t \leqq 1)$ such that $K_0 = K$,

$$\begin{cases} K_t(\theta) = K_t(\pi) + a_1(t)(\pi - \theta)^{n-2} + R_1^t(\theta), \\ K_t(\theta) = K_t(0) + a_2(t)\theta^{n-2} + R_2^t(\theta), \end{cases}$$

where $K_t(\pi)$, $K_t(0)$, $-a_1(t)$, $-a_2(t)$ are positive continuous functions on the interval $0 \leqq t \leqq 1$,

$$K_t(0)K_t(\pi) > c_1 a_1(t)a_2(t), \quad 0 \leqq t \leqq 1,$$

$$K_1(0) = K_1(\pi) = 1, \quad |a_1(1) + |a_2(1)| \ll 1,$$

$R_1^t(\theta) = \circ((\pi - \theta)^{n-2})$, $\frac{dR_1}{d\theta}(\theta) = \circ((\pi-\theta)^{n-3})$ as $\theta \to \pi$, and $R_2^t(\theta) = \circ(\theta^{n-2})$, $\frac{dR_2}{d\theta}(\theta) = \circ(\theta^{n-3})$ as $\theta \to 0$ uniformly for $0 \leqq t \leqq 1$. Using Propositions 4.2 and 4.3 and Proposition 3.2 (keeping track of the dependence of the constants) we conclude that the degree for $K = K_0$ is the same as the degree for K_1 that satisfies (4.7).

Once K satisfies (4.7), we consider $K_\mu = \mu K + (1-\mu)R_0$ $(0 \leqq \mu \leqq 1)$. It follows that for $\varepsilon > 0$ small and all sufficiently large $C = C(\varepsilon)$,

$$\begin{aligned}
&\deg\left(v + L_{g_0}^{-1}\left(c(n)Kv^{\frac{n+2}{n-2}}\right), \left\{v \in C_r^{2,\alpha} : \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C,\ 1/C < v < C\right\}, 0\right) \\
&\quad = \deg\Big(v + L_{g_0}^{-1}\left(c(n)K_\varepsilon v^{\frac{n+2}{n-2}}\right), \\
&\qquad \left\{v \in C_r^{2,\alpha} : \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C, 1/C < v < C\right\}, 0\Big) \\
&\quad = -1.
\end{aligned}$$

Case II. $K(0)K(\pi) < c_1 a_1 a_2$.

In [8] Bianchi and Egnell constructed an axisymmetric function $K^*(\theta) > 0$ with the properties that $\lim_{\theta \to 0} \theta^{2-n}K^*(\theta) < 0$, $\lim_{\theta \to \pi^-} (\pi-\theta)^{2-n}K^*(\theta) < 0$. In addition, (0.1) has no axisymmetric solution for this function. We can easily construct a nice family of functions K_t, keeping $K_t(0)K_t(\pi) > c_1 a_1(t)a_2(t)$ $(0 \leqq t \leqq 1)$ and connecting K to K^*. It follows as before that for all C large,

$$\begin{aligned}
&\deg\left(v + L_{g_0}^{-1}\left(c(n)Kv^{\frac{n+2}{n-2}}\right), \left\{v \in C_r^{2,\alpha} : \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C,\ 1/C < v < C\right\}, 0\right) \\
&\quad = \deg\Big(v + L_{g_0}^{-1}\left(c(n)K^* v^{\frac{n+2}{n-2}}\right), \\
&\qquad \left\{v \in C_r^{2,\alpha} : \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} < C, 1/C < v < C\right\}, 0\Big) \\
&\quad = 0.
\end{aligned}$$

Proof of Corollaries 0.22 and 0.24: These corollaries follow from the homotopy invariance of the Leray-Schauder degree and the results we have established.

5. A Simpler Proof of a Sobolev-Aubin-Type Inequality in [16]

For $u \in H^1(\mathbb{S}^n)$, $a > 0$, set

$$I_a(u) = a \fint_{\mathbb{S}^n} |\nabla u|^2 + c(n)R_0 \fint_{\mathbb{S}^n} u^2 ,$$
$$\mathcal{S}_p = \left\{ u \in H^1(\mathbb{S}^n) : \fint_{\mathbb{S}^n} |u|^p x = 0 \right\} ,$$
$$\mathcal{S}_p^0 = \left\{ u \in \mathcal{S}_p : \fint_{\mathbb{S}^n} |u|^p = 1 \right\} .$$

The Sobolev Inequality

For $n \geqq 3$,

$$\min_{u \in H^1(\mathbb{S}^n)\setminus\{0\}} \frac{\fint_{\mathbb{S}^n} |\nabla u|^2 + c(n)R_0 \fint_{\mathbb{S}^n} u^2}{\left(\fint_{\mathbb{S}^n} |u|^{\frac{2n}{n-2}}\right)^{(n-2)/n}} = c(n)R_0 . \tag{5.1}$$

