Solutions to the $\sigma_k$-Loewner-Nirenberg problem on annuli are locally Lipschitz and not differentiable

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Dedicated to Alice Chang and Paul Yang on their 70th birthday

Abstract

We show for $k \geq 2$ that the locally Lipschitz viscosity solution to the $\sigma_k$-Loewner-Nirenberg problem on a given annulus $\{a < |x| < b\}$ is $C^{1,\frac{1}{k}}_{\text{loc}}$ in each of $\{a < |x| \leq \sqrt{ab}\}$ and $\{\sqrt{ab} \leq |x| < b\}$ and has a jump in radial derivative across $|x| = \sqrt{ab}$. Furthermore, the solution is not $C^{1,\gamma}_{\text{loc}}$ for any $\gamma > \frac{1}{k}$. Optimal regularity for solutions to the $\sigma_k$-Yamabe problem on annuli with finite constant boundary values is also established.

1 Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$. For a positive $C^2$ function $u$ defined on an open subset of $\mathbb{R}^n$, let $A^u$ denote its conformal Hessian, namely

$$A^u = -\frac{2}{n-2}u \frac{n+2}{n-2} \nabla^2 u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}} \nabla \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I,$$  \hspace{1cm} (1.1)

and let $\lambda(-A^u)$ denote the eigenvalues of $-A^u$. Note that $A^u$, considered as a $(0,2)$ tensor, is the Schouten curvature tensor of the metric $u^{\frac{4}{n-2}} \hat{g}$ where $\hat{g}$ is the Euclidean metric.

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For $1 \leq k \leq n$, let $\sigma_k: \mathbb{R}^n \to \mathbb{R}$ denote $k$-th elementary symmetric function

$$\sigma_k(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k},$$

and let $\Gamma_k$ denote the cone $\Gamma_k = \{ \lambda = (\lambda_1, \ldots, \lambda_n) : \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0 \}$. In [6], it was shown that the $\sigma_k$-Loewner-Nirenberg problem

$$\sigma_k(\lambda(-A^u)) = 2^{-k}\left(\frac{n}{k}\right), \quad \lambda(-A^u) \in \Gamma_k, \quad u > 0 \quad \text{in } \Omega, \quad (1.2)$$

$$u(x) \to \infty \text{ as dist}(x, \partial \Omega) \to 0. \quad (1.3)$$

has a unique continuous viscosity solution $u$ and such $u$ belongs to $C^{0,1}_{loc}(\Omega)$. (The uniqueness is a consequence of the principle of propagation of touching points [20, Theorem 3.2] – a form of comparison principle – and the boundary estimate [6, Lemma 3.4].)

Equation (1.2) is a fully nonlinear elliptic equation of the kind considered by Caffarelli, Nirenberg and Spruck [2]. We recall the following definition of viscosity solutions which follows Li [17, Definitions 1.1 and 1.1'] (see also [16]) where viscosity solutions were first considered in the study of nonlinear Yamabe problems.

Let

$$\Sigma_k := \left\{ \lambda \in \Gamma_k | \sigma_k(\lambda) \geq 2^{-k}\left(\frac{n}{k}\right) \right\}, \quad (1.4)$$

$$\underline{\Sigma}_k := \mathbb{R}^n \setminus \left\{ \lambda \in \Gamma_k | \sigma(\lambda) > 2^{-k}\left(\frac{n}{k}\right) \right\}. \quad (1.5)$$

**Definition 1.1.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq k \leq n$. We say that an upper (a lower semi-continuous) function $u: \Omega \to (0, \infty)$ is a sub-solution (super-solution) to (1.2) in the viscosity sense, if for any $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ satisfying $(u - \varphi)(x_0) = 0$ and $u - \varphi \leq 0$ ($u - \varphi \geq 0$) near $x_0$, there holds

$$\lambda(-A^\varphi(x_0)) \in \Sigma_k \quad (\lambda(-A^\varphi(x_0)) \in \underline{\Sigma}_k, \text{ respectively}).$$

We say that a positive function $u \in C^0(\Omega)$ satisfies (1.2) in the viscosity sense if it is both a sub- and a super-solution to (1.2) in the viscosity sense.

In the rest of this introduction, we assume that $\Omega$ is an annulus $\{a < |x| < b\} \subset \mathbb{R}^n$ with $0 < a < b < \infty$, unless otherwise stated. $C^2$ radially symmetric solutions to (1.2) were classified by Chang, Han and Yang [4, Theorems 1 and 2]. As a consequence,
when $2 \leq k \leq n$, there is no $C^2$ radially symmetric function satisfying \(1.2\)-\(1.3\). On the other hand, the aforementioned uniqueness result from \cite{6,20} implies that the solution $u$ to \(1.2\)-\(1.3\) is radially symmetric (since $u(R \cdot)$ is also a solution for any orthogonal matrix $R$). Therefore, \(1.2\)-\(1.3\) has no $C^2$ solutions.

Our first result improves on the above non-existence of $C^2$ solutions to \(1.2\)-\(1.3\), asserting that there is no $C^2$ sub-solution. In fact, we show that there is no function $u \in C^2(\Omega)$ which satisfies $\lambda(-Au) \in \bar{\Gamma}_2$ in $\Omega$ and tends to infinity at $\partial \Omega$.

**Theorem 1.2.** Suppose that $n \geq 3$. Let $\Omega = \{a < |x| < b\} \subset \mathbb{R}^n$ with $0 < a < b < \infty$ be an annulus. Then there exists no function $u \in C^2(\Omega)$ such that

\[
\lambda(-Au) \in \bar{\Gamma}_2, \quad u > 0 \quad \text{in } \Omega, \quad u(x) \to \infty \text{ as } \text{dist}(x, \partial \Omega) \to 0.
\]

Our next result shows that the locally Lipschitz solution $u$ is not $C^1$.

**Theorem 1.3.** Suppose that $n \geq 3$ and $2 \leq k \leq n$. Let $\Omega = \{a < |x| < b\} \subset \mathbb{R}^n$ with $0 < a < b < \infty$ be an annulus and $u$ be the unique locally Lipschitz viscosity solution to \(1.2\)-\(1.3\). Then $u$ is radially symmetric, i.e. $u(x) = u(|x|)$,

(i) $u$ is smooth in each of $\{a < |x| < \sqrt{ab}\}$ and $\{\sqrt{ab} < |x| < b\}$,

(ii) $u$ is $C^{1,\frac{k}{k}}$ but not $C^{1,\gamma}$ with $\gamma > \frac{1}{k}$ in each of $\{a < |x| \leq \sqrt{ab}\}$ and $\{\sqrt{ab} \leq |x| < b\}$,

(iii) and the first radial derivative $\partial_r u$ jumps across $\{|x| = \sqrt{ab}\}$:

$$
\left. \partial_r \ln u \right|_{r=\sqrt{ab}} = -\frac{n-2}{\sqrt{ab}} \quad \text{and} \quad \left. \partial_r \ln u \right|_{r=\sqrt{ab}} = 0.
$$

