Harnack inequalities and Bôcher-type theorems for conformally invariant fully nonlinear degenerate elliptic equations

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Abstract

We give a generalization of a theorem of Bôcher for the Laplace equation to a class of conformally invariant fully nonlinear degenerate elliptic equations. We also prove a Harnack inequality for locally Lipschitz viscosity solutions and a classification of continuous radially symmetric viscosity solutions.

1 Introduction

On a Riemannian manifold (M,g) of dimension $n \geq 3$, consider the Schouten tensor

$$A_g = \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{1}{2(n-1)} R_g\, g\Big),$$

where Ric_g denotes the Ricci curvature. Let $\lambda(A_g) = (\lambda_1, \cdots, \lambda_n)$ denote the eigenvalues of A_g with respect to g, and let

(1.1) $\quad \Gamma \subset \mathbb{R}^n$ be an open convex symmetric cone with vertex at the origin,

$$\Big\{\lambda \in \mathbb{R}^n | \lambda_i > 0, 1 \leq i \leq n\Big\} \subset \Gamma \subset \Big\{\lambda \in \mathbb{R}^n | \sum_{i=1}^n \lambda_i > 0\Big\}, \tag{1.2}$$

(1.3) $\quad f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ be concave, homogeneous of degree one, and symmetric in λ_i,

$$f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \partial\Gamma; \quad f_{\lambda_i} > 0 \text{ in } \Gamma\ \forall 1 \leq i \leq n. \tag{1.4}$$

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The following fully nonlinear version of the Yamabe problem has received much attention in recent years:

$$f\Big(\lambda\big(A_{u^{\frac{4}{n-2}}g}\big)\Big) = 1, \quad u > 0 \quad \text{and} \quad \lambda(A_{\hat{g}}) \in \Gamma \quad \text{on } M. \tag{1.5}$$

For $1 \le k \le n$, let $\sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, denote the k-th elementary symmetric function, and let Γ_k denote the connected component of $\{\lambda \in \mathbb{R}^n | \sigma_k(\lambda) > 0\}$ containing the positive cone $\{\lambda \in \mathbb{R}^n | \lambda_1, \cdots, \lambda_n > 0\}$. Then $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ satisfies (1.1)-(1.4). When $(f, \Gamma) = (\sigma_1, \Gamma_1)$, (1.5) is the Yamabe problem in the so-called positive case.

When M is a Euclidean domain and $g = g_{\text{flat}}$ is the flat metric, equation (1.5) takes the form

$$f(\lambda(A^u)) = 1, \quad u > 0, \tag{1.6}$$

where $\lambda(A^u)$ denotes the eigenvalues of the matrix A^u with entries

$$(A^u)_{ij} := -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla_{ij} u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla_i u \nabla_j u - \frac{2}{(n-2)^{-2}} u^{-\frac{2n}{n-2}} |\nabla u|^2 \delta_{ij}.$$

Equation (1.5) is a second order fully nonlinear elliptic equation of u. Fully nonlinear elliptic equations involving $f(\lambda(\nabla^2 u))$ was investigated in the classic paper of Caffarelli, Nirenberg and Spruck [5].

Another equation which is closely related to (1.6) is

$$\lambda(A^u) \in \partial\Gamma, \quad u > 0. \tag{1.7}$$

Equation (1.7) is equivalent to

$$f(\lambda(A^u)) = 0, \quad u > 0 \quad \text{and} \quad \lambda(A^u) \in \bar{\Gamma}.$$

Both equations (1.6) and (1.7) arise naturally in studying blow-up sequences of solutions of (1.5).

There have been many works on equations (1.6) and (1.7), which include Liouville-type theorems for solutions of (1.6) and (1.7) in R^n, Harnack-type inequalities, symmetry of solutions of (1.6) and (1.7) on $R^n \setminus \{0\}$, and behaviors of solutions of (1.6) near isolated singularities; see e.g. [6, 7, 9, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 25, 26].