The Aubin Inequality [1]

For $n \geqq 3$ and given any $\varepsilon > 0$, there exists some constant C_ε such that

$$\inf_{u \in \mathcal{S}^0_{\frac{2n}{n-2}}} \left\{ (2^{-(n-2)/n} + \varepsilon) \fint_{\mathbb{S}^n} |\nabla u|^2 + C_\varepsilon \fint_{\mathbb{S}^n} u^2 \right\} \geqq c(n)R_0 .$$

The following lemma is pointed out in [16] and can be proved in the same way as the above Aubin inequality.

LEMMA 5.1. *For $n \geqq 3$, $2 < p \leqq \frac{2n}{n-2}$, given any $\varepsilon > 0$, there exists some constant C_ε such that*

$$\inf_{u \in \mathcal{S}_p^0} \left\{ (2^{-(n-2)/n} + \varepsilon) \fint_{\mathbb{S}^n} |\nabla u|^2 + C_\varepsilon \fint_{\mathbb{S}^n} u^2 \right\} \geqq c(n)R_0 . \tag{5.2}$$

Sobolev-Aubin-Type Inequality [16]

For $n \geqq 3$, there exist some constants $a^\star(n) < 1$ and $p^\star(n) < \frac{2n}{n-2}$ such that for all $p^\star(n) \leqq p \leqq \frac{2n}{n-2}$

$$\inf_{u \in \mathcal{S}_p} \frac{a^\star(n) \fint_{\mathbb{S}^n} |\nabla u|^2 + c(n)R_0 \fint_{\mathbb{S}^n} u^2}{\left(\fint_{\mathbb{S}^n} |u|^p\right)^{2/p}} \geqq c(n)R_0 . \tag{5.3}$$

The above Sobolev-Aubin type inequality is established in [16]. The rest of this section will be devoted to providing a simpler proof of it.

Set

$$\mathcal{M}_{a,p} = \inf_{u \in \mathcal{S}_p^0} I_a(u).$$

It is well-known that for $a > 0$ and $2 \leqq p < \frac{2n}{n-2}$, $\mathcal{M}_{a,p}$ is achieved.

LEMMA 5.2.

$$\mathcal{M}_{a,p} \leqq c(n)R_0 \quad \textit{for all } 0 \leqq a \leqq 1,\ 2 \leqq p \leqq \frac{2n}{n-2},$$

$$\lim_{a \to 1} \mathcal{M}_{a,p} = c(n)R_0 \quad \textit{uniformly for } 2 \leqq p \leqq \frac{2n}{n-2}.$$

Proof: The first inequality follows easily by taking the test function $u \equiv 1$. The second inequality follows from the Sobolev inequality and the Hölder inequality.

Suppose that (5.3) does not hold. Then there exist sequences $\{a_k\}, \{p_k\} \in \mathbb{R}$, $\{u_k\} \in \mathcal{S}_{p_k}^0$, such that $a_k < 1$, $a_k \to 1$, $p_k < \frac{2n}{n-2}$, $p_k \to \frac{2n}{n-2}$, $u_k \geqq 0$, and

$$I_{a_k}(u_k) = \mathcal{M}_{a_k,p_k} < c(n)R_0. \tag{5.4}$$

It follows from (5.4) and (5.2) that for some positive constant $C(n)$ (independent of k) we have

$$\|u_k\|_{H^1(\mathbb{S}^n)} \leqq C(n), \qquad \fint_{\mathbb{S}^n} u_k^2 \geqq 1/C(n).$$

It follows, after passing to a subsequence, that $u_k \to \bar{u}$ weakly in $H^1(\mathbb{S}^n)$ for some $\bar{u} \in H^1(\mathbb{S}^n) \setminus \{0\}$.

The Euler-Lagrange equation satisfied by u_k is

$$-a_k \Delta u_k + c(n)R_0 u_k = \mathcal{M}_k u_k^{p_k-1} + \Lambda_k \cdot x u_k^{p_k-1}, \tag{5.5}$$

where $\mathcal{M}_k = \mathcal{M}_{a_k,p_k}$ and $\Lambda_k \in \mathbb{R}^{n+1}$.

Multiplying (5.5) by u_k and integrating over $\mathbb{S}^n$ we have, by using Lemma 5.2, that

$$\lim_{k \to \infty} \left\{ \fint_{\mathbb{S}^n} |\nabla u_k|^2 + c(n)R_0 \fint_{\mathbb{S}^n} u_k^2 \right\} = c(n)R_0. \tag{5.6}$$

LEMMA 5.3. $|\Lambda_k| = O(1)$.