A related problem in manifold settings is to solve on a given closed Riemannian manifold $(M, g)$ the equation

\[
\sigma_k \left( \lambda \left( -A_{u^{\frac{4}{n-2}} g} \right) \right) = 2^{-k} \left( \begin{array}{c} n \\ k \end{array} \right), \quad \lambda \left( -A_{u^{\frac{4}{n-2}} g} \right) \in \Gamma_k, \quad u > 0 \quad \text{in } M, \quad (1.8)
\]

where $A_{u^{\frac{4}{n-2}} g}$ is the so-called Schouten tensor of the metric $u^{\frac{4}{n-2}} g$. Equations \(1.2\) and \(1.8\) are fully non-linear and non-uniformly elliptic equations of Hessian type, usually referred to as the $\sigma_k$-Yamabe equation in the ‘negative case’, which is a generalization of the Loewner-Nirenberg problem \cite{22}. This equation and its variants
have been studied in Chang, Han and Yang [4], Gonzalez, Li and Nguyen [6], Gurksy
and Viaclovsky [11], Li and Sheng [15], Guan [8], Gursky, Streets and Warren [10],
and Sui [24]. For further studies on the counterpart of (1.2) in the positive case, see
[3, 5, 9, 12, 13, 14, 17, 18, 23, 25, 26] and the references therein.

We observe the following result, which is essentially due to Gurksy and Viaclovsky [11]. We provide in the appendix the detail for the piece which is not directly available from [11].

**Theorem 1.4.** Suppose that \( n \geq 3, 2 \leq k \leq n \), and \((M^n, g)\) is a compact Riemannian
manifold such that \( \lambda(-A_g) \in \Gamma_k \) on \( M \). Then (1.8) has a Lipschitz viscosity solution.

Here viscosity solution is defined analogously as in Definition 1.1.

**Definition 1.5.** Let \((M^n, g)\) be a Riemannian manifold, \( 1 \leq k \leq n \), and \( S_k \) be given by (1.4) and (1.5). We say that an upper semi-continuous (a lower semi-
continuous) function \( u : M \to (0, \infty) \) is a sub-solution (super-solution) to (1.8) in
the viscosity sense, if for any \( x_0 \in M \), \( \varphi \in C^2(M) \) satisfying \((u - \varphi)(x_0) = 0 \) and
\( u - \varphi \leq 0 \) (\( u - \varphi \geq 0 \)) near \( x_0 \), there holds

\[
\lambda \left( -A_{\varphi \eta^g} \right)(x_0) \in S_k \quad \left( \lambda \left( -A_{\varphi \eta^g} \right)(x_0) \in S_k \right),
\]

We say that a positive function \( u \in C^0(M) \) satisfies (1.8) in the viscosity sense if
it is both a sub- and a super-solution to (1.8) in the viscosity sense.

In both contexts, it is an interesting open problem to understand relevant conditions on \( \Omega \), or on \((M, g)\), which would ensure that (1.2)-(1.3), or (1.8) respectively,
admits a smooth solution.

**Question 1.6.** Suppose that \( n \geq 3, 2 \leq k \leq n \), and \( \Omega \subset \mathbb{R}^n \) is a smooth strictly
convex (non-empty) domain. Is the locally Lipschitz viscosity solution to (1.2)-(1.3)
smooth?

If \( \Omega \) is a ball, then the solution to (1.2)-(1.3) is smooth and corresponds to the
Poincaré metric.

**Question 1.7.** Suppose that \( n \geq 3, 2 \leq k \leq n \), and \( \Omega = \Omega_2 \setminus \Omega_1 \neq \emptyset \) where
\( \Omega_1 \subseteq \Omega_2 \subset \mathbb{R}^n \) are smooth bounded strictly convex domains. Is the locally Lipschitz
viscosity solution to (1.2)-(1.3) not \( C^2 \)?

In the case \( \Omega_1 \) and \( \Omega_2 \) are balls, \( \Omega = \Omega_2 \setminus \Omega_1 \) is conformally equivalent to an
annulus, and so, by Theorem 1.3 the solution to (1.2)-(1.3) is not \( C^2 \).
Question 1.8. Suppose that $n \geq 3$, $2 \leq k \leq n$, and $(M^n, g)$ is a Riemannian manifold such that $\lambda(-A_g) \in \Gamma_k$ on $M$. Does (1.8) have a unique Lipschitz viscosity solution?

It is clear that (1.8) has at most one $C^2$ solution by the maximum principle. In fact, if (1.8) has a $C^2$ solution, then that solution is also the unique continuous viscosity solution in view of the strong maximum principle [1, Theorem 3.1]. Equivalently, if (1.8) has two viscosity solutions, then it has no $C^2$ solution.

Question 1.9. Suppose that $n \geq 3$ and $2 \leq k \leq n$. Does there exist a Riemannian manifold $(M^n, g)$ such that $\lambda(-A_g) \in \Gamma_k$ on $M$ and (1.8) has a Lipschitz viscosity solution which is not $C^2$?

One point which we would like to remark is that in solving (1.2)-(1.3), one does not need to make a global assumption similar to the condition $\lambda(-A_g) \in \Gamma$ in the manifold setting. Note that the requirement that $\lambda(-A_g) \in \Gamma$ on $M$ is equivalent, after a conformal change of the metric, that (1.8) has a smooth sub-solution. Whether some such global condition is sufficient for the existence of a smooth solution to (1.2)-(1.3) remains for further investigation. For example,

Question 1.10. Suppose that $n \geq 3$, $2 \leq k \leq n$, and $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. If (1.2)-(1.3) has a smooth sub-solution, must (1.2)-(1.3) have a smooth solution?

Recall that, by Theorem 1.2 if $\Omega$ is (conformally equivalent to) an annulus, (1.2)-(1.3) has no smooth sub-solution. We note here a result which gives a negative answer to an analogous question in a closely related setting. This concerns the case where (1.3) is replaced by finite constant boundary conditions

$$u|_{\{|x|=a\}} = c_1 \text{ and } u|_{\{|x|=b\}} = c_2.$$  \quad (1.9)

In this case, we show that, for large $c_1$ and $c_2$, the existence of a smooth sub-solution is insufficient to ensure the smoothness of the solution; see Corollary 1.14 below. This contrasts the result of Bo Guan [7] on the $\sigma_k$-Yamabe problem on positive $\Gamma_k$-cones where the existence of a smooth sub-solution implies the existence of a smooth solution.

In fact, we completely determine in the following theorem the regularity of the solution to (1.2) and (1.9) depending on whether $\ln \frac{b}{a}$ is larger, equal to, or smaller than $2T(a, b, c_1, c_2)$ where

$$T(a, b, c_1, c_2) := \frac{1}{2} \int_{-|p_b - p_a|}^0 \left\{ 1 + e^{-2\eta - 2\max(p_a, p_b)} \left[ 1 - e^{n\eta} \right]^{1/k} \right\}^{-1/2} d\eta, \quad (1.10)$$
Remark 1.13. Suppose that \( n \geq 3 \) and \( 2 \leq k \leq n \). Let \( \Omega = \{ a < |x| < b \} \subset \mathbb{R}^n \) with \( 0 < a < b < \infty \) be an annulus, and \( c_1, c_2 \) be two positive constants and let \( T(a, b, c_1, c_2) \) be given by (1.10). Then there exists a unique continuous viscosity solution to (1.2) and (1.9). Furthermore, \( u \) is radially symmetric, i.e. \( u(x) = u(|x|) \), and exactly one of the following four alternatives holds.