The main focus of the present paper concerns solutions of (1.7) with isolated singularies. When $\Gamma = \Gamma_1$, (1.7) is $\Delta u = 0$. A classical theorem of Bôcher [2] asserts that any positive harmonic function in the punctured ball $B_1 \setminus \{0\} \subset \mathbb{R}^n$ can be expressed as the sum of a multiple of the fundamental solution of the Laplace equation and a harmonic function in the whole unit ball B_1. This can be viewed as a statement on the asymptotic behavior of a positive harmonic function near its isolated singularities. Our goal is to establish a generalization of this result for (1.7).

Equation (1.7) is a fully nonlinear *degenerate elliptic* equation. For example, when $\Gamma = \Gamma_k$ with $k \ge 2$, the strong maximum principle and the Hopf lemma fail for

(1.7) (see the discussion after (1.9) below). For fully nonlinear *uniformly elliptic* equations, extensions of Bôcher's theorem have been established in the literature. See Labutin [17], Felmer and Quass [10] and Armstrong, Sirakov and Smart [1].

In the case of the non-degenerate elliptic equation (1.6) with $(f,\Gamma) = (\sigma_k^{1/k},\Gamma_k)$, local behavior at an isolated singularity is fairly well-understood: It was proved by Caffarelli, Gidas and Spruck [3] for $k = 1$ and by Han, Li and Teixeira [16] for $2 \le k \le n$ that $u(x) = u_*(|x|)(1+O(|x|^\alpha))$ where u_* is some radial solution of (1.6) on $R^n \setminus \{0\}$ and α is some positive number. This statement is complemented by the classification of radial solutions of (1.6) by Chang, Han and Yang [8]. For (1.7) with Γ satisfying (1.1) and (1.2), it was proved by the first author in [22] that a locally Lipschitz viscosity solution in $R^n \setminus \{0\}$ must be radially symmetric about $\{0\}$. We also note that Gonzalez showed in [13] that isolated singularities of C^3 solutions of (1.6) with finite volume are bounded, among other statements. See also [12] for related work in the subcritical case.

As mentioned above, solutions of (1.7) arise as (rescaled) limits of blow-up sequence of solutions of (1.5), along which one may lose uniform ellipticity. For this reason, it is of interest to consider solutions u of (1.7) which is not C^2. We adopt the following definition for less regular solutions of (1.7). For $\Omega \subset \mathbb{R}^n$, we use $LSC(\Omega)$ and $USC(\Omega)$ to denote respectively the set of lower and upper semi-continuous (real valued) functions on Ω.

Definition 1.1. Let Ω be an open subset of $\mathbb{R}^n$, Γ satisfy (1.1) and (1.2), and u be a positive function in $LSC(\Omega)$ ($USC(\Omega)$). We say that

$$\lambda(A^u) \in \bar\Gamma \qquad (\lambda(A^u) \in \mathbb{R}^n \setminus \Gamma)$$

in Ω in the viscosity sense if for any $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$, $(u-\varphi)(x_0) = 0$ and

$$u - \varphi \ge 0 \qquad (u - \varphi \le 0) \text{ near } x_0,$$

there holds

$$\lambda(A^\varphi(x_0)) \in \bar\Gamma \qquad (\lambda(A^\varphi(x_0)) \in \mathbb{R}^n \setminus \Gamma).$$

We say that a positive continuous function u satisfies $\lambda(A^u) \in \partial\Gamma$ in Ω in the viscosity sense if $\lambda(A^u)$ belongs to both $\bar\Gamma$ and $\mathbb{R}^n \setminus \bar\Gamma$ in the viscosity sense thereof.

It is well known that if a C^2 function satisfies the above differential relations in the viscosity sense then it satisfies them in the classical sense.

In our discussion, the constant μ_Γ^+ defined by

(1.8) $\quad \mu_\Gamma^+ \in [0,n-1]$ is the unique number such that $(-\mu_\Gamma^+,1,\dots 1) \in \partial\Gamma$

plays an important role. Note that μ_Γ^+ is well-defined thanks to (1.1) and (1.2). For $\Gamma = \Gamma_k$, we have $\mu_{\Gamma_k}^+ = \frac{n-k}{k}$ for $1 \le k \le n$. In particular,

$$\begin{cases} \mu_{\Gamma_k}^+ > 1 & \text{if } k < \frac{n}{2}, \\ \mu_{\Gamma_k}^+ = 1 & \text{if } k = \frac{n}{2}, \\ \mu_{\Gamma_k}^+ < 1 & \text{if } k > \frac{n}{2}. \end{cases}$$

As another example, for the so-called θ-convex cone

$$\Sigma_\theta = \Big\{\lambda \ : \ \lambda_i + \theta \sum_{j=1}^n \lambda_j > 0 \text{ for all } i\Big\}, \quad \theta \geq 0,$$

we have $\mu^+_{\Sigma_\theta} = \frac{(n-1)\theta}{1+\theta} \in [0, n-1)$.