Proof: Suppose the contrary; let $\xi_k = \Lambda_k/|\Lambda_k|$ and (after passing to a subsequence) $\xi = \lim_{k \to \infty} \xi_k \in \mathbb{S}^n$. Let $\eta \in C^\infty(\mathbb{S}^n)$ be any test function. We multiply (5.5) by $|\Lambda_k|^{-1}\eta$, integrate it over $\mathbb{S}^n$, and then send k to ∞. It follows immediately that $⨍_{\mathbb{S}^n}\xi \cdot x\bar{u}^{(n+2)/(n-2)}\eta = 0$. Hence $\bar{u} \equiv 0$, which is a contradiction.

Clearly $\bar{u}$ satisfies

$$-\Delta\bar{u} + c(n)R_0\bar{u} = c(n)R_0\bar{u}^{(n+2)/(n-2)} + \Lambda \cdot x\bar{u}^{(n+2)/(n-2)}, \tag{5.7}$$

where $\Lambda = \lim_{k\to\infty} \Lambda_k$.

The Kazdan-Warner identity [27] gives

$$⨍_{\mathbb{S}^n} \nabla(c(n)R_0 + \Lambda \cdot x)\nabla x\bar{u}^{\frac{2n}{n-2}} = 0.$$

It follows that

$$\Lambda = \lim_{k\to\infty} \Lambda_k = 0. \tag{5.8}$$

It follows from (5.1), (5.7), and (5.8) that

$$c(n)R_0 \leqq \frac{⨍_{\mathbb{S}^n}|\nabla\bar{u}|^2 + c(n)R_0⨍_{\mathbb{S}^n}\bar{u}^2}{\left(⨍_{\mathbb{S}^n}|\bar{u}|^{\frac{2n}{n-2}}\right)^{(n-2)/n}} = c(n)R_0\left(⨍_{\mathbb{S}^n}|\bar{u}|^{\frac{2n}{n-2}}\right)^{2/n}.$$

Therefore $⨍_{\mathbb{S}^n}|\bar{u}|^{\frac{2n}{n-2}} \geqq 1$.

On the other hand, $⨍_{\mathbb{S}^n}|\bar{u}|^{\frac{2n}{n-2}} \leqq \liminf_{k\to\infty} ⨍_{\mathbb{S}^n}|u_k|^{p_k} = 1$. It follows (using also (5.7) and (5.8)) that

$$\begin{cases} ⨍_{\mathbb{S}^n}|\bar{u}|^{\frac{2n}{n-2}} = 1, \\ ⨍_{\mathbb{S}^n}|\nabla\bar{u}|^2 + c(n)R_0⨍_{\mathbb{S}^n}\bar{u}^2 = c(n)R_0. \end{cases} \tag{5.9}$$

From (5.6) and (5.9) we have that

$$\lim_{k\to\infty} \|u_k - \bar{u}\|_{H^1(\mathbb{S}^n)} = 0.$$

Clearly $\bar{u} \in \mathscr{S}^0_{\frac{2n}{n-2}}$ and hence by Obata's result (see [33]) that $\bar{u} \equiv 1$.

In the following we expand $I_a(u)$ for $u \in \mathscr{S}^0_p$ and $\|u - 1\|_{H^1(\mathbb{S}^n)}$ small. It is easy to see that

$$\begin{aligned} T_1\mathscr{S}^0_p &= \left\{\varphi : ⨍_{\mathbb{S}^n}\varphi = 0,\ ⨍_{\mathbb{S}^n}\varphi x = 0\right\} \\ &= \text{span}\{\text{spherical harmonics of degree} \geqq 2\}. \end{aligned}$$

Here $T_1\mathscr{S}^0_p$ denotes the tangent space of $\mathscr{S}^0_p$ at $u \equiv 1$. The following lemma is elementary.

LEMMA 5.4. *For* $\frac{2n-2}{n-2} \leqq p \leqq \frac{2n}{n-2}$, $\tilde{u} \in T_1\mathcal{S}_p^0$, $\tilde{u}$ *close to* 0, *there exist* C^2 *functions* $\mu(\tilde{u}) \in \mathbb{R}$ *and* $\eta = \eta(\tilde{u}) \in \mathbb{R}^{n+1}$ *such that*

$$\begin{cases} \displaystyle\fint_{\mathbb{S}^n} |1 + \tilde{u} + \mu + \eta \cdot x|^p = 1\,, \\ \displaystyle\fint_{\mathbb{S}^n} |1 + \tilde{u} + \mu + \eta \cdot x|^p x = 0\,. \end{cases}$$

Furthermore, $\mu(0) = 0$, $\eta(0) = 0$, $D\mu(0) = 0$, $D\eta(0) = 0$, *and* μ *and* η *have uniform (with respect to p)* C^2 *modulo of continuity near* 0.