Case 1: \( \ln \frac{b}{a} < 2T(a, b, c_1, c_2) \), and \( u \) is smooth in \( \{ a \leq |x| \leq b \} \).

Case 2: \( \ln \frac{b}{a} = 2T(a, b, c_1, c_2) \), \( b^\frac{n-2}{2}c_2 < a^\frac{n-2}{2}c_1 \), and \( u \) is smooth in \( \{ a \leq |x| < b \} \), is \( C^{1,\frac{1}{k}} \) but not \( C^{1,\gamma} \) with \( \gamma > \frac{1}{k} \) in \( \{ a \leq |x| \leq b \} \).

Case 3: \( \ln \frac{b}{a} = 2T(a, b, c_1, c_2) \), \( b^\frac{n-2}{2}c_2 > a^\frac{n-2}{2}c_1 \), and \( u \) is smooth in \( \{ a < |x| \leq b \} \), is \( C^{1,\frac{1}{k}} \) but not \( C^{1,\gamma} \) with \( \gamma > \frac{1}{k} \) in \( \{ a \leq |x| \leq b \} \).

Case 4: \( \ln \frac{b}{a} > 2T(a, b, c_1, c_2) \), and there is some \( m \in (a, b) \) such that

\( \text{ (i) } u \) is smooth in each of \( \{ a \leq |x| < m \} \) and \( \{ m < |x| \leq b \} \),

\( \text{ (ii) } u \) is \( C^{1,\frac{1}{k}} \) but not \( C^{1,\gamma} \) with \( \gamma > \frac{1}{k} \) in each of \( \{ a \leq |x| \leq m \} \) and \( \{ m \leq |x| \leq b \} \),

\( \text{ (iii) and the first radial derivative } \partial_r u \text{ jumps across } \{|x| = m\}: \)

\[ \partial_r \ln u \big|_{r=m^-} = -\frac{n-2}{m} \text{ and } \partial_r \ln u \big|_{r=m^+} = 0. \]

Note that when \( \ln \frac{b}{a} = 2T(a, b, c_1, c_2) \), we have in view of the definition of \( T(a, b, c_1, c_2) \), \( p_a \) and \( p_b \) that \( b^\frac{n-2}{2}c_2 \neq a^\frac{n-2}{2}c_1 \).

Remark 1.12. It is clear from Theorem 1.11 (in Cases 1–3) that if \( u \) is a \( C^1 \) and radially symmetric solution to (1.2) in the viscosity sense in some open annulus \( \Omega \) then \( u \in C^\infty(\Omega) \).

Remark 1.13. In Case 4, the exact value of \( m \) is

\[ m = \sqrt{ab} \exp \left( \frac{1}{2} \int_{p_b-p}^{p_a-p} \left( 1 + e^{-2\eta-2p}[1-e^{\eta}]^{1/k} \right)^{-1/2} d\eta \right) \]

where \( p \) is the solution to

\[ \ln \frac{b}{a} = \int_{p_b-p}^{0} \left( 1 + e^{-2\eta-2p}[1-e^{\eta}]^{1/k} \right)^{-1/2} d\eta \]

and

\[ + \int_{p_a-p}^{0} \left( 1 + e^{-2\eta-2p}[1-e^{\eta}]^{1/k} \right)^{-1/2} d\eta. \]
Corollary 1.14. Suppose that $n \geq 3$ and $2 \leq k \leq n$. Let $\Omega = \{a < |x| < b\} \subset \mathbb{R}^n$ with $0 < a < b < \infty$ be an annulus. For every given $c > 0$, there exist positive constants $c_1, c_2 > c$ such that there is a smooth function $u \in C^\infty(\bar{\Omega})$ satisfying

$$\sigma_k(\lambda(-A^u)) \geq 2^{-k}\left(\frac{n}{k}\right), \quad \lambda(-A^u) \in \Gamma_k, \quad u > 0 \quad \text{in} \ \Omega,$$  \hspace{1cm} (1.11)

$$u|_{\{|x|=a\}} = c_1 \quad \text{and} \quad u|_{\{|x|=b\}} = c_2,$$  \hspace{1cm} (1.12)

while the viscosity solution to (1.2) and (1.9) belongs to $C^{0,1}_{\text{loc}}(\Omega)$ but not $C^1(\Omega)$.

We conclude the introduction with one more question.

Question 1.15. Let $n \geq 3$, $2 \leq k \leq n$ and $m \neq n - 1$. Does there exist a smooth domain $\Omega \subset \mathbb{R}^n$ such that the locally Lipschitz solution to (1.2)-(1.3) is $C^2$ away from a set $\Sigma$ which has Hausdorff dimension $m$?

In the next section, we prove all the results above except Theorem 1.14, whose proof is done in the appendix. Theorem 1.2 is proved first in Subsection 2.1. We then prove a lemma on the existence and uniqueness a non-standard boundary value problem for the ODE related to (1.2) in Subsection 2.2 and use it to prove Theorem 1.3 in Subsection 2.3 and Theorem 1.11 in Subsection 2.4. Corollary 1.14 is proved in Subsection 2.5.

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2 Proofs

By the uniqueness result in [6, 20], the solutions $u$ in Theorems 1.3 and 1.11 are radially symmetric, $u(x) = u(r)$ where $r = |x|$.

Let

$$t = \ln r - \frac{1}{2}\ln(ab), \quad \xi(t) = -\frac{2}{n-2}\ln u(r) - \ln r.$$

A direct computation gives that, at points where $u$ is twice differentiable,

$$\sigma_k(\lambda(-A^u)) = \frac{(-1)^k}{2^{k-1}}\left(\frac{n-1}{k-1}\right)e^{2k\xi}(1 - |\xi'|^2)^{k-1}[\xi'' + \frac{n-2k}{2k}(1 - |\xi'|^2)],$$  \hspace{1cm} (2.1)
where here and below \( ' \) denotes differentiation with respect to \( t \).

Note that, for \( k \geq 2 \), at points where \( u \) is twice differentiable, \( \lambda(-A^u) \in \Gamma_k \) if and only if \( \sigma_k(\lambda(-A^u)) > 0 \) and \( |\xi'| > 1 \). Indeed, if \( \sigma_k(\lambda(-A^u)) > 0 \) and \( |\xi'| > 1 \), then (2.1) implies \( \sigma_i(\lambda(-A^u)) > 0 \) for \( 1 \leq i \leq k \) and so \( \lambda(-A^u) \in \Gamma_k \). Conversely, if \( \lambda(-A^u) \in \Gamma_k \) for some \( k \geq 2 \), then \( \sigma_1(\lambda(-A^u)) > 0 \), \( \sigma_2(\lambda(-A^u)) > 0 \) and \( \sigma_k(\lambda(-A^u)) > 0 \). Using (2.1), we see that the first two inequalities imply \( |\xi'| > 1 \).

