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Prescribing Scalar Curvature on $\mathbf{S}^3$, $\mathbf{S}^4$ and Related Problems

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We show that for the prescribing scalar curvature problem on $\mathbf{S}^n$ $(n=3,4)$, we can perturb (in an explicit way) any given positive continuous function in any neighborhood of any given point on $\mathbf{S}^n$ such that for the perturbed function there exist many solutions. The related critical exponent equations $-\Delta u = K(x)\, u^{(n+2)/(n-2)}$ in $\mathbf{R}^n$ $(n=3,4)$, with $K(x)$ being asymptotically periodic in one of the variables, are also studied and infinitely many positive solutions (modulo translations by its periods) are obtained under some additional mild hypotheses on $K(x)$. Some a priori estimates to solutions of the prescribing scalar curvature problem on $\mathbf{S}^3$ are as given.

0. INTRODUCTION

Let $(\mathbf{S}^n, g_0)$ be the standard n-sphere. The prescribing scalar curvature problem is the problem of finding suitable conditions on a function $K(x)$ on $\mathbf{S}^n$ such that $K(x)$ is the scalar curvature of a metric g on $\mathbf{S}^n$ conformally equivalent to g_0. For $n \geqslant 3$, we write $g = u^{4/(n-2)} g_0$; the problem (Nirenberg's problem) is equivalent to finding a positive function u on $\mathbf{S}^n$ which satisfies the following partial differential equation.

$$-\Delta_{g_0} u + \frac{n(n-2)}{4} u = \frac{n-2}{4(n-1)} K(x)\, u^{(n+2)/(n-2)}, \tag{1}$$

where Δ_{g_0} denotes the Laplace–Beltrami operator associated with the metric g_0.

For $n=2$, we write $g = e^{2u} g_0$; the problem is equivalent to finding a function u on $\mathbf{S}^2$ which satisfies the following equation

$$-\Delta_{g_0} u + 1 = K(x)\, e^{2u}. \tag{1'}$$

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It is easy to see, by multiplying (1) for $n \geqslant 3$) by u and interpreting by parts, that a necessary condition for solving the problem is that K has to be positive somewhere. For $n=2$, this follows from the Gauss–Bonnet theorem.

It turns out that there is at least another obstruction (Kazdan–Warner obstruction) to solving the problem. In particular, if $\mathbf{S}^n$ is embedded as usual in $\mathbf{R}^{n+1}$ and $K(x)$ is strictly monotone in one direction, then (1) ((1′) for $n=2$) is not solvable. For instance, (1) ((1′) for $n=2$) has no solution for $K(x)=x^{n+1}+2$.

There have been many works devoted to the existence and multiplicity results, trying to understand under what conditions (1) ((1′) for $n=2$) is solvable. For details please see the works of Moser, Kazdan and Warner, Bahri and Coron, Chang and Yang, Chen and Ding, Ding and Ni, Ni, Han, Escobar, Schoen, Zhang, Chang and Liu, ... ([Mo, KW, BC2, CY1–3, CD, DN, Ni2, E1–2, ES, Sc1–3, Ha, Z, SZ, CL]), and the references therein.

We state one of our results on (1) for dimension $n=3, 4$.

THEOREM 0.1. *Let* $K \in L^{\infty}(\mathbf{S}^n)$ $(n=3,4)$. *If for some* $\tilde{x} \in \mathbf{S}^n$, $\tilde{\varepsilon}>0$, $K \in C^0(B_{\tilde{\varepsilon}}(\tilde{x}))$ *($B_{\tilde{\varepsilon}}(\tilde{x})$ denotes the geodesic ball in* $\mathbf{S}^n$ *of radius* $\tilde{\varepsilon}$ *and centered at* $\tilde{x}$*),* $K(\tilde{x})>0$, *then for any* $k=1,2,3,\ldots$, $m=2,3,\ldots$, $\varepsilon>0$, *there exists* $K_{\varepsilon,k,m} \in L^{\infty}(\mathbf{S}^n)$, $K_{\varepsilon,k,m}-K \in C^0(\mathbf{S}^n)$, $\|K_{\varepsilon,k,m}-K\|_{C^0(\mathbf{S}^n)}<\varepsilon$, $K_{\varepsilon,k,m} \equiv K$ *in* $\mathbf{S}^n \backslash B_{\varepsilon}(\tilde{x})$, *such that, for each* $2 \leqslant s \leqslant m$,

$$-\Delta_{g_0} u+\frac{n(n-2)}{4} u=\frac{n-2}{4(n-1)} K_{\varepsilon,k,m}(x)\, u^{(n+2)/(n-2)}$$

has at least k positive solutions with s bumps.

Remark 0.1. The perturbation $K_{\varepsilon,k,m}$ can be constructed explicitly (Section 5) and it is a way of gluing approximate solutions into genuine solutions. The method is variational, rather than through implicit function theorems as in [Ta, Sc3, Ka, Sm, Po, ...]. Theorem 0.1 is derived in Section 5 as a corollary of Theorem 5.1. In fact, we know much more about the structure of the solutions we produced. This becomes clear in Section 5.

Remark 0.2. From the proof (see Section 5) we can see that if dimension $n=3$ and $K \in C^1(B_{\tilde{\varepsilon}}(\tilde{x}))$, then we can choose $K_{\varepsilon,k,m}$ to further satisfy $K_{\varepsilon,k,m}-K \in C^{\infty}(\mathbf{S}^3)$.

Remark 0.3. One cannot expect to perturb any $K(x)$ near any point $\tilde{x} \in \mathbf{S}^n$ in the C^1 sense to obtain the existence of solutions. This is evident if we take $K(x)=x^{n+1}+2$ and $\tilde{x}$ to be different from the north and the south poles, since the perturbed function would still violate the Kazdan–Warner conditions.

Next we look at a related problem,

$$\begin{aligned} -\Delta u &= K(x)\, u^{(n+2)/(n-2)} \qquad \text{in } \mathbf{R}^n, \\ u &> 0 \end{aligned} \tag{2}$$

There have been many works on (2), see Ni [N1–N2], Ding and Ni [DN], Li and Ni [LN], etc., and the references therein.

We should point out that (1) is exactly (2) (up to a harmless positive constant in front of $K(x)$) after making a stereographic projection. Therefore, if K behaves well at infinity and a solution u of (2) decays to zero at infinity (hence u must decay to zero at the rate of $|x|^{2-n}$), then u is actually a solution of (1).

In [Li1], the author has studied

$$-\Delta u = K(x)\, u^5 \qquad \text{in } \mathbf{R}^3, \tag{3}$$

and has proved for a large class of positive functions $K(x)$ that (3) has infinitely many positive solutions which decay at infinity at the rate of $1/|x|$. In this paper, we prove some further results concerning both (1) and (2) in dimensions $n = 3, 4$. Many of the results in this have been announced and briefly described in [Li2].

Let E be the closure of $C_c^\infty(\mathbf{R}^n)$ (set of all smooth functions with compact support) under the norm $\|u\|_E = (\int_{\mathbf{R}^n} |\nabla u|^2)^{1/2}$. E is clearly a Hilbert space. We now simply use $\|\cdot\|$ to denote $\|\cdot\|_E$.

We define the Sobolev constant S_n by

$$S_n = \min_{u \in E} \frac{(\int_{\mathbf{R}^n} |\nabla u|^2)^{1/2}}{(\int_{\mathbf{R}^n} |u|^{2n/(n-2)})^{(n-2)/2n}}.$$

For any ψ in $L^\infty(\mathbf{R}^n)$, $0 \leqslant \tau < 4/(n-2)$, we define

$$\begin{aligned} I_{\psi,\tau}(u) &= \frac{1}{2} \int |\nabla u|^2 - \left(\frac{2n}{n-2} - \tau\right)^{-1} \\ &\quad \times \int \psi \left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau} |u|^{(2n/(n-2)) - \tau}, \qquad u \in E, \\ I_\psi &= I_{\psi,0}. \end{aligned}$$

Clearly, $I_{\psi,\tau} \in C^2(E, \mathbf{R})$.

THEOREM 0.2. *Suppose that $K(x) \in L^\infty(\mathbf{R}^3)$ satisfies*

1. *For some positive constant $T > 0$, $R > 0$, $K(x_1 + lT, x_2, x_3) = K(x_1, x_2, x_3)$ for any integer l with $|x_1|$, $|x_1 + lT| > R$, $x = (x_1, x_2, x_3) \in \mathbf{R}^3$.*

2. *There exists* $x^* \in \{x=(x_1, x_2, x_3) \mid |x_1| > R\}$, *such that* K *is* C^2 *in a neighborhood of* x^* *with*

$$K(x^*) = \sup_{x \in \mathbf{R}^3,\, |x_1| > R} K(x) > 0,$$

$$\Delta K(x^*) \neq 0.$$

Then for any $m = 2, 3, 4, \ldots$, (2) *has infintely many m-bump solutions in E.*

More precisely, for any $\varepsilon > 0$, $m = 2, 3, 4, \ldots$, *there exists some positive constant* $l^* > R$, *such that, for any integers* $l^{(1)}, \ldots, l^{(k)}$ *satisfying* $2 \leqslant k \leqslant m$, $\min_{1 \leqslant i \leqslant k} |l^{(i)}|$, $\min_{i \neq j} |l^{(i)} - l^{(j)}| \geqslant l^*$, *there is at least one solution u of* (2) *in* $V(k, \varepsilon, B_\varepsilon(x^{(1)}), \ldots, B_\varepsilon(x^{(k)}))$ *with*

$$kc - \varepsilon \leqslant I_K(u) \leqslant kc + \varepsilon,$$

where $x^{(i)} = x^* + (l^{(i)}T, 0, 0)$, $c = (1/3)(K(x^*))^{-1/2} (S_3)^3$ *and* $V(k, \varepsilon, B_\varepsilon(x^{(1)}), \ldots, B_\varepsilon(x^{(k)}))$ *is some subset of E defined in Appendix A.*

Remark 0.4. $u \in V(k, \varepsilon, B_\varepsilon(x^{(1)}), \ldots, B_\varepsilon(x^{(k)}))$ implies that u has most of its mass concentrated in $B_\varepsilon(x^{(1)}), \ldots, B_\varepsilon(x^{(k)})$. In particular, if $(l^{(1)}, \ldots, l^{(k)}) \neq (\tilde{l}^{(1)}, \ldots, \tilde{l}^{(k)})$, the corresponding solutions u and $\tilde{u}$ are different.

Remark 0.5. Let $K \in C^2(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$ satisfy the hypotheses of Theorem 0.2. Then the conclusion of Theorem 0.2 holds not only for K, but also for any functions in a suitably small C^2 neighborhood of K. This can be seen easily from the proof.

THEOREM 0.3. *Suppose that* $K(x) \in L^\infty(\mathbf{R}^n)$ $(n = 3, 4)$ *satisfies*

(i) *For some positive constant* $T > 0$, $K(x_1 + lT, x_2, \ldots, x_n) = K(x_1, x_2, \ldots, x_n)$ *for any integer l and* $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$.

(ii) *There exist some positive constants* K_1, K_2, $\delta > 0$, *a bounded open set* $O \subset \mathbf{R}^n$, $x^* \in O$, *such that*

$$\begin{aligned} &K \in C^1(\bar{O}) \qquad \text{for} \quad n = 3 \\ &K \in C^2(\bar{O}) \qquad \text{for} \quad n = 4 \end{aligned} \tag{4}$$

$$K_1 \leqslant K(x) \leqslant K_2 \qquad x \in \bar{O}, \tag{5}$$

$$K(x^*) = K_{\max} \equiv \sup_{x \in \mathbf{R}^n} K(x) \geqslant \max_{x \in \partial O} K(x) + \delta, \tag{6}$$

$$x \in O,\ K_{\max} - \delta \leqslant K(x) \leqslant K_{\max},\ \nabla K(x) = 0 \textit{ implies } \Delta K(x) \geqslant 0 \qquad \textit{if } n = 4. \tag{7}$$

Then for any $\varepsilon > 0$, (2) *has infinitely many solutions in E satisfying*

$$c \leqslant I_K(u) \leqslant c + \varepsilon \qquad \text{or} \qquad 2c - \varepsilon \leqslant I_K(u) \leqslant 2c + \varepsilon \tag{8}$$

and

$$\sup\{\|u\|_{L^\infty} \mid I'_K(u) = 0, u > 0, u \in E, u \text{ satisfies } (8)\} = \infty, \tag{9}$$

where

$$c = \frac{1}{n}(K(x^*))^{(2-n)/2}(S_n)^n.$$

More precisely, for any $\varepsilon > 0$, *there exists* $l^* > 0$, *such that, for any integers* $l^{(1)}$, $l^{(2)}$ *satisfying* $|l^{(1)} - l^{(2)}| \geqslant l^*$, *there is at least one solution u of* (2) *in* $V(1, \varepsilon, O) \cup V(2, \varepsilon, O_l^{(1)}, O_l^{(2)}, K)$ *with* (8), *where*

$$O_l^{(1)} = O + (l^{(1)}T, 0, ..., 0),\ O_l^{(2)} = O + (l^{(2)}T, 0, ..., 0),$$

and $V(1, \varepsilon, O)$, $V(2, \varepsilon, O_l^{(1)}, O_l^{(2)}, K)$ *are some subsets of E defined in Section* 1 *and Appendix A.*

One can easily derive from Theorem 0.3 the following corollary,

COROLLARY 0.1. *Suppose that* $K(x) \in C^1(\mathbf{R}^3)$ *or* $C^2(\mathbf{R}^4)$ *satisfies* (i) *and*

(ii)′ $K_{\max} \equiv \max_{x \in \mathbf{R}^n} K(x) > 0$ *is achieved and the set* $K^{-1}(K_{\max}) \equiv \{x \in \mathbf{R}^n \mid K(x) = K_{\max}\}$ *has at least one bounded connected component. If* $n = 4$, *we further require that* $\Delta K \geqslant 0$ *on critical points of K in the bounded component.*

Then for any $\varepsilon > 0$, (2) $(n = 3, 4)$ *has infinitely many solutions in E satisfying* (8) *and* (9).

Remark 0.6. Corollary 0.1 is not true any more if one drops the hypothesis (ii)′. See [Li1] for the examples.

Theorems 0.1, 0.2, and 0.3, and some other corollaries are derived in Section 5 from Theorem 5.1, a more general result on (2).

To derive Theorem 5.1, we first study a compactified problem (Theorem 3.1) in Section 1–4. Then we derive Theorem 5.1 by using Theorem 3.1 and some recent results of Schoen [Sc2]. Theorem 3.1 holds for all dimensions $n \geqslant 3$. The dimension restriction $n = 3, 4$ comes in only when we do the blow up analysis to derive Theorem 5.1.

To prove Theorem 3.1, we first establish some a priori estimates in Section 1 (Propositions 1.1–1.4). Next we adopt and modify a minimax procedure as in Coti Zelati and Rabinowitz [CR1, CR2]. The minimax procedure mentioned above was introduced first in study's of the existence of homoclinic orbits to Hamiltonian systems. See [Se] for a closely related minimax procedure introduced earlier by Séré, as well as a related and even earlier paper by Coti Zelati, Ekeland, and Séré [CES].

In Appendix A (Proposition A.1) we rule out, for $3 \leqslant n \leqslant 6$, the possibility of "one point blow up" near any critical point of K with $\Delta K \neq 0$. This is needed in establishing Theorem 0.2. Using Proposition A.1 together with some recent works of Schoen and Zhang (see [Z], [SZ]), we immediately obtain the following result.

THEOREM 0.4. *Let $K \in C^2(\mathbf{S}^3)$ satisfy, for some constants K_1, $d > 0$, that*

$$K_1 \leqslant K(x), \qquad \forall x \in \mathbf{S}^3,$$

$$\min_{\{x \in \mathbf{S}^3, |\nabla K(x)| \leqslant d\}} |\Delta_{g_0} K(x)| \geqslant d.$$

Then for all positive solutions u of (1), $0 < \alpha < 1$,

$$\|u\|_{C^{3,\alpha}(\mathbf{S}^3)}, \|u^{-1}\|_{C^{3,\alpha}(\mathbf{S}^3)} \leqslant C,$$

where C depends only on K_1, d, $\|K\|_{C^2(\mathbf{S}^3)}$, α and the modulo of continuity of $\nabla^2_{g_0} K$ on $\mathbf{S}^3$.

In [HL], Han and the present author have given some existence results as an application of such a priori estimates.

The author has been informed by Professor Chang that results similar to Theorem 0.4 have been obtained in [CGY] by a somewhat different approach.

1. Some a Priori Estimates

In this section we present some a priori estimates of solutions to the equation

$$-\Delta u = K(x) |u|^{(n+2)/(n-2)-\tau} u, \qquad |x| \geqslant 1,$$

for $\tau \geqslant 0$ small.

PROPOSITION 1.1. *Suppose that* $K \in L^\infty(\mathbf{R}^n \setminus B_1(0))$ $(n \geqslant 3)$ *with* $\|K\|_{L^\infty} \leqslant A_0$ *for some constant* A_0. *Then there exists some positive constant* $\mu_1 = \mu_1(n, A_0) > 0$, $C(n, A_0) > 0$, *such that for any solutions u of*

$$-\Delta u = K(x)\, |u|^{4/(n-2)}\, u, \qquad |x| \geqslant 1,$$

with $\nabla u \in L^2(\mathbf{R}^n \setminus B_1(0))$, $u \in L^{2n/(n-2)}(\mathbf{R}^n \setminus B_1(0))$, *and*

$$\int_{|x| \geqslant 1} |\nabla u|^2 \leqslant \mu_1,$$

we have

$$|u(x)| \leqslant C(n, A_0)\, |x|^{2-n}, \qquad \forall\, |x| \geqslant 2.$$

Proof. It follows from the proof of Proposition 3.1 in [Li1] that

$$\int_{|x| \geqslant 1} |u|^{2n/(n-2)} \leqslant C_0(n) \int |\nabla u|^2 \leqslant C_0(n)\, \mu_1.$$

Throughout the paper, $C_0 = C_0(n) > 1$ denotes some universal constant whose value may change from line to line.

We perform a Kelvin transformation on u by letting

$$y = \frac{x}{|x|^2}, \qquad |x| \geqslant 1,$$

$$v(y) = \frac{1}{|y|^{n-2}}\, u\left(\frac{y}{|y|^2}\right).$$

It is not difficult to see that $v(y)$ satisfies

$$-\Delta v(y) = K\left(\frac{y}{|y|^2}\right) |v(y)|^{4/(n-2)}\, v(y), \qquad 0 < |y| \leqslant 1.$$

Furthermore,

$$\int_{|y| \leqslant 1} |\nabla v|^2 + \int_{|y| \leqslant 1} |v|^{2n/(n-2)} \leqslant C_0(n)\, \mu_1.$$

A theorem by Brezis and Kato [BK] implies that $v \in L^\infty(B_{9/10}(0))$. It follows then from the standard elliptic theories that $v \in C^{1,\alpha}$. To prove Proposition 1.1, we only have to prove

$$|v|_{L^\infty(B_{1/2}(0))} \leqslant C(n, A_0). \tag{10}$$

We prove (10) by contradiction argument. If (10) is not true, then there exist some functions $K_j(x)$, $u_j(x)$, $j = 1, 2, 3, \ldots$, such that, for $|x| \geqslant 1$,

$$\|K_j\|_{L^\infty} \leqslant A_0,$$

$$-\Delta u_j = K_j(x)\, |u_j|^{4/(n-2)}\, u_j,$$

$$\int_{|y| \leqslant 1} |\nabla v_j|^2 + \int_{|y| \leqslant 1} |v_j|^{2n/(n-2)} \leqslant C_0(n)\, \mu_1,$$

$$|v_j|_{L^\infty(B_{1/2}(0))} \geqslant j,$$

where v_j is obtained as before by a Kelvin transformation on u_j.

Let $y_j \in B_{9/10}(0)$ be the maximum of the quantity

$$(0.9 - |y_j|)^{(n-2)/2}\, |v_j(y_j)| = \max_{|y| \leqslant 0.9} (0.9 - |y|)^{(n-2)/2}\, |v_j(y)|.$$

Let $\sigma_j = \frac{1}{2}(0.9 - |y_j|) > 0$; clearly we have

$$\begin{aligned}
\sigma_j^{(n-2)/2} \sup_{B_{\sigma_j}(y_j)} |v_j| &\geqslant (\tfrac{1}{3})^{(n-2)/2} (0.9 - |y_j|)^{(n-2)/2}\, |v_j(y_j)| \\
&= (\tfrac{1}{3})^{(n-2)/2} \max_{|y| \leqslant 0.9} (0.9 - |y|)^{(n-2)/2}\, |v_j(y)| \\
&\geqslant (\tfrac{1}{3})^{(n-2)/2} \max_{|y| \leqslant 1/2} (0.9 - |y|)^{(n-2)/2}\, |v_j(y)| \\
&\geqslant (\tfrac{1}{3})^{(n-2)/2} (0.9 - \tfrac{1}{2})^{(n-2)/2}\, j \\
&\to \infty
\end{aligned}$$

$$\begin{aligned}
|v_j(y_j)| &= (0.9 - |y_j|)^{(2-n)/2} \max_{|y| \leqslant 0.9} (0.9 - |y|)^{(n-2)/2}\, |v_j(y)| \\
&\geqslant (0.9 - |y_j|)^{(2-n)/2} \max_{|y - y_j| \leqslant \sigma_j} (0.9 - |y|)^{(n-2)/2}\, |v_j(y)| \\
&\geqslant (2\sigma_j)^{(2-n)/2} (\sigma_j)^{(n-2)/2} \max_{|y - y_j| \leqslant \sigma_j} |v_j(y)| \\
&\geqslant (\tfrac{1}{2})^{(n-2)/2} \max_{|y - y_j| \leqslant \sigma_j} |v_j(y)|.
\end{aligned}$$

Therefore we have

$$|y_j| < 0.9,$$

$$(\sigma_j)^{(n-2)/2} \max_{|y - y_j| \leqslant \sigma_j} |v_j(y)| \to \infty,$$

$$|v_j(y_j)| \geqslant (\tfrac{1}{2})^{(n-2)/2} \max_{|y - y_j| \leqslant \sigma_j} |v_j(y)|.$$

Without loss of generality, we assume that

$$v_j(y_j) > 0.$$

Let

$$w_j(z) = \frac{1}{v_j(y_j)} v_j\left(\left(\frac{1}{v_j(y_j)}\right)^{2/(n-2)} z + y_j\right), \ |z| < v_j(y_j)^{2/(n-2)} \sigma_j \to \infty.$$

Clearly,

$$\int_{|z| \leqslant v_j(y_j)^{2/(n-2)}\sigma_j} |\nabla w_j|^2 + |w_j|^{2n/(n-2)} \leqslant C_0(n)\, \mu_1.$$

$$-\Delta w_j(z) = K_j\left(\left(\frac{1}{v_j(y_j)}\right)^{2/(n-2)} z + y_j\right) |w_j(z)|^{4/(n-2)} w_j(z),$$

$$\forall\ |z| < v_j(y_j)^{2/(n-2)} \sigma_j,$$

$$w_j(0) = 1,$$

$$|w_j(z)| \leqslant 2^{(n-2)/2}, \qquad \forall\ |z| \leqslant v_j(y_j)^{2/(n-2)} \sigma_j.$$

By the standard elliptic theories, $\{w_j\}$ is bounded in $W^{2,q}_{\text{loc}}$ $(q > 1)$. Therefore, by passing to a subsequence, we can assume that $w_j \rightharpoonup w$ in $W^{2,q}_{\text{loc}}$. Let $\bar{K}$ be the weak * limit of $\{K_j((1/v_j(y_j))^{2/(n-2)} z + y_j)\}$ in $L^{\infty}_{\text{loc}}(\mathbf{R}^n)$. One sees easily that $\|\bar{K}\|_{L^\infty} \leqslant A_0$. Sending $j \to \infty$, we obtain

$$-\Delta w = \bar{K}\, |w|^{4/(n-2)} w \qquad \text{in} \quad \mathbf{R}^n.$$

We see easily that $w(0) = 1$ and

$$\int_{\mathbf{R}^n} |\nabla w|^2 + |w|^{2n/(n-2)} \leqslant C_0(n)\, \mu_1. \tag{11}$$

Multiplying (11) by w and integrating by parts, we have

$$\int_{\mathbf{R}^n} |\nabla w|^2 = \int_{\mathbf{R}^n} \bar{K}\, |w|^{2n/(n-2)}$$

$$\leqslant A_0 \left(\int_{\mathbf{R}^n} |\nabla w|^2\right)^{n/(n-2)} \left(\frac{1}{S_n}\right)^{2n/(n-2)}.$$

Therefore we have

$$1 \leqslant A_0 \left(\int_{\mathbf{R}^n} |\nabla w|^2\right)^{2/(n-2)} \left(\frac{1}{S_n}\right)^{2n/(n-2)}$$

$$\leqslant A_0 (C_0(n)\, \mu_1)^{2/(n-2)} \left(\frac{1}{S_n}\right)^{2n/(n-2)}$$

This is a contradiction if we choose $\mu_1 = \mu_1(n, A_0)$ to satisfy

$$A_0(C_0(n)\,\mu_1(n, A_0))^{2/(n-2)} \left(\frac{1}{S_n}\right)^{2n/(n-2)} < 1. \quad \blacksquare$$

We can deduce from Proposition 1.1 the following result.

PROPOSITION 1.2. *Let $\mu_1 = \mu_1(n, A_0) > 0$, $C(n, A_0) > 0$ be the constants in Proposition 1.1, then for any $2 < l_1 < l_2 < \infty$, there exists a positive constant, $R_1 = R_1(n, A_0, \mu_1, l_1, l_2) > l_2$, such that, for any $K \in L^\infty(B_{R_1}(0)\backslash B_1(0))$ with $\|K\|_{L^\infty} \leqslant A_0$ and any solutions u of*

$$-\Delta u = K(x)\,|u|^{4/(n-2)} u, \qquad 1 < |x| < R_1,$$

with

$$\int_{1<|x|<R_1} |\nabla u|^2 + \int_{1<|x|<R_1} |u|^{2n/(n-2)} \leqslant \mu_1,$$

we have

$$|u(x)| \leqslant \frac{2C(n, A_0)}{|x|^{n-2}}, \qquad \forall l_1 \leqslant |x| \leqslant l_2.$$

Proof. Suppose the contrary, that for $R_j = l_2 + j$, 3, 4, 5, ..., there exists u_j such that, for $1 < |x| < R_j$, we have

$$-\Delta u_j = K_j(x)\,|u|_j^{4/(n-2)} u_j,$$

$$\|K_j\|_{L^\infty} \leqslant A_0,$$

$$\int_{1<|x|<R_j} |\nabla u_j|^2 + \int_{1<|x|<R_j} |u_j|^{2n/(n-2)} \leqslant \mu_1,$$

but

$$\max_{l_1 \leqslant |x| \leqslant l_2} |x|^{n-2}\,|u_j(x)| > 2C(n, A_0).$$

Arguing as in the proof of Proposition 1.1, we know that for any $0 < \mu < 1$, $\|u_j\|_{L^\infty(B_{R_j/2}(0)\backslash B_{1+\mu}(0))}$ is bounded by a constant independent of j. Let u be the $W^{2,q}_{\mathrm{loc}}$ weak limit of u_j (passing to a subsequence); we know that

$$-\Delta u = \bar{K}\,|u|^{4/(n-2)} u,$$

$$\max_{l_1 \leqslant |x| \leqslant l_2} |x|^{n-2}\,|u(x)| \geqslant 2C(n, A_0), \tag{12}$$

where $\bar{K}(x)$ is the weak * limit of $K_j(x)$ in L^∞, $\|\bar{K}\|_{L^\infty} \leqslant A_0$.