For simplicity, in most of this introduction, we restrict ourselves to the case where

$$(1,0,\dots,0) \in \partial\Gamma. \tag{1.9}$$

We note that when (1.9) holds, the range for μ^+_Γ is $[0, n-2]$.

Clearly, the cone Γ_k for $2 \leq k \leq n$ satisfies (1.9). See Theorems 1.8, 2.2 and 4.6 for the case where (1.9) does not hold. Note that, by (1.2), $(1,0,\dots,0) \in \bar\Gamma$, and by (1.1) and (1.2), (1.9) is equivalent to

$$(\lambda_1, -1, \dots, -1) \in \mathbb{R}^n \setminus \bar\Gamma \text{ for all } \lambda_1 \in \mathbb{R}. \tag{1.10}$$

In [24], it was shown that, under (1.9), the strong maximum principle and the Hopf lemma fail for a large class of nonlinear degenerate elliptic equations including (1.7). Conversely, if (1.9) does not hold, then the strong maximum principle and the Hopf lemma hold.

Our first two main theorems (which cover the case $\Gamma = \Gamma_k$ for $2 \leq k \leq \frac{n}{2}$) are as follows.

Theorem 1.2. *Assume that Γ satisfies* (1.1), (1.2), (1.9), *and* $\mu^+_\Gamma > 1$. *Let* $u \in C^{0,1}_{\rm loc}(B_1 \setminus \{0\})$ *be a positive viscosity solution of* (1.7) *in* $B_1 \setminus \{0\}$. *Then*

$$u(x)^{\frac{\mu^+_\Gamma - 1}{n-2}} = a^{\frac{\mu^+_\Gamma - 1}{n-2}} |x|^{-\mu^+_\Gamma + 1} + \mathring{w}(x),$$

where

$$a = \inf_{x \in B_1 \setminus \{0\}} |x|^{n-2} u(x) \geq 0,$$

and $\mathring{w}$ is a non-negative function in $L^\infty_{\rm loc}(B_1)$. Moreover,

$$\text{either } \mathring{w} \equiv 0 \text{ in } B_1 \setminus \{0\} \text{ and } u(x) \equiv a|x|^{-(n-2)} > 0 \text{ in } B_1 \setminus \{0\}, \tag{1.11}$$

$$\text{or } 0 < \min_{\partial B_r} \mathring{w} \leq \mathring{w} \leq \max_{\partial B_r} \mathring{w} \text{ in } B_r \setminus \{0\} \qquad \forall\, 0 < r < 1. \tag{1.12}$$

Finally, if $a = 0$ then u can be extended to a positive function in $C^{0,\beta}_{\rm loc}(B_1)$ and

$$\|u\|_{C^{0,\beta}(B_{1/2})} \leq C(\Gamma, \beta) \sup_{B_{1/2}} u \ \forall\, \beta \in (0,1).$$

Theorem 1.3. *Assume that Γ satisfies* (1.1), (1.2), (1.9), *and* $\mu^+_\Gamma = 1$. *Let* $u \in C^{0,1}_{\rm loc}(B_1 \setminus \{0\})$ *be a positive viscosity solution of* (1.7) *in* $B_1 \setminus \{0\}$. *Then*

$$\ln u(x) = -\alpha \ln |x| + \mathring{w}(x),$$

where $\alpha \in [0, n-2]$ *and* $\mathring{w} \in L^\infty_{\rm loc}(B_1)$ *satisfying*

$$\min_{\partial B_r} \mathring{w} \leq \mathring{w} \leq \max_{\partial B_r} \mathring{w} \text{ in } B_r \setminus \{0\} \quad \forall\, 0 < r < 1. \tag{1.13}$$