By the same reasoning, we have, at points where \( u \) is twice differentiable, if \( \lambda(-A^u) \in \bar{\Gamma}_2 \), then \( |\xi'| \geq 1 \).

We are thus led to study the differential equation

\[
e^{2k\xi}(1 - |\xi'|^2)^{k-1}[\xi'' + \frac{n-2k}{2k}(1 - |\xi'|^2)] = \frac{(-1)^k n}{2k}.
\]

(2.2)

under the constraint that \( |\xi'| > 1 \).

It is well known (see [4, 26]) that (2.2) has a first integral, namely

\[
H(\xi, \xi') := e^{(2k-n)\xi}(1 - |\xi'|^2)^k - (-1)^k e^{-n\xi} \text{ is (locally) constant along } C^2 \text{ solutions.}
\]

A plot of the contours of \( H \) for \( k = 2, n = 7 \) is provided in Figure 1. See [4] for a more complete catalog.

Before moving on with the proofs of our results, we note the following statement.

**Remark 2.1.** As a consequence of Theorem 1.11, we have in fact that \( H(\xi, \xi') \) is (locally) constant along viscosity solutions.

**Proof.** Fix \( a < b \) in the domain of \( \xi \) and apply Theorem 1.11 with \( c_1 = \xi(a) \) and \( c_2 = \xi(b) \). If we are in cases 1–3, \( \xi \) is \( C^2(a, b) \) and so \( H(\xi, \xi') \) is constant in \( (a, b) \). Suppose we are in case 4. We have that \( \xi \) is \( C^2 \) in \( (a, m) \cup (m,b) \) and so \( H(\xi, \xi') \) is constant in each of \( (a, m) \) and \( (m,b) \). Also, as \( \xi \) is \( C^1 \) in each of \( (a, m) \) and \( [m,b) \), we have by assertion (iii) in case 4 that

\[
\lim_{t \to m^-} H(\xi(t), \xi'(t)) = H(\xi(m), 1) = H(\xi(m), -1) = \lim_{t \to m^+} H(\xi(t), \xi'(t)).
\]

Hence \( H(\xi, \xi') \) is also constant in \( (a, b) \).

\[\square\]

### 2.1 Proof of Theorem 1.2

Argue by contradiction, assume that there exists \( u \in C^2(\Omega) \) satisfying (1.6)-(1.7) (which may or may not be radially symmetric).
Figure 1: The contours of $H$ for $k = 2$, $n = 7$. Each radially symmetric viscosity solution to (1.2) lies on a single contour of $H$ but avoid the shaded region, i.e. the dotted parts of the contours of $H$ are excluded. Every smooth solution stays on one side of the shaded region. Every non-smooth solution jumps (on one contour) from the part below the shaded region to the part above the shaded region at a single non-differentiable point.

Let \( w = u^{-\frac{2}{n-2}} \) and let

\[
A_w = w^{-1} A_u = \nabla^2 w - \frac{1}{2w} |\nabla w|^2 I.
\]

As \( \lambda(-A_w) \in \tilde{\Gamma}_2 \) in \( \Omega \), we have that \( \lambda(-A_u) \in \tilde{\Gamma}_2 \) in \( \Omega \).

We will use the following lemma on the concavity of \( A_w \) with respect to \( w \).

**Lemma 2.2** \([21]\). Suppose that \( 0 < w_1, w_2 \in C^2(\Omega) \). Then

\[
A_{\frac{1}{2}(w_1 + w_2)} \geq \frac{1}{2} (A_{w_1} + A_{w_2}) \text{ in } \Omega.
\]

Let \( G = SO(n) \) and \( \mu \) denotes the Haar measure on \( G \). For \( g \in G \), let \( w_g(x) = w(gx) \). It is clear that \( \lambda(-A_{w_g}) \in \tilde{\Gamma}_2 \) in \( \Omega \) for every \( g \in G \). For \( x \in \Omega \), let

\[
\bar{w}(x) = \frac{1}{\mu(G)} \int_G w_g(x) \, d\mu(g).
\]
By Lemma 2.2 above, \( \lambda(-A_{\overline{w}}) \in \Gamma_2 \) in \( \Omega \).

(For readers’ convenience, we provide here a direct proof of this fact. We have

\[
A_{\overline{w}}(x) = \frac{1}{\mu(G)} \int_G \nabla^2 w_g(x) \, d\mu(g) - \frac{1}{2\overline{w}} \left| \frac{1}{\mu(G)} \int_G \nabla w_g(x) \, d\mu(g) \right|^2 I
\]

\[
= \frac{1}{\mu(G)} \int_G A_{w_g}(x) \, d\mu(g)
\]

\[
+ \frac{1}{2|\mu(G)|^2 \overline{w}} \left\{ \int_G w_g(x) \, d\mu(g) \int_G \frac{1}{\mu(G)} |\nabla w_g(x)|^2 \, d\mu(g)
\right.
\]

\[
- \left| \int_G \nabla w_g(x) \, d\mu(g) \right|^2 \bigg\} I.
\]

Noting that the term in the curly braces is non-negative thanks to Cauchy-Schwarz' inequality, we deduce that

\[
A_{\overline{w}} \geq \frac{1}{\mu(G)} \int_G A_{w_g}(x) \, d\mu(g).
\]

Since the set of symmetric matrices whose eigenvalues belong to \( \overline{\Gamma}_2 \) is convex (see e.g. [19, Lemma B.1]), it hence follows that \( \lambda(-A_{\overline{w}}) \in \overline{\Gamma}_2 \).

Now replacing \( u \) by \( \overline{w} - \frac{\ln r}{2} \), we may assume from the beginning that \( u \) is radially symmetric, i.e. \( u(x) = u(|x|) \). As noted at the beginning of the section, the condition that \( \lambda(-A_{\overline{w}}) \in \Gamma_2 \) implies that the function \( \xi(t) = -\frac{2}{\pi^2} \ln u(r) - \ln r \) satisfies \( |\xi'(t)| \geq 1 \) in \((-T, T)\) where \( t = \ln r - \frac{1}{2} \ln(ab) \), \( T = \frac{1}{2} \ln \frac{b}{a} \), and prime denotes differentiation with respect to \( t \). In particular, \( \xi' \) is nowhere vanishing in \((-T, T)\). On the other hand, in view of (1.7), \( \xi \to -\infty \) as \( t \to \pm T \), which implies that \( \xi \) has a local maximum somewhere at which \( \xi' \) necessarily vanishes. This contradiction finishes the proof. \( \square \)

2.2 A lemma

**Lemma 2.3.** For any \( T > 0 \), there exists a unique classical solution \( \xi \in C^\infty(0, T) \cap C^{1,1}_{\text{loc}}([0, T)) \) to (2.2) in \((0, T)\) such that

\[
\lim_{t \to T^-} \xi(t) = -\infty, \quad \xi'(0) = -1, \quad \xi'(t) < -1 \quad \text{in} \quad (0, T).
\]

Furthermore, for every \( \gamma \in (\frac{1}{k}, 1] \), \( \xi \notin C^{1,\gamma}_{\text{loc}}([0, T)) \).
Proof. We use ideas from [4].