On the other hand, it follows from Proposition 1.1 that

$$|u(x)| \leqslant \frac{C(n, A_0)}{|x|^{n-2}}, \qquad \forall\ |x| \geqslant 2.$$

This contradicts (12). ▮

PROPOSITION 1.3. *Suppose that* $K \in L^\infty(B_{l_2}(0)\backslash B_{l_1}(0))$, $l_1 > 1$, $l_2 > 100 l_1$, *u is a solution of*

$$-\Delta u(z) = K(z)\,|u(z)|^{4/(n-2)}\,u, \qquad l_1 \leqslant |z| \leqslant l_2,$$

satisfying

$$|u(z)| \leqslant \frac{A}{|z|^{n-2}}, \qquad l_1 \leqslant |z| \leqslant l_2.$$

Then we have

$$|\nabla u(z)| \leqslant \frac{C(A, \|K\|_{L^\infty}, n)}{|z|^{n-1}}, \qquad 4l_1 \leqslant |z| \leqslant \frac{1}{4} l_2,$$

where $C(A, \|K\|_{L^\infty}, n)$ *is some positive constant depending only on* A, $\|K\|_{L^\infty}$, n.

Proof. For any $r \in (4l_1, \frac{1}{4} l_2)$, we have

$$-\Delta u(z) = K(z)\,|u(z)|^{4/(n-2)}\,u, \qquad \frac{r}{2} \leqslant |z| \leqslant 2r,$$

$$|u(z)| \leqslant \frac{A}{|z|^{n-2}} \leqslant \left(\frac{2}{r}\right)^{n-2} A, \qquad \frac{r}{2} \leqslant |z| \leqslant 2r.$$

Let

$$v(x) = r^{n-2} u(rx), \qquad \tfrac{1}{2} \leqslant |x| \leqslant 2,$$

we have

$$-\Delta v(x) = \frac{1}{r^2} K(rx)\,|v(x)|^{4/(n-2)}\,v.$$

Using the fact that $|v(x)| \leqslant 2^{n-2} A$ and $|-\Delta v(x)| \leqslant \|K\|_{L^\infty} 2^{n+2} A^{(n+2)/(n-2)}$ in the annulus $\{x \in \mathbf{R}^n \mid \frac{1}{2} \leqslant x \leqslant 2\}$, we deduce from the standard elliptic theories that

$$|\nabla v(x)| \leqslant C(A, \|K\|_{L^\infty}, n), \qquad |x| = 1,$$

namely,

$$|\nabla u(z)| \leqslant \frac{C(A, \|K\|_{L^\infty}, n)}{|z|^{n-1}}, \qquad |z| = r. \quad \blacksquare$$

PROPOSITION 1.4. *For any* $2 < l_1 < l_2 < \infty$, $A_0 > 0$, *there exists* $0 < \mu_2 = \mu_2(n, A_0) \leqslant \mu_1(n, A_0)$, $\bar{\tau} = \bar{\tau}(l_1, l_2, n, A_0) > 0$, *such that, for any* $0 \leqslant \tau \leqslant \bar{\tau}$, $K \in L^\infty(B_{R_1}(0) \backslash B_1(0))$ *with* $\|K\|_{L^\infty} \leqslant A_0$ *and any solutions u of*

$$-\Delta u = K(x)\, |u|^{4/(n-2)-\tau}\, u, \qquad 1 < |x| < 2R_1,$$

with

$$\int_{1 < |x| < 2R_1} |\nabla u|^2 + \int_{1 < |x| < 2R_1} |u|^{2n/(n-2)} \leqslant \mu_2,$$

we have

$$|u(x)| \leqslant \frac{3C(n, A_0)}{|x|^{n-2}}, \qquad \forall l_1 \leqslant |x| \leqslant l_2,$$

where R_1, $C(n, A_0)$ *are the constants in Proposition* 1.2.

Proof. Suppose Proposition 1.4 is false, there exists a sequence $\tau_j \to 0$, and u_{τ_j} satisfying

$$-\Delta u_{\tau_j} = K(x)\, |u_{\tau_j}|^{4/(n-2)} u_{\tau_j}, \qquad 1 < |x| < 2R_1$$

with

$$\int_{1 < |x| < 2R_1} |\nabla u_{\tau_j}|^2 + \int_{1 < |x| < 2R_1} |u_{\tau_j}|^{2n/(n-2)} \leqslant \mu_2,$$

and

$$\max_{l_1 \leqslant |x| \leqslant l_2} \{ |x|^{n-2}\, |u_{\tau_j}(x)| \} \geqslant 3C(n, A_0). \tag{13}$$

Choosing $\mu_2 \in (0, \mu_1)$ to be small, we use an argument similar to that used in the proof of Proposition 1.1 to obtain (by passing to a subsequence)

$$u_{\tau_j} \to u \qquad \text{in } C^{1,\alpha}_{\text{loc}}(\{x \in \mathbf{R}^n \mid 1 < |x| < 2R_1\}).$$

It follows that u satisfies all the hypotheses of Proposition 1.2 and therefore satisfies

$$\max_{l_1 \leqslant |x| \leqslant l_2} \{ |x|^{n-2}\, |u(x)| \} \leqslant 2C(n, A_0). \tag{14}$$

Equations (13) and (14) contradict each other after sending j to ∞. $\blacksquare$

2. A Minimization Problem

Let $x_1, x_2 \in \mathbf{R}^n$, $|x_1 - x_2| \geqslant 10$, $\Omega = \mathbf{R}^n \setminus \{B_1(x_1) \cup B_1(x_2)\}$. We define E_Ω by taking the closure of $\{u \mid u \in C^\infty(\bar{\Omega}), \operatorname{supp} u \text{ is compact}\}$ under the norm

$$\|u\|_{E_\Omega} = \left(\int_\Omega |\nabla u|^2\right)^{1/2} + \left(\int_\Omega |u|^{2n/(n-2)}\right)^{(n-2)/2n}.$$

Clearly, E_Ω is a Banach space.

It has been proved in [Li1] (see Proposition 3.1 there) that $u \in E_\Omega$ iff there exists $\tilde{u} \in E$, such that $u = \tilde{u}|_\Omega$ and $\forall u \in E_\Omega$,

$$\left(\int_\Omega |u|^{2n/(n-2)}\right)^{(n-2)/2n} \leqslant C_0(n) \left(\int_\Omega |\nabla u|^2\right)^{1/2}. \tag{15}$$

In the following we study a minimization problem. Let Ω be as before and let $K \in L^\infty(\Omega)$ satisfy

$$\|K\|_{L^\infty} \leqslant A_0,$$

for some constant A_0.

We define, for $0 \leqslant \tau \leqslant 2/(n-2)$, a functional on E_Ω by

$$I_\tau(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \left(\frac{2n}{n-2} - \tau\right)^{-1} \int_\Omega K \left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau} \times |u|^{(2n/(n-2))-\tau}, \qquad u \in E_\Omega.$$

It follows from Hölder inequalities and (15) that for any $u \in E_\Omega$ we have

$$\left|I_\tau(u) - \tfrac{1}{2}\int_\Omega |\nabla u|^2\right| \leqslant A_0 C_0(n)\, \|u\|_{L^{2n/(n-2)}(\Omega)}^{(2n/(n-2))-\tau}$$
$$\leqslant A_0 C_0(n) \left(\int_\Omega |\nabla u|^2\right)^{(1/2)(2n/(n-2)-\tau)}. \tag{16}$$

Proposition 2.1. *There exist some constants, $0 < r_0 = r_0(n, A_0) < 1$, $C_3 = C_3(n) > 1$, such that, for any x_1, $x_2 \in \mathbf{R}^n$, $|x_1 - x_2| \geqslant 10$, $\varphi \in H^{1/2}(\partial(B_1(x_1) \cup B_1(x_2)))$ with $r = \|\varphi\|_{H^{1/2}(\partial(B_1(x_1) \cup B_1(x_2)))} \leqslant r_0$, the following minimum is achieved:*

$$\min\left\{I_\tau(u) \mid u \in E_\Omega,\ u|_{\partial(B_1(x_1) \cup B_1(x_2))} = \varphi,\ \int_\Omega |\nabla u|^2 \leqslant C_3(n)\, r_0^2\right\}.$$

The minimizer is unique (denoted u_φ) and satisfies $\int_\Omega |\nabla u_\varphi|^2 \leqslant \frac{1}{2} C_3(n)\, r^2$. Furthermore, the map $\varphi \to u_\varphi$ is continuous from $H^{1/2}(\partial(B_1(x_1) \cup B_1(x_2)))$ to E_Ω.

Proof. It is well known that there exists constant $C_3 = C_3(n) > 0$ and $\Phi \in E_\Omega$, such that

$$\int_\Omega |\nabla \Phi|^2 \leqslant \frac{C_3(n)}{8} r^2$$

and

$$\Phi|_{\partial(B_1(x_1) \cup B_1(x_2))} = \varphi.$$

The value of C_3 had been chosen now. As for the value of $r_0(n, A_0)$, it is determined in the following.

First we require $r_0(n, A_0) > 0$ to be small enough that

$$\begin{aligned} I_\tau(\Phi) &\leqslant \frac{1}{2} \int_\Omega |\nabla \Phi|^2 + A_0 C_0(n) \left(\int_\Omega |\nabla \Phi|^2 \right)^{(1/2)(2n/(n-2)-\tau)} \\ &\leqslant \frac{4}{5} \int_\Omega |\nabla \Phi|^2 \\ &\leqslant \frac{C_3(n)}{10} r^2. \end{aligned}$$

Then we choose $r_0(n, A_0) > 0$ to further satisfy

$$A_0 C_0(n) (20 r_0^2 C_3(n))^{(1/2)(4/(n-2)-\tau)} \leqslant \tfrac{1}{4}. \tag{17}$$

We observe first that for any u in $\{u \in E_\Omega \mid C_3 r^2/2 \leqslant \int_\Omega |\nabla u|^2 \leqslant 20 C_3 r_0^2\}$ we can derive from (16) and (17) that

$$\begin{aligned} I_\tau(u) &\geqslant \frac{1}{4} \int_\Omega |\nabla u|^2 \\ &\geqslant \frac{1}{4} \left(\frac{C_3 r^2}{2} \right) \\ &> I_\tau(\Phi). \end{aligned}$$

Next we prove the existence of the minimizer.

Write $u = v + \Phi$, $v|_{\partial\Omega} = 0$, $J_\tau(v) = I_\tau(u) = I_\tau(v + \Phi)$.

We only need to minimize $J_\tau(v)$ for $\int_\Omega |\nabla v|^2 \leqslant 2C_3(n)\, r_0^2$ due to the above argument. It is easy to see that if r_0 is small enough then J_τ is strictly convex in the ball $\{v \in E_\Omega \mid v|_{\partial\Omega} = 0, \int_\Omega |\nabla v|^2 \leqslant 2C_3(n)\, r_0^2\}$. Therefore it is standard to conclude the existence of a unique local minimizer v_φ. Set $u_\varphi = v_\varphi + \Phi$, then u_φ is a local minimizer. As discussed above, u_φ satisfies $\int_\Omega |\nabla u_\varphi|^2 \leqslant (C_3(n)/2)\, r^2$. The uniqueness and the continuity of the map $\varphi \to u_\varphi$ follows from the strict local convexity of J_τ. An alternative proof can be made by integration by parts. See [Li1] for the details.

3. Uniform Lower Bounds of the Gradient Vectors in Certain Regions of E

We introduce some notation which is used throughout the paper.

From now on, $B_r(x) \subset \mathbf{R}^n$ denotes a ball of radius $r > 0$ centered at $x \in \mathbf{R}^n$. Let $\{K_l(x)\}$ be a sequence of functions in $L^\infty(\mathbf{R}^n)$, satisfying the following conditions.

(i) There exists some positive constant $A_1 > 1$, such that, for any $l = 1, 2, 3, \ldots$,

$$|K_l(x)| \leqslant A_1, \qquad \forall x \in \mathbf{R}^n. \tag{18}$$

(ii) For some integer $m \geqslant 2$, there exist $x_l^{(i)} \in \mathbf{R}^n$, $1 \leqslant i \leqslant m$, $R_l \leqslant \frac{1}{2} \min_{i \neq j} |x_l^{(i)} - x_l^{(j)}|$, such that K_l is continuous near $x_l^{(i)}$.

$$\lim_{l \to \infty} R_l = \infty, \tag{19}$$

$$K_l(x_l^{(i)}) = \max_{x \in B_{R_l}(x_l^{(i)})} K_l(x), \qquad 1 \leqslant i \leqslant m, \tag{20}$$

$$\lim_{l \to \infty} K_l(x_l^{(i)}) = a^{(i)}, \qquad 1 \leqslant i \leqslant m, \tag{21}$$

$$K_\infty^{(i)}(x) := (\text{weak } *) \lim_{l \to \infty} K_l(x + x_l^{(i)}), \qquad 1 \leqslant i \leqslant m. \tag{22}$$

(iii) There exist some positive constants A_2, $A_3 > 1$, δ_0, $\delta_1 > 0$, and some bounded open sets $O_l^{(1)}, \ldots, O_l^{(m)} \subset \mathbf{R}^n$, such that, if we define for $1 \leqslant i \leqslant m$,

$$\tilde{O}_l^{(i)} = \{x \in \mathbf{R}^n \mid \operatorname{dist}(x, O_l^{(i)}) < \delta_0\},$$

$$O_l = \bigcup_{i=1}^m O_l^{(i)}, \qquad \tilde{O}_l = \bigcup_{i=1}^m \tilde{O}_l^{(i)},$$

we have

$$\begin{aligned} &x_l^{(i)} \in O_l^{(i)}, \operatorname{diam}(O_l^{(i)}) < \frac{1}{10} R_l, \\ &K_l \in C^1\left(\tilde{O}_l, \left[\frac{1}{A_2}, A_2\right]\right), \end{aligned} \tag{23}$$

$$K_l(x_l^{(i)}) \geqslant \max_{x \in \partial O_l^{(i)}} K_l(x) + \delta_1, \tag{24}$$

$$\max_{x \in \bar{O}_l} |\nabla K_l(x)| \leqslant A_3. \tag{25}$$

For $\varepsilon > 0$, we define $V_l(m, \varepsilon) = V(m, \varepsilon, O_l^{(1)}, ..., O_l^{(m)}, K_l)$ as in Appendix A. For the reader's convenience, we repeat the definition here for $m = 2$.

$u \in V_l(2, \varepsilon)$ if $u \in E$ and there exist $\alpha_1, \alpha_2 \in \mathbf{R}$, $x = (x_1, x_2) \in O_l^{(1)} \times O_l^{(2)}$, $\lambda = (\lambda_1, \lambda_2)$, such that

$$\lambda_1, \lambda_2 > \frac{1}{\varepsilon},$$

$$|\alpha_i - K_l(x_i)^{(2-n)/4}| < \varepsilon, \qquad i = 1, 2,$$

$$\|u - \varphi(\alpha, x, \lambda)\| < \varepsilon,$$

where

$$\varphi(\alpha, x, \lambda) = \alpha_1 \delta(x_1, \lambda_1) + \alpha_2 \delta(x_2, \lambda_2),$$

$$\delta(x_i, \lambda_i)(y) = (n(n-2))^{(n-2)/4} \left(\frac{\lambda_i}{1 + \lambda_i^2 |y - x_i|^2} \right)^{(n-2)/2}.$$

It is well known that for any $x_i \in \mathbf{R}^n$, $\lambda_i > 0$,

$$\delta(x_i, \lambda_i) \in E,$$

$$-\Delta\delta(x_i, \lambda_i) = \delta(x_i, \lambda_i)^{(n+2)/(n-2)} \text{ in } \mathbf{R}^n,$$

$$\|\delta(x_i, \lambda_i)\| = \|\delta(0, 1)\|,$$

$$S_n = \left(\int |\nabla\delta(0, 1)|^2 \right)^{1/n}.$$

Let $\bar{\tau}_l > 0$ be a sequence satisfying

$$\lim_{l \to \infty} \bar{\tau}_l = 0, \qquad \lim_{l \to \infty} (|x_l^{(1)}| + |x_l^{(2)}|)^{\bar{\tau}_l} = 1. \tag{26}$$

PROPOSITION 3.1. *There exists $\varepsilon_0 \in (0, 1)$ depending only on A_1, A_2, A_3, n, δ_0, and m, but independent of l, such that, for any $0 < \varepsilon \leqslant \varepsilon_0$, $u \in V_l(2, \varepsilon)$,*

$$\min_{(\alpha, x, \lambda) \in B_{4\varepsilon}} \|u - \varphi(\alpha, x, \lambda)\| \tag{27}$$

has a unique solution modulo permutation, and it is achieved in $B_{2\varepsilon}$ *with* $x_1 \in \tilde{O}_l^{(1)}$, $x_2 \in \tilde{O}_l^{(2)}$ *for large* l, *where*

$$B_\varepsilon = \Big\{ (\alpha, x, \lambda) \mid x = (x_1, x_2), x_1, x_2 \in \mathbf{R}^n,$$

$$\frac{1}{2A_2^{(n-2)/4}} \leqslant \alpha_1, \alpha_2 \leqslant 2A_2^{(n-2)/4}, \lambda = (\lambda_1, \lambda_2), \lambda_1, \lambda_2 \geqslant \frac{1}{\varepsilon} \Big\}.$$

Proposition 3.1 follows immediately from Lemma A.1 in Appendix A.

For $u \in V_l(2, \varepsilon)$, $0 < \varepsilon \leqslant \varepsilon_0$, we always make the minimization (27) and denote u as

$$u = \alpha_1^l \, \delta(x_1^l, \lambda_1^l) + \alpha_2^l \, \delta(x_2^l, \lambda_2^l) + v^l,$$

where $\alpha_i = \alpha_i^l = \alpha_i^l(u)$, $x_i = x_i^l = x_i^l(u)$, ..., are well defined by the uniqueness of the minimizer in (27).

LEMMA 3.1. *There exists some constant* $A_4 = A_4(\delta_1, A_2) > 1$ *such that, for* $\varepsilon_1 \in (0, \varepsilon_0)$ *small enough and for* l *large enough,* $0 \leqslant \tau \leqslant \bar{\tau}_l$, *any* $u \in V_l(2, \varepsilon_1)$ *with* $x_1(u) \in \tilde{O}_l^{(1)}$, $x_2(u) \in \tilde{O}_l^{(2)}$, *and* $\operatorname{dist}(x_1(u), \partial O_l^{(1)}) < \delta_1/2A_3$ *or* $\operatorname{dist}(x_2(u), \partial O_l^{(2)}) < \delta_1/2A_3$ *satisfies*

$$I_{K_{l,\tau}}(u) \geqslant c^{(1)} + c^{(2)} + \frac{1}{A_4},$$

where

$$c^{(i)} = \frac{1}{n} (a^{(i)})^{(2-n)/2} (S_n)^n. \tag{28}$$

Proof of Lemma 3.1. First we can prove that

$$|\alpha_i - K_l(x_i)^{(2-n)/4}| = \circ_{\varepsilon_1}(1),$$

where $\circ_{\varepsilon_1}(1)$ denotes some quantity which goes to zero as ε_1 goes to zero. See [BC1] for the proof of the above (see also Lemma 4.2 of [Li1]).

Without loss of generality, we assume that for $x_1 = x_1(u)$, $\operatorname{dist}(x_1, \partial O_l^{(1)}) < \delta_1/2A_3$. Then we have the following for $\varepsilon_1 > 0$ small and l large.

$$\begin{aligned} &I_{K_{l,\tau}}(u) \\ &\quad = \sum_{i=1}^{2} I_{K_{l,\tau}}(\alpha_i \, \delta(x_i, \lambda_i)) + \circ_{\varepsilon_1}(1) \\ &\quad = \sum_{i=1}^{2} I_{K_{l,\tau}}(K_l(x_i)^{(2-n)/4} \, \delta(x_i, \lambda_i)) + \circ_{\varepsilon_1}(1) \end{aligned}$$

$$= \sum_{i=1}^{2} \left\{ \frac{1}{2} K_l(x_i)^{(2-n)/2} \int |\nabla\delta(0,1)|^2 - \left(\frac{2n}{n-2} - \tau\right)^{-1} \right.$$

$$\times K_l(x_i)^{-(n/2)+((n-2)/4)\tau} \int K_l \left(\frac{1}{1+|\cdot|^2}\right)^{((n-2)/2)\tau}$$

$$\left. \times \delta(x_i, \delta_i)^{(2n/(n-2))-\tau} \right\} + o_{\varepsilon_1}(1)$$

$$= \sum_{i=1}^{2} \left\{ \frac{1}{2} K_l(x_i)^{(2-n)/2} \int |\nabla\delta(0,1)|^2 - \frac{n-2}{2n} K_l(x_i)^{-n/2} \right.$$

$$\left. \times \int K_l \delta(x_i, \lambda_i)^{2n/(n-2)-\tau} \right\} + o_{\varepsilon_1}(1) + o(1)$$

$$\geqslant \sum_{i=1}^{2} \left\{ \frac{1}{2} K_l(x_i)^{(2-n)/2} \int |\nabla\delta(0,1)|^2 - \frac{n-2}{2n} K_l(x_i)^{-n/2} (n(n-2))^{n/2} \right.$$

$$\left. \times \int K_l \frac{(\lambda_i)^n}{(1+\lambda_i^2 |\cdot - x_i|^2)^{n-((n-2)/2)\tau}} \right\} + o_{\varepsilon_1}(1) + o(1)$$

$$= \sum_{i=1}^{2} \left\{ \frac{1}{2} K_l(x_i)^{(2-n)/2} \int |\nabla\delta(0,1)|^2 - \frac{n-2}{2n} K_l(x_i)^{(2-n)/2} \right.$$

$$\left. \times \int \delta(0,1)^{2n/(n-2)} \right\} + o_{\varepsilon_1}(1) + o(1)$$

$$= \sum_{i=1}^{2} \frac{1}{n} K_l(x_i)^{(2-n)/2} (S_n)^n + o_{\varepsilon_1}(1) + o(1)$$

$$\geqslant \frac{1}{n} (K_l(x_l^{(1)}) - \delta_1/2)^{(2-n)/2} (S_n)^n + \frac{1}{n} K_l(x_l^{(2)})^{(2-n)/2} (S_n)^n + o_{\varepsilon_1}(1) + o(1)$$

$$\geqslant \sum_{i=1}^{2} c^{(i)} + \frac{1}{A_4}.$$

The choice of A_4 and the last two inequalities are evident (see (21), (20), (28), (25), and (24)).

From now on, the value of A_4 is fixed and the value of ε_1 is also fixed.

THEOREM 3.1. *Suppose that $\{K_l\}$ is a sequence of functions in $L^\infty(\mathbf{R}^n)$ satisfying* (i), (ii), *and* (iii) *of Section* 3. *If there exist some bounded open sets $O^{(1)}, ..., O^{(m)} \subset \mathbf{R}^n$ and some positive constants $\delta_2, \delta_3 > 0$, such that, for all $1 \leqslant i \leqslant m$,*

$$\tilde{O}_l^{(i)} - x_l^{(i)} \subset O^{(i)}, \qquad \text{for all } l,$$

$$\{u \mid I'_{K_\infty^{(i)}}(u) = 0, u > 0, u \in E, c^{(i)} \leqslant I_{K_\infty^{(i)}}(u) \leqslant c^{(i)} + \delta_2\}$$

$$\cap V(1, \delta_3, O^{(i)}, K_\infty^{(i)}) = \phi. \tag{29}$$

Then for any $\varepsilon > 0$, *there exists integer* $\bar{l}_{\varepsilon,m} > 0$, *such that, for all* $l \geq \bar{l}_{\varepsilon,m}$, $0 < \tau < \bar{\tau}_l$, *there exists* $u_l \equiv u_{l,\tau} \in V_l(m,\varepsilon)$ *which solves*

$$\begin{aligned} -\Delta u_l &= K_l(x)\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau} u_l^{(n+2)/(n-2)-\tau}, \qquad \text{in } \mathbf{R}^n, \\ u_l &\in E, \\ u_l &> 0. \end{aligned} \tag{30}$$

Furthermore, u_l *satisfies*

$$\sum_{i=1}^{m} c^{(i)} - \varepsilon \leq I_{K_{l,\tau}}(u_l) \leq \sum_{i=1}^{m} c^{(i)} + \varepsilon. \tag{31}$$

Remark 3.1. Equation (29) can be replaced by

$$\sup\{\|u\|_{L^\infty} \mid I'_{K_x^{(i)}}(u) = 0, u > 0, u \in E, c^{(i)} \leq I_{K_x^{(i)}}(u) \leq c^{(i)} + \delta_2\} \leq 1/\delta_3.$$

We prove Theorem 3.1 by contradiction argument and, for simplicity, only for $m = 2$. The changes for $m > 2$ are evident. From now on we suppose the contrary of Theorem 3.1, namely, that for some $\varepsilon^* > 0$, a sequence of $l \to \infty$, $0 < \tau_l < \bar{\tau}_l$ (without loss of generality, we assume that the following statement is true for the whole sequence), such that (30), for $\tau = \tau_l$, has no solution in $V_l(2, \varepsilon^*)$ satisfying (31) with $\varepsilon = \varepsilon^*$. The above is always referred to as "*the contrary of Theorem* 3.1."

We define, for $\varepsilon_2 > 0$, $\tilde{V}_l(2, \varepsilon_2) \subset E$ by the following.

$u \in \tilde{V}_l(2, \varepsilon_2)$ if there exist $\alpha_1, \alpha_2 \in \mathbf{R}$, $x = (x_1, x_2) \in O_l^{(1)} \times O_l^{(2)}$, $\lambda = (\lambda_1, \lambda_2)$, such that

$$\lambda_1, \lambda_2 > \frac{1}{\varepsilon_2},$$

$$|\lambda_i^{\tau_l} - 1| < \varepsilon_2, \qquad i = 1, 2,$$

$$|\alpha_i - K_l(x_i)^{(2-n)/4}| < \varepsilon_2, \qquad i = 1, 2,$$

$$\left\| u - \sum_{i=1}^{2} \alpha_i \left(\frac{\lambda_i}{1 + \lambda_i^2 |\cdot - x_i|^2}\right)^{2/(p_l - 1)} \right\| < \varepsilon_2.$$

Throughout the paper, $p_l = (n+2)/(n-2) - \tau_l$.

LEMMA 3.2. *For* $\varepsilon_2 > 0$ *small enough, depending only on* ε_1, ε^*, n *(but independent of* l*), we have, for* l *large enough,*

$$\tilde{V}_l(2, \varepsilon_2) \subset V_l(2, o_{\varepsilon_2}(1)) \subset V_l(2. \varepsilon_1) \cap V_l(2, \varepsilon^*), \tag{32}$$

where $o_{\varepsilon_2}(1)$ *denotes some quantity which is independent of* l *and tends to zero as* ε_2 *tends to zero.*

Proof of Lemma 3.2. For $u \in \tilde{V}_l(2, \varepsilon_2)$, let α_1, α_2, $x = (x_1, x_2)$, $\lambda = (\lambda_1, \lambda_2)$ be as in the definition of $\tilde{V}_l(2, \varepsilon_2)$, and we can check easily that

$$\left\| u - \sum_{i=1}^{2} \alpha_i \left(\frac{\lambda_i}{1 + \lambda_i^2 |\cdot - x_i|^2} \right)^{(n-2)/2} \right\| \leqslant \circ_{\varepsilon_2}(1) + O(\tau_l).$$

Lemma 3.2 follows immediately.