If $\alpha = n-2$*, then* $\mathring{w}$ *is constant, i.e.* $u(x) = \frac{C}{|x|^{n-2}}$ *for some positive constant* C*. If* $\alpha = 0$*, then* u *can be extended to a positive function in* $C^{0,\beta}(B_1)$ *and* $\|u\|_{C^{0,\beta}(B_{1/2})} \leq C(\Gamma,\beta) \sup_{B_{1/2}} u$ *for all* $\beta \in (0,1)$*.*

When $0 \leq \mu_\Gamma^+ < 1$, which is the case for $\Gamma = \Gamma_k$ with $k > \frac{n}{2}$, there have been works in the literature. In this case, Γ is closely related to the so-called θ-convex cone Σ_θ for some $0 \leq \theta < \frac{1}{n-2}$ (see Appendix B for a definition). For such Γ, Gursky and Viaclovsky [15] showed that classical solutions of (1.7) in a punctured ball either extends to a Hölder continuous function or is pinched between two multiples of $|x|^{2-n}$. For $\Gamma = \Gamma_k$ with $\frac{n}{2} < k \leq n$, Li showed in [20] that bounded classical solutions in a punctured ball extends to a Hölder continuous function in the ball, and Trudinger and Wang showed in [25] that solutions of $\lambda(A^u) \in \bar{\Gamma}_k$ in B_1 in some appropriate weak sense is either Hölder continuous or is a multiple of $|x-x_0|^{2-n}$ for some x_0. Using a result of Caffarelli, Li and Nirenberg [4, Theorem 1.1] (see also Proposition 4.1), we prove:

Theorem 1.4. *Assume that* Γ *satisfies* (1.1)*,* (1.2)*,* (1.9) *and* $0 \leq \mu_\Gamma^+ < 1$*. Let* $u \in LSC(B_1 \setminus \{0\}) \cap L^\infty_{\rm loc}(B_1 \setminus \{0\})$ *be a positive viscosity solution of* $\lambda(A^u) \in \bar{\Gamma}$ *in* $B_1 \setminus \{0\}$*. Then either* $u = \frac{C}{|x|^{n-2}}$ *for some* $C > 0$*, or* u *can be extended to a positive function in* $C^{0,1-\mu_\Gamma^+}_{\rm loc}(B_1)$*. Moreover, in the latter case, there holds*

$$\|u^{\frac{\mu_\Gamma^+ - 1}{n-2}}\|_{C^{0,1-\mu_\Gamma^+}(B_{1/2})} \leq C(\Gamma) \sup_{\partial B_{3/4}} u^{\frac{\mu_\Gamma^+ - 1}{n-2}}. \tag{1.14}$$

The Hölder exponent obtained in Theorem 1.4 is optimal (see Theorem 2.2). If Γ does not satisfy (1.9), the rigidity assertion about singular solutions of (1.7) in Theorem 1.4 is false. For example, for $\Gamma = \Sigma_\theta$ with $0 < \theta < \frac{1}{n-2}$, the function

$$u(x) = u(|x|) = \left(|x|^{-(n-2+\theta^{-1})} - \frac{1}{2}\right)^{\frac{n-2}{n-2+\theta^{-1}}}$$

is a positive radially symmetric solution of (1.7) in $B_1 \setminus \{0\}$, which is singular at the origin but is not a multiple of $|x|^{2-n}$.

For Γ_k, $k > \frac{n}{2}$, if u is a weak solution of $\lambda(A^u) \in \bar{\Gamma}_k$ in B_1 in the sense of [25], then u is a viscosity solution of $\lambda(A^u) \in \bar{\Gamma}_k$ in B_1. On the other hand, it is unclear to us that the converse is true.

A key technical step of our proof of the Bôcher-type theorems is the following Harnack inequality for $C^{0,1}$ viscosity solutions of (1.7) which is of independent interest.