Step 1: We start by collecting relevant facts from [4] about the classical solution \( \xi_{p,q} \) to (2.2) satisfying the initial condition \( \xi_{p,q}(0) = p \) and \( \xi_{p,q}'(0) = q \) for \( p \in \mathbb{R}, q \in (-\infty, -1) \) on its maximal interval of unique existence \( I_{p,q} = (\overline{T}_{p,q}, \overline{T}_{p,q}) \subset \mathbb{R} \).

Note that, since \( \xi_{p,q}'(0) = q < -1 \), it follows from (2.2) that, for as long as \( \xi_{p,q} \) remains \( C^2 \), \( \xi_{p,q}' < -1 \). Thus, as \( H(\xi_{p,q}, \xi_{p,q}') = H(p, q) \), we have in \( I_{p,q} \) that

\[
\xi_{p,q}' = -\left\{ 1 + e^{-2\xi_{p,q}} \left[ 1 + (-1)^k H(p, q)e^{n\xi_{p,q}} \right]^{1/k} \right\}^{1/2}. \tag{2.5}
\]

By [4] (Theorem 1, Cases II.2 and II.3 for even \( k \) and Theorem 2, Cases II.2 and II.3 for odd \( k \)), we have that \( \overline{T}_{p,q} \) is finite. Furthermore,

\[
\lim_{t \to \overline{T}_{p,q}} \xi_{p,q}(t) = -\infty. \tag{2.6}
\]

By (2.5) we thus have

\[
\overline{T}_{p,q} = \int_{-\infty}^{p} \left\{ 1 + e^{-2\xi} \left[ 1 + |H(p, q)|e^{n\xi} \right]^{1/k} \right\}^{-1/2} d\xi. \tag{2.7}
\]

In this proof, we will only need to consider the case that \( (-1)^k H(p, q) < 0 \). Then by [4] (Theorem 1, Case II.2 for even \( k \) and Theorem 2, Case II.2 for odd \( k \)), we have that \( \overline{T}_{p,q} \) is also finite and

\[
\lim_{t \to \overline{T}_{p,q}^+} \xi_{p,q}(t) \text{ is finite}, \quad \lim_{t \to \overline{T}_{p,q}^-} \xi_{p,q}'(t) = -1, \quad \text{and} \quad \lim_{t \to \overline{T}_{p,q}^-} \xi_{p,q}''(t) = -\infty. \tag{2.8}
\]

Using (2.8) as well as the fact that \( H(\xi_{p,q}, \xi_{p,q}') = H(p, q) \) and \( \xi_{p,q} \) is decreasing, we have in \( I_{p,q} \) that

\[
\xi_{p,q} < \lim_{t \to \overline{T}_{p,q}^+} \xi_{p,q}(t) = -\frac{1}{n} \ln |H(p, q)|. \tag{2.9}
\]

Differentiating (2.5), we see that, as \( t \to \overline{T}_{p,q}^+ \),

\[
\lim_{t \to \overline{T}_{p,q}^+} (t - \overline{T}_{p,q})^{-\frac{n+1}{k}} \xi_{p,q}(t) \text{ exists and belongs to } (-\infty, 0).
\]

Thus \( \xi_{p,q} \) extends to a \( C^{1,\frac{1}{k}} \) function in a neighborhood of \( \overline{T}_{p,q} \) and \( \xi_{p,q} \) does not extend to a \( C^{1,\gamma} \) function in any neighborhood of \( \overline{T}_{p,q} \).
Before moving on to the next stage, we note that, in view of (2.5),
\[
\overline{T}_{p,q} - T_{p,q} = \int_{-\infty}^{0} \left\{ 1 + e^{-2\xi} [1 - |H(p,q)| e^{n\xi}]^{1/k} \right\}^{-1/2} d\xi
\]
\[
= \int_{-\infty}^{0} \left\{ 1 + |H(p,q)|^{2} e^{-2\eta} [1 + e^{n\eta}]^{1/k} \right\}^{-1/2} d\eta.
\] (2.10)

In particular, then length of \(I_{p,q}\) depends only on \(n, k\) and the value of \(H(p,q)\), rather than \(p\) and \(q\) themselves.

**Step 2:** We now define for each given \(p \in \mathbb{R}\) a unique classical solution \(\xi_{p}\) to (2.2) in some maximal interval \((0, T_{p})\) satisfying \(\xi_{p}(0) = p, \xi_{p}'(0) = -1\) and \(\xi_{p}' < -1\) in \((0, T_{p})\).

It is clear that \((-1)^{k}H(p, -1) = -e^{-np} < 0\), and as \(\partial_{p}H(p, -1) = (-1)^{k}ne^{-np} \neq 0\). By the implicit function theorem, there exist \(\hat{p}\) and \(\hat{q} < -1\) such that \(H(\hat{p}, \hat{q}) = H(p, -1)\). Note that this implies

\[-e^{-np} = (-1)^{k}H(\hat{p}, \hat{q}) > -e^{-n\hat{p}}\] and so \(\hat{p} < p\).

Let

\[\xi_{p}(t) = \xi_{\hat{p}, \hat{q}}(t + T_{\hat{p}, \hat{q}})\] and \(T_{p} = T_{\hat{p}, \hat{q}} - T_{\hat{p}, \hat{q}}\).

By Step 1, it is readily seen that \(\xi_{p}\) is smooth in \((0, T_{p})\), belongs to \(C^{1,1}_{\text{loc}}([0, T_{p}])\) and no \(C^{1,\gamma}_{\text{loc}}([0, T_{p}])\) with \(\gamma > \frac{1}{k}\), satisfies (2.2) and \(\xi_{p}' < -1\) in \((0, T_{p})\),

\[
\lim_{t \to T_{p}} \xi_{p}(t) = -\infty,
\] (2.11)

\[
\xi_{p}(0) = -\frac{1}{n} \ln |H(\hat{p}, \hat{q})| = p, \quad \xi_{p}'(0) = -1,
\] (2.12)

\[
0 > \xi_{p,q}''(t) = O(t^{\frac{k-1}{k}}) \quad \text{as} \quad t \to 0^{+},
\] (2.13)

\[
\text{and} \quad T_{p} = \int_{-\infty}^{0} \left\{ 1 + e^{-2\eta - 2p} [1 - e^{-n\eta}]^{1/k} \right\}^{-1/2} d\eta.
\] (2.14)