PROPOSITION 3.2. *Under the hypotheses of Theorem* 3.1 *and the contrary of Theorem* 3.1, *there exists* $\varepsilon_2 \in (0, \min\{\varepsilon_0, \varepsilon_1, \varepsilon^*, \delta_3\})$, $\varepsilon_3 \in (0, \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon^*, \delta_3\})$ *which are independent of* l, *such that* (32) *holds for* ε_2 *and there exists* $\delta_4 = \delta_4(\varepsilon_2, \varepsilon_3) > 0$, $l'_{\varepsilon_2, \varepsilon_3} > 1$, *such that, for any* $l \geqslant l'_{\varepsilon_2, \varepsilon_3}$, $u \in \tilde{V}_l(2, \varepsilon_2) \setminus \tilde{V}_l(2, \varepsilon_2/2)$, $|I_{K_l, \tau_l}(u) - (c^{(1)} + c^{(2)})| < \varepsilon_3$, *we have*

$$\|I'_{K_l, \tau_l}(u)\| \geqslant \delta_4,$$

where I'_{K_l, τ_l} *denotes the Fréchet derivative and as usual we often identify the dual space of* E *as* E *through the inner product.*

Proof. Evidently we have, under the contrary of Theorem 3.1, that for each l,

$$\inf\{ \|I'_{K_l, \tau_l}(u)\| \mid u \in \tilde{V}_l(2, \varepsilon_2) \setminus \tilde{V}_l(2. \varepsilon_2/2), |I_{K_l, \tau_l}(u) - (c^{(1)} + c^{(2)})| < \varepsilon_3\} > 0.$$

We prove Proposition 3.2 by contradiction argument. Suppose the contrary; then no matter how small ε_2, $\varepsilon_3 > 0$ are, there exists a sequence of $l \to \infty$ (without loss of generality we assume the following statement holds for the whole sequence), such that

$$\{u_l\} \in \tilde{V}_l(2, \varepsilon_2) \setminus \tilde{V}_l(2, \varepsilon_2/2), \tag{33}$$

$$|I_{K_l, \tau_l}(u_l) - (c^{(1)} + c^{(2)})| < \varepsilon_3, \tag{34}$$

$$\lim_{l \to \infty} \|I'_{K_l, \tau_l}(u_l)\| = 0. \tag{35}$$

We write

$$u_l = \alpha_1^l\, \delta(x_1^l, \lambda_1^l) + \alpha_2^l\, \delta(x_2^l, \lambda_2^l) + v_l \tag{36}$$

after making the minimization (27). It is elementary to derive for $\varepsilon_2 > 0$ small enough that

$$\frac{1}{\lambda_1^l}, \frac{1}{\lambda_2^l} \leqslant \circ_{\varepsilon_2}(1), \tag{37}$$

$$|\alpha_i^l - K_l(x_i^l)^{(2-n)/4}| \leqslant \circ_{\varepsilon_2}(1), \qquad i=1,2, \tag{38}$$

$$\|v_l\| \leqslant \circ_{\varepsilon_2}(1), \tag{39}$$

$$\operatorname{dist}(x_1^l, O_l^{(1)}), \operatorname{dist}(x_2^l, O_l^{(2)}) \leqslant \circ_{\varepsilon_2}(1). \tag{40}$$

See [BC1] (also [Li1]) for helpful arguments to derive the above.

Claim 1.

$$\lim_{l\to\infty} \lambda_1^l = \lim_{l\to+\infty} \lambda_2^l = \infty, \qquad \text{provided } \varepsilon_2 > 0 \text{ is small enough.}$$

Proof of Claim 1. Suppose the contrary, $\lambda_1^l = \lambda_1 + \circ(1)$ along a subsequence $l \to \infty$. Without loss of generality, we assume the above for the whole sequence.

For $x \in \mathbf{R}^n$, we define a linear isometry $T_x: E \to E$ by

$$(T_x u)(\cdot) = u(\cdot + x).$$

It follows from (36) that

$$T_{x_1^l} u_l = \alpha_1^l\, \delta(0, \lambda_1^l) + \alpha_2^l\, \delta(x_2^l - x_1^l, \lambda_2^l) + T_{x_1^l} v_l.$$

By passing to a sequence, we have

$$\lim_{l\to\infty} \alpha_1^l = \alpha_1 \in [\tfrac{1}{2}(A_2)^{(2-n)/4} - \circ_{\varepsilon_2}(1), 2(A_2)^{(n-2)/4} + \circ_{\varepsilon_2}(1)],$$

$$T_{x_1^l} v_l \rightharpoonup w_0 \text{ weakly in } E.$$

It follows from standard functional analysis arguments that

$$\|w_0\| \leqslant \varliminf_{l\to\infty} \|T_{x_1^l} v_l\| \leqslant \circ_{\varepsilon_2}(1). \tag{41}$$

It follows from (ii) that

$$\lim_{l\to\infty} \|x_2^l - x_1^l\| \geqslant \lim_{l\to\infty} R_l = \infty. \tag{42}$$

Therefore we have the following weak convergence in E,

$$T_{x_1^l} u_l \rightharpoonup w := \alpha_1\, \delta(0, \lambda_1) + w_0. \tag{43}$$

It is obvious that $w \neq 0$ if ε_2 is small enough.

Next we prove that w is a solution of the following equation,

$$-\Delta w = T_\zeta K_\infty^{(1)}(x)\, |w|^{4/(n-2)} w \qquad \text{in } \mathbf{R}^n, \tag{44}$$

where $\zeta \in O^{(1)}$, $\operatorname{dist}(\zeta, \partial O^{(1)}) > \delta_0/2$.

For any test function $\varphi \in C_c^\infty(\mathbf{R}^n)$, it follows from (35) that

$$I'_{K_l,\tau_l}(u_l)(T_{-x_1^l}\varphi) = \circ(1)\,\|T_{-x_1^l}\varphi\|$$
$$= \circ(1)\,\|\varphi\|$$
$$= \circ(1).$$

Therefore, by using (26), (43), (22), and (25), we obtain

$$\circ(1) = \int \nabla u_l \nabla T_{-x_1^l}\varphi - \int K_l \left(\frac{1}{1+|\cdot|^2}\right)^{((n-2)/2)\tau_l} |u_l|^{4/(n-2)-\tau_l} u_l T_{-x_1^l}\varphi$$
$$= \int \nabla(T_{x_1^l}u_l)\nabla\varphi - \int (T_{x_1^l}K_l)\left(\frac{1}{1+|\cdot+x_1^l|^2}\right)^{((n-2)/2)\tau_l}$$
$$\times |T_{x_1^l}u_l|^{(4/(n-2)-\tau_l}(T_{x_1^l}u_l)\varphi$$
$$= \int \nabla w\nabla\varphi - \int T_\zeta K_\infty^{(1)}(x)\,|w|^{4/(n-2)}w\varphi + \circ(1),$$

where $\zeta = \lim_{l\to\infty}(x_1^l - x_l^{(1)})$ along a subsequence.

We have thus established (44).

The positivity of w follows from the following argument.

Let $w = w^+(x) - w^-(x)$, $w^+(x) = \max\{w(x), 0\}$, $w^-(x) = \max\{-w(x), 0\}$. It follows from (43) and (41) that $\int (w^-)^{2n/(n-2)} \leqslant \circ_2(1)$. Multiplying (44) by w^- and integrating by parts, we have

$$\int |\nabla w^-|^2 \leqslant \int T_\zeta K_\infty^{(1)}(x)(w^-)^{2n/(n-2)}$$
$$\leqslant \circ_{\varepsilon_2}(1)\left(\int (w^-)^{2n/(n-2)}\right)^{(n-2)/n}$$
$$\leqslant \circ_{\varepsilon_2}(1)\int |\nabla w^-|^2.$$

If $\varepsilon_2 > 0$ is small enough, we immediately obtain $w^- \equiv 0$; namely, $w \geqslant 0$. It follows from (44) and the strong maximum principle that $w > 0$.

We also need to estimate the value of $I_{T_\zeta K_\infty^{(1)}}(w)$ in order to obtain a contradiction. The estimate we are going to establish is

$$c^{(1)} \leqslant I_{T_\zeta K_\infty^{(1)}}(w) \leqslant c^{(1)} + \circ_{\varepsilon_2}(1), \tag{45}$$

where $\circ_{\varepsilon_2}(1)$ denotes some quantity which goes to zero as $\circ_{\varepsilon_2}(1)$ goes to zero.

The first part of (45) can be proved by multiplying w on (44), integrating by part, using the definition of S_n and the fact that $K_\infty^{(1)} \leqslant a^{(1)}$. For the details please see Lemma 5.1 in [Li1].

It is clear that (see (18))

$$|K_{\infty}^{(1)}(x)| \leqslant A_1, \qquad \forall x \in \mathbf{R}^n.$$

Using (36) and noting the facts

$$\frac{1}{\lambda_2^l} \leqslant \circ_{\varepsilon_2}(1) \leqslant 1, \tag{46}$$

$$\lim_{l \to \infty} |x_1^l - x_2^l| = \infty,$$

$$\|v_l\| \leqslant \circ_{\varepsilon_2}(1),$$

we have, for large l, that

$$\begin{aligned} I_{K_l, \tau_l}(u_l) &= I_{K_l, \tau_l}(\alpha_1^l\, \delta(x_1^l, \lambda_1^l)) + I_{K_l, \tau_l}(\alpha_2^l\, \delta(x_2^l, \lambda_2^l)) + \circ_{\varepsilon_2}(1) \\ &= I_{T_{x_1^l} K_l, \tau_l}(\alpha_1^l\, \delta(0, \lambda_1^l)) + I_{K_l, \tau_l}(\alpha_2^l\, \delta(x_2^l, \lambda_2^l)) + \circ_{\varepsilon_2}(1) + \circ(1) \\ &= I_{T_{x_1^l} K_l, \tau_l}(\alpha_1\, \delta(0, \lambda_1)) + I_{K_l, \tau_l}(\alpha_2^l\, \delta(x_2^l, \lambda_2^l)) + \circ_{\varepsilon_2}(1) + \circ(1) \\ &= I_{T_\zeta K_\infty^{(1)}}(\alpha_1\, \delta(0, \lambda_1)) + I_{K_l, \tau_l}(\alpha_2^l\, \delta(x_2^l, \lambda_2^l)) + \circ_{\varepsilon_2}(1) + \circ(1) \\ &= I_{T_\zeta K_\infty^{(1)}}(w) + I_{K_l, \tau_l}(\alpha_2^l\, \delta(x_2^l, \lambda_2^l)) + \circ_{\varepsilon_2}(1) + \circ(1), \end{aligned}$$

where $\circ(1)$ denotes some quantity which, for fixed $\varepsilon_2, \varepsilon_3$, goes to zero as $l \to \infty$. Therefore

$$I_{T_\zeta K_\infty^{(1)}}(w) = I_{K_l, \tau_l}(u_l) - I_{K_l, \tau_l}(\alpha_2^l\, \delta(x_2^l, \lambda_2^l)) + \circ_{\varepsilon_2}(1) + \circ(1). \tag{47}$$

Using (35) and (36), we have

$$\begin{aligned} \circ(1) &= I'_{K_l, \tau_l}(u_l)(\alpha_2^l\, \delta(x_2^l, \lambda_2^l)) \\ &= I'_{K_l, \tau_l}(\alpha_2^l\, \delta(x_2^l, \lambda_2^l))(\alpha_2^l\, \delta(x_2^l, \lambda_2^l)) + \circ(1); \end{aligned}$$

namely,

$$\begin{aligned} \int |\nabla(\alpha_2^l\, \delta(x_2^l, \lambda_2^l))|^2 &= \int K_l \left(\frac{2}{1 + |\cdot|^2}\right)^{((n-2)/2)\tau_l} (\alpha_2^l\, \delta(x_2^l, \lambda_2^l))^{2n/(n-2) - \tau_l} \\ &\quad + \circ_{\varepsilon_2}(1) + \circ(1). \end{aligned} \tag{48}$$

$$I_{K_l, \tau_l}(\alpha_2^l\, \delta(x_2^l, \lambda_2^l)) = \frac{1}{n} \int |\nabla(\alpha_2^l\, \delta(x_2^l, \lambda_2^l))|^2 + \circ_{\varepsilon_2}(1) + \circ(1). \tag{49}$$

Note that

$$\begin{aligned} \int |\nabla(\alpha_2^l\, \delta(x_2^l, \lambda_2^l))|^2 &\geqslant (\tfrac{1}{2}(A_2)^{(2-n)/4} - \circ_{\varepsilon_2}(1))(S_n)^n \\ &> \tfrac{1}{4}(A_2)^{(2-n)/4} (S_n)^n > 0. \end{aligned} \tag{50}$$

By the definition of S_n, we have

$$
\begin{aligned}
S_n &\leqslant \frac{\{\int |\nabla(\alpha_2^l\,\delta(x_2^l,\lambda_2^l))|^2\}^{1/2}}{\{\int [\alpha_2^l\,\delta(x_2^l,\lambda_2^l)]^{2n/(n-2)}\}^{(n-2)/2n}} \\
&= \frac{\{\int |\nabla(\alpha_2^l\,\delta(x_2^l,\lambda_2^l))|^2\}^{1/2}}{\{\int_{|x-x_l^{(2)}|\leqslant R_l} [(\alpha_2^l\,\delta(x_2^l,\lambda_2^l)]^{2n/(n-2)}\}^{(n-2)/2n}+o(1)} \\
&\qquad \text{(see (46) and (19))} \\
&\leqslant \frac{\{\int |\nabla(\alpha_2^l\,\delta(x_2^l,\lambda_2^l))|^2\}^{1/2}}{\{\int_{|x-x_l^{(2)}|\leqslant R_l} [(\alpha_2^l\,\delta(x_2^l,\lambda_2^l))]^{2n/(n-2)-\tau_l}\}^{(n-2)/2n}+o(1)} \\
&\qquad \text{(Hölder inequality and (26))} \\
&\leqslant \frac{\{\int |\nabla(\alpha_2^l\,\delta(x_2^l,\lambda_2^l))|^2\}^{1/2}\cdot K_l(x_l^{(2)})^{(n-2)/2n}}{\left(\begin{array}{l}\{\int_{|x-x_l^{(2)}|\leqslant R_l} K_l(\cdot)(2/(1+|\cdot|^2))^{((n-2)/2)\tau_l} \\ \quad\times [(\alpha_2^l\,\delta(x_2^l,\lambda_2^l))]^{2n/(n-2)-\tau_l}\}^{(n-2)/2n}+o(1)\end{array}\right)} \qquad \text{(see (26), (20))} \\
&= \frac{\{\int |\nabla(\alpha_2^l\,\delta(x_2^l,\lambda_2^l))^2\}^{1/2}\cdot(a^{(2)})^{(n-2)/2n}+o(1)}{\{\int K_l(\cdot)(2/(1+|\cdot|^2))^{((n-2)/2)\tau_l}\,[\alpha_2^l\,\delta(x_2^l,\lambda_2^l)]^{2n/(n-2)-\tau_l}\}^{(n-2)/2n}+o(1)} \\
&\qquad \text{(see (21), (46), and (19)).}
\end{aligned}
$$

Using (50) and (48), we have

$$
S_n \leqslant \left\{\int |\nabla(\alpha_2^l\,\delta(x_2^l,\lambda_2^l))|^2\right\}^{1/n}\cdot(a^{(2)})^{(n-2)/2n}+o(1).
$$

Together with (49) we obtain

$$
\begin{aligned}
I_{K_l,\tau_l}(\alpha_2^l\,\delta(x_2^l,\lambda_2^l)) &\geqslant \frac{1}{n}\,(a^{(2)})^{(2-n)/2}\,(S_n)^n + o_{\varepsilon_2}(1)+o(1) \\
&= c^{(2)} + o_{\varepsilon_2}(1)+o(1).
\end{aligned}
$$

Putting (47), (34), and the above estimate together, we obtain the second part of (45).

It is very easy to see from (44), (45), (29), (43), and the positivity of w that for $\varepsilon_2>0$ small enough we obtain a contradiction. Hence we have proved that $\lim_{l\to\infty}\lambda_1^l=\infty$. Similarly we can prove that $\lim_{l\to\infty}\lambda_2^l=\infty$. Claim 1 has been established. ▮

For $\lambda>0$, $x\in\mathbf{R}^n$, we define $\mathscr{T}_{l,\lambda,x}: E\to E$ by $(\mathscr{T}_{l,\lambda,x}u)(\cdot)=\lambda^{2/(1-p_l)}u(\cdot/\lambda+x)$, $u\in E$. It is clear that $\mathscr{T}^{-1}_{l,\lambda,x}u(\cdot)=\lambda^{2/(p_l-1)}u(\lambda(\cdot-x))$.

LEMMA 3.3. *There exists some constant $C=C(A_2,n)$, such that, for small ε_2 and large l, we have*

$$
(\lambda_1^l)^{\tau_l},\ (\lambda_2^l)^{\tau_l}\leqslant C.
$$

Proof of Lemma 3.3. Using (35) we have

$$I'_{K_l,\tau_l}(u_l)(\delta(x_1^l, \lambda_1^l)) = \circ(1). \tag{51}$$

It follows from Claim 1, (42), and (39) that

$$\int \nabla\,\delta(x_2^l, \lambda_2^l)\,\nabla\,\delta(x_1^l, \lambda_1^l) = \circ(1),$$

$$\int \nabla v_l\,\nabla\,\delta(x_1^l, \lambda_1^l) = \circ_{\varepsilon_2}(1),$$

$$\int K_l\left(\frac{1}{1+|\cdot|^2}\right)^{((n-2)/2)\tau_l} \delta(x_2^l, \lambda_2^l)^{p_l}\,\delta(x_1^l, \lambda_1^l) = \circ(1),$$

$$\int K_l\left(\frac{1}{1+|\cdot|^2}\right)^{((n-2)/2)\tau_l} v_l^{p_l}\,\delta(x_1^l, \lambda_1^l) = \circ_{\varepsilon_2}(1).$$

Using the above estimates, we can derive from (51) that

$$\begin{aligned}(\alpha_1^l)^{p_l}\int K_l\left(\frac{1}{1+|\cdot|^2}\right)^{((n-2)/2)\tau_l} & \delta(x_1^l, \lambda_1^l)^{p_l+1} \\ &= \alpha_1^l\int|\nabla\,\delta(x_1^l, \lambda_1^l)|^2 + \circ(1) + \circ_{\varepsilon_2}(1).\end{aligned} \tag{52}$$

It follows immediately from (52), (38), (26), (23), (40), and Claim 1 that $(\lambda_1^l)^{\tau_l} \leqslant C$. Similarly we can prove that $(\lambda_2^l)^{\tau_l} \leqslant C$.

Without loss of generality, we assume that

$$\lambda_1^l \leqslant \lambda_2^l. \tag{53}$$

A straightforward computation yields (see (36))

$$\mathscr{T}_{l,\lambda_1^l,x_1^l} u_l = \tilde{\alpha}_1^l\,\delta(0,1) + \tilde{\alpha}_2^l\,\delta(\lambda_1^l(x_2^l - x_1^l), \lambda_2^l/\lambda_1^l) + \mathscr{T}_{l,\lambda_1^l,x_1^l} v_l, \tag{54}$$

where

$$\tilde{\alpha}_1^l = \alpha_1^l(\lambda_1^l)^{(n-2)/2 - 2/(p_l-1)},$$

$$\tilde{\alpha}_2^l = \alpha_2^l(\lambda_1^l)^{(n-2)/2 - 2/(p_l-1)}.$$

Passing to a subsequence, we have, for some $u_1 \in E$, $\xi_1 \in O^{(1)}$, that

$$\mathscr{T}_{l,\lambda_1^l,x_1^l} u_l \rightharpoonup u_1, \text{ weakly in } E, \tag{55}$$

$$\lim_{l\to\infty}(x_1^l - x_l^{(1)}) = \xi_1. \tag{56}$$

It then follows from (22), (25), (56), and (40) that

$$\lim_{l\to\infty} K_l(x_1^l)^{(2-n)/4} = K_\infty^{(1)}(\xi_1)^{(2-n)/4}. \tag{57}$$

For any test function $\varphi \in C_c^\infty(\mathbf{R}^n)$, it follows from (35) that

$$\begin{aligned} \circ(1) &= I'_{K_l,\tau_l}(u_l)(\mathscr{T}^{-1}_{l,\lambda_1^l,x_1^l}\varphi) \\ &= (\lambda_1^l)^{2(p_l+1)/(p_l-1)-n}\left\{\int \nabla \mathscr{T}_{l,\lambda_1^l,x_1^l}u_l\,\nabla\varphi - \int T_{x_l^{(1)}}K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l-x_l^{(1)}\right)\right. \\ &\quad\left.\times\left(\frac{1}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l}|\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l|^{4/(n-2)-\tau_l}(\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l)\varphi\right\}. \end{aligned}$$

Let $l\to\infty$, we obtain, using (55), (26), (56), (22), (40), and Lemma 3.3, that

$$\int \nabla u_1\,\nabla\varphi - \int K_\infty^{(1)}(\xi_1)\,|u_1|^{4/(n-2)}\,u_1\varphi = 0.$$

Namely,

$$-\Delta u_1 = K_\infty^{(1)}(\xi_1)\,|u_1|^{4/(n-2)}\,u_1. \tag{58}$$

It is clear from (54) that u_1 is not identically zero if ε_2 is small enough. We then argue as before to obtain that $u_1 > 0$.

According to the uniqueness (up to a conformal group) of positive solutions of (58), there exists $\lambda^* > 0$, $x^* \in \mathbf{R}^n$, such that

$$u_1 = K_\infty^{(1)}(\xi_1)^{(2-n)/4}\,\delta(x^*,\lambda^*). \tag{59}$$

Claim 2. For l large enough, we have $|x^*| \leqslant \circ_{\varepsilon_2}(1)$, $|\lambda^*-1| \leqslant \circ_{\varepsilon_2}(1)$, $(\lambda_1^l)^{\tau_l} = 1+\circ_{\varepsilon_2}(1)$.

Proof. Letting $(\lambda_1^l)^{\tau_l} = B_{\varepsilon_2,\varepsilon_3} + \circ(1)$ (we need to pass to a subsequence, of course), we recall that

$$\tilde{\alpha}_1^l = \alpha_1^l(\lambda_1^l)^{(n-2)/2-2/(p_l-1)} = \alpha_1^l(\lambda_1^l)^{-((n-2)^2/8)\tau_l)+O(\tau_l^2)}.$$

It follows from (38) and (57) that

$$\alpha_1^l = K_\infty^{(1)}(\xi_1)^{(2-n)/4} + \circ_{\varepsilon_2}(1) + \circ(1), \tag{60}$$

Therefore

$$\tilde{\alpha}_1^l = K_\infty^{(1)}(\xi_1)^{(2-n)/4}(B_{\varepsilon_2,\varepsilon_3})^{-(n-2)^2/8} + \circ_{\varepsilon_2}(1) + \circ(1). \tag{61}$$

From (54), (53), (42), (37), and (55), we see that

$$\tilde{\alpha}_1^l\,\delta(0,1)+\mathscr{T}_{l,\lambda_1^l,x_1^l}v_l\rightharpoonup u_1 \qquad \text{weakly in } E. \tag{62}$$

It follows from (61), (62), (39), and Lemma 3.3 that

$$\begin{aligned}\|K_\infty^{(1)}(\xi_1)^{(2-n)/4}\cdot(B_{\varepsilon_2,\varepsilon_3})^{-(n-2)^2/8}\,\delta(0,1)\\ -K_\infty^{(1)}(\xi_1)^{(2-n)/4}\,\delta(x^*,\lambda^*)\|=\circ_{\varepsilon_2}(1)+\circ(1).\end{aligned}$$

It follows immediately by sending l to ∞ that

$$|x^*|=\circ_{\varepsilon_2}(1), \qquad \lambda^*=1+\circ_{\varepsilon_2}(1), \qquad B_{\varepsilon_2,\varepsilon_3}=1+\circ_{\varepsilon_2}(1).$$

Claim 2 has been established. ▮

We define $\xi_l\in E$ by

$$\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l=u_1+\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l. \tag{63}$$

It follows from (55) that

$$\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l\rightharpoonup 0 \qquad \text{weakly in } E.$$

Claim 3. $\|I'_{K_l,\tau_l}(\xi_l)\|=\circ(1)$, provided ε_2 is small enough.

Proof. For any $\varphi\in E$, it follows from (35), (58), (63), and Lemma 3.3 that

$$\begin{aligned}
&\circ(1)\,\|\varphi\|\\
&\quad=I'_{K_l,\tau_l}(u_l)(\mathscr{T}^{-1}_{l,\lambda_1^l,x_1^l}\varphi)\\
&\quad=\int\nabla u_l\nabla(\mathscr{T}^{-1}_{l,\lambda_1^l,x_1^l}\varphi)-\int K_l\left(\frac{1}{1+|\cdot|^2}\right)^{((n-2)/2)\tau_l}|u_l|^{p_l-1}(u_l)\,\mathscr{T}^{-1}_{l,\lambda_1^l,x_1^l}\varphi\\
&\quad=(\lambda_1^l)^{2(p_l+1)/(p_l-1)-n}\left\{\int\nabla(\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l)\,\nabla\varphi-\int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\right.\\
&\qquad\left.\times\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l}|\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l|^{p_l-1}(\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l)\,\varphi\right\}\\
&\quad=(\lambda_1^l)^{2(p_l+1)/(p_l-1)-n}\left\{\int\nabla u_1\nabla\varphi+\int\nabla(\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l)\,\nabla\varphi-\int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\right.\\
&\qquad\times\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l}|\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l|^{p_l-1}(\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l)\,\varphi
\end{aligned}$$

$$
\begin{aligned}
&= (\lambda_1^l)^{2(p_l+1)/(p_l-1)-n}\left\{\int K_\infty^{(1)}(\xi_1)(u_1)^{(n+2)/(n-2)}\varphi + \int \nabla(\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l)\nabla\varphi\right.\\
&\quad - \int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l}\\
&\quad \times |\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l|^{p_l-1}(\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l)\varphi\\
&\quad + \int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l}\\
&\quad \times |\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l|^{p_l-1}(\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l)\varphi\\
&\quad - \int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l}\\
&\quad \left.\times |\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l|^{p_l-1}(\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l)\varphi\right\}\\
&= I'_{K_l,\tau_l}(\xi_l)(\mathscr{T}^{-1}_{l,\lambda_1^l,x_1^l}\varphi) + (\lambda_1^l)^{2(p_l+1)/(p_l-1)-n}\left\{\int K_\infty^{(1)}(\xi_1)(u_1)^{(n+2)/(n-2)}\varphi\right.\\
&\quad + \int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l}\\
&\quad \times |\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l|^{p_l-1}(\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l)\varphi\\
&\quad - \int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l}\\
&\quad \left.\times |\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l|^{p_l-1}(\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l)\varphi\right\}. \qquad (64)
\end{aligned}
$$

It follows from (26), (57), (59), Claim 2, Hölder inequalities, and the Sobolev embedding theorems that

$$
\left|\int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l}(u_1)^{p_l}\varphi - \int K_\infty^{(1)}(\xi_1)(u_1)^{(n+2)/(n-2)}\varphi\right| = \circ(1)\,\|\varphi\|. \qquad (65)
$$

Using (63), (64), Lemma 3.3, and some elementary inequalities, we have

$$
\begin{aligned}
&|I'_{K_l,\tau_l}(\xi_l)(\mathscr{T}^{-1}_{l,\lambda_1^l,x_1^l}\varphi)|\\
&\quad \leqslant \circ(1)\,\|\varphi\| + O(1)\int\left\{|\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l|^{p_l-1}u_1 + |\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l|\,(u_1)^{p_l-1}\right\}|\varphi|\\
&\quad \leqslant \circ(1)\,\|\varphi\|.
\end{aligned}
$$

The last inequality follows from the fact that $\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l \rightharpoonup 0$ weakly in E, (59), Claim 2, Hölder inequalities, and the Sobolev embedding theorems. Claim 3 has been established (Lemma 3.3 is needed here). ∎

Claim 4.

$$I_{K_l,\tau_l}(\xi_l) \leqslant c^{(2)} + \varepsilon_3 + \circ(1).$$

Proof. Using Claim 2, we have

$$\begin{aligned}
&I_{K_l,\tau_l}(u_l)\\
&= \frac{1}{2}\int |\nabla u_l|^2 - \frac{1}{p_l+1}\int K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau_l} |u_l|^{p_l+1}\\
&= (\lambda_1^l)^{2(p_l+1)/(p_l-1)-n}\left\{\frac{1}{2}\int |\nabla \mathscr{T}_{l,\lambda_1^l,x_1^l}u_l|^2 - \frac{1}{p_l+1}\int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\right.\\
&\quad\left.\times\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l} |\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l|^{p_l+1}\right\}\\
&= (\lambda_1^l)^{2(p_l+1)/(p_l-1)-n}\left\{\frac{1}{2}\int |\nabla \mathscr{T}_{l,\lambda_1^l,x_1^l}u_l|^2 - \frac{n-2}{2n}\int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\right.\\
&\quad\left.\times\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l} |\mathscr{T}_{l,\lambda_1^l,x_1^l}u_l|^{p_l+1}\right\} + \circ(1).
\end{aligned}$$

Using (63), we obtain

$$\begin{aligned}
&I_{K_l,\tau_l}(u_l)\\
&= (\lambda_1^l)^{2(p_l+1)/(p_l-1)-n}\left\{\frac{1}{2}\int |\nabla u_1|^2 + \int \nabla u_1\, \nabla(\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l)\right.\\
&\quad + \frac{1}{2}\int |\nabla(\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l)|^2\\
&\quad - \frac{n-2}{2n}\int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l} |u_1|^{p_l+1}\\
&\quad - \frac{n-2}{2n}\int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l} |\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l|^{p_l+1}\\
&\quad\left. - O(1)\int \left\{|\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l|^{p_l} u_1 + |\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l\, |u_1|^{p_l}\right\}\right\} + \circ(1)\\
&= I_{K_l,\tau_l}(\xi_l) + (\lambda_1^l)^{2(p_l+1)/(p_l-1)-n}\left\{\frac{1}{2}\int |\nabla u_1|^2 - \frac{n-2}{2n}\int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\right.\\
&\quad\left.\times\left(\frac{2}{1+|\cdot/\lambda_1^l}+x_1^l|^2\right)^{((n-2)/2)\tau_l} (u_1)^{p_l+1}\right\} + \circ(1).
\end{aligned} \tag{66}$$

The last equality follows from the fact that $\mathscr{T}_{l,\lambda_1^l,x_1^l}\xi_l \rightharpoonup 0$ weakly in E and (59).