It is not difficult to see that for $\tilde{u} \in T_1\mathcal{S}_p^0$

$$\mu(\tilde{u}) = -\frac{p-1}{2}\fint_{\mathbb{S}^n} \tilde{u}^2 + \circ(\|\tilde{u}\|^2)\,.$$

Let us use $\tilde{u} \in T_1\mathcal{S}_p^0$ as local coordinates of $u \in \mathcal{S}_p^0$ near $u \equiv 1$. $\tilde{u} = 0$ corresponds to $u \equiv 1$.

Let

$$E(\tilde{u}) \equiv I_a(u) = a\fint_{\mathbb{S}^n} |\nabla u|^2 + c(n)R_0u^2\,,$$

where $\tilde{u} \in T_1\mathcal{S}_p^0$ and $u = 1 + \tilde{u} + \mu(\tilde{u}) + \eta(\tilde{u}) \cdot x$.

A straightforward computation yields

$$E(\tilde{u}) = c(n)R_0(1 + 2\mu(\tilde{u})) + a\fint_{\mathbb{S}^n} |\nabla\tilde{u}|^2 + c(n)R_0\fint_{\mathbb{S}^n} \tilde{u}^2 + \circ(\|\tilde{u}\|^2)\,.$$

Hence

$$E(\tilde{u}) = c(n)R_0 + a\fint_{\mathbb{S}^n} |\nabla\tilde{u}|^2 - \frac{n(n-2)(p-2)}{4}\fint_{\mathbb{S}^n} \tilde{u}^2 + \circ(\|\tilde{u}\|^2)\,. \tag{5.10}$$

It follows that there exists some positive constant $C(n)$ (determined by the difference of the first and the second eigenvalue of $-\Delta_{g_0}$) such that for a close to 1 and p close to $\frac{2n}{n-2}$ we have

$$a\fint_{\mathbb{S}^n} |\nabla\tilde{u}|^2 - \frac{n(n-2)(p-2)}{4}\fint_{\mathbb{S}^n} \tilde{u}^2 \geqq \frac{1}{C(n)}\fint_{\mathbb{S}^n} |\nabla\tilde{u}|^2. \tag{5.11}$$

It follows from (5.10) and (5.11) that for k large we have $I_{a_k}(u_k) \geqq c(n)R_0$. This is a contradiction. The Sobolev-Aubin-type inequality is thus established.

Appendix

Let $d(P,x)$ denote the geodesic distance between $P, x \in \mathbb{S}^n$; it is not difficult to see that

$$\delta_{P,t}(x) = \left(\frac{t}{1 + \frac{t^2-1}{2}(1 - \cos d(P,x))}\right)^{\frac{n-2}{2}}\,.$$

Let P be the south pole of $\mathbb{S}^n$ and make a stereographic projection with respect to the equatorial plane; we then have

$$\delta_{P,t}(y) = \left(\frac{t(1+|y|^2)}{1+t^2|y|^2}\right)^{\frac{n-2}{2}}, \quad \forall y \in \mathbb{R}^n.$$

It is not difficult to see that

$$-\int_{\mathbb{S}^n}(L_{g_0}\delta_{P,t})\delta_{P,t} = c(n)R_0\int_{\mathbb{S}^n}\delta_{P,t}^{\frac{2n}{n-2}} = c(n)R_0|\mathbb{S}^n|,$$

$$G_P(x) = \frac{2^{\frac{2-n}{2}}|\mathbb{S}^n|}{(n-2)|\mathbb{S}^{n-1}|}\frac{1}{(1-\cos d(P,x))^{\frac{n-2}{2}}}, \quad \forall P, x \in \mathbb{S}^n.$$

For $n = 4$, it is easy to see that

$$\left|\frac{\partial \delta_{P,t}}{\partial t}(y)\right| \leqq \frac{2(1+|y|^2)}{(1+t^2|y|^2)}, \quad \forall y \in \mathbb{R}^4.$$

We list below some estimates that can be verified by elementary calculations. The above formulae are helpful in verifying Lemmas A.4 through A.7.