We claim that \(\xi_{p}\) is unique in the sense that if \(\hat{\xi}_{p} \in C^{2}(0, \hat{T}_{p}) \cap C^{1}([0, \hat{T}_{p}])\) is a solution to (2.2) in some maximal interval \((0, \hat{T}_{p})\) satisfying \(\hat{\xi}_{p}(0) = p, \hat{\xi}_{p}'(0) = -1\) and \(\hat{\xi}_{p}' < -1\) in \((0, \hat{T}_{p})\), then \(T_{p} = \hat{T}_{p}\) and \(\xi_{p} \equiv \hat{\xi}_{p}\). To see this, note that \(\hat{\xi}_{p}(t) = \xi_{\xi_{p}(s), \xi_{p}'(s)}(t - s)\) for all \(t, s \in (0, \hat{T}_{p})\). By Step 1, \(\hat{\xi}_{p}(t) \to -\infty\) as \(t \to \hat{T}_{p}^{-}\), and so, as \(\hat{p} < p\) and \(\hat{\xi}_{p}(0) = 0\), there exists \(t_{0} \in (0, \hat{T}_{p})\) such that \(\hat{\xi}_{p}(t_{0}) = \hat{p}\). This implies that

\(H(p, \hat{\xi}_{p}'(t_{0})) = H(\hat{p}, \hat{\xi}_{p}') = H(p, -1) = H(\hat{p}, \hat{q})\) and so \(\xi_{p}'(t_{0}) = \hat{q}\). We deduce that

\(t_{0} = -\overline{T}_{\hat{p}, \hat{q}}, \hat{T}_{p} = T_{p}\) and \(\hat{\xi}_{p} \equiv \xi_{\hat{p}, \hat{q}}(\cdot - t_{0}) \equiv \xi_{p}\), as claimed.
Step 3: From (2.14), we see that, as a function of $p$, $T_p$ is continuous and increasing and satisfies
\[
\lim_{p \to -\infty} T_p = 0 \quad \text{and} \quad \lim_{p \to \infty} T_p = \infty.
\]
Thus, for any given $T > 0$, there is a unique $p(T)$ such that $T_{p(T)} = T$. The solution $\xi_{p(T)}$ to (2.2) gives the desired solution.

2.3 Proof of Theorem 1.3

Let $T = \frac{1}{2} \ln \frac{b}{a}$ and $t = \ln r - \frac{1}{2} \ln(ab)$. We need to exhibit a function $\xi : (-T, T) \to \mathbb{R}$ such that $\xi$ is smooth in each of $(0, T)$ and $(-T, 0)$, is $C^1_{loc}$ but not $C^{1,\gamma}_{loc}$ for any $\gamma > \frac{1}{k}$ in each of $[0, T)$ and $(-T, 0]$, the function $u$ defined by
\[
u(r) = \exp \left[ -\frac{n-2}{2} \left( \xi(t) + \ln r \right) \right]
\]
solves (1.2)-(1.3) in $\{a < r = |x| < b\}$ in the viscosity sense, and

(i) $\lim_{t \to \pm T} \xi(t) = -\infty$,

(ii) $\xi'(0^-) = 1$, $\xi'(0^+) = -1$,

(iii) and $|\xi'| > 1$ in $(-T, 0) \cup (0, T)$.

Indeed, let $\xi^T : [0, T) \to \mathbb{R}$ be the solution obtain in Lemma 2.3 and define
\[
\xi(t) = \begin{cases} 
\xi^T(t) & \text{if } 0 \leq t < T, \\
\xi^T(-t) & \text{if } -T < t < 0.
\end{cases}
\]

It is clear that $\xi$ satisfies all the listed requirements except for the statement that $u$ satisfies (1.2) in the viscosity sense at $r = \sqrt{ab}$. It remains to demonstrate, for any given $x_0$ with $|x_0| = \sqrt{ab}$, that

(a) if $\varphi$ is $C^2$ near $x_0$ and satisfies $\varphi \geq u$ near $x_0$ and $\varphi(x_0) = u(x_0)$, then $\lambda(-A^\varphi(x_0)) \in \Gamma_k$ and $\sigma_k(\lambda(A^\varphi(x_0))) \geq 2^{-k} \left( \frac{n}{k} \right)$,

(b) and if $\varphi$ is $C^2$ near $x_0$ and satisfies $\varphi \leq u$ near $x_0$ and $\varphi(x_0) = u(x_0)$, then either $\lambda(-A^\varphi(x_0)) \notin \Gamma_k$ or $\lambda(-A^\varphi(x_0)) \in \Gamma_k$ but $\sigma_k(\lambda(A^\varphi(x_0))) \leq 2^{-k} \left( \frac{n}{k} \right)$. 

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Without loss of generality, we may assume that \( x_0 = (\sqrt{a^2}, 0, \ldots, 0) \).

Since \( \partial_t \ln u\big|_{r=\sqrt{ab}} = -\frac{n-2}{\sqrt{ab}} < 0 = \partial_r u\big|_{r=\sqrt{ab}} \), there is no \( C^2 \) function \( \varphi \) such that \( \varphi \geq u \) near \( x_0 \) and \( \varphi(x_0) = u(x_0) \). Therefore (a) holds.

Suppose now that \( \varphi \) is a \( C^2 \) function such that \( \varphi \leq u \) near \( x_0 \) and \( \varphi(x_0) = u(x_0) \). As \( u \) is radial, this implies that

\[
-\frac{n-2}{\sqrt{ab}} = \partial_x_1 \ln u\big|_{r=\sqrt{ab}} \leq \partial_x_1 \ln \varphi(x_0) \leq \partial_x_1 \ln u\big|_{r=\sqrt{ab}} = 0, \tag{2.15}
\]

\[
\partial_{x_2} \ln \varphi(x_0) = \ldots = \partial_{x_n} \ln \varphi(x_0) = 0, \tag{2.16}
\]

\[
\left( \partial_{x_i} \partial_{x_j} \varphi(x_0) - \frac{1}{\sqrt{ab}} \partial_{x_1} \varphi(x_0) \delta_{ij} \right)_{2 \leq i,j \leq n} \leq 0. \tag{2.17}
\]

Now define \( \bar{\varphi}(x) = \varphi(|x|) = \varphi(|x|, 0, \ldots, 0), t = \ln r - \frac{1}{2} \ln(ab) \) and \( \bar{\xi}(t) = -\frac{2}{n-2} \ln \varphi(r) - \ln r \). By (2.15), we have that \( |\bar{\varphi}'(0)| \leq 1 \) and so \( \lambda(-A^\varphi(x_0)) \notin \Gamma_k \).