It follows easily from (56), (22), and the definition of S_n that

$$\begin{aligned}
&\frac{1}{2}\int |\nabla u_1|^2 - \frac{n-2}{2n}\int K_l\left(\frac{\cdot}{\lambda_1^l}+x_1^l\right)\left(\frac{2}{1+|\cdot/\lambda_1^l+x_1^l|^2}\right)^{((n-2)/2)\tau_l}(u_1)^{p_l+1}\\
&\quad = I_{K_\infty^{(1)}(\xi_1)}(u_1)+\circ(1)\\
&\quad \geqslant \frac{1}{n}K_\infty^{(1)}(\xi_1)^{(2-n)/2}(S_n)^n+\circ(1)\\
&\quad \geqslant c^{(1)}+\circ(1).
\end{aligned} \tag{67}$$

It is easy to check that

$$(\lambda_1^l)^{2(p_l+1)/(p_l-1)-n}\geqslant 1. \tag{68}$$

Claim 4 follows from (66), (67), (68), and (34). ∎

From (63), (36), and (59) we have

$$\begin{aligned}
\xi_l &= u_l - \mathscr{T}^{-1}_{l,\lambda_1^l,x_1^l}u_1\\
&= \alpha_2^l\,\delta(x_2^l,\lambda_2^l)+w_l,
\end{aligned}$$

where

$$\begin{aligned}
w_l = {}&\alpha_1^l\,\delta(x_1^l,\lambda_1^l)-K_\infty^{(1)}(\xi_1)^{(2-n)/4}(\lambda_1^l)^{2/(p_l-1)-(n-2)/2}\\
&\times\delta\left(\frac{x^*}{\lambda_1^l}+x_1^l,\lambda^*\lambda_1^l\right)+v_l.
\end{aligned}$$

Using Claim 2 and (60), we have, for large l, that

$$\|w_l\|\leqslant \circ_{\varepsilon_2}(1).$$

To sum up, we have obtained the following for large l.

$$u_l=\mathscr{T}^{-1}_{l,\lambda_1^l,x_1^l}u_1+\xi_l,$$

$$\xi_l=\alpha_2^l\,\delta(x_2^l,\lambda_2^l)+w_l, \tag{69}$$

$$\|w_l\|\leqslant \circ_{\varepsilon_2}(1), \tag{70}$$

$$I_{K_l,\tau_l}(\xi_l)\leqslant c^{(2)}+\varepsilon_3+\circ(1),$$

$$\|I'_{K_l,\tau_l}(\xi_l)\|=\circ(1). \tag{71}$$

We can simply repeat the previous argument on ξ_l instead of on u_l. For the reader's convenience, we carry out the details.

A simple calculation yields (see (69))

$$\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l = \bar{\alpha}_2^l\,\delta(0,1) + \mathscr{T}_{l,\lambda_2^l,x_2^l}w_l, \tag{72}$$

where

$$\bar{\alpha}_2^l = \alpha_2^l(\lambda_2^l)^{(n-2)/2-2/(p_l-1)}. \tag{73}$$

By passing to a subsequence, we have, for some $u_2 \in E$, $\xi_2 \in O^{(2)}$, that

$$\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l \rightharpoonup u_2 \qquad \text{weakly in } E, \tag{74}$$

$$\lim_{l\to\infty}(x_2^l - x_l^{(2)}) = \xi_2. \tag{75}$$

It follows easily from (25), (22), and (75) that

$$\lim_{l\to\infty} K_l(x_2^l)^{(2-n)/4} = K_\infty^{(2)}(\xi_2)^{(2-n)/4}. \tag{76}$$

For any test function $\varphi \in C_c^\infty(\mathbf{R}^n)$, it follows from (71) and Lemma 3.3 that

$$\begin{aligned} \circ(1) &= I'_{K_l,\tau_l}(\xi_l)(\mathscr{T}^{-1}_{l,\lambda_2^l,x_2^l}\varphi) \\ &= (\lambda_2^l)^{2(p_l+1)/(p_l-1)-n}\left\{\int \nabla\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l\,\nabla\varphi - \int T_{x_l^{(2)}}K_l\left(\frac{\cdot}{\lambda_2^l} + x_2^l - x_l^{(2)}\right)\right. \\ &\quad \left.\times\left(\frac{1}{1+|\cdot/\lambda_2^l + x_2^l|^2}\right)^{((n-2)/2)\tau_l} |\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l|^{4/(n-2)-\tau_l}(\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l)\varphi\right\}. \end{aligned}$$

Letting $l\to\infty$ and arguing as before, we have

$$\int \nabla u_2\,\nabla\varphi - \int K_\infty^{(2)}(\xi_2)\,|u_2|^{4/(n-2)}\,u_2\varphi = 0.$$

Therefore

$$-\Delta u_2 = K_\infty^{(2)}(\xi_2)\,|u_2|^{4/(n-2)}\,u_2. \tag{77}$$

Arguing as before, we can prove, for ε_2 small enough, that $u_2 > 0$ and for some $x^{**} \in \mathbf{R}^n$, $\lambda^{**} > 0$,

$$u_2 = K_\infty^{(2)}(\xi_2)^{(2-n)/4}\,\delta(x^{**},\lambda^{**}). \tag{78}$$

Claim 5. For l large enough, we have $|x^{**}| \leqslant \circ_{\varepsilon_2}(1)$, $|\lambda^{**}-1| \leqslant \circ_{\varepsilon_2}(1)$, $(\lambda_2^l)^{\tau_l} = 1 + \circ_{\varepsilon_2}(1)$.

Proof. It follows from (22), (25), (75), and (40) that

$$\lim_{l\to\infty} K_l(x_2^l)^{(2-n)/4} = K_\infty^{(2)}(\xi_2)^{(2-n)/4}. \tag{79}$$

From (38) and (79), we obtain

$$\alpha_2^l = K_\infty^{(2)}(\xi_2)^{(2-n)/4} + \circ_{\varepsilon_2}(1) + \circ(1). \tag{80}$$

We write (along a subsequence) $(\lambda_2^l)^{\tau_l} = A_{\varepsilon_2,\varepsilon_3} + \circ(1)$ and we derive from (73) and (80) that

$$\bar{\alpha}_2^l = K_\infty^{(2)}(\xi_2)^{(2-n)/4}(A_{\varepsilon_2,\varepsilon_3})^{-(n-2)^2/8} + \circ_{\varepsilon_2}(1) + \circ(1). \tag{81}$$

From (72), (81), (74), (70), and (78) we see that

$$\begin{aligned}\|K_\infty^{(2)}(\xi_2)^{(2-n)/4}\cdot(A_{\varepsilon_3,\varepsilon_4})^{-(n-2)^2/8}\,\delta(0,1)\\ - K_\infty^{(2)}(\xi_2)^{(2-n)/4}\,\delta(x^{**},\lambda^{**})\| = \circ_{\varepsilon_2}(1) + \circ(1).\end{aligned}$$

It follows immediately by sending l to ∞ that

$$|x^{**}| = \circ_{\varepsilon_2}(1), \qquad \lambda^{**} = 1 + \circ_{\varepsilon_2}(1), \qquad A_{\varepsilon_3,\varepsilon_3} = 1 + \circ_{\varepsilon_2}(1).$$

Claim 5 has been established.

We define $\eta_l \in E$ by

$$\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l = u_2 + \mathscr{T}_{l,\lambda_2^l,x_2^l}\eta_l. \tag{82}$$

Clearly,

$$\mathscr{T}_{l,\lambda_2^l,x_2^l}\eta_l \rightharpoonup 0 \qquad \text{weakly in } E.$$

Claim 6. $\|I'_{K_l,\tau_l}(\eta_l)\| = \circ(1)$, provided ε_2 is small enough.

Proof. For any $\varphi \in E$, it follows from Claim 3, (77), (82), and Lemma 3.3 that

$$\begin{aligned}
&\circ(1)\,\|\varphi\| \\
&= I'_{K_l,\tau_l}(\xi_l)(\mathscr{T}^{-1}_{l,\lambda_2^l,x_2^l}\varphi) \\
&= \int \nabla\xi_l\,\nabla(\mathscr{T}^{-1}_{l,\lambda_2^l,x_2^l}\varphi) - \int K_l\left(\frac{1}{1+|\cdot|^2}\right)^{((n-2)/2)\tau_l} |\xi_l|^{p_l-1}(\xi_l)\,\mathscr{T}^{-1}_{l,\lambda_2^l,x_2^l}\varphi \\
&= (\lambda_2^l)^{2(p_l+1)/(p_l-1)-n}\left\{\int \nabla(\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l)\,\nabla\varphi - \int K_l\left(\frac{\cdot}{\lambda_2^l} + x_2^l\right)\right. \\
&\quad \left.\times\left(\frac{1}{1+|\cdot/\lambda_2^l + x_2^l|^2}\right)^{((n-2)/2)\tau_l} |\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l|^{p_l-1}(\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l)\varphi\right\} \\
&= (\lambda_2^l)^{2(p_l+1)/(p_l-1)-n}\left\{\int \nabla u_2\,\nabla\varphi + \int \nabla(\mathscr{T}_{l,\lambda_2^l,x_2^l}\eta_l)\,\nabla\varphi - \int K_l\left(\frac{\cdot}{\lambda_2^l} + x_2^l\right)\right. \\
&\quad \left.\times\left(\frac{1}{1+|\cdot/\lambda_2^l + x_2^l|^2}\right)^{((n-2)/2)\tau_l} |\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l|^{p_l-1}(\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l)\varphi\right\}
\end{aligned}$$

$$= (\lambda_2^l)^{2(p_l+1)/(p_l-1)-n} \left\{ \int K_\infty^{(2)}(\xi_2)(u_2)^{(n+2)/(n-2)} \varphi + \int \nabla(\mathcal{T}_{l,\lambda_2^l,x_2^l}\eta_l) \nabla\varphi \right.$$

$$- \int K_l\left(\frac{\cdot}{\lambda_2^l} + x_2^l\right)\left(\frac{1}{1+|\cdot/\lambda_2^l + x_2^l|^2}\right)^{((n-2)/2)\tau_l}$$

$$\times |\mathcal{T}_{l,\lambda_2^l,x_2^l}\eta_l|^{p_l-1}(\mathcal{T}_{l,\lambda_2^l,x_2^l}\eta_l)\varphi$$

$$+ \int K_l\left(\frac{\cdot}{\lambda_2^l} + x_2^l\right)\left(\frac{1}{1+|\cdot/\lambda_2^l + x_2^l|^2}\right)^{((n-2)/2)\tau_l}$$

$$\times |\mathcal{T}_{l,\lambda_2^l,x_2^l}\eta_l|^{p_l-1}(\mathcal{T}_{l,\lambda_2^l,x_2^l}\eta_l)\varphi$$

$$- \int K_l\left(\frac{\cdot}{\lambda_2^l} + x_2^l\right)\left(\frac{1}{1+|\cdot/\lambda_2^l + x_2^l|^2}\right)^{((n-2)/2)\tau_l}$$

$$\left. \times |\mathcal{T}_{l,\lambda_2^l,x_2^l}\xi_l|^{p_l-1}(\mathcal{T}_{l,\lambda_2^l,x_2^l}\xi_l)\varphi \right\}$$

$$= I'_{K_l,\tau_l}(\eta_l)(\mathcal{T}^{-1}_{l,\lambda_2^l,x_2^l}\varphi) + (\lambda_2^l)^{2(p_l+1)/(p_l-1)-n} \left\{ \int K_\infty^{(2)}(\xi_2)(u_2)^{(n+2)/(n-2)} \varphi \right.$$

$$+ \int K_l\left(\frac{\cdot}{\lambda_2^l} + x_2^l\right)\left(\frac{1}{1+|\cdot/\lambda_2^l + x_2^l|^2}\right)^{((n-2)/2)\tau_l}$$

$$\times |\mathcal{T}_{l,\lambda_2^l,x_2^l}\eta_l|^{p_l-1}(\mathcal{T}_{l,\lambda_2^l,x_2^l}\eta_l)\varphi$$

$$- \int K_l\left(\frac{\cdot}{\lambda_2^l} + x_2^l\right)\left(\frac{1}{1+|\cdot/\lambda_2^l + x_2^l|^2}\right)^{((n-2)/2)\tau_l}$$

$$\left. \times |\mathcal{T}_{l,\lambda_2^l,x_2^l}\xi_l|^{p_l-1}(\mathcal{T}_{l,\lambda_2^l,x_2^l}\xi_l)\varphi \right\}. \tag{83}$$

It follows from (26), (76), (78), Claim 2, Hölder inequalities, and the Sobolev embedding theorems that

$$\left| \int K_l\left(\frac{\cdot}{\lambda_2^l} + x_2^l\right)\left(\frac{1}{1+|\cdot/\lambda_2^l + x_2^l|^2}\right)^{((n-2)/2)\tau_l} (u_2)^{p_l}\varphi \right.$$

$$\left. - \int K_\infty^{(2)}(\xi_2)(u_2)^{(n+2)/(n-2)}\varphi \right| = \circ(1)\,\|\varphi\|. \tag{84}$$

Using (83), (84), Lemma 3.3, and some elementary inequalities, we have

$$|I'_{K_l,\tau_l}(\eta_l)(\mathcal{T}^{-1}_{l,\lambda_2^l,x_2^l}\varphi)|$$

$$\leqslant \circ(1)\,\|\varphi\| + O(1)\int \{|\mathcal{T}_{l,\lambda_2^l,x_2^l}\eta_l|^{p_l-1}u_2 + |\mathcal{T}_{l,\lambda_2^l,x_2^l}\eta_l|\,(u_2)^{p_l-1}\}\,|\varphi|$$

$$\leqslant \circ(1)\,\|\varphi\|.$$

The last inequality follows from the fact that $\mathscr{T}_{l,\lambda_2^l,x_2^l}\eta_l \rightharpoonup 0$ weakly in E, (78), Claim 2, Hölder inequalities, and the Sobolev embedding theorems. Claim 6 has been established (Lemma 3.3 is needed here). ∎

Claim 7. $I_{K_l,\tau_l}(\eta_l) \leqslant \varepsilon_3 + \circ(1)$, provided ε_2 is small enough.

Proof. Using Claim 5, we have

$$
\begin{aligned}
&I_{K_l,\tau_l}(\xi_l)\\
&= \frac{1}{2}\int |\nabla \xi_l|^2 - \frac{1}{p_l+1}\int K_l\left(\frac{2}{+|\cdot|^2}\right)^{((n-2)/2)\tau_l} |\xi_l|^{p_l+1}\\
&= (\lambda_2^l)^{2(p_l+1)/(p_l-1)-n}\left\{\frac{1}{2}\int |\nabla \mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l|^2 - \frac{1}{p_l+1}\int K_l\left(\frac{\cdot}{\lambda_2^l}+x_2^l\right)\right.\\
&\quad\left.\times\left(\frac{1}{1+|\cdot/\lambda_2^l+x_2^l|^2}\right)^{((n-2)/2)\tau_l} |\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l|^{p_l+1}\right\}\\
&= (\lambda_2^l)^{2(p_l+1)/(p_l-1)-n}\left\{\frac{1}{2}\int |\nabla \mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l|^2 - \frac{n-2}{2n}\int K_l\left(\frac{\cdot}{\lambda_2^l}+x_2^l\right)\right.\\
&\quad\left.\times\left(\frac{1}{1+|\cdot/\lambda_2^l+x_2^l|^2}\right)^{((n-2)/2)\tau_l} |\mathscr{T}_{l,\lambda_2^l,x_2^l}\xi_l|^{p_l+1}\right\} + \circ(1).
\end{aligned}
$$

Using (82), we obtain

$$
\begin{aligned}
&I_{K_l,\tau_l}(\xi_l)\\
&= (\lambda_2^l)^{2(p_l+1)/(p_l-1)-n}\left\{\frac{1}{2}\int |\nabla u_2|^2 + \int \nabla u_2\, \nabla(\mathscr{T}_{l,\lambda_2^l,x_2^l}\eta_l)\right.\\
&\quad + \frac{1}{2}\int |\nabla(\mathscr{T}_{l,\lambda_2^l,x_2^l}\eta_l)|^2 - \frac{n-2}{2n}\int K_l\left(\frac{\cdot}{\lambda_2^l}+x_2^l\right)\\
&\quad \times\left(\frac{1}{1+|\cdot/\lambda_2^l+x_2^l|^2}\right)^{((n-2)/2)\tau_l} |u_2|^{p_l+1} - \frac{n-2}{2n}\int K_l\left(\frac{\cdot}{\lambda_2^l}+x_2^l\right)\\
&\quad \times\left(\frac{1}{1+|\cdot/\lambda_2^l+x_2^l|^2}\right)^{((n-2)/2)\tau_l} |\mathscr{T}_{l,\lambda_2^l,x_2^l}\eta_l|^{p_l+1}\\
&\quad \left. - O(1)\int \{|\mathscr{T}_{l,\lambda_2^l,x_2^l}\eta_l|^{p_l} u_2 + |\mathscr{T}_{l,\lambda_2^l,x_2^l}\eta_l|\, |u_2|^{p_l}\}\right\} + \circ(1)\\
&= I_{K_l,\tau_l}(\eta_l) + (\lambda_2^l)^{2(p_l+1)/(p_l-1)-n}\left\{\frac{1}{2}\int |\nabla u_2|^2 - \frac{n-2}{2n}\int K_l\left(\frac{\cdot}{\lambda_2^l}+x_2^l\right)\right.\\
&\quad \left.\times\left(\frac{1}{1+|\cdot/\lambda_2^l+x_2^l|^2}\right)^{((n-2)/2)\tau_l} (u_2)^{p_l+1}\right\} + \circ(1). \qquad (85)
\end{aligned}
$$

The last equality follows from the fact that $\mathscr{T}_{l,\lambda_2^l,x_2^l}\eta_l \rightharpoonup 0$ weakly in E and (78).

It follows easily from (75), (22), and the definition of S_n that

$$\begin{aligned}
&\frac{1}{2}\int |\nabla u_2|^2 - \frac{n-2}{2n}\int K_l\left(\frac{\cdot}{\lambda_2^l}+x_2^l\right)\left(\frac{1}{1+|\cdot/\lambda_2^l+x_2^l|^2}\right)^{((n-2)/2)\tau_l}(u_2)^{p_l+1}\\
&\quad = I_{K_\infty^{(2)}(\xi_2)}(u_2)+\circ(1)\\
&\quad \geqslant \frac{1}{n}K_\infty^{(2)}(\xi_2)^{(2-n)/2}(S_n)^n+\circ(1)\\
&\quad \geqslant c^{(2)}+\circ(1).
\end{aligned}\tag{86}$$

It is easy to check that

$$(\lambda_2^l)^{2(p_l+1)/(p_l-1)-n}\geqslant 1.\tag{87}$$

Claim 7 follows from Claim 4, (85), (86), and (87). ∎

Claim 8. If $\varepsilon_2>0$ is small enough, we have $\eta_l\to 0$ strongly in E.

Proof. It follows from Claim 6 and Claim 7 that

$$\int |\nabla\eta_l|^2-\int K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau_l}|\eta_l|^{2n/(n-2)-\tau_l}=\circ(1),\tag{88}$$

$$\frac{1}{2}\int |\nabla\eta_l|^2-\frac{1}{p_l+1}\int K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau_l}|\eta_l|^{2n/(n-2)-\tau_l}\leqslant\varepsilon_3+\circ(1).$$

Therefore we have

$$\int |\nabla\eta_l|^2\leqslant n\varepsilon_3+\circ(1).\tag{89}$$

Suppose that Claim 8 does not hold, then along a subsequence we have

$$\|\eta_l\|^{\tau_l}=1+\circ(1).\tag{90}$$

We derive from (88) and (90), by using Hölder inequalities, that

$$\int |\nabla\eta_l|^2\leqslant C(n,A_1)\left(\int |\eta_l|^{2n/(n-2)}\right)^{(p_l+1)(n-2)/(2n)}+\circ(1),$$

and then

$$\int |\nabla\eta_l|^2\leqslant C(n,A_1)\int |\eta_l|^{2n/(n-2)}+\circ(1).\tag{91}$$

It follows from (91) and the definition of S_n that

$$S_n \leqslant \frac{(\int |\nabla \eta_l|^2)^{1/2}}{(\int |\eta_l|^{2n/(n-2)})^{(n-2)/2n}} \leqslant \frac{(\int |\nabla \eta_l|^2)^{1/2} (C(n, A_1))^{(n-2)/2n}}{(\int \nabla \eta_l|^2 + \circ(1))^{(n-2)/2n}}.$$

Thus

$$S_n \leqslant (C(n, A_1))^{(n-2)/2n} \left(\int |\nabla \eta_l|^2 \right)^{1/n}. \tag{92}$$

Because of (89), (92) cannot hold if $\varepsilon_2 > 0$ is small enough. Claim 8 has been established. ∎

Rewriting (63) and (82), we have

$$u_l = \mathscr{T}^{-1}_{l, \lambda_2^l, x_2^l} u_1 + \mathscr{T}^{-1}_{l, \lambda_2^l, x_2^l} u_2 + \eta_l. \tag{93}$$

Claim 9. For $\varepsilon_2 > 0$ small enough, we have

$$(\lambda_1^l)^{\tau_l} = 1 + \circ_{\varepsilon_3}(1) + \circ(1), \qquad (\lambda_2^l)^{\tau_l} = 1 + \circ_{\varepsilon_3}(1) + \circ(1).$$

Proof. The following holds for ε_2 small only.

First, we deduce from (66), (67), and Lemma 3.3 that

$$I_{K_l, \tau_l}(u_l) \geqslant I_{K_l, \tau_l}(\xi_l) + (\lambda_1^l)^{2(p_l+1)/(p_l-1)-n} c^{(1)} + \circ(1). \tag{94}$$

Second, we deduce from (85), (86), and Lemma 3.3 that

$$I_{K_l, \tau_l}(\xi_l) \geqslant I_{K_l, \tau_l}(\eta_l) + (\lambda_2^l)^{2(p_l+1)/(p_l-1)-n} c^{(2)} + \circ(1). \tag{95}$$

Third, we deduce from Claim 8 that

$$I_{K_l, \tau_l}(\eta_l) = \circ(1). \tag{96}$$

Finally, we put together (34), (94), (95), and (96) and obtain

$$\sum_{i=1}^{2} \{(\lambda_i^l)^{2(p_l+1)/(p_l-1)-n} - 1\} c^{(i)} \leqslant \varepsilon_3 + \circ(1).$$

Hence the conclusion of Claim 9.