LEMMA A.1. *There exists some universal constant $C > 0$ such that for any $2 \leqq \alpha \leqq 3$ and any $a, b \geqq 0$ we have*

$$|(a+b)^\alpha - a^\alpha - b^\alpha - \alpha a^{\alpha-1}b| \leqq Ca^{\alpha-2}b^2,$$

$$|(a+b)^\alpha - a^\alpha - b^\alpha| \leqq C(a^{\alpha-1}b + ab^{\alpha-1}).$$

LEMMA A.2. *For $2 \leqq \alpha \leqq \beta$, there exists some constant $C = C(\beta)$ depending only on α such that for any $a \geqq 0$, $b \in \mathbb{R}$, we have*

$$\left||a+b|^{\alpha-1}(a+b) - a^\alpha - \alpha a^{\alpha-1}b - \frac{\alpha(\alpha-1)}{2}a^{\alpha-2}b^2\right| \leqq C(|b|^\alpha + a^\gamma|b|^{\alpha-\gamma}),$$

where $\gamma = \max\{0, \alpha-3\}$.

For $1 \leqq \alpha \leqq 2$, there exists some universal constant $C > 0$ such that for any $a, b \geqq 0$, we have

$$|(a+b)^\alpha - a^\alpha| \leqq C(a^{\alpha-1}b + b^\alpha).$$

LEMMA A.3.

$$\int_{\mathbb{R}^4}\frac{dz}{(1+|z|^2)^3} = \frac{|\mathbb{S}^3|}{4}, \qquad \int_{\mathbb{R}^4}\frac{dz}{(1+|z|^2)^4} = \frac{|\mathbb{S}^3|}{12},$$

$$\int_{\mathbb{R}^4} \frac{|z|^2 dz}{(1+|z|^2)^4} = \frac{|\mathbb{S}^3|}{6}, \qquad \int_{\mathbb{R}^4} \frac{(1-|z|^2) dz}{(1+|z|^2)^4} = -\frac{|\mathbb{S}^3|}{12}.$$

LEMMA A.4. *For any* $\varepsilon_0 > 0, A > 0, 0 < \tau \ll 1, P_1, P_2 \in \mathbb{S}^4, |P_1 - P_2| \geqq \varepsilon_0$, $A^{-1}\tau^{-1/2} < t_1, t_2 < A\tau^{-1/2}$, *we have*

$$\int_{\mathbb{S}^4} \delta^3_{P_1,t_1} \delta_{P_2,t_2} = \frac{2^7|\mathbb{S}^3|}{|\mathbb{S}^4|} \left(\int_{\mathbb{R}^4} \frac{dz}{(1+|z|^2)^3} \right) \frac{G_{P_1}(P_2)}{t_1 t_2} + O(\tau^2 |\log \tau|), \tag{A.1}$$

$$\int_{\mathbb{S}^4} \delta^{3-\tau}_{P_1,t_1} \delta_{P_2,t_2} = O(\tau), \tag{A.2}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^4} \delta_{P_2,t_2} \delta^3_{P_1,t_1} = -\frac{2^7|\mathbb{S}^3|}{|\mathbb{S}^4|} \left(\int_{\mathbb{R}^4} \frac{dz}{(1+|z|^2)^3} \right) \frac{G_{P_1}(P_2)}{t_1^2 t_2} + O(\tau^2), \tag{A.3}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^4} \delta_{P_2,t_2} \delta^{3-\tau}_{P_1,t_1} = \frac{\partial}{\partial t_1} \int_{\mathbb{S}^4} \delta_{P_2,t_2} \delta^3_{P_1,t_1} + O(\tau^{5/2} |\log \tau|), \tag{A.4}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^4} \delta^{3-\tau}_{P_2,t_2} \delta_{P_1,t_1} = \frac{\partial}{\partial t_1} \int_{\mathbb{S}^4} \delta^3_{P_2,t_2} \delta_{P_1,t_1} + O(\tau^{5/2} |\log \tau|), \tag{A.5}$$

$$\int_{\mathbb{S}^4} |\cdot - P_1|^2 \delta^{4-\tau}_{P_1,t_1} = \frac{2^6}{t_1^2} \int_{\mathbb{R}^4} \frac{|z|^2}{(1+|z|^2)^4} dz + O(\tau^{3/2}), \tag{A.6}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^4} \delta^{4-\tau}_{P_1,t_1} = -\frac{\tau}{t_1} \int_{\mathbb{R}^4} \frac{2^4}{(1+|z|^2)^4} dz + O(\tau^{5/2} |\log \tau|), \tag{A.7}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^4} |\cdot - P_1|^2 \delta^{4-\tau}_{P_1,t_1} = -\frac{2^7}{t_1^3} \int_{\mathbb{R}^4} \frac{|z|^2}{(1+|z|^2)^4} dz + O(\tau^{5/2} |\log \tau|). \tag{A.8}$$

For all the equations above,

$$|O(\tau^2 |\log \tau|)| \leqq C\tau^2 |\log \tau|, \ |O(\tau^{5/2} |\log \tau|)| \leqq C\tau^{5/2} |\log \tau|, \ \ldots,$$

for some constant C *depending only on* ε_0 *and* A.