Let \( O \) denote the diagonal matrix with diagonal entries 1, -1, ..., -1. Note that, in block form,

\[
\nabla^2 \varphi(x_0) + O' \nabla^2 \varphi(x_0) O = 2 \begin{pmatrix} \partial_{x_1}^2 \bar{\varphi}(x_0) & \frac{1}{\sqrt{ab}} \partial_{x_1} \varphi(x_0) \bar{\varphi}(x_0) & 0 \\ \partial_{x_1} \varphi(x_0) & 0 & 0 \\ 0 & 0 & (\partial_{x_i} \partial_{x_j} \varphi(x_0))_{2 \leq i,j \leq n} \end{pmatrix}. 
\]

Thus, by (2.17),

\[
\nabla^2 \varphi(x_0) + O' \nabla^2 \varphi(x_0) O \leq 2 \begin{pmatrix} \frac{1}{\sqrt{ab}} \partial_{x_1} \varphi(x_0) & 0 & 0 \\ 0 & \frac{1}{\sqrt{ab}} \partial_{x_1} \varphi(x_0) \bar{\varphi}(x_0) & 0 \\ 0 & 0 & (\partial_{x_i} \partial_{x_j} \varphi(x_0))_{2 \leq i,j \leq n} \end{pmatrix} = 2 \nabla^2 \bar{\varphi}(x_0).
\]

Also, \( \varphi(x_0) = \bar{\varphi}(x_0) \) and, in view of (2.16), \( \nabla \bar{\varphi}(x_0) = \nabla \varphi(x_0) \). Hence

\[
-A^\varphi(x_0) - O' A^\varphi(x_0) O \leq -2A^\bar{\varphi}(x_0).
\]

As \( \lambda(-A^\varphi(x_0)) \notin \Gamma_k \), it follows that \( \lambda(-A^\varphi(x_0) - O' A^\varphi(x_0) O) \notin \Gamma_k \). Since the set of matrices with eigenvalues belonging to \( \Gamma_k \) is a convex cone (see e.g. [19] Lemma B.1]), we thus have that \( \lambda(-A^\varphi(x_0)) \notin \Gamma_k \) or \( \lambda(-O' A^\varphi(x_0) O) \notin \Gamma_k \). Since \( O \) is orthogonal, we deduce that \( \lambda(-A^\varphi(x_0)) \notin \Gamma_k \). We have verified (b) and thus completed the proof.

\[ \square \]

2.4 Proof of Theorem 1.11

As mentioned before, the uniqueness of solution follows from [6, 20]. We proceed to construct a radially symmetric solution with the indicated properties.
Let \( T = \frac{1}{2} \ln \frac{b}{a}, \quad p_a = -\frac{2}{n-2} \ln c_1 - \ln a \) and \( p_b = -\frac{2}{n-2} \ln c_2 - \ln b \). We will only consider the case that \( p_a \geq p_b \) (which is equivalent to \( b^{\frac{n-2}{2}} c_2 \geq a^{\frac{n-2}{2}} c_1 \)). (The case \( p_a < p_b \) can be treated using an inversion about \( |x| = \sqrt{ab} \).) We then have

\[
T = \frac{1}{2} \int_{p_a}^{p_b} \left\{ 1 + e^{-2\eta - 2p_a} \left[ 1 - e^{n\eta} \right]^{1/k} \right\}^{-1/2} d\eta
\]

(i) Suppose that \( T < T(a, b, c_1, c_2) \). We show that Case 1 holds.

Note that \( H(p_a, -1) = -(-1)^k e^{-np_a} \). Thus as \( T < T(a, b, c_1, c_2) \) and \((-1)^k H(p_a, \cdot)\) is decreasing in \((-\infty, -1)\), we can find \( q_a < -1 \) such that

\[
T = \frac{1}{2} \int_{p_a}^{p_b} \left\{ 1 + e^{-2\xi \left[ 1 - e^{n\left(\xi - p_a\right)} \right]^{1/k}} \right\}^{-1/2} d\xi
\]

(2.18)

Recall the solution \( \xi_{p_a, q_a} \) to (2.2) considered in the proof of Lemma 2.3. By (2.7), we have that \( 2T < T_{p_a, q_a} \). We then deduce from (2.5) and (2.18) that

\[
\xi_{p_a, q_a}(2T) = p_b.
\]

It thus follows that \( \xi(t) = \xi_{p_a, q_a}(t + T) \) is smooth in \([-T, T]\), satisfies (2.2) and \( \xi' < -1 \) in \((-T, T)\), as well as \( \xi(-T) = p_a \) and \( \xi(T) = p_b \). Returning to \( u = \exp \left( -\frac{n-2}{2} (\xi(\ln r - \frac{1}{2} \ln(ab)) + \ln r) \right) \) we obtain the conclusion.

(ii) Suppose that \( T = T(a, b, c_1, c_2) \). We show that Case 3 holds.

Recalling the definition of \( T(a, b, c_1, c_2) \), we see that as \( T > 0 \), we have \( p_a \neq p_b \).

As \( p_a \geq p_b \), we have \( p_a > p_b \). We can now follow the argument in (i) with \( \xi_{p_a, q_a} \) replaced by \( \xi_{p_a} \) (defined in the proof of Lemma 2.3) to reach the conclusion. We omit the details.

(iii) Suppose that \( T > T(a, b, c_1, c_2) \). We show that Case 4 holds.

In this case, we select \( p \geq p_a (> p_b) \) such that

\[
T = \frac{1}{2} \int_{p_a - p}^{0} \left\{ 1 + e^{-2\eta - 2p} \left[ 1 - e^{n\eta} \right]^{1/k} \right\}^{-1/2} d\eta
\]

\[
+ \frac{1}{2} \int_{p_b - p}^{0} \left\{ 1 + e^{-2\eta - 2p} \left[ 1 - e^{n\eta} \right]^{1/k} \right\}^{-1/2} d\eta
\]

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Such as $p$ exists as the right hand side tends to $T(a, b, c_1, c_2)$ when $p \to p_a$ and diverges to $\infty$ as $p \to \infty$. Recall the solution $\xi_p$ defined in the proof of Lemma 2.3. Let

$$T_+ = \frac{1}{2} \int_{p_a-p}^{p_0} \left\{ 1 + e^{-2\eta-2p} \left[ 1 - e^{n\eta} \right]^{1/k} \right\}^{-1/2} d\eta$$

and

$$T_- = \frac{1}{2} \int_{p_a-p}^{p_0} \left\{ 1 + e^{-2\eta-2p} \left[ 1 - e^{n\eta} \right]^{1/k} \right\}^{-1/2} d\eta.$$

Then $2T_+ < T_p$ and the function $\xi_p$ satisfies $\xi_p(2T_+) = p_b$ and $\xi_p(2T_-) = p_a$.

We then let

$$\xi(t) = \begin{cases} \xi_p(T_+ - T_- + t) & \text{if } -T_+ + T_- \leq t < T, \\ \xi_p(-T_+ + T_- - t) & \text{if } -T < t < -T_+ + T_- \end{cases}$$

We can then proceed as in the proof of Theorem 1.3 to show that $\xi$ is the desired solution.