Claim 10. Let $\delta_5 = \delta_1 / 2A_3 > 0$. Then if $\varepsilon_2 > 0$ is chosen to be small enough, we have, for large l, that

$$\operatorname{dist}(x_1^l, \partial O_l^{(1)}) \geqslant \delta_5, \tag{97}$$

and

$$\operatorname{dist}(x_2^l, \partial O_l^{(2)}) \geqslant \delta_5. \tag{98}$$

Proof. Suppose the contrary, that for a sequence of $l \to \infty$, either (97) or (98) fails. Without loss of generality, we assume that $\operatorname{dist}(x_1^l, \partial O_l^{(1)}) < \delta_5$. Using (24) and (25), we know that

$$K_l(x_1^l) \leqslant K_l(x_l^{(1)}) - \delta_1 + A_3 \delta_5 = K_l(x_l^{(1)}) - \frac{\delta_1}{2}. \tag{99}$$

Using (36), (39), (37), (38), Claim 1, Claim 2, Claim 5, (26), (99), and (21), we obtain

$$\begin{aligned}
&I_{K_l, \tau_l}(u_l)\\
&= I_{K_l,\tau_l}(\alpha_1^l\, \delta(x_1^l, \lambda_1^l)) + I_{K_l,\tau_l}(\alpha_2^l\, \delta(x_2^l, \lambda_2^l)) + \circ_{\varepsilon_2}(1)\\
&= I_{K_l,\tau_l}(K_l(x_1^l)^{(2-n)/4}\, \delta(x_2^l, \lambda_1^l)) + I_{K_l,\tau_l}(K_l(x_2^l)^{(2-n)/4}\, \delta(x_2^l, \lambda_2^l)) + \circ_{\varepsilon_2}(1)\\
&= I_{K_l}(K_l(x_2^l)^{(2-n)/4}\, \delta(x_1^l, \lambda_1^l)) + I_{K_l}(K_l(x_2^l)^{(2-n)/4}\, \delta(x_2^l, \lambda_2^l)) + \circ_{\varepsilon_2}(1)\\
&= \sum_{i=1}^{2} \left\{ \frac{1}{2} \int |\nabla (K_l(x_i^l)^{(2-n)/4}\, \delta(x_i^l, \lambda_i^l))|^2 \right.\\
&\qquad \left. - \frac{n-2}{2n} \int K_l(x_i^l)\, |K_l(_i^l)^{(2-n)/4}\, \delta(x_i^l, \lambda_i^l)|^{2n/(n-2)} \right\} + \circ_{\varepsilon_2}(1) + \circ(1)\\
&= \sum_{i=1}^{2} K_l(x_i^l)^{(2-n)/2} \frac{1}{n} \int |\nabla\, \delta(0, 1)|^2 + \circ_{\varepsilon_2}(1) + \circ(1)\\
&= \sum_{i=1}^{2} K_l(x_i^l)^{(2-n)/2} \frac{1}{n} (S_n)^n + \circ_{\varepsilon_2}(1) + \circ(1)\\
&\geqslant \left(K_l(x_l^{(1)}) - \frac{\delta_1}{2} \right)^{(2-n)/2} \frac{1}{n} (S_n)^n + K_l(x_l^{(2)})^{(2-n)/2} \frac{1}{n} (S_n)^n + \circ_{\varepsilon_2}(1) + \circ(1)\\
&= \left(\alpha^{(1)} - \frac{\delta_1}{2} \right)^{(2-n)/2} \frac{1}{n} (S_n)^n + (a^{(2)}) \frac{1}{n} (S_n)^n + \circ_{\varepsilon_2}(1) + \circ(1)\\
&= c^{(1)} \left(\frac{\alpha^{(1)}}{a^{(1)} - \delta_1/2} \right)^{(n-2)/2} + c^{(2)} + \circ_{\varepsilon_2}(1) + \circ(1).
\end{aligned} \tag{100}$$

If $\varepsilon_2 > 0$ is small enough and l large enough, (100) contradicts (34), Claim 10 has been established. ∎

Using (59), Claim 9, (56), (22), and (25), we obtain that

$$\begin{aligned}
\mathscr{T}^{-1}_{l, \lambda_2^l, x_2^l} u_l &= (\lambda_1^l)^{2/(p_l - 1)}\, u_l(\lambda_1^l(\cdot - x_1^l))\\
&= (\lambda_1^l)^{2/(p_l - 1) - (n-2)/2}\, K_\infty^{(1)}(\xi_1)^{(2-n)/4}\, \delta\left(x_1^l + \frac{x^*}{\lambda_1^l}, \lambda^* \lambda_1^l \right)
\end{aligned}$$

$$= K_\infty^{(1)}(\xi_1)^{(2-n)/4}\,\delta\left(x_1^l + \frac{x^*}{\lambda_1^l}, \lambda^*\lambda_1^l\right) + \circ_{\varepsilon_3}(1)$$

$$= K_l\left(x_1^l + \frac{x^*}{\lambda_1^l}\right)^{(2-n)/4}\delta\left(x_1^l + \frac{x^*}{\lambda_1^l}, \lambda^*\lambda_1^l\right) + \circ_{\varepsilon_3}(1) + \circ(1).$$

Similarly, we obtain

$$\mathscr{T}^{-1}_{l,\lambda_2^l,x_2^l}u_2 = K_l\left(x_2^l + \frac{x^{**}}{\lambda_2^l}\right)^{(2-n)/4}\delta\left(x_2^l + \frac{x^{**}}{\lambda_2^l}, \lambda^{**}\lambda_2^l\right) + \circ_{\varepsilon_3}(1) + \circ(1).$$

Therefore we can rewrite (93) as (using Claim 8 and the above)

$$u_l = K_l\left(x_1^l + \frac{x^*}{\lambda_1^l}\right)^{(2-n)/4}\delta\left(x_1^l + \frac{x^*}{\lambda_1^l}, \lambda^*\lambda_1^l\right)$$
$$+ K_l\left(x_2^l + \frac{x^{**}}{\lambda_2^l}\right)^{(2-n)/4}\delta\left(x_2^l + \frac{x^{**}}{\lambda_2^l}, \lambda^{**}\lambda_2^l\right) + \circ_{\varepsilon_3}(1) + \circ(1). \quad (101)$$

We now fix the value of ε_2 to be small to make all the previous arguments hold and then make $\varepsilon_3 > 0$ small (depending on ε_2, in particular) to make the following hold (using Claim 9).

$$|(\lambda^*\lambda_1^l)^{\tau_l} - 1| \leqslant \circ_{\varepsilon_3}(1) + \circ(1) < \varepsilon_2/2, \qquad |(\lambda^{**}\lambda_2^l)^{\tau_l} - 1| \leqslant \circ_{\varepsilon_3}(1) + \circ(1) < \varepsilon_2/2. \quad (102)$$

From (101), Claim 1, Claim 9, and (102), we se that for $\varepsilon_3 > 0$ small, we have

$$u_l \in \tilde{V}_l(2, \varepsilon_2/2) \qquad \text{for large } l.$$

This contradicts (33).

Proposition 3.2 has been established.

4. Completion of the Proof of Theorem 3.1.

We complete the proof of Theorem 3.1 in this section.

In this section, $\tau = \tau_l$. We define

$$\gamma_{l,\tau}^{(1)} = \{g^{(1)} \in C([0,1], H_0^1(B_{R_l}(x_l^{(1)}))) \mid g^{(1)}(0) = 0, I_{K_l,\tau}(g^{(1)}(1)) < 0\},$$

$$\gamma_{l,\tau}^{(2)} = \{g^{(2)} \in C([0,1], H_0^1(B_{R_l}(x_l^{(2)}))) \mid g_{(0)}^{(2)} = 0, I_{K_l,\tau}(g^{(2)}(1)) < 0\},$$

$$c_{l,\tau}^{(1)} = \inf_{g^{(1)} \in \gamma_{l,\tau}^{(1)}} \max_{0 \leqslant \theta_1 \leqslant 1} I_{K_l,\tau}(g^{(1)}(\theta_1)),$$

$$c_{l,\tau}^{(2)} = \inf_{g^{(2)} \in \gamma_{l,\tau}^{(2)}} \max_{0 \leqslant \theta_2 \leqslant 1} I_{K_l,\tau}(g^{(2)}(\theta_2)).$$

Here we have abused the notation of $I_{K_l,\tau}$. We consider it as $I_{K_l,\tau}\colon H_0^1(B_{R_l}(x_l^{(1)})) \to \mathbf{R}$ and also $I_{K_l,\tau}\colon H_0^1(B_{R_l}(x_l^{(2)})) \to \mathbf{R}$.

PROPOSITION 4.1. *Letting $\{K_l\}$ be a sequence of $L^\infty(\mathbf{R}^n)$ functions satisfying* (20) *and* (21), *we have*

$$c_{l,\tau}^{(1)} = c^{(1)} + \circ(1), \tag{103}$$

$$c_{l,\tau}^{(2)} = c^{(2)} + \circ(1), \tag{104}$$

where $\circ(1)$ denotes, as usual, some quantity which goes to zero as l goes to ∞.

Proof. It is easy to derive from the definitions of $c_{l,\tau}^{(1)}$ and $c_{l,\tau}^{(2)}$ that for some constant $C(A_1, n) > 0$, we have

$$\frac{1}{C(A_1, n)} + \circ(1) \leqslant c_{l,\tau}^{(1)}, c_{l,\tau}^{(2)} \leqslant C(A_1, n). \tag{105}$$

Let η be a cutoff function defined as

$$\eta \in C_c^\infty(B_1(0)),$$
$$\eta(x) = \begin{cases} 1, & |x| \leqslant \frac{1}{2}, \\ 0, & |x| \geqslant 1, \\ \geqslant 0, & \text{elsewhere}, \end{cases}$$

and λ_l be a sequence satisfying

$$\lim_{l \to \infty} \lambda_l = \infty, \qquad (\lambda_l)^{\tau_l} = 1 + \circ(1), \tag{106}$$

for any $u \in H_0^1(B_{R_l}(x_l^{(1)})) \backslash \{0\}$, it follows from the definition of $c_{l,\tau}^{(1)}$ that

$$c_{l,\tau}^{(1)} \leqslant \max_{0 \leqslant t < \infty} I_{K_l,\tau}(tu) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \frac{(\int |\nabla u|^2)^{(p+1/(p-1)}}{(\int K_l(2/(1+|\cdot|^2)^{((n-2)2)\tau}\, |u|^{p+1})^{2/(p-1)}}.$$

Throughout the paper, $p = (n+2)/(n-2) - \tau$.

Picking $u = \eta(\cdot + x_l^{(1)})\, \delta(x_l^{(1)}, \lambda_l)$, we obtain

$$c_{l,\tau}^{(1)} \leqslant \frac{1}{n} (a^{(1)})^{(2-n)/2} (S_n)^n + \circ(1).$$

The other side of the inequality can be proved as the following.

For any l, τ fixed, it is well known that there exists $\{u_k\} \subset H_0^1(B_{R_l}(x_l^{(1)}))$, such that

$$\lim_{k\to\infty} I_{K_l,\tau}(u_k) = c_{l,\tau}^{(1)},$$

$$I'_{K_l,\tau}(u_k) \to 0 \text{ strongly in } H^{-1}(B_{R_l}(x_l^{(1)})) \text{ as } k \to \infty,$$

where $H^{-1}(B_{R_l}(x_l^{(1)}))$ denotes, as usual, the dual space of $H_0^1(B_{R_l}(x_l^{(1)}))$.

It follows from some standard arguments that

$$\int_{B_{R_l}(x_l^{(1)})} |\nabla u_k|^2 = \int_{B_{R_l}(x_l^{(1)})} K_l \left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau} |u_k|^{p+1} + \circ_k(1),$$

$$\frac{1}{2}\int_{B_{R_l}(x_l^{(1)})} |\nabla u_k|^2 - \frac{1}{p+1}\int_{B_{R_l}(x_l^{(1)})} K_l \left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau} |u_k|^{p+1} = c_{l,\tau}^{(1)} + \circ_k(1). \tag{107}$$

Here $\circ_k(1)$ denotes some quantity which tends to zero as k tends to infinity.

It follows that

$$c_{l,\tau}^{(1)} = \frac{1}{n}\int_{B_{R_l}(x_l^{(1)})} |\nabla u_k|^2 + \circ(1) + \circ_k(1). \tag{108}$$

By (26), (105), (108), (107), and the definition of S_n, we have

$$\begin{aligned} S_n &\leqslant \frac{\{\int |\nabla u_k|^2\}^{1/2}}{\{\int |u_k|^{2n/(n-2)}\}^{(n-2)/2n}} \\ &\leqslant \frac{\{\int |\nabla u_k|^2\}^{1/2}}{(1+\circ(1))\{\int |u_k|^{p+1}\}^{1/(p+1)}} \\ &\leqslant \frac{\{\int |\nabla u_k|^2\}^{1/2} K_l(x_l^{(1)})^{1/(p+1)}}{(1+\circ(1))\{\int K_l(2/(1+|\cdot|^2))^{((n-2)/2)\tau} |u_k|^{p+1}\}^{1/(p+1)}} \\ &= \left\{\int |\nabla u_k|^2\right\}^{1/n} K(x_l^{(1)})^{(n-2)/2n} + \circ(1) + \circ_k(1). \end{aligned}$$

Namely,

$$\varliminf_{k\to\infty} \int |\nabla u_k|^2 \geqslant K_l(x_l^{(1)})^{(2-n)/2} (S_n)^n + \circ(1). \tag{109}$$

One deduces from (108), (21), and (109) that $c_{l,\tau}^{(1)} \geqslant (1/n)(a^{(1)})^{(2-n)/2} (S_n)^n + \circ(1)$. One can establish (104) in the same way.

We define

$$\Gamma_l = \{G = g^{(1)} + g^{(2)} \mid g^{(1)}, g^{(2)} \text{ satisfy } (110)\text{–}(114)\}.$$

$$g^{(1)}, g^{(2)} \in C([0,1]^2, E), \tag{110}$$

$$g^{(1)}(0, \theta_2) = g^{(2)}(\theta_1, 0) = 0, \qquad 0 \leqslant \theta_1, \theta_2 \leqslant 1, \tag{111}$$

$$I_{K_l,\tau}(g^{(1)}(1, \theta_2)) < 0, \; I_{K_l,\tau}(g^{(2)}(\theta_1, 1)) < 0, \qquad 0 \leqslant \theta_1, \theta_2 \leqslant 1, \tag{112}$$

$$\operatorname{supp} g^{(1)} \subset B_{R_l}(x_l^{(1)}), \qquad \theta \in [0,1]^2, \tag{113}$$

$$\operatorname{supp} g^{(2)} \subset B_{R_l}(x_l^{(2)}), \qquad \theta \in [0,1]^2. \tag{114}$$

We define also

$$b_{l,\tau} = \inf_{G \in \Gamma_l} \max_{\theta \in [0,1]^2} I_{K_l,\tau}(G(\theta)). \quad \blacksquare$$

PROPOSITION 4.2. *Letting $\{K_l\}$ be a sequence of $L^\infty(\mathbf{R}^n)$ functions satisfying* (20) *and* (21), *we have* $b_{l,\tau} = c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} + \circ(1)$.

Proof. Let $g^{(1)}, g^{(2)}$ satisfy (110)–(114). Since any curve joining $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$, after being composed with $g^{(1)}$, lies in $\gamma_{l,\tau}^{(1)}$, we know that $(I_{K_l,\tau}(g^{(1)}))^{-1}(c_{l,\tau}^{(1)}) \subset [0,1]^2$ is a nonempty compact set. For any $\varepsilon > 0$, by the continuity of $g^{(1)}$, there exists an open cover D_ε of $(I_{K_l,\tau}(g^{(1)}))^{-1}(c_{l,\tau}^{(1)})$, which is a union of finite open balls, such that $I_{K_l,\tau}(g^{(1)}) > c_{l,\tau}^{(1)} - \varepsilon$ on D_ε. As mentioned before, as any curve joining $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$, after being composed with $g^{(1)}$, lies in $\gamma_{l,\tau}^{(1)}$, it has to intersect D_ε by the definition of $c_{l,\tau}^{(1)}$. This means that we can find a curve in D_ε connecting $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$. This curve, after being composed with $g^{(2)}$, lies in $\gamma_{l,\tau}^{(2)}$ because of (111) and (112); therefore there exists at least one point $\bar{\theta}_\varepsilon$ on the curve such that $I_{K_l,\tau}(g^{(2)}(\bar{\theta}_\varepsilon)) \geqslant c_{l,\tau}^{(2)}$. Noting that $\bar{\theta}_\varepsilon \in D_\varepsilon$, we have $I_{K_l,\tau}(g^{(1)}(\bar{\theta}_\varepsilon)) \geqslant c_{l,\tau}^{(1)} - \varepsilon$. By the compactness of $[0,1]^2$, we have, for some $\bar{\theta} \in [0,1]^2$, $\lim_{\varepsilon \to 0} \bar{\theta}_\varepsilon = \bar{\theta}$ along a subsequence. Clearly, $\sum_{i=1}^2 I_{K_l,\tau}(g^{(i)}(\bar{\theta})) \geqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)}$. Therefore $b_{l,\tau} \geqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)}$.

For $0 \leqslant \theta_1, \theta_2 \leqslant 1$, let

$$g_l^{(1)}(\theta_1) = \theta_1 \, C_1 K_l(x_l^{(1)})^{(2-n)/4} \, \eta(\cdot + x_l^{(1)}) \, \delta(x_l^{(1)}, \lambda_l),$$

$$g_l^{(2)}(\theta_2) = \theta_2 \, C_1 K_l(x_l^{(2)})^{(2-n)/4} \, \eta(\cdot + x_l^{(2)}) \, \delta(x_l^{(2)}, \lambda_l),$$

where $C_1 = C_1(A_1, A_2, n) > 1$ is some constant chosen large enough that for large l we have,

$$I_{K_l,\tau}(g_l^{(1)}(1)) < 0,$$

$$I_{K_l,\tau}(g_l^{(2)}(1)) < 0.$$

It is easy to see that such a $C_1(A_1, A_2, n)$ exists. We fix the value of $C_1(A_1, A_2, n)$ from now on.

For $\theta = (\theta_1, \theta_2) \in [0, 1]^2$, let

$$G_l(\theta) = g_l^{(1)}(\theta_1) + g_l^{(2)}(\theta_2).$$

It is clear that $G_l \in \Gamma_l$ and

$$\begin{aligned}
&\max_{\theta \in [0,1]^2} I_{K_l,\tau}(G_l(\theta)) \\
&= \max_{\theta_1 \in [0,1]} I_{K_l,\tau}(g_l^{(1)}(\theta_1)) + \max_{\theta_2 \in [0,1]} I_{K_l,\tau}(g_l^{(1)}(\theta_2)) \\
&\leqslant \max_{0 \leqslant t < \infty} I_{K_l,\tau}(t\eta(\cdot + x_l^{(1)})\, \delta(x_l^{(1)}, \lambda_l)) \\
&\quad + \max_{0 \leqslant t < \infty} I_{K_l,\tau}(t\eta(\cdot + x_l^{(2)})\, \delta(x_l^{(2)}, \lambda_l)) \\
&= \frac{1}{n} (a^{(1)})^{(2-n)/2} (S_n)^n + \frac{1}{n} (a^{(2)})^{(2-n)/2} (S_n)^n + \circ(1) \\
&= c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} + \circ(1).
\end{aligned}$$

The last equality follows from Proposition 4.1. Therefore, $b_{l,\tau} \leqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} + \circ(1)$. ∎

In the following, we show that if we suppose the contrary of Theorem 3.1, we can construct $H_l \in \Gamma_l$ for large l, such that

$$\max_{\theta \in [0,1]^2} I_{K_l,\tau}(H_l(\theta)) < b_{l,\tau},$$

which contradicts the definition of $b_{l,\tau}$.

We achieve this by the following three steps.

Step 1: Choose some suitably small number $\varepsilon_4 > 0$ and construct $G_l \in \Gamma_l$, which satisfies

$$\max_{\theta \in [0,1]^2} I_{K_l,\tau}(G_l(\theta)) \leqslant b_{l,\tau} + \varepsilon_4$$

and some further properties.

Step 2: We follow the negative gradient flow of $I_{K_l,\tau}$ to deform G to U_l with

$$\max_{\theta \in [0,1]^2} I_{K_l,\tau}(U_l(\theta)) \leqslant b_{l,\tau} - \varepsilon_4.$$

However, U_l is not necessarily in Γ_l any more since the deformation may not preserve properties (113)–(114).

Step 3: Applying Propositions 3.2, 1.3, 1.4 and 2.1, we modify U_l to obtain $H_l \in \Gamma_l$ with

$$\max_{\theta \in [0,1]^2} I_{K_l,\tau}(H_l(\theta)) \leqslant b_{l,\tau} - \varepsilon_4/2.$$

All three steps are completed for large l only. Now we start to establish these three steps.

Step 1: Let G_l be the one we have just defined. We establish some properties of it which are needed.

LEMMA 4.1. *For any* $0<\varepsilon<1$, *there exists* $A_1 = A_1(\varepsilon, A_1, A_3) > 1$, *such that, for any* $l \geqslant A_1$, $0 \leqslant \theta_1, \theta_2 \leqslant 1$, *we have that*

$$I_{K_l,\tau}(g_l^{(1)}(\theta_1)) \geqslant c_{l,\tau}^{(1)} - \varepsilon \qquad \textit{implies } |C_1\theta_1 - 1| \leqslant C_0(n)\sqrt{\varepsilon}, \tag{115}$$

$$I_{K_l,\tau}(g_l^{(2)}(\theta_2)) \geqslant c_{l,\tau}^{(2)} - \varepsilon \qquad \textit{implies } |C_1\theta_2 - 1| \leqslant C_0(n)\sqrt{\varepsilon}, \tag{116}$$

Proof. Let $s = C_1\theta_1$.

$$\begin{aligned}
&I_{K_l,\tau}(g_l^{(1)}(\theta_1)) \\
&= \frac{1}{2} s^2 K_l(x_l^{(1)})^{(2-n)/2} \int |\nabla(\eta(\cdot + x_l^{(1)})\, \delta(x_l^{(1)}, \lambda_l))|^2 \\
&\quad - \frac{1}{p+1} s^{p+1} K_l(x_l^{(1)})^{-((n-2)/4)(p+1)} \\
&\quad \times \int K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau} |\eta(\cdot + x_l^{(1)})\, \delta(x_l^{(1)}, \lambda_l)|^{p+1} \\
&= \left(\frac{1}{2} + \circ(1)\right) s^2 K_l(x_l^{(1)})^{(2-n)/2} \int |\nabla\delta(0,1)|^2 \\
&\quad - \left(\frac{n-2}{2n} + \circ(1)\right) s^{p+1} K_l(x_l^{(1)})^{(2-n)/2} \int \delta(0,1)^{2n/(n-2)} \\
&= \left(\left(\frac{n}{2} + \circ(1)\right) s^2 - \left(\frac{n-2}{2} + \circ(1)\right) s^{p+1}\right) c_{l,}^{(1)}.
\end{aligned}$$

It is not difficult to see that there exists some universal constant $C_0(n) > 1$ and $A_1 = A_1(\varepsilon_1, A_1, A_3)$ such that, for $l \geqslant A_1$, (115) and (116) hold. ∎

LEMMA 4.2. *For any* $0<\varepsilon<1$, *there exists* $A_2 = A_2(\varepsilon, A_1, A_3) > A_1$, *such that, for* $l \geqslant A_2$, $0 \leqslant \theta_1, \theta_2 \leqslant 1$, *we have*

$$I_{K_l,\tau}(g_l^{(1)}(\theta_1)) \leqslant c_{l,\tau}^{(1)} + \frac{\varepsilon}{10} \tag{117}$$

and

$$I_{K_l,\tau}(g_l^{(2)}(\theta_2)) \leqslant c_{l,\tau}^{(2)} + \frac{\varepsilon}{10}. \tag{118}$$

Proof.

$$\begin{aligned}
&I_{K_l,\tau}(g_l^{(1)}(\theta_1))\\
&\leqslant \max_{0<t<\infty} I_{K_l,\tau}(tg_l^{(1)}(\theta_1))\\
&=\frac{1}{n}\frac{\{\int \nabla(\eta(\cdot+x_l^{(1)})\,\delta(x_l^{(1)},\lambda_l))|^2\}^{n/2}}{\{\int K_l(2/(1+|\cdot|^2))^{((n-2)/2)\tau}\,|\eta(\cdot+x_l^{(1)})\,\delta(x_l^{(1)},\lambda_l)|^{p+1}\}^{2/(p-1)}}\\
&=\frac{1}{n}\frac{\{\int \nabla\delta(x_l^{(1)},\lambda_l)|^2+\circ(1)\}^{n/2}}{\{\int K_l(x_l^{(1)})\,\delta(x_l^{(1)},\lambda_l)^{2n/(n-2)}+\circ(1)\}^{(n-2)/2}}\\
&=c_{l,\tau}^{(1)}+\circ(1).
\end{aligned}$$

Therefore, there exists $A_2 = A_2(\varepsilon, A_1, A_3) > A_1$, such that, for $l \geqslant A_2$, $0 \leqslant \theta_1 \leqslant 1$, we have (117). It is clear that (118) follows in a similar way. ∎

LEMMA 4.3. *For any* $0<\varepsilon<1$, *there exists* $A_3 = A_3(\varepsilon, A_1, A_3) > A_2$, *such that, for* $l \geqslant A_3$, *we have*

$$I_{K_l,\tau}(G_l(\theta))|_{\theta\in\partial[0,1]^2} \leqslant \max\{c^{(1)}+\varepsilon, c^{(2)}+\varepsilon\}.$$

Proof. Lemma 4.3 follows easily from (117) and (118). ∎

LEMMA 4.4. *There exists some universal constant* $C_0 = C_0(n) > 1$, *such that, for any* $0<\varepsilon<\frac{1}{2}$, $l \geqslant A_3(\varepsilon, A_1, A_2)$, $\theta \in [0,1]^2$,

$$I_{K_l,\tau}(G_l(\theta)) \geqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} - \varepsilon \text{ implies } |C_1\theta_i - 1| \leqslant C_0\sqrt{\varepsilon}, \qquad i=1,2.$$

Proof. Since $g_l^{(1)}(\theta_1)$ *and* $g_l^{(2)}(\theta_2)$ have disjoint supports, we have

$$I_{K_l,\tau}(G_l(\theta)) = I_{K_l,\tau}(g_l^{(1)}(\theta_1)) + I_{K_l,\tau}(g_l^{(2)}(\theta_2)).$$

It follows from Lemma 4.2 that

$$I_{K_l,\tau}(g_l^{(1)}(\theta_1)) \leqslant c_{l,\tau}^{(1)} + \varepsilon/10,$$
$$I_{K_l,\tau}(g_l^{(2)}(\theta_2)) \leqslant c_{l,\tau}^{(2)} + \varepsilon/10.$$

Therefore, $I_{K_l,\tau}(G_l(\theta)) \geqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} - \varepsilon$ implies that

$$I_{K_l,\tau}(g_l^{(1)}(\theta_1)) \geqslant c_{l,\tau}^{(1)} - 2\varepsilon,$$
$$I_{K_l,\tau}(g_l^{(2)}(\theta_2)) \geqslant c_{l,\tau}^{(2)} - 2\varepsilon.$$

Lemma 4.4 follows immediately from Lemma 4.1 and the above two inequalities.

Let

$$M_l = \sup\{\|I'_{K_l,\tau}(u)\| \mid u \in V_l(2, \varepsilon_1)\},$$
$$\beta_l = \mathrm{dist}(\partial \tilde{V}_l(2, \varepsilon_2), \partial \tilde{V}_l(2, \varepsilon_2/2)).$$

It is very easy to see that there exists some constant $C_2 = C_2(A_1, \varepsilon_2) > 1$, such that

$$M_l \leqslant C_2(A_1, \varepsilon_2). \tag{119}$$

It is also clear from the definition of $\tilde{V}_l(2, \varepsilon_2)$ that

$$\beta_l \geqslant \frac{\varepsilon_2}{4}. \tag{120}$$

Using Lemma 4.4, we choose ε_4 to satisfy, for l large, that

$$\varepsilon_4 < \varepsilon_3, \qquad \varepsilon_4 < 1/2A_4, \tag{121}$$

$$\varepsilon_4 < \frac{\varepsilon_2\, \delta_2(\varepsilon_2, \varepsilon_3)^2}{8C_2(A_1, \varepsilon_2)}, \tag{122}$$

$$\begin{aligned} &I_{K_l,\tau}(G_l(\theta)) \geqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} - \varepsilon_4 \text{ implies that} \\ &G_l(\theta) \in \tilde{V}_l(2, \varepsilon_2/2),\, x_1(G_l(\theta)) \in O_l^{(1)},\, x_2(G_l(\theta)) \in O_l^{(2)}. \end{aligned} \tag{123}$$

We know from Lemma 4.2 that for l large enough

$$\max_{\theta \in [0,1]^2} I_{K_l}(G_l(\theta)) \leqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} + \varepsilon_4.$$

We consider the negative gradient of $I_{K_l,\tau}$,

$$\begin{aligned} &\frac{d}{ds}\xi(s, u_0) = -I'_{K_l,\tau}(\xi(s, u_0)), \qquad s \geqslant 0 \\ &\xi(0, u_0) = u_0. \end{aligned} \tag{124}$$

It is well known that $I_{K_l,\tau}$ satisfies the Palais Smale condition and, due to the contrary of Theorem 3.1, the flow defined by (124) never stops before exiting $V_l(2, \varepsilon^*)$.