LEMMA A.5. *Under the hypotheses of Lemma* A.4 *and* $\ell \neq m$,

$$\langle \delta_{P_1,t_1}, \delta_{P_1,t_1} \rangle = 2|\mathbb{S}^4|, \qquad \left\langle \frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(l)}}, \frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(l)}} \right\rangle = \Gamma_1 t_1^2 + O(\tau), \tag{A.9}$$

$$\left\langle \frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(l)}}, \frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(m)}} \right\rangle = 0, \qquad \left\langle \frac{\partial \delta_{P_1,t_1}}{\partial t_1}, \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\rangle = \Gamma_2 t_1^{-2}, \tag{A.10}$$

$$\langle \delta_{P_1,t_1}, \delta_{P_2,t_2} \rangle = O(\tau), \qquad \left\langle \delta_{P_1,t_1}, \frac{\partial \delta_{P_2,t_2}}{\partial P_2} \right\rangle = O(\tau), \tag{A.11}$$

$$\langle \delta_{P_1,t_1}, \frac{\partial \delta_{P_2,t_2}}{\partial t_2} \rangle = O(\tau^{3/2}), \qquad \left\langle \frac{\partial \delta_{P_1,t_1}}{\partial t_1}, \frac{\partial \delta_{P_2,t_2}}{\partial P_2} \right\rangle = O(\tau^{3/2}), \tag{A.12}$$

$$\left\langle \frac{\partial \delta_{P_1,t_1}}{\partial P_1}, \frac{\partial \delta_{P_2,t_2}}{\partial P_2} \right\rangle = O(\sqrt{\tau}), \qquad \left\langle \frac{\partial \delta_{P_1,t_1}}{\partial t_1}, \frac{\partial \delta_{P_2,t_2}}{\partial t_2} \right\rangle = O(\tau^2), \tag{A.13}$$

$$\left\langle \delta_{P_1,t_1}, \frac{\partial^2 \delta_{P_1,t_1}}{\partial t_1 \partial P_1} \right\rangle = 0, \qquad \left\| \frac{\partial^2 \delta_{P_1,t_1}}{\partial t_1 \partial P_1} \right\| \leqq C, \tag{A.14}$$

$$\|\delta_{P_1,t_1}^{2-\tau} \delta_{P_2,t_2}\|_{L^{4/3}(\mathbb{S}^4)} \leqq C\tau, \qquad \|\delta_{P_1,t_1}^{1-\tau} \delta_{P_2,t_2}^2\|_{L^{4/3}(\mathbb{S}^4)} \leqq C\tau, \tag{A.15}$$

$$\begin{aligned} \left\| \delta_{P_2,t_2}^{2-\tau} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^{4/3}(\mathbb{S}^4)} &\leqq C\tau^{3/2}, \\ \left\| \delta_{P_1,t_1}^{1-\tau} \delta_{P_2,t_2} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^{4/3}(\mathbb{S}^4)} &\leqq C\tau^{3/2}, \end{aligned} \tag{A.16}$$

$$\left\| \delta_{P_1,t_1}^{1-\tau} \delta_{P_2,t_2}^2 \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^1(\mathbb{S}^4)} \leqq C\tau^{5/2} |\log \tau|, \tag{A.17}$$

$$\begin{aligned} \|\delta_{P_1,t_1}^{3-\tau} - \delta_{P_1,t_1}^3\|_{L^{4/3}(\mathbb{S}^4)} &\leqq C\tau |\log \tau|, \\ \|\delta_{P_1,t_1}^{2-\tau} - \delta_{P_1,t_1}^2\|_{L^2(\mathbb{S}^4)} &\leqq C\tau |\log \tau|, \end{aligned} \tag{A.18}$$

$$\|\delta_{P_1,t_1}^{4-\tau} - \delta_{P_1,t_1}^4\|_{L^1(\mathbb{S}^4)} \leqq C\tau |\log \tau|, \tag{A.19}$$

$$\text{(A.20)}\qquad \| |\cdot - P_1| \delta^3_{P_1,t_1} \|_{L^{4/3}(\mathbb{S}^4)} \leqq C\sqrt{\tau}\,, \qquad \| |\cdot - P_1|^2 \delta^3_{P_1,t_1} \|_{L^{4/3}(\mathbb{S}^4)} \leqq C\tau\,,$$

$$\text{(A.21)}\qquad \left\| |\cdot - P_1| \delta^{2-\tau}_{P_1,t_1} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^2(\mathbb{S}^4)} \leqq C\sqrt{\tau}\,,$$

where $\Gamma_1, \Gamma_2 > 0$ *are some universal constants and* C *depends only on* ε_0, A.