### 2.5 Proof of Corollary 1.14

It is readily seen from (1.10) that

$$\lim_{c_3 \to 0, c_4 \to \infty} T(a, b, c_3, c_4) = \infty.$$ 

Thus, by Theorem 1.11 there exist positive constants $c_3$ and $c_4$ and a smooth function $v \in C^\infty(\Omega)$ such that

$$\sigma_k(\lambda(-A^v)) = 2^{-k} \binom{n}{k}, \quad \lambda(-A^v) \in \Gamma_k, \quad v > 0 \quad \text{in } \{ a < |x| < b \},$$

$$v|_{\{ |x| = a \}} = c_3, \quad v|_{\{ |x| = b \}} = c_4.$$

Now, by (1.10),

$$\lim_{s \to \infty} T(a, b, sc_3, sc_4) = 0.$$ 

We can thus select some $s > 1$ such that $\ln \frac{b}{a} > 2T(a, b, sc_3, sc_4), \ s c_3 > c$ and $s c_4 > c$. We then let $c_1 = sc_3$ and $c_2 = sc_4$. It is now readily seen that $u := sv$ satisfies (1.11)-(1.12), while, by Theorem 1.11 the solution to (1.2) and (1.9) belongs to $C^{0,1}_0(\Omega)$ but not $C^1(\Omega)$. \qed
A Appendix: Proof of Theorem 1.4

We abbreviate \( u^{4-n/2}g \) as \( g_u \). For small \( \tau > 0 \), let

\[
A_{g_u}^\tau = A_{g_u} + \tau \text{tr}_{g_u}(A_{g_u})g_u.
\]

By [11, Theorem 1.4], we have for all sufficiently small \( \tau > 0 \) that the problem

\[
\sigma_k \left( \lambda \left( -A_{g_u}^\tau \right) \right) = 2^{-k} \left( \frac{n}{k} \right), \quad \lambda \left( -A_{g_u}^\tau \right) \in \Gamma_k, \quad u_\tau > 0 \quad \text{in} \ M, \quad (A.1)
\]

has a unique smooth solution \( u_\tau \). Furthermore, by [11, Propositions 3.2 and 4.1], the family \( \{u_\tau\} \) is bounded in \( C^1(M) \) as \( \tau \to 0 \). Hence, along some sequence \( \tau_i \to 0 \), \( u_{\tau_i} \) converges uniformly to some \( u \in C^{0,1}(M) \). To conclude, we show that \( u \) is a viscosity solution to (1.8).

For notational convenience, we rename \( u_{\tau_i} \) as \( u_i \).

Fix some \( \bar{x} \in M \).

Step 1: We show that \( u \) is a sub-solution to (1.8) at \( \bar{x} \). More precisely, we show that for every \( \varphi \in C^2(M) \) such that \( \varphi \geq u \) on \( M \) and \( \varphi(\bar{x}) = u(\bar{x}) \) there holds that

\[
\lambda \left( -A_{g_u}(\bar{x}) \right) \in \left\{ \lambda \in \Gamma_k \mid \sigma(\lambda) \geq 2^{-k} \left( \frac{n}{k} \right) \right\} = \overline{S}_k =: \overline{S}. \quad (A.2)
\]

Here \( d_g \) denotes the distance function of \( g \) and \( B_\delta(\bar{x}) \) denote the open geodesic ball of radius \( \delta \) and centered at \( \bar{x} \) with respect to \( g \). Fix some arbitrary small \( \nu > 0 \) so that \( \varphi_\nu := \varphi + \nu d_g(\cdot, \bar{x})^2 \) is \( C^2 \) in \( B_\delta(\bar{x}) \).

Note that

\[
\varphi_\nu = \varphi + \nu^3 \geq u + \nu^3 \quad \text{on} \ \partial B_\delta(\bar{x}) \quad \text{and} \quad \varphi_\nu(\bar{x}) = u(\bar{x}). \quad (A.3)
\]

Select \( x_{i,\delta} \in \overline{B_\delta(\bar{x})} \) such that

\[
(\varphi_\nu - u_i)(x_{i,\delta}) = \inf_{B_\delta(\bar{x})} (\varphi_\nu - u_i) =: m_{i,\delta}.
\]

By (A.3) and the uniform convergence of \( u_i \) to \( u \), we have that \( x_{i,\delta} \in B_\delta(\bar{x}) \). It follows that

\[
\nabla_g(\varphi_\nu - u_i)(x_{i,\delta}) = 0, \quad \nabla_g^2(\varphi_\nu - u_i)(x_{i,\delta}) \geq 0
\]

and so

\[
-A_{g_{\varphi_\nu - \nu^3}}^\tau (x_{i,\delta}) \geq -A_{g_u}^\tau (x_{i,\delta}).
\]

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Recalling (A.1), we hence have
\[
\lambda \left( -A_g \varphi_{\delta - m_i, \delta} (x_{i, \delta}) \right) \in \overline{S}. \tag{A.4}
\]

On the other hand, as \( \bar{x} \) is the unique minimum point of \( \varphi_{\delta} - u \) in \( B_\delta(\bar{x}) \), we have \( x_{i, \delta} \to \bar{x} \) and \( m_{i, \delta} \to 0 \) as \( i \to \infty \). We can now pass \( i \to \infty \) in (A.4) to obtain
\[
\lambda \left( -A_g \varphi_{\delta} (\bar{x}) \right) \in \overline{S}.
\]

Since \( \delta \) is arbitrary, this proves (A.2) after sending \( \delta \to 0 \).

**Step 2:** We show that \( u \) is a super-solution to (1.8) at \( \bar{x} \), i.e. if \( \varphi \in C^2(M) \) is such that \( \varphi \leq u \) on \( M \) and \( \varphi(\bar{x}) = u(\bar{x}) \), then
\[
\lambda \left( -A_g \varphi(\bar{x}) \right) \in \mathbb{R}^n \setminus \left\{ \lambda \in \Gamma_k \mid \sigma(\lambda) > 2^{-k} \left( \frac{n}{k} \right) \right\} = S_k =: \overline{S}. \tag{A.5}
\]

The proof is analogous to that in Step 1. Fix some arbitrary small \( \delta > 0 \) so that \( \hat{\varphi}_\delta := \varphi_{-\delta} = \varphi - \delta d_g(\cdot, \bar{x})^2 \) is \( C^2 \) in \( B_\delta(\bar{x}) \). Clearly
\[
\hat{\varphi}_\delta \leq u - \delta^3 \text{ on } \partial B_\delta(\bar{x}) \text{ and } \hat{\varphi}_\delta(\bar{x}) = u(\bar{x}).
\]

We next select \( \hat{x}_{i, \delta} \in \overline{B_\delta(\bar{x})} \) such that
\[
(\hat{\varphi}_\delta - u_i)(\hat{x}_{i, \delta}) = \sup_{B_\delta(\bar{x})} (\hat{\varphi}_\delta - u_i) =: \hat{m}_{i, \delta}.
\]

As before, we have \( \hat{x}_{i, \delta} \in B_\delta(\bar{x}), \nabla_g(\hat{\varphi}_\delta - u_i)(\hat{x}_{i, \delta}) = 0, \nabla_g^2(\hat{\varphi}_\delta - u_i)(\hat{x}_{i, \delta}) \leq 0 \) and
\[
-A_{g_{\hat{\varphi}_\delta - m_{i, \delta}}} (\hat{x}_{i, \delta}) \leq -A_{g_{u_i}} (\hat{x}_{i, \delta}).
\]

By (A.1), we hence have
\[
\lambda \left( -A_{g_{\hat{\varphi}_\delta - m_{i, \delta}}} (\hat{x}_{i, \delta}) \right) \in \overline{S}. \tag{A.6}
\]

Also, as \( \hat{x}_{i, \delta} \to \bar{x} \) and \( \hat{m}_{i, \delta} \to 0 \) as \( i \to \infty \), we can first pass \( i \to \infty \) and then \( \delta \to 0 \) in (A.6) to reach (A.5). \( \square \)
References