We define $U_l \in C([0,1]^2, E)$ by the following. If $I_{K_l,\tau}(G_l(\theta)) \leqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} - \varepsilon_4$, we define $s_l^*(\theta) = 0$. If $I_{K_l,\tau}(G_l(\theta)) > c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} - \varepsilon_4$, then, according to (123), $G_l(\theta) \in \tilde{V}_l(2, \varepsilon_2/2)$, $x_1(G_l(\theta)) \in O_l^{(1)}$, and $x_2(G_l(\theta)) \in O_l^{(2)}$. We define $s_l^*(\theta) = \min\{s > 0 \mid I_{K_l,\tau}(\xi(s, G_l(\theta)) = c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} - \varepsilon_4\}$ and $U_l(\theta) = \xi(s_l^*(\theta), G_l(\theta))$.

The above definition is justified in the following.

LEMMA 4.5. *For any* $u_0 \in \tilde{V}_l(2, \varepsilon_2/2)$ *with* $x_1(u_0) \in O_l^{(1)}$, $x_2(u_0) \in O_l^{(2)}$, *and* $c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} - \varepsilon_4 < I_{K_l,\tau}(u_0) \leqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} + \varepsilon_4$, *the flow line* $\xi(s, u_0)$ $(s \geqslant 0)$ *cannot leave* $\tilde{V}_l(2, \varepsilon_2)$ *before reaching* $I_{K_l,\tau}^{-1}(c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} - \varepsilon_4)$.

Proof. Suppose Lemma 4.5 is false, then we can find $0 < s_1 < s_2$ with

$$I_{K_l,\tau}(\xi(x, u_0)) \geqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} - \varepsilon_4, \qquad s_1 \leqslant s \leqslant s_2,$$

$$\xi(s_1, u_0) \in \partial\tilde{V}_l(2, \varepsilon_2/2),$$

$$\xi(s_2, u_0) \in \partial\tilde{V}_l(2, \varepsilon_2),$$

$$\xi(s, u_0) \in \tilde{V}_l(2, \varepsilon_2)\backslash\tilde{V}_l(2, \varepsilon_2/2), \qquad s_1 \leqslant s \leqslant s_2.$$

Because of (124) we have, for $s \geqslant 0$ and l large, that

$$\begin{aligned} I_{K_l,\tau}(\xi(s, u_0)) &\leqslant I_{K_l,\tau}(u_0) \\ &\leqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} + \varepsilon_4 \\ &\leqslant c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} + \frac{1}{2A_4} \\ &< c^{(2)} + c^{(2)} + \frac{1}{A_4}. \end{aligned}$$

The last inequality holds for l sufficiently large. It then follows from the definition of A_4 (See Lemma 3.1) that

$$\operatorname{dist}(x_1(\xi(s, u_0)), \partial O_l^{(1)}) \geqslant \delta_1/2A_3, \qquad 0 \leqslant s \leqslant s_2,$$

$$\operatorname{dist}(x_2(\xi(s, u_0)), \partial O_l^{(2)}) \geqslant \delta_1/2A_3, \qquad 0 \leqslant s \leqslant s_2.$$

Consequently,

$$x_1(\xi(s, u_0)) \in O_l^{(1)}, \qquad 0 \leqslant s \leqslant s_2,$$

$$x_2(\xi(s, u_0)) \in O_l^{(2)}, \qquad 0 \leqslant s \leqslant s_2.$$

Using Proposition 3.2, we have

$$\begin{aligned} 2\varepsilon_4 &\geqslant I_{K_l,\tau}(\xi(s_1, u_0)) - I_{K_l,\tau}(\xi(s_2, u_0)) \\ &= \int_{s_2}^{s_1} \frac{d}{ds} I_{K_l,\tau}(\xi(s, u_0))\, ds \\ &= \int_{s_1}^{s_2} \|I'_{K_l,\tau}(\xi(s, u_0))\|^2\, ds \\ &\geqslant \int_{s_1}^{s_2} \delta_4(\varepsilon_2, \varepsilon_3)^2\, ds \\ &= \delta_4(\varepsilon_2, \varepsilon_3)^2\,(s_2 - s_1). \end{aligned}$$

On the other hand,

$$\begin{aligned}\beta_l &\leqslant \|\xi(s_1, u_0) - \xi(s_2, u_0)\| \\ &\leqslant \int_{s_1}^{s_2} \left\| \frac{d}{ds} \xi(s, u_0) \right\| ds \\ &\leqslant M_l(s_2 - s_1).\end{aligned}$$

Using (120) and (119), we obtain that

$$\begin{aligned}\frac{\varepsilon_2}{4} &\leqslant M_l(s_2 - s_1) \\ &\leqslant C_2(A_1, \varepsilon_2) \frac{2\varepsilon_4}{\delta_4(\varepsilon_2, \varepsilon_3)^2},\end{aligned}$$

which contradicts (122). ∎

We can see from Lemma 4.5 that $s_l^*(\theta)$ is well defined. Since $I_{K_l,\tau}$ has no critical point in $\tilde{V}_l(2, \varepsilon_2) \cap \{u \mid u \in E,\ |I_{K_l,\tau}(u) - c^{(1)} - c^{(2)}| \leqslant \varepsilon_4\} \subset V_l(2, \varepsilon^*) \cap \{u \mid u \in E,\ |I_{K_l,\tau}(u) - c^{(1)} - c^{(2)}| \leqslant \varepsilon^*\}$ under the contradiction hypothesis, $s_l^*(\theta)$ is continuous in θ, hence $U_l \in C([0, 1]^2, E)$.

It follows that if $I_{K_l,\tau}(G_l(\theta)) > c_{l,\tau}^{(1)} + c_{l,\tau}^{(2)} - \varepsilon_4$, then $U_l(\theta) \in \tilde{V}_l(2, \varepsilon_2) \subset V_l(2, \circ_{\varepsilon_2}(1))$, $x_1(U_l(\theta)) \in O_l^{(1)}$, $x_2(U_l(\theta)) \in O_l^{(2)}$.

This implies that

$$\int_{\Omega_l} |\nabla U_l(\theta)|^2 + |U_l(\theta)|^{2n/(n-2)} \leqslant \circ_{\varepsilon_2}(1), \tag{125}$$

$$\|U_l(\theta)\|_{H^{1/2}(\partial(B_r(x_l^{(1)}) \cup B_r(x_l^{(2)})))} \leqslant \circ_{\varepsilon_2}(1), \tag{126}$$

where

$$\begin{aligned}\Omega_l &= \mathbf{R}^n \setminus \{B_r(x_l^{(1)}) \cup B_r(x_l^{(2)})\}, \\ r &= 4(\operatorname{diam} O^{(1)} + \operatorname{diam} O^{(2)}), \\ \operatorname{diam} O^{(1)} &= \sup\{|x - y| \mid x, y \in O^{(1)}\}, \\ \operatorname{diam} O^{(2)} &= \sup\{|x - y| \mid x, y \in O^{(2)}\}.\end{aligned}$$

Without loss of generality we can assume that $\varepsilon_2 > 0$ has been so small that we can apply Proposition 2.1. We modify $U_l(\theta)$ in Ω_l after making the following minimization.

Let

$$\varphi_l(\theta) = U_l(\theta)|_{\partial(B_r(x_l^{(1)}) \cup B_r(x_l^{(2)}))}.$$

Because of (125) and (126), we can apply Proposition 2.1 to obtain the minimizer $u_{\varphi_l}(\theta)$ to

$$\min\left\{I_{K_l,\tau}(u) \mid u\in E_{\Omega_l},\ u|_{\partial(B_r(x_l^{(1)})\cup B_r(x_l^{(2)}))}=\varphi_l(\theta),\ \int_{\Omega_l}|\nabla u|^2\leqslant C_0 r_0^2\right\},$$

where E_{Ω_l} is the closure of $\{u \mid u\in C^\infty(\bar{\Omega}_l),\ \operatorname{supp} u \text{ is compact}\}$ under the norm $\|u\|_{E_{\Omega_l}}=(\int_{\Omega_l}|\nabla u|^2)^{1/2}+(\int_{\Omega_l}|u|^{2n/(n-2)})^{(n-2)/2n}$, and $r_0=r_0(n, A_1)>0$ is the constant given by Proposition 2.1.

We define for $\theta\in[0,1]^2$ that

$$W_l(\theta)(x)=\begin{cases}U_l(\theta)(x), & x\in\mathbf{R}^n\setminus\Omega_l,\\ u_{\varphi_l(\theta)}(x), & x\in\Omega_l.\end{cases}$$

It follows from Proposition 2.1 that $W_l\in C([0,1]^2, E)$ and satisfies

$$\max_{\theta\in[0,1]^2} I_{K_l,\tau}(W_l(\theta))\leqslant\max_{\theta\in[0,1]^2} I_{K_l,\tau}(U_l(\theta))\leqslant c_{l,\tau}^{(1)}+c_{l,\tau}^{(2)}-\varepsilon_4, \tag{127}$$

$$\int_{\Omega_l}|\nabla W_l(\theta)|^2+|W_l(\theta)|^{2n/(n-2)}\leqslant o_{\varepsilon_2}(1),$$

$$-\Delta W_l(\theta)=K_l(\cdot)\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau}|W_l(\theta)|^{p-1}W_l(\theta),\qquad \text{in } \Omega_l. \tag{128}$$

For $l_2>100 l_1$, $l_1>10r$ (we determine the values of l_1, l_2 at the end), we introduce cutoff functions $\eta_l\in C_c^\infty(\mathbf{R}^n)$ for large l,

$$\eta_l(x)=\begin{cases}1, & |x-x_l^{(1)}|\leqslant l_1 \text{ or } |x-x_l^{(2)}|\leqslant l_1,\\ 0, & |x-x_l^{(1)}|\geqslant l_2 \text{ and } |x-x_l^{(2)}|\geqslant l_2,\\ \geqslant 0, & \text{elsewhere.}\end{cases}$$

$$|\nabla\eta_l(x)|\leqslant\frac{10}{l_2-l_1},\qquad x\in\mathbf{R}^n.$$

Let

$$H_l(\theta)=\eta_l W_l(\theta).$$

Multiplying (128) by $(1-\eta_l)\,W_l(\theta)$ and integrating by parts, we have

$$\int\nabla((1-\eta_l)\,W_l(\theta))\,\nabla W_l(\theta)=\int K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau}(1-\eta_l)\,|W_l(\theta)|^{p+1}.$$

Therefore

$$\int_{(\mathbf{R}^n\setminus B_{l_2}(x_l^{(1)}))\cap(\mathbf{R}^n\setminus B_{l_2}(x_l^{(2)}))}\left[|\nabla W_l(\theta)|^2-K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau}|W_l(\theta)|^{p+1}\right]$$

$$=\int_{(B_{l_2}(x_l^{(1)})\setminus B_{l_1}(x_l^{(1)}))\cup(B_{l_2}(x_l^{(2)})\setminus B_{l_1}(x_l^{(2)}))}\left[-\nabla((1-\eta_l)\,W_l(\theta))\,\nabla W_l(\theta)\right.$$

$$\left.+K_l(1-\eta_l)\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)}|W_l(\theta)|^{p+1}\right]$$

$$\geqslant-\int_{(B_{l_2}(x_l^{(1)})\setminus B_{l_2})(x_l^{(1)}))\cup(B_{l_2}(x_l^{(2)})\setminus B_{l_1}(x_l^{(2)}))}\left[\frac{10}{l_2-l_1}|W_l(\theta)|\,|\nabla W_l(\theta)|\right.$$

$$\left.+|\nabla W_l(\theta)|^2+A_1\,|W_l(\theta)|^{p+1}\right].$$

It follows from Proposition 1.4 that there exists some constant $C_3=C_3(A_1,n)>1$, such that for l large enough,

$$|W_l(\theta)(x)|\leqslant\frac{C_3(A_1,n)}{|x-x_l^{(1)}|^{n-2}},\qquad l_1\leqslant|x-x_l^{(1)}|\leqslant l_2 \tag{129}$$

$$|W_l(\theta)(x)|\leqslant\frac{C_3(A_1,n)}{|x-x_l^{(2)}|^{n-2}},\qquad l_1\leqslant|x-x_l^{(2)}|\leqslant l_2, \tag{130}$$

$$|\nabla W_l(\theta)(x)|\leqslant\frac{C_3(A_1,n)}{|x-x_l^{(1)}|^{n-1}},\qquad l_1\leqslant|x-x_l^{(1)}|\leqslant l_2, \tag{131}$$

$$|\nabla W_l(\theta)(x)|\leqslant\frac{C_3(A_1,n)}{|x-x_l^{(2)}|^{n-1}},\qquad l_1\leqslant|x-x_l^{(2)}|\leqslant l_2, \tag{132}$$

Consequently, we obtain

$$\int_{(\mathbf{R}^n\setminus B_{l_2}(x_l^{(1)}))\cap(\mathbf{R}^n\setminus B_{l_2}(x_l^{(2)}))}\left[|\nabla W_l(\theta)|^2-K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau}|W_l(\theta)|^{p+1}\right]$$

$$\geqslant-C_0C_3(A_1,n)\left\{\frac{\log(l_2)}{l_2-l_1}+\frac{1}{l_1}\right\}. \tag{133}$$

Using (129)–(133), we obtain

$$I_{K_l,\tau}(H_l(\theta))$$

$$=\frac{1}{2}\int|\nabla(\eta_lW_l(\theta))|^2-\frac{1}{p+1}\int K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau}|\eta_lW_l(\theta)|^{p+1}$$

$$=\frac{1}{2}\int|\nabla\eta_l|^2\,|W_l(\theta)|^2+\int\eta_lW_l(\theta)\,\nabla\eta_l\,\nabla W_l(\theta)$$

$$+\frac{1}{2}\int|\eta_l|^2\,|\nabla W_l(\theta)|^2-\frac{1}{p+1}\int K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau}|\eta_lW_l(\theta)|^{p+1}$$

$$\leqslant I_{K_l,\tau}(W_l(\theta))+C_0C_3(A_1,\varepsilon_3)\frac{\log l_2}{l_2-l_1}-\frac{1}{2}\int(1-|\eta_l|^2)\,|\nabla W_l(\theta)|^2$$

$$+\frac{1}{p+1}\int K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau}(1-|\eta_l|^{p+1})\,|W_l(\theta)|^{p+1}$$

$$\leqslant I_{K_l,\tau}(W_l(\theta))+C_0C_3(A_1,\varepsilon_3)\frac{\log l_2}{l_2-l_1}$$

$$+\int_{(\mathbf{R}^n\setminus B_{l_2}(x_l^{(1)}))\cap(\mathbf{R}^n\setminus B_{l_2}(x_l^{(2)}))}\left[-\frac{1}{2}|\nabla W_l(\theta)|^2\right.$$

$$\left.+\frac{1}{p+1}K_l\left(\frac{2}{1+|\cdot|^2}\right)^{((n-2)/2)\tau}|W_l(\theta)|^{p+1}\right]$$

$$\leqslant c_{l,\tau}^{(1)}+c_{l,\tau}^{(2)}-\varepsilon_4+C_0C_3(A_1,\varepsilon_3)\frac{\log l_2}{l_2-l_1}+C_0C_3(A_1)\cdot\frac{1}{l_1}.$$

The last inequality follows from (127) and (133).

Therefore we have, for large l, that

$$I_{K_l,\tau}(H_l(\theta))\leqslant c_{l,\tau}^{(1)}+c_{l,\tau}^{(2)}-\varepsilon_4+C_0C_3(A_1,\varepsilon_3)\left\{\frac{\log l_2}{l_2-l_1}+\frac{1}{l_1}\right\}. \tag{134}$$

Now we choose $l_1>10r$, $l_2>100l_1$ to be so large that

$$C_0C_3(A_1,\varepsilon_3)\left\{\frac{\log l_2}{l_2-l_1}+\frac{1}{l_1}\right\}<\frac{\varepsilon_4}{2}. \tag{135}$$

Then for l large enough (depending on l_1, l_2, ε's, C's), we have

$$H_l\in\Gamma_l. \tag{136}$$

It follows from (134) and (135) that for l sufficiently large,

$$\max_{\theta\in[0,1]^2} I_{K_l,\tau}(H_l(\theta))\leqslant c_{l,\tau}^{(1)}+c_{l,\tau}^{(2)}-\frac{\varepsilon_4}{2}<b_{l,\tau}. \tag{137}$$

Equalities (136) and (137) contradict the definition of $b_{l,\tau}$.

Theorem 3.1 has been established. Remark 3.1 can be established without much change; we leave the details to the reader.

5. Main Theorem and Its Applications

In this section we present our main theorem from which we deduce Theorems 0.1, 0.2, and 0.3.

THEOREM 5.1. *Assume that* $\{K_l\}$ *is a sequence of functions in* $L^{\infty}(\mathbf{R}^n)$ $(n=3,4)$ *satisfying conditions* (i), (ii), (iii) (*see Section* 3). *When* $n=4$ *we further assume that* $K_l \in C^2(\bigcup_{i=1}^m O_l^{(i)})$ *and* $\Delta K_l(x) \geqslant 0$ *for all*

$$x \in O_l^{(i)}, K_l(x_l^{(i)}) - \delta_0 \leqslant K_l(x) \leqslant K_l(x_l^{(i)}), \qquad \nabla K_l(x) = 0.$$

Assume also that there exist some bounded open sets $O^{(1)}, ..., O^{(m)} \subset \mathbf{R}^n$ *and some positive constants* $\delta_2, \delta_3 > 0$, *such that, for all* $1 \leqslant i \leqslant m$,

$$\tilde{O}_l^{(i)} - x_l^{(i)} \subset O^{(i)}, \qquad \text{for all } l,$$

$$\{u \mid I'_{K_\infty^{(i)}}(u) = 0, u > 0, u \in E, c^{(i)} \leqslant I_{K_\infty^{(i)}}(u) \leqslant c^{(i)} + \delta_2\}$$
$$\cap V(1, \delta_3, O^{(i)}, K_\infty^{(i)}) = \phi. \tag{138}$$

Then for any $\varepsilon > 0$, *there exists integer* $\bar{l}_{\varepsilon,m} > 0$, *such that, for all* $l \geqslant \bar{l}_{\varepsilon,m}$, *there exists* $u_l \in V_l(m, \varepsilon)$ *which solves*

$$\begin{aligned} -\Delta u_l &= K_l(x)\, u_l^{(n+2)/(n-2)}, \qquad \text{in } \mathbf{R}^n, \\ u_l &\in E, \\ u_l &> 0. \end{aligned} \tag{139}$$

Furthermore, u_l *satisfies*

$$\sum_{i=1}^{m} c^{(i)} - \varepsilon \leqslant I_{K_l}(u_l) \leqslant \sum_{i=1}^{m} c^{(i)} + \varepsilon. \tag{140}$$

Remark 5.1. Equation (138) can be replaced by

$$\sup\{\|u\|_{L^\infty} \mid I'_{K_\infty^{(i)}}(u) = 0, u > 0, u \in E, c^{(i)} \leqslant I_{K_\infty^{(i)}}(u) \leqslant c^{(i)} + \delta_2\} \leqslant 1/\delta_3.$$

To prove Theorem 5.1, we apply Theorem 3.1 to obtain (for large l) $u_{l,\tau}$ $(0 < \tau < \bar{\tau}_l)$ as solutions of (30). Noting that $u_{l,\tau}$ belongs to $V_l(m, \circ_{\varepsilon_2}(1))$, which consist of functions with m $(m \geqslant 2)$ "bumps," we can apply some recent results of Schoen to conclude that, as τ tends to zero, there is no blow up occurring under the hypotheses of Theorem 5.1. Namely, $u_{l,\tau}$ remains bounded as τ tends to zero. Therefore it follows from the standard elliptic theories that there exists $u_l \in V_l(m, \circ_{\varepsilon_2}(1))$, such that, as $\tau \to 0$ along a subsequence, $u_{l,\tau} \to u_l$ in $W^{2,q}_{\text{loc}}(\mathbf{R}^n)$ $(\forall q > 0)$ and u_l satisfies (139) and (140). Theorem 5.1 is then established.

We establish the boundedness of $\{u_{l,\tau}\}$ (as $\tau \to \infty$) in the following.

First, by some standard blow up arguments, the blow up point (see Definition B.1 in Appendix B) cannot occur in $\mathbf{R}^n \backslash (\bigcup_{i=1}^m \tilde{O}_l^{(i)})$ since the energy of $\{u_{l,\tau}\}$ in the region is small by the fact that $u_{l,\tau} \in V_l(m, \circ_{\varepsilon_2}(1))$ and the definition of $V_l(m, \circ_{\varepsilon_2}(1))$.

Therefore the blow up points can occur only in $\bigcup_{i=1}^m \tilde{O}_l^{(i)}$. By the structure of functions in $V_l(m, \circ_{\varepsilon_2}(1))$ and some blow up arguments (see the proof of Proposition 2.1 in [Li1]) we obtain that there are at most m isolated blow up points (See Definition B.2 in Appendix B); namely, the blow up occurs in $\{\bar{y}_1, ..., \bar{y}_m\}$, for some $\bar{y}_i \in \tilde{O}_l^i$.

In the following we apply some results of Schoen.

Using the fact that we are in dimension $n = 3, 4$, we conclude (see Theorems B.2, B.3 in Appendix B) that an isolated blow up has to be an isolated simple blow up (See Definition B.3 in Appendix B). From the structure of functions in $V_l(m, \circ_{\varepsilon_2}(1))$ we know that if the blow up does occur, there have to be exactly m isolated simple blow up points. Let us consider this situation only namely, $\{\bar{y}_1, ..., \bar{y}_m\}$ is the blow up set and they are all isolated simple blow up points.

In the following we assume that $p_i = (n+2)/(n-2) - \tau_i$, $\tau_i \to 0^+$ is the subsequence such that the blow up occurs at $u_i = u_{l,\tau_i}$ and all the results in Appendix B apply to u_i. Here and in the following we suppress the dependence of l in the notation since l is held fixed in the blow up analysis.

In our situation, $K_i(x) = K(x)(2/(1+|x|^2))^{((n-2)/2)\tau_i}$ is the sequence of functions in Appendix B.

Let $\bar{y} = \bar{y}_1$ and $\{y_i\}$ be the sequence as in Definitions B.2 and B.3. Applying (164) to u_i, we obtain

$$\begin{aligned}
\frac{2n}{n-2} \int_{\partial B_\sigma(y_i)} \sigma^n u_i^2 T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) d\xi
&= c(n)^{-1} \left(1 + \frac{\tau_i}{p_i+1}\right) \int_{B_\sigma(y_i)} r \frac{\partial K_i}{\partial r} u_i^{p_i+1}\, dx \\
&\quad + \frac{n\tau_i}{p_i+1} c(n)^{-1} \int_{B_\sigma(y_i)} K_i u_i^{p_i+1} \\
&\quad - \frac{\tau_i \sigma^n}{p_i+1} c(n)^{-1} \int_{\partial B_\sigma(y_i)} K_i u_i^{p_i+1}\, d\xi,
\end{aligned} \tag{141}$$

where $(n) = (n-2)/(4(n-1))$ and

$$T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = (n-2)\, u_i^{2/(n-2)} \left\{\frac{\partial^2}{\partial r^2}(u_i^{-2/(n-2)}) - \frac{1}{n} \Delta(u_i^{-2/(n-2)})\right\}.$$

Arguing as in Appendix B (see (182)), we have

$$\lim_{i\to\infty} u_i(y_i)^2 \frac{\tau_i \sigma^n}{p_i+1} c(n)^{-1} \int_{\partial B_\sigma} K_i u_i^{p_i+1}\, d\xi = 0. \tag{142}$$

LEMMA 5.1. *Under the hypotheses of Theorem 5.1, we have, for each* $0<\sigma<1$, *that*

$$\varliminf_{i\to\infty} u_i(y_i)^2 \int_{B_\sigma} (y-y_i)\cdot\nabla K_i(y)\, u_i^{p_i+1}(y)\, dy \geq 0.$$

Proof. First we look at the case $n=3$, Note that $\{K_i\}$ has uniform $C^1_{\rm loc}$ continuity in our situation, therefore we have

$$\begin{aligned}
&\int_{B_\sigma(y_i)} (y-y_i)\cdot\nabla K_i u_i^{p_i+1}\\
&\quad=\int_{B_\sigma(y_i)} (y-y_i)\cdot\nabla K_i(y_i)\, u_i^{p_i+1}+\int_{B_\sigma(y_i)} \circ(|y-y_i|)\, u_i^{p_i+1}\\
&\quad=\circ(u_i(y_i)^{-(p_i-1)/2}).
\end{aligned}\tag{143}$$

The last equality follows from Propositions B.1 and B.3 in Appendix B. In particular we have used the evenness of the function $(1+k_i\,|y|^2)^{(2-n)/2}$.

When $n=3$, Lemma 5.1 follows from the above estimate, Corollary B.1 in Appendix B, and the fact that in dimension $n=3$, $p_i=5-\tau_i$.

Next we look at the case when $n=4$. It is easy to see that the only thing we need to do is to obtain a better estimate than (143).

Choosing an orthonormal coordinate (depending on i) such that the Hessian of K_i at y_i is diagonal, we have

$$\begin{aligned}
&\int_{B_\sigma(y_i)} (y-y_i)\cdot\nabla K_i u_i^{p_i+1}\\
&\quad=\int_{B_\sigma(y_i)} (y-y_i)\cdot\nabla K_i(y_i)\, u_i^{p_i+1}+\int_{B_\sigma(y_i)} \sum_{k=1}^{n} \frac{\partial^2 K_i}{\partial x_k^2}(y_i)(y-y_i)_k^2\, u_i^{p_i+1}\\
&\qquad+\int_{B_\sigma(y_i)} \circ(|y-y_i|^2)\, u_i^{p_i+1}.
\end{aligned}$$

Applying Proposition B.5, we obtain

$$|\nabla K_i(y_i)| = O(u_i(y_i)^{-(p_i-1)/2}).$$

Applying Proposition B.3 and the evenness of $(1+k_i\,|y|^2)^{(2-n)/2}$, we have

$$\int_{B_\sigma(y_i)} (y-y_i)\, u_i^{p_i+1} = \circ(u_i(y_i)^{-(p_i-1)/2}).$$

Therefore

$$\int_{B_\sigma(y_i)} (y-y_i)\cdot\nabla K_i(y_i)\, u_i^{p_i+1} = \circ(u_i(y_i)^{-(p_i-1)}).$$

Applying Proposition B.3 and Corollary B.1 we have (here $p_i = 3 - \tau_i$ since $n = 4$)

$$\int_{B_\sigma(y_i)} \sum_{k=1}^{n} \frac{\partial^2 K_i}{\partial x_k^2}(y_i)(y-y_i)_k^2 u_i^{p_i+1}$$

$$= \sum_{k=1}^{n} \frac{\partial^2 K_i}{\partial x_k^2}(y_i) \int_{B_\sigma(y_i)} (y-y_i)_k^2 u_i^{p_i+1}$$

$$= \Delta K_i(y_i)\, u_i(y_i)^{-2} \left(\frac{1}{4} \int_{\mathbf{R}^4} \frac{|y|^2}{(1+k_i|y|^2)^4}\, dy + \circ(1) \right).$$

Therefore (using $\underline{\lim}_{i\to\infty} \Delta K_i(y_i) \geqslant 0$)

$$\lim_{i\to\infty} u_i(y_i)^2 \int_{B_\sigma(y_i)} \sum_{k=1}^{n} \frac{\partial^2 K_i}{\partial x_k^2}(y_i)(y-y_i)_k^2\, u_i^{p_i+1} \geqslant 0.$$

Applying Proposition B.3 and Corollary B.1, we have

$$\int_{B_\sigma((y_i)} \circ(|y-y_i|^2)\, u_i^{p_i+1} = \circ(u_i^{-(p_i-1)}).$$

Note that we are in the case $n = 4$ and therefore Lemma 5.1 follows from the above estimates, Corollary B.1 in Appendix B, and $p_i = 3 - \tau_i$.