LEMMA A.6. *For any* $\varepsilon_0 > 0$, $A > 0$, $0 < \tau \ll 1$, $P_1, P_2, P_3 \in \mathbb{S}^4$, $|P_1 - P_2|$, $|P_1 - P_3|$, $|P_2 - P_3| \geqq \varepsilon_0$, $A^{-1}\tau^{-1/2} < t_1, t_2, t_3 < A\tau^{-1/2}$, *we have*

$$\text{(A.22)}\qquad \left| \frac{\partial}{\partial P_1} \int_{\mathbb{S}^4} \delta^{3-\tau}_{P_2,t_2} \delta_{P_1,t_1} \right| \leqq C\tau\,, \qquad \left| \frac{\partial}{\partial P_1} \int_{\mathbb{S}^4} \delta^3_{P_2,t_2} \delta_{P_1,t_1} \right| \leqq C\tau\,,$$

$$\text{(A.23)}\qquad \int_{\mathbb{S}^4} \left| \delta^{3-\tau}_{P_2,t_2} \frac{\partial \delta_{P_1,t_1}}{\partial P_1} \right| \leqq C\tau\,, \qquad \int_{\mathbb{S}^4} \delta^2_{P_2,t_2} \delta^{1-\tau}_{P_1,t_1} \left| \frac{\partial \delta_{P_1,t_1}}{\partial P_1} \right| \leqq C\tau^{3/2}\,,$$

$$\text{(A.24)}\qquad \left\| \delta^{2-\tau}_{P_2,t_2} \delta_{P_3,t_3} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^1(\mathbb{S}^4)} = o(\tau^{3/2})\,, \qquad \int_{\mathbb{S}^4} |\cdot - P_1|^2 \left| \frac{\partial \delta_{P_1,t_1}}{\partial P_1} \right| \leqq C\sqrt{\tau}\,,$$

where $C = C(\varepsilon_0, A)$.

LEMMA A.7. *In addition to the hypotheses of Lemma* A.4, *we assume that* $K \in C^1(\mathbb{S}^4)$. *Then*

$$\text{(A.25)}\qquad \begin{aligned} &\frac{\partial}{\partial t_1} \int_{\mathbb{S}^4} [K - K(P_1)] \delta_{P_2,t_2} \delta^{3-\tau}_{P_1,t_1} = O(\tau^2)\,, \\ &\frac{\partial}{\partial t_1} \int_{\mathbb{S}^4} [K - K(P_2)] \delta^{3-\tau}_{P_2,t_2} \delta_{P_1,t_1} = O(\tau^2)\,. \end{aligned}$$

where $|O(\tau^2)| \leqq C\tau^2$ *and* C *denotes some constant depending only on* ε_0, C_0, $\|K\|_{L^\infty(\mathbb{S}^4)}$, *and* $\|\nabla K\|_{L^\infty(\mathbb{S}^4)}$.

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Bibliography

[1] Aubin, T., *Meilleures constantes dans le thérèm d'inclusion de Sobolev st un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. 32, 1979, pp. 149–179.

[2] Aubin, T., *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*, Springer-Verlag, New York, 1982.

[3] Bahri, A., *Critical Points at Infinity in Some Variational Problems*, Research Notes in Mathematics No. 182, Longman-Pitman, London, 1989.

[4] Bahri, A., and Coron, J. M., *The scalar-curvature problem on standard three-dimensional sphere*, J. Funct. Anal. 95, 1991, pp. 106–172.

[5] Bahri, A., Li, Y. Y., and Rey, O., *On a variational problem with lack of compactness: The topological effect of the critical points at infinity*, Calc. Var. Partial Differential Equations 3, 1995, pp. 67–93. Also in the preprint series of Centre de Mathématiques, Ecole Polytechnique, No. 1017, 1991.

[6] Benayed, M., Chen, Y., Chtioui, H., and Hammami, M., *On the prescribed scalar curvature problem on four-manifolds*, Duke Math. J., to appear.

[7] Besse, A., *Einstein Manifolds*, Springer-Verlag, Berlin–New York, 1987.

[8] Bianchi, G., and Egnell, H., *An ODE approach to the equation* $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$, *in* $\mathbb{R}^n$, Math. Z. 210, 1992, pp. 137–166.

[9] Bourguignon, J. P., and Ezin, J. P., *Scalar curvature functions in a conformal class of metric and conformal transformations*, Tran. Amer. Math. Soc. 301, 1987, pp. 723–736.

[10] Brezis, H., and Nirenberg, L., *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36, 1983, pp. 437–477.