Using (141), (142), and Lemma 5.1, we can deduce as in Appendix B that

$$\frac{2n}{n-2}\sigma^n \int_{\partial B_\sigma} (\lim_{i\to\infty} u_i(y_i)\, u_i(y))^2\, T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)(\lim_{i\to\infty} u_i(y_i)\, u_i(y))\, d\xi \geqslant 0, \tag{144}$$

where

$$\begin{aligned} T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)&(\lim_{i\to\infty} u_i(y_i)\, u_i(y)) \\ &= (n-2)(\lim_{i\to\infty} u_i(y_i)\, u_i)^{2/(n-2)} \\ &\quad \times \left\{ \frac{\partial^2}{\partial r^2}((\lim_{i\to\infty} u_i(y_i)\, u_i)^{-2/(n-2)} - \frac{1}{n}\Delta((\lim_{i\to\infty} u_i(y_i)\, u_i)^{-2/(n-2)}) \right\}. \end{aligned} \tag{145}$$

Applying Proposition B.2 and the maximum principle, we deduce that for some positive constants $a_i > 0$ $(1 \leqslant i \leqslant m)$ and some nonnegative harmonic function $\bar{\alpha}(y)$ in $\mathbf{R}^n$, we have, in $C^0_{\mathrm{loc}}(\mathbf{R}^n \setminus \{\bar{y}, \bar{y}_2, ..., \bar{y}_m\})$, that

$$\lim_{i\to\infty} u_i(y_i)\, u_i(y) = \frac{a_1}{|y-\bar{y}|^{n-2}} + \sum_{k=2}^{m} \frac{a_k}{|y-\bar{y}_k|^{n-2}} + \bar{\alpha}(y).$$

In particular, in a small punctured disc centered at $\bar{y}$, we have

$$\lim_{i \to \infty} u_i(y_i)\, u_i(y) = \frac{a_1}{|y-\bar{y}|^{n-2}} + A + \alpha(y), \tag{146}$$

where $a_1, A > 0$ are positive constants, α is a smooth function near $y = \bar{y}$ with $\alpha(\bar{y}) = 0$.

Choosing $\sigma > 0$ small enough, it follows from a direct computation (using (146) and (145)) that (144) leads to a contradiction. This implies that there will be no blow up under the hypotheses of Theorem 5.1 and hence Theorem 5.1 is established.

Proof of Theorem 0.1. Let $\psi \in C^\infty(\mathbf{R}^n)$ satisfy

$$\|\psi\|_{C^2(\mathbf{R}^n)} < \infty, \tag{147}$$

$$\lim_{|x| \to \infty} \psi(x) =: \psi_\infty > 0, \tag{148}$$

$$\Delta\psi(0) = 0, \tag{149}$$

and

$$\sum_{i=1}^{n} \psi_{x_i}(x)\, x_i < 0 \qquad \forall x \neq 0. \tag{150}$$

Remark 5.1. It is well known that under (147), (148), and (150), ψ violates the Kazdan–Warner conditions and therefore

$$-\Delta u = \psi\, |u|^{4/(n-2)}\, u \text{ in } \mathbf{R}^n \tag{151}$$

has no nontrivial solution in E. Recall that the Kazdan–Warner conditions are

$$\int \sum_{i=1}^{n} \psi_{x_i}(x)\, x_i\, |u|^{2n/(n-2)} = 0$$

for any solution $u \in E$ of (151).

Let $K \in L^\infty(\mathbf{R}^n)$ satisfy, for some positive constants $A_1 > 1$, $R > 1$, and $K_\infty > 0$,

$$\|K\|_{L^\infty} \leqslant A_1,$$

$$K \in C^0(\mathbf{R}^n \backslash B_R),$$

$$\lim_{|x| \to \infty} K(x) = K_\infty. \tag{152}$$

For any $0 < \varepsilon < 1$, $k = 1, 2, 3, ..., m = 2, 3, 4, ...,$ we choose an integer $\bar{k}$ such that for any $2 \leqslant s \leqslant m$, $\binom{\bar{k}}{s} \geqslant k$. We then choose $e_1, e_2, ..., e_{\bar{k}} \in \partial B_1$ to be $\bar{k}$ distinct points.

Let

$$\delta_R = \max_{|x| \geqslant R} |K(x) - K_\infty| + \max_{|x| \geqslant R} |\psi(x) - \psi_\infty|, \qquad R > 1,$$

and $\tilde{\Omega}_l^{(i)}$ be the connected component of

$$\{x \mid \varepsilon(\psi(x - le_i) - \psi_\infty) + K_\infty - \delta_{\sqrt{l}} > K(x)\}$$

which contains $x = le_i$.

We define

$$R_l = \min_{1 \leqslant i \leqslant m} \sup\{|x - le_i| \mid x \in \tilde{\Omega}_l^{(i)}\},$$

and

$$K_{\varepsilon,k,m,l} = \begin{cases} \varepsilon(\psi(x - le_i) - \psi_\infty) + K_\infty - \delta_{\sqrt{l}}, & \text{if } \ x \in \tilde{\Omega}_l^{(i)}, \\ K(x) & \text{otherwise.} \end{cases}$$

It is easy to prove that

$$\operatorname{diam}(\tilde{\Omega}_l^{(i)}) \leqslant \sqrt{l}, \qquad \text{for large } l,$$

and

$$\lim_{l \to \infty} R_l = \infty.$$

With $K_{\varepsilon,k,m,l}$ defined as above, for any $2 \leqslant s \leqslant m$, we prove that for l large enough,

$$\begin{aligned} & -\Delta u = K_{\varepsilon,k,m,l} u^{(n+2)/(n-2)}, \qquad \text{in } \mathbf{R}^n, \\ & u \in E, \\ & u > 0 \end{aligned} \tag{153}$$

has at least k solutions with s bumps.

Let $e_{j_1}, ..., e_{j_s}$ be any s distinct point among $e_1, ..., e_{\bar{k}}$, we define for $i = 1, 2, ..., s$ that

$$x_l^{(i)} = le_{j_i},$$

$$O_l^{(i)} = B_1(x_l^{(i)}), \qquad \tilde{O}_l^{(i)} = B_2(x_l^{(i)}),$$

$$K_\infty^{(i)} = \varepsilon(\psi - \psi_\infty) + K_\infty,$$

$$a^{(i)} = \varepsilon(\psi(0) - \psi_\infty) + K_\infty.$$

Clearly we can apply Theorem 5.1 to conclude that there exists at least a solution in $V_l(s, \varepsilon)$ for large l. It is also easy to see that if we choose a different set of s points among $\{e_1, ..., e_{\bar{k}}\}$, we get different solutions since their mass are distributed in different regions by the definition of $V_l(s, \varepsilon)$. By the choice of $\bar{k}$, there are at least k different sets of such points. Therefore (153) has at least k solutions for large l. Now we fix l large enough to make the above argument work for all $2 \leqslant s \leqslant m$ and set $K_{\varepsilon,k,m} = K_{\varepsilon,k,m,l}$. Clearly there are at least k solutions with s $(2 \leqslant s \leqslant m)$ bumps to the following equations

$$\begin{aligned} &-\Delta u = K_{\varepsilon,k,m,} u^{(n+2)/(n-2)}, \qquad \text{in } \mathbf{R}^n, \\ &u \in E, \\ &u > 0. \end{aligned}$$

Theorem 0.1 follows from the above after performing a sterographic projection on the original equation. We leave the details to the reader. ∎

Proof of Theorem 0.2. We prove it by contradiction argument. Suppose Theorem 0.2 is false. Then for some $\bar{\varepsilon} > 0$ (without loss of generality, we can assume $\bar{\varepsilon}$ to be very small), $k \in \{2, 3, 4, ...\}$, there exists a sequence of integers $I_l^{(1)}, ..., I_l^{(k)}$, such that

$$\lim_{l \to \infty} |I_l^{(i)}| = \infty,$$

$$\lim_{l \to \infty} |I_l^{(i)} - I_l^{(j)}| = \infty, \qquad i \neq j,$$

but (2) has no solution in $V(k, \bar{\varepsilon}, B_{\bar{\varepsilon}}(x_l^{(1)}), ..., B_{\bar{\varepsilon}}(x_l^{(k)}))$ which satisfies $kc - \bar{\varepsilon} \leqslant I_K \leqslant kc + \bar{\varepsilon}$, where $c = (1/n)(K(x^*))^{(2-n)/2} (S_n)^n$, $x_l^{(i)} = x^* + (I_l^{(i)}T, 0, 0)$.

We now define, for $\varepsilon > 0$ small, that

$$K_l(x_1, x_2, x_3) = K(x_1, x_2, x_3),$$

$$O_l^{(i)} = B_\varepsilon(x_l^{(i)}), \qquad \tilde{O}_l^{(i)} = B_{2\varepsilon}(x_l^{(i)}),$$

$$R_l = \min_{i \neq j} \{\sqrt{|I_l^{(i)}|}, \sqrt{|I_l^{(i)} - I_l^{(j)}|}\},$$

$$K_\infty^{(i)}(x_1, x_2, x_3) = K_\infty(x_1, x_2, x_3) = \lim_{x_1 \to \infty} K(x_1 + lT, x_2, x_3),$$

$$a^{(i)} = K(x^*).$$

It is easy to see that K_∞ is T-periodic in x_1 and satisfies

$$K_\infty \text{ is } C^2 \text{ in a neighborhood of } x^*,$$

$$K_\infty(x^*) = \sup_{x \in \mathbf{R}^n} K_\infty(x) > 0,$$

$$\Delta K_\infty(x^*) \neq 0.$$

Applying Proposition A.1 in Appendix A and Theorem 5.1, we immediately get a contradiction.

Theorem 0.3 can be proved in a similar way; we leave it to the reader.

Proof of Theorem 0.4. Under the hypotheses of the theorem, it has been proved by Schoen and Zhang that there is no more than one point blow up. See [Z] for the details. However, it follows from Proposition A.1 in Appendix A that one point blow up may not occur either. Hence the conclusion of the theorem.

Appendix A

Let $K(x)$ be some L^∞ function defined on $\mathbf{R}^n$, $O^{(1)}, O^{(2)}, ..., O^{(m)} \subset \mathbf{R}^n$ open sets with $\operatorname{dist}(O^{(i)}, O^{(j)}) \geqslant 1$ $\forall i \neq j$. Suppose that $K \in C^0(\bigcup_{i=1}^m O^{(i)})$ and we define $V(m, \varepsilon) = V(m, \varepsilon, O^{(1)}, ..., O^{(m)}, K)$, for $\varepsilon > 0$, as the following open set in E.

$u \in V(m, \varepsilon)$ if $u \in E$ and there exist $\alpha = (\alpha_1, ..., \alpha_m)$, $\alpha_1, ..., \alpha_m \in \mathbf{R}^n$, $x = (x_1, ..., x_m)$, $x_i \in O^{(i)}$, $\lambda = (\lambda_1, ..., \lambda_m)$, such that

$$\lambda_1, ..., \lambda_m > \frac{1}{\varepsilon},$$

$$|\alpha_i - K(x_i)^{(2-n)/4}| < \varepsilon,\ 1 \leqslant i \leqslant m,$$

$$\|u - \varphi(\alpha, x, \lambda)\| < \varepsilon.$$

where

$$\varphi(\alpha, x, \lambda) := \sum_{i=1}^m \alpha_i\, \delta(x_i, \lambda_i),$$

$$\delta(x_i, \lambda_i)(y) = (n(n-2))^{(n-2)/4} \left(\frac{\lambda_i}{1 + \lambda_i^2\, |y - x_i|^2}\right)^{(n-2)/2}.$$

It is well known that for any $x_i \in \mathbf{R}^n$, $\lambda_i > 0$,

$$\delta(x_i, \lambda_i) \in E,$$

$$-\Delta\delta(x_i, \lambda_i) = \delta(x_i, \lambda_i)^{(n+2)/(n-2)} \text{ in } \mathbf{R}^n,$$

$$\|\delta(x_i, \lambda_i)\| = \|\delta(0, 1)\|,$$

$$S_n = \left(\int |\nabla\delta(0, 1)|^2\right)^{1/n}.$$

LEMMA A.1. *Suppose further that for some* $\delta_0>0$, $K(x)$ *is* C^1 *in* $\{x \mid \operatorname{dist}(x, \bigcup_{i=1}^m O^{(i)}) \leqslant \delta_0\}$ *and satisfies*

$$0<K_1 \leqslant K(x) \leqslant K_2<\infty \qquad \text{if} \quad \operatorname{dist}\left(x, \bigcup_{i=1}^m O^{(i)}\right) \leqslant \delta_0,$$

$$|K(x)| \leqslant K_3<\infty \qquad \forall x \in \mathbf{R}^n,$$

$$|\nabla K(x)| \leqslant K_4<\infty \qquad \text{if} \quad \operatorname{dist}\left(x, \bigcup_{i=1}^m O^{(i)}\right) \leqslant \delta_0,$$

Then there exists $\varepsilon_0=\varepsilon_0(K_1, K_2, K_3, K_4, n, \delta_0, m)>0$, *such that, for any* $0<\varepsilon \leqslant \varepsilon_0$, $u \in V(m, \varepsilon)$,

$$\min_{(\alpha, x, \lambda) \in B_{4\varepsilon}} \|u-\varphi(\alpha, x, \lambda)\|$$

has a unique minimum, and it is achieved in $B_{2\varepsilon}$, *where*

$$B_\varepsilon=\left\{(\alpha, x, \lambda) \mid x=(x_1, \ldots, x_m), x_i \in \mathbf{R}^n, \operatorname{dist}(x_i, O^{(i)}) \leqslant \delta_0/2,\right.$$
$$\left.\frac{1}{2K_2^{(n-2)/4}} \leqslant \alpha_i \leqslant \frac{2}{K_1^{(n-2)/4}}, \lambda=(\lambda_1, \ldots, \lambda_m), \lambda_1, \ldots, \lambda_m \geqslant 1/\varepsilon\right\}.$$

The proof of Lemma A.1 only requires some modification of Proposition 3.1 in [Li1]. See also [BC1] and Appendix A of [BC2] for the original argument of this type.

PROPOSITION A.1. *Let* $K(x) \in L^\infty(\mathbf{R}^n)$ *be* C^2 *and positive near* $x^* \in \mathbf{R}^n$, *where* x^* *is a critical point of* K *with* $\Delta K(x^*) \neq 0$. *Then for* $3 \leqslant n \leqslant 6$, *there exists* $\varepsilon_1>0$ (ε_1 *depends on* K_1, K_2, K_3, K_4, n, δ_0, *the modulo of continuity of* D^2K) *and an open neighborhood* O *of* x^*, *such that*

$$\begin{aligned} &-\Delta u=K(x)\, u^{(n+2)/(n-2)}, \qquad \text{in } \mathbf{R}^n,\\ &u \in E, \\ &u>0. \end{aligned} \tag{154}$$

has no solution in $V(1, \varepsilon_1, O, K)$.

Let O be a small neighborhood of x^* where $|\nabla K| << |\Delta K|$.

According to Lemma A.1, if $\varepsilon>0$ is small enough, any function $u \in V(1, \varepsilon, O, K)$ can be represented uniquely as

$$u=\alpha \delta(x, \lambda)+v, \tag{155}$$

where $\alpha = K(x)^{(2-n)/4} + \circ_\varepsilon(1) \in \mathbf{R}$, $x \in \mathbf{R}^n$, $\lambda > 0$ (where $\circ_\varepsilon(1)$ denotes some quantity which tends to zero as ε tends to zero), and $v \in E$ satisfies

$$\begin{aligned} &\int \nabla\delta \, \nabla v = 0, \\ &\int \nabla\delta_x \, \nabla v = 0, \\ &\int \nabla\delta_\lambda \, \nabla v = 0, \end{aligned} \tag{156}$$

where δ_x, δ_λ denotes the partial derivatives in x, λ, respectively.

LEMMA A.2. *For $\beta > 1$, there exists $C(\beta) > 1$, such that, for any $a, b \in \mathbf{R}$, we have*

$$\begin{aligned} &||a+b|^\beta - (|a|^\beta + \beta\,|a|^{\beta-2}\,ab)| \\ &\qquad \leqslant \begin{cases} C(\beta)\,|b|^\beta & \beta < 2 \\ C(\beta)(|b|^\beta + |a|^{\beta-2}\,|b|^2) & \beta \geqslant 2 \end{cases} \end{aligned}$$

Proof of Lemma A.2. Set $x = b/a$ (if $a \neq 0$); it is equivalent to prove that for any $x \in \mathbf{R}$, we have

$$\begin{aligned} &||1+x|^\beta - (1+\beta x)| \\ &\qquad \leqslant \begin{cases} C(\beta)\,|x|^\beta & \beta < 2 \\ C(\beta)(|x|^\beta + |x|^2) & \beta \geqslant 2. \end{cases} \end{aligned}$$

The above is obvious (check it for x small and large).

LEMMA A.3. *If $u \in V(1, \varepsilon, O, K)$ is a solution of* (154), *then v in* (155) *satisfies*

$$\|v\| \leqslant C\left(\frac{|DK(x)|}{\lambda} + \frac{1}{\lambda^2}\right). \tag{157}$$

Proof of Lemma A.3. Multiply (154) by v and integrate by parts. We obtain, by using Lemma A.2, that

$$\begin{aligned} &\int |\nabla v|^2 \\ &\quad = \int K(\alpha\delta(x,\lambda) + v)^{(n+2)/(n-2)}\, v \\ &\quad = \int K(\alpha\delta(x,\lambda))^{(n+2)/(n-2)}\, v \\ &\qquad + \frac{n+2}{n-2}\int K(\alpha\delta(x,\lambda))^{4/(n-2)}\, v^2 + O(1)(\|v\|^{2n/(n-2)} + \|v\|^3) \end{aligned}$$

$$= \int (K - K(x))(\alpha\delta(x, \lambda))^{(n+2)/(n-2)} v$$
$$+ \frac{n+2}{n-2} \int K(\alpha\delta(x, \lambda))^{4/(n-2)} v^2 + O(1)(\|v\|^{2n/(n-2)} + \|v\|^3).$$

The last equality follows from (156).

It follows that

$$(1 + \circ_\varepsilon(1)) \|v\|^2 \leqslant C \int |K - K(x)| \, \delta(x, \lambda)^{(n+2)/(n-2)} |v|$$
$$+ \frac{n+2}{n-2} (1 + \circ_\varepsilon(1)) \int \delta(x, \lambda)^{4/(n-2)} v^2$$
$$+ C \int |K - K(x)| \, \delta(x, \lambda)^{4/(n-2)} v^2.$$

Using Lemma A.2 of [BC2] and the Hölder inequality, we have

$$\|v\|^2 \leqslant C \int |K - K(x)| \, \delta(x, \lambda)^{(n+2)/(n-2)} |v|$$
$$\leqslant C \left\{ \int |K - K(x)|^{2n/(n+2)} \delta(x, \lambda)^{2n/(n-2)} \right\}^{(n+2)/2n} \|v\|$$
$$\leqslant C \left\{ \frac{|DK(x)|}{\lambda} + \frac{1}{\lambda^2} \right\} \|v\|.$$

Proof of Proposition A.1. Let $u_0 = \alpha_0 \, \delta(x_0, \lambda_0) + v_0$ be a solution of (154) in $V(1, \varepsilon, O, K)$. Introduce $u_t = \alpha_0 \, \delta(x_0, \lambda_0 + t) + v_0$ and we have (since u_0 is a solution)

$$\frac{d}{dt} I(u_t)|_{t=0} = 0.$$

Recall that

$$I(u_t) = \frac{1}{2} \int |\nabla u_t|^2 - \frac{n-2}{2n} \int K \, |u_t|^{2n/(n-2)}.$$

Therefore we have

$$0 = \frac{d}{dt} I(u_t)|_{t=0}$$
$$= \int \nabla u_0 \, \nabla(\alpha_0 \, \delta_\lambda(x_0, \lambda_0)) - \int K \, |u_0|^{4/(n-2)} u_0(\alpha_0 \, \delta_\lambda(x_0, \lambda_0))$$
$$= -\int K \, |\alpha_0 \, \delta(x_0, \lambda_0) + v_0|^{(n+2)/(n-2)} (\alpha_0 \, \delta_\lambda(x_0, \lambda_0)).$$

The last equality follows from (156) and the fact that $\int \nabla\delta(x_0, \lambda_0)\, \nabla\delta_\lambda(x_0, \lambda_0) = 0$.

Applying Lemma A.2, we have

$$\begin{aligned}
&\int K\, |\alpha_0\, \delta(x_0, \lambda_0) + v_0|^{(n+2)/(n-2)}\, (\alpha_0\, \delta_\lambda(x_0, \lambda_0)) \\
&= \int K(\alpha_0\, \delta(x_0, \lambda_0))^{(n+2)/(n-2)}\, (\alpha_0\, \delta_\lambda(x_0, \lambda_0)) \\
&\quad + \frac{n+2}{n-2} \int K(\alpha_0\, \delta(x_0, \lambda_0))^{4/(n-2)}\, (\alpha_0\, \delta_\lambda(x_0, \lambda_0))\, v_0 \\
&\quad + \begin{cases} O(1) \int |v_0|^{(n+2)/(n-2)}\, |\delta_\lambda(x_0, \lambda_0)| & n > 6 \\ O(1) \int |v_0|^2\, \delta(x_0, \lambda_0)^{(6-n)/(n-2)}\, |\delta_\lambda(x_0, \lambda_0)| & \\ \qquad + O(1) \int |v_0|^{(n+2)/(n-2)}\, |\delta_\lambda(x_0, \lambda_0)| & 3 \leqslant n \leqslant 6 \end{cases} \\
&= I + II + III
\end{aligned}$$

We estimate I, II, and III one by one.

$$\begin{aligned}
I &= \int \{K - K(x_0)\}\, (\alpha_0\, \delta(x_0, \lambda_0))^{(n+2)/(n-2)}\, (\alpha_0\, \delta_\lambda(x_0, \lambda_0)) \\
&= \int \{\nabla K(x_0)(y - x_0) \\
&\quad + \sum_{i,j} K_{ij}(x_0)(y - x_0)_i\, (y - x_0)_j + \circ(|y - x_0|^2)\} \\
&\quad \times (\alpha_0\, \delta(x_0, \lambda_0))^{(n+2)/(n-2)}\, (\alpha_0\, \delta_\lambda(x_0, \lambda_0))\, dy \\
&= \frac{\Delta K(x_0)}{n} \int (|y - x_0|^2 + \circ(|y - x_0|^2)) \\
&\quad \times (\alpha_0\, \delta(x_0, \lambda_0))^{(n+2)/(n-2)}\, (\alpha_0\, \delta_\lambda(x_0, \lambda_0))\, dy \\
&= (C^{-1} + \circ_\varepsilon(1))\, \Delta K(x_0) \left(\frac{1}{\lambda_0^3} + \circ\left(\frac{1}{\lambda_0^3}\right)\right),
\end{aligned}$$

where $C > 1$ is some constant depending on K_1, K_2, n.

Note that

$$\begin{aligned}
\frac{n+2}{n-2}\int \delta(x_0, \lambda_0)^{4/(n-2)}\,\delta_\lambda(x_0, \lambda_0)\,v_0
&= \frac{d}{d\lambda}\int \delta(x_0, \lambda)^{(n+2)/(n-2)}\,v_0\big|_{\lambda=\lambda_0}\\
&= \frac{d}{d\lambda}\int (-\Delta\delta(x_0, \lambda))\,v_0\big|_{\lambda=\lambda_0}\\
&= \int (-\Delta\delta_\lambda(x_0, \lambda_0))\,v_0\\
&= \int \nabla\delta_\lambda(x_0, \lambda_0)\,\nabla v_0\\
&= 0.
\end{aligned}$$

The last equality follows from (156).

We have

$$\begin{aligned}
|II| &\leqslant O(1)\int |K-K(x_0)|\,\delta(x_0, \lambda_0)^{4/(n-2}\,|\delta_\lambda(x_0, \lambda_0))|\,|v_0|\\
&\leqslant O(1)\frac{1}{\lambda_0}\int |K-K(x_0)|\,\delta(x_0, \lambda_0)^{(n+2)/(n-2)}\,|v_0|\\
&\leqslant O(1)\frac{1}{\lambda_0}\left(\int |K-K(x_0)|^{2n/(n+2)}\,\delta(x_0, \lambda_0)^{2n/(n-2)}\right)^{(n+2)/2n}\|v_0\|\\
&\leqslant O(1)\frac{1}{\lambda_0^2}\|v_0\|\\
&\leqslant O(1)\left(\frac{|DK(x_0)|}{\lambda_0^3}+\frac{1}{\lambda_0^4}\right).
\end{aligned}$$

The last inequality follows from Lemma A.3.

For $3 \leqslant n \leqslant 6$, we have (noticing that $|\lambda_0\,\delta_\lambda(x_0, \lambda_0)| \leqslant C(n)\,\delta(x_0, \lambda_0)$),

$$\begin{aligned}
|III| &\leqslant O(1)\frac{1}{\lambda_0}\int |v_0|^2\,\delta(x_0, \lambda_0)^{4/(n-2)} + O(1)\frac{1}{\lambda_0}\int |v_0|^{(n+2)/(n-2)}\,\delta(x_0, \lambda_0)\\
&\leqslant O(1)\frac{1}{\lambda_0}\|v_0\|^2 + O(1)\frac{1}{\lambda_0}\|v_0\|^{(n+2)/(n-2)}\\
&\leqslant O(1)\left(\frac{|DK(x_0)|^2}{\lambda_0^3}+\frac{1}{\lambda_0^5}\right) + O(1)\left(\frac{|DK(x_0)|^{(n+2)/(n-2)}}{\lambda_0^{2n/(n-2)}}+\frac{1}{\lambda_0^{(3n+2)/(n-2)}}\right)\\
&\leqslant O(1)\left(\frac{|DK(x_0)|^2+|DK(x_0)|^{(n+2)/(n-2)}}{\lambda_0^3}+\circ\left(\frac{1}{\lambda_0^3}\right)\right).
\end{aligned}$$

For $n > 6$, we have

$$
\begin{aligned}
|III| &\leqslant O(1)\left|\int |v_0|^{(n+2)/(n-2)}\,\delta_\lambda(x_0,\lambda_0)\right| \\
&\leqslant O(1)\frac{1}{\lambda_0}\int |v_0|^{(n+2)/(n-2)}\,\delta(x_0,\lambda_0) \\
&\leqslant O(1)\frac{1}{\lambda_0}\,\|v_0\|^{(n+2)/(n-2)} \\
&\leqslant O(1)\left(\frac{|DK(x_0)|^{(n+2)/(n-2)}}{\lambda_0^{2n/(n-2)}}+\frac{1}{\lambda_0^{(3n+2)/(n-2)}}\right).
\end{aligned}
$$

It follows from the above estimates that for $3 \leqslant n \leqslant 6$, if $\varepsilon > 0$ is small enough, then λ_0 is very large and we have (using $|\nabla K(x_0)| << |DK(x_0)|$)

$$|I| > |II| + |III|.$$

This contradicts (157).

Appendix B

For the convenience of readers we present in this appendix some results of Schoen (see [Sc2], also [Z]).

We first recall a local form of the Pohozaev identity (see [Sc3]).

Theorem B.1. *Let (N, g) be an n-dimensional compact Riemannian manifold with smooth boundary. Let $R = R(g)$ be the scalar curvature function of N, and suppose X is a conformal killing vector field on N. Then*

$$\int_N (\mathscr{L}_X R)\,dv_g = \frac{2n}{n-2}\int_{\partial N} T(X, v_g)\,d\sigma_g, \tag{158}$$

where $T = Ric - (1/n)\,Rg$ is the trace free Ricci tensor, $\mathscr{L}_X$ denotes the Lie derivative.

Letting $u > 0$ be a C^2 function defined in a neighborhood of the origin of $\mathbf{R}^n$, set

$$g = u^{4/(n-2)} \sum_{i=1}^n (dx^i)^2, \tag{159}$$

$$r^2 = \sum_{i=1}^n (x^i)^2, \qquad B_\sigma = \{x \in \mathbf{R}^n \mid |x| < \sigma\}.$$

We can derive from (158) (taking $X = r(\partial/\partial r)$) that (see [Be], [Sc1])

$$\int_{B_\sigma} r \frac{\partial R(g)}{\partial r} u^{2n/(n-2)}\, dx = \frac{2n}{n-2} \int_{\partial B_\sigma} \sigma^n u^2 T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) d\xi, \tag{160}$$

where dx is the volume element of $\sum (dx^i)^2$, $d\xi$ is the surface element of the unit $(n-1)$-sphere,

$$T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = (n-2)\, u^{2/(n-2)} \left\{\frac{\partial^2}{\partial r^2}(u^{-2/(n-2)} - \frac{1}{n} \Delta(u^{-2/(n-2)})\right\}. \tag{161}$$

Here Δ is with respect to the flat metric $\sum_{i=1}^n (dx^i)^2$.

Let K be a positive C^2 function near the origin of $\mathbf{R}^n$, $1 < p \leqslant (n+2)/(n-2)$ and $u > 0$ be a solution of

$$\Delta u + K u^p = 0 \tag{162}$$

near the origin.

Let g be defined as in (159); it follows from a straightforward computation that (see [Be])

$$R(g) = -c(n)^{-1} u^{-(n+2)/(n-2)} \Delta u = c(n)^{-1} K u^{-\tau}, \tag{163}$$

where $\tau = (n+2)/(n-2) - p$ and $(n) = (n-2)/(4(n-1))$.

It follows from (160) and (163) that

$$\begin{aligned} &\frac{2n}{n-2} \int_{\partial B_\sigma} \sigma^n u^2 T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) d\xi \\ &\quad = c(n)^{-1} \left(1 + \frac{\tau}{p+1}\right) \int_{B_\sigma} r \frac{\partial K}{\partial r} u^{p+1}\, dx \\ &\quad\quad + \frac{n\tau}{p+1} c(n)^{-1} \int_{B_\sigma} K u^{p+1} \\ &\quad\quad - \frac{\tau \sigma^n}{p+1} c(n)^{-1} \int_{\partial B_\sigma} K u^{p+1}\, d\xi, \end{aligned} \tag{164}$$

where $T(\partial/\partial r, \partial/\partial r)$ is as in (161).

Let $\tau_i \geqslant 0$ satisfy $\lim_{i\to\infty} \tau_i = 0$ and $p_i = (n+2)/(n-2) - \tau_i$. Consider

$$\begin{aligned} -\Delta u_i &= K_i(x)\, u_i^{(n+2)/(n-2) - \tau_i} \quad \text{in } \Omega, \\ u_i &> 0 \end{aligned} \tag{165}$$

where $\Omega \subset \mathbf{R}^n$ is an open set,

$\{K_i\}$ are bounded in $C^1_{\text{loc}}(\Omega)$ and have a positive lower bound. (166)

We are mainly concerned with what happens to u_i when τ_i tends to 0. It follows from standard elliptic theories that if $\{u_i\}$ remain bounded, then for some $u \in C^2_{\text{loc}}(\Omega)$, u_i tends to u in C^2_{loc} along a subsequence as τ_i tends to 0. Otherwise, we say that u_i blows up.

DEFINITION B.1. Suppose $\{u_i\}$ satisfy (165). $\bar{y} \in \Omega$ is called a blow up point of $\{u_i\}$ if there exists a sequence y_i tending to $\bar{y}$, such that $u_i(y_i) \to \infty$.

DEFINITION B.2. Suppose $\{u_i\}$ satisfy (165). $\bar{y} \in \Omega$ is called an isolated blow up point of $\{u_i\}$ if there exists $0 < \bar{r} < \text{dist}(\bar{y}, \partial\Omega)$, $C > 0$, and a sequence y_i tending to y, such that $\bar{y}$ is a local maximum of u_i, $u_i(y_i) \to \infty$, and

$$u_i(y) \leqslant \bar{C}\,\text{dist}(y, y_i)^{\;2/(p_i - 1)} \qquad \text{for all} \quad y \in B_{\bar{r}}(y_i). \tag{167}$$

Let $\bar{y}$ be an isolated blow up of $\{u_i\}$; we define

$$\bar{u}_i(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(y_i)} u, \qquad r > 0,$$

and

$$\bar{w}_i(r) = r^{2/(p_i - 1)} \bar{u}_i(r), \qquad r > 0.$$

DEFINITION B.3. $\bar{y} \in \Omega$ is called an isolated simple blow up point, if $\bar{y}$ is an isolated blow up point, such that for some $\rho > 0$ (independent of i) $\bar{w}_i$ has only one critical point in $(0, \rho)$.

PROPOSITION B.1. *Suppose $\{K_i\}$ satisfy* (166) *and $\{u_i\}$ satisfy* (165). *Let $\bar{y} \in \Omega$ be an isolated blow up point of $\{u_i\}$ be the sequence of points as in Definition* B.2. *Then for any $R_i \to \infty$, $\varepsilon_i \to 0^+$, we have, after passing to a subsequence (still denoted as $\{u_i\}$, $\{y_i\}$, etc.), that*

$$\|u_i(y_i)^{-1} u_i(u_i(y_i)^{-(p_i - 1)/2} \cdot + y_i) - (1 + k_i |\cdot|^2)^{(2-n)/2}\|_{C^2(B_{2R_i}(0))} \leqslant \varepsilon_i,$$
$$R_i u_i(y_i)^{-(p_i - 1)/2} \to 0 \qquad \text{as} \quad i \to \infty,$$

where $k_i = (n(n-2))^{-1} K_i(y_i)$.

Proof. Consider

$$\zeta_i(x) = u_i(y_i)^{-1} u_i(u_i(y_i)^{-(p_i - 1)/2} x + y_i), \; |x| < r u_i(y_i)^{(p_i - 1)/2}.$$

We have

$$\begin{aligned}
&-\Delta\zeta_i(x)=K_i(u_i(y_i)^{-(p_i-1)/2}x+y_i)\,\zeta_i(x)^{p_i}, && |x|<\bar{r}u_i(y_i)^{(p_i-1)/2},\\
&\zeta_i(0)=1,\ \nabla\zeta_i(0)=0, && (168)\\
&0<\zeta_i(x)\leqslant \bar{C}\,|x|^{-2/(p_i-1)}, && |x|<\bar{r}u_i(y_i)^{(p_i-1)/2}.
\end{aligned}$$

Applying the Harnack inequality to ζ_i (using (168)), we obtain for $0<r<1$ that

$$\max_{\partial B_r}\zeta_i\leqslant C\min_{\partial B_r}\zeta_i, \tag{169}$$

where $C>0$ is some constant depending only on n, $\bar{C}$ and the upper bound of $\|K_i\|_{L^\infty}$.

Noting that $-\Delta\zeta_i\geqslant 0$, $\zeta_i(0)=1$, and (169), we obtain (using the maximum principle)

$$\max_{B_1}\zeta_i\leqslant C, \tag{170}$$

for some positive constant C.

It follows from (167), (170), and the standard elliptic theories that there exists $\zeta>0$, such that (after passing to a subsequence),

$$\begin{gathered}
\zeta_i\to\zeta \qquad \text{in } C^2_{\mathrm{loc}}(\mathbf{R}^n),\\
-\Delta\zeta=\lim_{i\to\infty}K_i(y_i)\,\zeta^{(n+2)/(n-2)} \qquad \text{in } \mathbf{R}^n,\\
\zeta\geqslant 0,\ \zeta(0)=1, \qquad \nabla\zeta(0)=0.
\end{gathered}$$

It follows that (see [CGS])

$$\zeta(x)=(1+\lim_{i\to\infty}k_i\,|x|^2)^{(2-n)/2}.$$

Proposition B.1 follows immediately.

PROPOSITION B.2. *Suppose* $\{K_i\}$ *satisfy* (166) *and* $\{u_i\}$ *satisfy* (165). *Let* $\bar{y}\in\Omega$ *be an isolated blow up point of* $\{u_i\}$ *and* $\{y_i\}$ *be the sequence of points as in Definition* B.2. *Then for any* $\delta>0$ *we have* (*after passing to a subsequence*)

$$C_{\delta,i}^{-1}u_i(y_i)(1+k_iu_i(y_i)^{4/(n-2)}\,|x-y_i|^2)^{-(n-2)/2}\leqslant u_i(x), \qquad |x-y_i|<\delta,$$

where $k_i=(n(n-2))^{-1}K_i(y_i)$ *and* $C_{\delta,i}\to 1$ *as* $\delta\to 0^+$, $i\to\infty$.

PROPOSITION B.3. *Suppose* $\{K_i\}$ *satisfy* (166) *and* $\{u_i\}$ *satisfy* (165). *Let* $\bar{y} \in \Omega$ *be an isolated simple blow up point of* $\{u_i\}$ *and* $\{y_i\}$ *be the sequence of points as in Definition* B.2-B.3. *Then for any* $\delta > 0$ *we have* (*after passing to a subsequence*)

$$u_i(x) \leqslant C_{\delta,i} u_i(y_i)(1 + k_i u_i(y_i)^{p_i - 1} |x - y_i|^2)^{-(n-2)/2},$$
$$|x - y_i| \leqslant R_i u_i(y_i)^{-(p_i - 1)/2},$$
$$u_i(x) \leqslant C_{\delta,i} u_i(y_i)^{-1} |x - y_i|^{2-n},$$
$$R_i u_i(y_i)^{-(p_i - 1)/2} \leqslant |x - y_i| \leqslant \delta,$$

where

$$R_i u_i(y_i)^{-(p_i - 1)/2} \to 0, \ R_i \to \infty \quad \textit{as} \quad i \to \infty,$$
$$C_{\delta,i} \to 1 \quad \text{as} \quad \delta \to 0^+ \quad \textit{and} \quad i \to \infty.$$

For the proof of Proposition B.2-B.3, see [Sc2] and [Z].

PROPOSITION B.4. *Suppose* $\{K_i\}$ *satisfy* (166) *and* $\{u_i\}$ *satisfy* (165). *Let* $y \in \Omega$ *be an isolated simple blow up point of* $\{u_i\}$ *and* $\{y_i\}$ *be the sequence of points as in Definition* B.2-B.3. *Then as* $i \to \infty$, *we have*

$$\tau_i = O\left(\int_{B_{\bar{r}}(y_i)} r \frac{\partial K_i}{\partial r} u_i^{p_i + 1}\, dx\right) + O(u_i(y_i)^{-2})$$
$$= O(u_i(y_i)^{-(p_i - 1)/2}).$$

Proof. Applying (164), we obtain

$$\frac{2n}{n-2} \int_{\partial B_{\bar{r}}(y_i)} \bar{r}^n u_i^2 T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) d\xi$$
$$= c(n)^{-1}\left(1 + \frac{\tau_i}{p_i + 1}\right) \int_{B_{\bar{r}}(y_i)} r \frac{\partial K_i}{\partial r} u_i^{p_i + 1}\, dx$$
$$+ \frac{n\tau_i}{p_i + 1} c(n)^{-1} \int_{B_{\bar{r}}(y_i)} K_i u_i^{p_i + 1}$$
$$- \frac{\tau_i \bar{r}^n}{p_i + 1} c(n)^{-1} \int_{\partial B_{\bar{r}}(y_i)} K_i u_i^{p_i + 1}\, d\xi,$$

where

$$r^2 = \sum_{k=1}^{n} (y - y_i)_k^2,$$

$$T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = (n-2)\, u_i^{2/(n-2)} \left\{\frac{\partial^2}{\partial r^2} (u_i^{-2/(n-2)}) - \frac{1}{n} \Delta(u_i^{-2/(n-2)})\right\}.$$

Proposition B.4 follows immediately from Propositions B.2 and B.3, and the above.

COROLLARY B.1. *Under the hypotheses of Proposition* B.4, $\lim_{i\to\infty} u_i(y_i)^{\tau_i} = 1$.

PROPOSITION B.5. *Suppose* $\{K_i\}$ *are bounded in* $C^2(B_{\bar r}(\bar y))$ *and* $\{u_i\}$ *satisfy* (165). *Let* $\bar y \in \Omega$ *be an isolated simple blow up point of* $\{u_i\}$ *and* $\{y_i\}$ *be the sequence of points as in Definition B.3. Then*

$$|\nabla K_i(y_i)| = O(u_i(y_i)^{-(p_i-1)/2}).$$

Proof. Define a cutoff function $\eta \in C_c^\infty(B_{1/2})$, satisfying

$$\eta(x) = 1, \qquad |x| \leqslant \tfrac{1}{4},$$
$$\eta(x) = 0, \qquad |x| \geqslant \tfrac{1}{2}.$$

Multiplying (165) by $\eta(\partial u_i/\partial x_1)$ and integrating by parts, it follows from Proposition B.3 and the standard elliptic theories that

$$\begin{aligned}
\frac{1}{p_i+1}\int_{B_1} \frac{\partial K_i}{\partial x_1} u_i^{p_i+1}\eta
&= \frac{1}{2}\int_{B_1} |\nabla u_i|^2 \frac{\partial \eta}{\partial x_1} - \int_{B_1} \frac{\partial u_i}{\partial x_1} \nabla u_i \nabla \eta - \frac{1}{p_i+1}\int_{B_1} K_i u_i^{p_i+1} \frac{\partial \eta}{\partial x_1} \\
&= \frac{1}{2}\int_{B_{1/2}\setminus B_{1/4}} |\nabla u_i|^2 \frac{\partial \eta}{\partial x_1} - \int_{B_{1/2}\setminus B_{1/4}} \frac{\partial u_i}{\partial x_1} \nabla u_i \nabla \eta \\
&\quad - \frac{1}{p_i+1}\int_{B_{1/2}\setminus B_{1/4}} K_i u_i^{p_i+1} \frac{\partial \eta}{\partial x_1} \\
&= O(u_i(y_i)^{-2}).
\end{aligned}$$

Therefore (see Proposition B.3 and Corollary B.1)

$$\begin{aligned}
\frac{\partial K_i}{\partial x_1}(y_i)\frac{1}{p_i+1}\int_{B_1} u_i^{p_i+1}\eta
&= \frac{1}{p_i+1}\int_{B_1} \left\{\frac{\partial K_i}{\partial x_1}(y_i) - \frac{\partial K_i}{\partial x_1}\right\} u_i^{p_i+1}\eta + \frac{1}{p_i+1}\int_{B_1} \frac{\partial K^i}{\partial x^1} u_i^{p_i+1}\eta \\
&= \frac{1}{p_i+1}\int_{B_1} \left\{\frac{\partial K_i}{\partial x_1}(y_i) - \frac{\partial K_i}{\partial x_1}\right\} u_i^{p_i+1}\eta + O(u_i(y_i)^{-2}) \\
&= O\left(\int_{B_1} |y-y_i|\, u_i(y)^{p_i+1}\, dy\right) + O(u_i(y_i)^{-2}) \\
&= O(u_i(y_i)^{-(p_i-1)/2}).
\end{aligned}$$

Clearly we can estimate $(\partial K_i/\partial x_k)(y_i)$ $(2 \leqslant k \leqslant n)$ in a similar way. Proposition B.5 follows immediately.

THEOREM B.2. *Assume* $\{K_i\}$ *satisfy* (166) *and* $\{u_i\}$ *satisfy* (165). *Assume also that* $\bar{y} \in \Omega$ *is an isolated blow up point of* $\{u_i\}$. *If* $n=3$, *then* $\bar{y}$ *is an isolated simple blow up point.*

Proof. We argue by contradiction. Assume the contrary, $\bar{y} \in \Omega$ is an isolated blow up point, but is not an isolated simple blow up point.

It follows from Proposition B.1 that the first critical point of $\bar{w}_i(r)$ $(r>0)$ occurs at $r = (\sqrt{n(n-2)/K_i(y_i)} + \circ(1))\, u_i(y_i)^{-(p_i-1)/2}$ and it is the only critical point in the interval $0 < r < R_i u_i(y_i)^{-(p_i-1)/2}$.

Let r_i be the second smallest critical point of $\bar{w}_i(r)$ in the interval $0<r<\bar{r}$. It follows from Proposition B.1 and the above that

$$r_i \geqslant R_i u_i(y_i)^{-(p_i-1)/2}. \tag{171}$$

Note that (see Proposition B.1) $\bar{w}_i(R_i u_i(y_i)^{-(p_i-1)/2}) \to 0$ as $i \to \infty$; thus we have (otherwise there would be another critical point of $\bar{w}_i(r)$ between $R_i u_i(y_i)^{-(p_i-1)/2}$ and r_i)

$$\bar{w}_i(r_i) \to 0 \qquad \text{as} \quad i \to \infty. \tag{172}$$

According to the contradiction hypothesis, we have

$$r_i \to 0 \qquad \text{as} \quad i \to \infty. \tag{173}$$

Let

$$v_i(y) = r_i^{2/(p_i-1)} u_i(r_i y + y_i), \qquad |y| \leqslant \bar{r}/r_i.$$

It follows from (171) that

$$v_i(0) = r_i^{2/(p_i-1)} u_i(y_i) \geqslant R_i^{2/(p_i-1)} \to \infty.$$

We derive from (165) that

$$-\Delta v_i(y) = K_i(r_i y + y_i)\, v_i(y)^{p_i}, \qquad |y| \leqslant \bar{r}/r_i. \tag{174}$$

It is easy to see that $y=0$ is an isolated simple blow up point of $\{v_i\}$. Therefore Propositions B.2 and B.3 apply to $\{v_i\}$.

Set

$$\tilde{v}_i(y) = \frac{v_i(y)}{\bar{w}_i(r_i)}, \qquad |y| \leqslant \bar{r}/r_i.$$

It follows from (174) that

$$-\Delta \tilde{v}_i(y) = \bar{w}_i(r_i)^{p_i-1} K_i(r_i y + y_i)\, \tilde{v}_i(y)^{p_i}, \qquad |y| \leqslant \bar{r}/r_i. \tag{175}$$

It follows from (167) that

$$|v_i(y)| \leqslant C\,|y|^{-2/(p_i-1)}, \; |y| \leqslant \bar{r}/r_i. \tag{176}$$

Applying the Harnack inequality to $\{v_i\}$ (see (174) and (176)) we obtain for some $C>0$ that

$$\frac{1}{C}\,\bar{w}_i(r_i) \leqslant \min_{|y|=1} v_i(y) \leqslant \max_{|y|=1} v_i(y) \leqslant C\bar{w}_i(r_i)$$

$$\frac{1}{C} \leqslant \tilde{v}_i(y) \leqslant C, \qquad |y|=1, \tag{177}$$

$\tilde{v}_i(y)$ is bounded on compact set of $\mathbf{R}^n\setminus\{0\}$.

It follows from (175), (177), and (172) that (after passing to a subsequence) for some $\tilde{v}(y)$,

$$\tilde{v}_i(y) \to \tilde{v}(y) \quad \text{as} \quad i\to\infty \text{ in } C^0_{\mathrm{loc}}(\mathbf{R}^n\setminus\{0\}),$$
$$-\Delta\tilde{v}=0,\ \tilde{v}>0, \text{ in } \mathbf{R}^n\setminus\{0\}.$$

It follows from standard elliptic theories that there exist some nonnegative constants b, c such that

$$\tilde{v}(y)=b\,|y|^{2-n}+c, \qquad y\in\mathbf{R}^n\setminus\{0\}. \tag{178}$$

Note that $\tilde{v}_i(y)=1$; we conclude that $\tilde{v}(y)=1$ for some $|y|=1$, namely, $b+c=1$.

Note also that $|y|^{2/(p_i-1)}\,\bar{v}_i(|y|)$ (here $\bar{v}_i(r)=(1/\partial B_r)\int_{\partial B_r} v_i$ for $r>0$) has a critical point at $|y|=1$; we conclude that $r^{(n-2)/2}\tilde{v}(r)$ has a critical point at $r=1$; namely, $b-c=0$.

Therefore we have

$$\tilde{v}(y)=\tfrac{1}{2}(|y|^{2-n}+1), \qquad y\in\mathbf{R}^n\setminus\{0\}.$$

To sum up, we have derived under the contradiction hypothesis that there exist $\{v_i\}$ such that

$$-\Delta v_i(y)=\hat{K}_i(y)\,v_i(y)^{p_i}, \qquad |y|\leqslant \bar{r}/r_i,$$

$$v_i>0,$$

$y=0$ is an isolated simple blow up point of $\{v_i\}$,

$$\hat{K}_i(y)=K_i(r_i y+y_i), \qquad |y|\leqslant \bar{r}/r_i,$$

$$r_i\to 0,\ \bar{w}_i(r_i)\to 0 \qquad \text{as} \quad i\to\infty,$$

$$\tilde{v}_i(y)\equiv\frac{v_i(y)}{\bar{w}_i(r_i)}\to\tilde{v}(y)\equiv\frac{1}{2}\,(|y|^{2-n}+1) \qquad \text{in } C^0_{\mathrm{loc}}(\mathbf{R}^n\setminus\{0\}). \tag{179}$$

Applying (164) to $\{v_i\}$ (see (179)), we obtain, for any $0<\sigma$,

$$\begin{aligned}\frac{2n}{n-2}\int_{B_\sigma}\sigma^n v_i^2\, T\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) d\xi\\ \geqslant -\frac{\tau_i\sigma^n}{p_i+1}c(n)^{-1}\int_{\partial B_\sigma}\hat{K}_i v_i^{p_i+1}\,d\xi\\ +c(n)^{-1}\left(1+\frac{\tau_i}{p_i+1}\right)\int_{B_\sigma} r\frac{\partial\hat{K}_i}{\partial r}v_i^{p_i+1}\,dx,\end{aligned} \tag{180}$$

where

$$T\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)=(n-2)\,v_i^{2/(n-2)}\left\{\frac{\partial^2}{\partial r^2}(v_i^{-2/(n-2)})-\frac{1}{n}\Delta(v_i^{-2/(n-2)})\right\}. \tag{181}$$

Applying Propositions B.2 and B.3, we obtain that

$$\frac{\tau_i\sigma^n}{p_i+1}c(n)^{-1}\int_{\partial B_\sigma}\hat{K}_i v_i^{p_i+1}\,d\xi=O(v_i(0)^{-(p_i+1)}).$$

Therefore

$$\lim_{i\to\infty} v_i(0)^2\frac{\tau_i\sigma^n}{p_i+1}c(n)^{-1}\int_{\partial B_\sigma}\hat{K}_i v_i^{p_i+1}\,d\xi=0. \tag{182}$$

It follows from Propositions B.2 and B.3 and (174) that

$$\left|\int_{B_\sigma} r\frac{\partial\hat{K}_i}{\partial r}v_i^{p_i+1}\right|\leqslant Cr_i\int_{B_\sigma} rv_i^{p_i+1}=o(v_i(0)^{-(p_i-1)/2}). \tag{183}$$

Multiplying (180) by $v_i(0)^2$ and letting i tend to ∞, we obtain for $n=3$, by using (182) and (183), that

$$\begin{aligned}\frac{2n}{n-2}\sigma^n\int_{\partial B_\sigma}(\lim_{i\to\infty} v_i(0)\,v_i(y))^2\, T\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)(\lim_{i\to\infty} v_i(0)\,v_i(y))\,d\xi\\ \geqslant c(n)^{-1}\lim_{i\to\infty} v_i(0)^2\int_{B_\sigma} r\frac{\partial\hat{K}_i}{\partial r}v_i^{p_i+1}\\ =0\end{aligned} \tag{184}$$

The last equality follows from (183), Corollary B.1, and the fact that $(p_i-1)/2=2-\tau_i/2$ when $n=3$.

Using (179), (177), and a direct computation, we see that if we take $\sigma>0$ small enough, (184) is violated. Theorem B.2 is thus established.

In the following we treat the case $n=4$ and obtain the following theorem.

THEOREM B.3. *Assume* $\{K_i\}$ *is bounded in* $C^2_{\text{loc}}(\Omega)$ *and* $\{u_i\}$ *satisfy* (165). *Assume also that* $\bar{y} \in \Omega$ *is an isolated blow up point of* $\{u_i\}$. *If* $n=4$, *then* $\bar{y}$ *is an isolated simple blow up point.*

Proof. We argue the same way as in the proof of Theorem B.2 until (182).Then we use the uniform $C^2_{\text{loc}}(\Omega)$ boundedness of $\{K_i\}$ and proceed as follows.

Choosing an orthonormal coordinate system (depending on i) such that the Hessian of K_i at y_i is diagonal, we have

$$\begin{aligned}
&\left|\int_{B_\sigma} r \frac{\partial \hat{K}_i}{\partial r} v_i^{p_i+1}\right| \\
&\quad = \left|\int_{B_\sigma} y \cdot \nabla \hat{K}_i v_i^{p_i+1}\right| \\
&\quad \leqslant \left|\int_{B_\sigma} y \cdot \nabla \hat{K}_i(0)\, v_i^{p_i+1}\right| + \max_{B_\sigma} |D^2 \hat{K}_i| \int_{B_\sigma} |y|^2 v_i^{p_i+1} \\
&\quad = \left|\int_{B_\sigma} y \cdot \nabla \hat{K}_i(0)\, v_i^{p_i+1}\right| + r_i^2 \max_{B_{\sigma r_i}(y_i)} |D^2 K_i| \int_{B_\sigma} |y|^2 v_i^{p_i+1}.
\end{aligned}$$

Applying Proposition B.5 to (179), we obtain

$$|\nabla \hat{K}_i(0)| = O(v_i(0)^{-(p_i-1)/2}).$$

Applying Proposition B.3 to (179), we have

$$\int_{B_\sigma} y v_i^{p_i+1} = o(v_i(0)^{-(p_i-1)/2}).$$

Therefore

$$\int_{B_\sigma} y \cdot \nabla \hat{K}_i(0)\, v_i^{p_i+1} = o(v_i(0)^{-(p_i-1)}).$$

Applying Proposition B.3 and Corollary B.1 we have (using $n=4$)

$$\int_{B_\sigma} |y|^2 v_i^{p_i+1} = v_i(0)^{-2}\left(\frac{1}{4}\int_{R^4} \frac{|y|^2}{(1+k_i|y|^2)^4}\, dy + o(1)\right).$$

Therefore (using $\lim_{i\to\infty} r_i = 0$)

$$\lim_{i\to\infty} v_i(0)^2 \int_{B_\sigma} \sum_{k=1}^{n} r \frac{\partial \hat{K}_i}{\partial r} v_i^{p_i+1} = 0. \tag{185}$$

Multiplying (180) by $v_i(0)^2$ and letting i tend to ∞, we obtain for $n=4$, using (182) and (185), that

$$\frac{2n}{n-2}\sigma^n \int_{\partial B_\sigma} (\lim_{i\to\infty} v_i(0)\, v_i(y))^2\, T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) (\lim_{i\to\infty} v_i(0)\, v_i(y))\, d\xi \geqslant 0. \quad (186)$$

As before, by using (179), (177), and a direct computation, we see that if we take $\sigma>0$ small enough, (186) is violated. Theorem B.3 is thus established.

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