[11] Caffarelli, L., Gidas, B., and Spruck, J., *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. 42, 1989, pp. 271–297.

[12] Chang, K. C., and Liu, J. Q., *On Nirenberg's problem*, Internat. J. Math. 4, 1993, pp. 35–58.

[13] Chang, S. Y., Gursky, M. J., and Yang, P., *The scalar curvature equation on 2- and 3-sphere*, Calc. Var. Partial Differential Equations 1, 1993, pp. 205–229.

[14] Chang, S. Y., and Yang, P., *Prescribing Gaussian curvature on* S^2, Acta Math. 159, 1987, pp. 215–259.

[15] Chang, S. Y., and Yang, P., *Conformal deformations of metrics on* S^2, J. Differential Geom. 27, 1988, pp. 256–296.

[16] Chang, S. Y., and Yang, P., *A perturbation result in prescribing scalar curvature on* S^n, Duke Math. J. 64, 1991, pp. 27–69.

[17] Chen, W. X., and Ding, W., *Scalar curvature on* S^2, Trans. Amer. Math. Soc. 303, 1987, pp. 365–382.

[18] Chen, W. X., and Li, C., *A necessary and sufficient condition for the Nirenberg problem*, Comm. Pure Appl. Math. 48, 1995, pp. 657–667.

[19] Cheng, K. S., and Smoller, J. A., *Conformal metrics with prescribed Gaussian curvature on* S^2, Trans. Amer. Math. Soc. 336, 1993, pp. 219–251.

[20] Ding, W., and Ni, W.-M., *On the elliptic equation* $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ *and related topics*, Duke Math. J. 52, 1985, pp. 485–506.

[21] Escobar, J., and Schoen, R., *Conformal metrics with prescribed scalar curvature*, Invent. Math. 86, 1986, pp. 243–254.

[22] Gidas, B., and Spruck, J., *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. 34, 1981, pp. 525–598.

[23] Gilbarg, D., and Trudinger, N. S., *Elliptic Partial Differential Equations*, 2nd ed., Grundlehoen der Mathematischen Wissenschaften 224, Springer-Verlag, 1983.

[24] Han, Z. C., *Prescribing Gaussian curvature on* S^2, Duke Math. J. 61, 1990, pp. 679–703.

[25] Han, Z. C., and Li, Y. Y., *A note on the Kazdan-Warner type condition*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.

[26] Hebey, H., and Vaugon, M., *Meilleures constantes dans le théorème d'inclusion de Sobolev et multiplicité pour les problèmes de Nirenberg et Yamabe*, Indiana Univ. Math. J. 41, 1992, pp. 377–407.

[27] Kazdan, J., and Warner, F., *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature*, Ann. Math. 101, 1975, pp. 317–331.

[28] Li, Y. Y., *On* $-\Delta u = K(x)u^5$ *in* $\mathbb{R}^3$, Comm. Pure Appl. Math. 46, 1993, pp. 303–340.

[29] Li, Y. Y., *Prescribing scalar curvature on* S^n *and related problems*, C. R. Acad. Sci. Paris Sér. I Math. 317, 1993, pp. 159–164.

[30] Li, Y. Y., *Prescribing scalar curvature on* S^n *and related problems, Part I*, J. Differential Equations 120, 1995, pp. 319–410.

[31] Moser, J., *On a nonlinear problem in differential geometry*, pp. 273–280 in: *Dynamical Systems (Proc. Sympos., Univ. Bahia, Salvador)*, M. Peixoto, ed., Academic Press, New York, 1973.

[32] Nirenberg, L., *Topics in Nonlinear Functional Analysis*, Lecture Notes, Courant Institute, New York University, 1974.

[33] Obata, M., *The conjecture on conformal transformations of Riemannian manifolds*, J. Differential Geom. 6, 1971, pp. 247–258.

[34] Onofri, E., *On the positivity of the effective action in a theory of random surface*, Comm. Math. Phys. 86, 1982, pp. 321–326.

[35] Pohozaev, S. I., *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6, 1965, pp. 1408–1411.

[36] Schoen, R., *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, pp. 120–154 in: *Topics in Calculus of Variations*, M. Giaquinta, ed., Lecture Notes in Mathematics No. 1365, Berlin–New York, Springer-Verlag, 1989.

[37] Schoen, R., *On the number of constant scalar curvature metrics in a conformal class*, pp. 311–320 in: *Differential Geometry*, Pitman Monographs Surveys Pure Appl. Math. 52, Longman Sci. Tech., Harlow, 1991.

[38] Trudinger, N., *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 22, 1968, pp. 265–274.

[39] Zhang, D., *New Results on Geometric Variational Problems*, Thesis, Stanford University, 1990.

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