Theorem 1.5. *Assume that* Γ *satisfies* (1.1) *and* (1.2). *Let* $u \in C^{0,1}(B_1)$ *be a positive viscosity solution of* $\lambda(A^u) \in \partial\Gamma$ *in* B_1. *Then, for every* $0 < \varepsilon < 1$, *there exists a constant* $C = C(\Gamma, \varepsilon)$ *such that*

$$|\nabla \ln u| \leq C \text{ a.e. in } B_{1-\varepsilon}.$$

Consequently,

$$\sup_{B_{1-\varepsilon}} u \leq e^C \inf_{B_{1-\varepsilon}} u.$$

We note that an analogue of Theorem 1.5 for the equation

$$f(\lambda(A^u)) = \psi, \quad \lambda(A^u) \in \Gamma \qquad \text{in } B_1$$

where ψ is a smooth positive function in B_1 was proved by the first author in [22]. For $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ and smooth $\psi \geq 0$, gradient estimates for C^3 solutions were obtained by Gursky and Viaclovsky [15] based on earlier work of Guan and Wang [14].

Beside Theorem 1.5, another ingredient in our proof of the Bôcher-type theorems is a classification of all C^0 positive radially symmetric viscosity solutions (1.7) in $\{a < |x| < b\} := \{x \in \mathbb{R}^n \mid a < |x| < b\}$, where $0 \leq a < b \leq \infty$.

Theorem 1.6. *Assume that* Γ *satisfies* (1.1), (1.2) *and* (1.9). *For* $0 \leq a < b \leq \infty$, *let* $u \in C^0(\{a < |x| < b\})$ *be radially symmetric and positive. Then* u *is a solution of* (1.7) *in* $\{a < |x| < b\}$ *in the viscosity sense if and only if*

$$u(x) = \begin{cases} C_1 |x|^{-C_2} \text{ with } C_1 > 0, 0 \leq C_2 \leq n-2 & \text{if } \mu_\Gamma^+ = 1, \\ (C_3 |x|^{-\mu_\Gamma^+ + 1} + C_4)^{\frac{n-2}{\mu_\Gamma^+ - 1}} \text{ with } C_3 \geq 0, C_4 \geq 0, C_3 + C_4 > 0 & \text{if } \mu_\Gamma^+ \neq 1. \end{cases}$$

An immediate consequence of Theorem 2.2 and [22, Theorem 1.18] is:

Corollary 1.7. *Assume that* Γ *satisfies* (1.1), (1.2), (1.9) *and* $\mu_\Gamma^+ \geq 1$. *If* $u \in C^{0,1}_{\rm loc}(B_1 \setminus \{0\})$ *is a positive viscosity solution of* (1.7) *in* $B_1 \setminus \{0\}$ *and if* u *is locally bounded near the origin, then* u *is constant.*

Note that in the above, u is not assumed to be a priori radial.

Last but not least, we have the following asymptotics for isolated singularities of (1.7) when (1.9) is not assumed.

Theorem 1.8. *Assume that* Γ *satisfies* (1.1) *and* (1.2). *Let* $u \in C^{0,1}_{\rm loc}(B_1 \setminus \{0\})$ *be a positive viscosity solution of* (1.7) *in* $B_1 \setminus \{0\}$. *Then*

$$\lim_{|x| \to 0} |x|^{n-2} u(x) = a \in [0, \infty).$$

The rest of the paper is organized as follows. We start with a study of radially symmetric solutions and super-solutions of (1.7) in Section 2. The key result of this section is a Theorem 2.2, which is more general than Theorem 1.6. Also in this section, we exhibit certain monotonicity properties which are used later on. In Section 3, we prove Theorem 1.5. Proofs of the Bôcher-type theorems are presented in Section 4.

2 Radially symmetric solutions and supersolutions

For a smooth radially symmetric function u, $\lambda(A^u)$ will take the form $(V,v,\dots,v)$ for some V and v. Thus, in studying radially symmetric solutions of (1.7), it is important to see which vectors of the above forms lie on $\partial\Gamma$. By homogeneity, it suffices to see which of

$$(\lambda_1,1,\dots,1), \qquad (1,0,\dots,0), \qquad (\lambda_1,-1,\dots,-1)$$

belong to $\partial\Gamma$. In this respect, the constant μ_Γ^+ defined in (1.8) and the condition (1.9) come naturally into our discussion. Recall that μ_Γ^+ is well-defined thanks to (1.1) and (1.2). If (1.9) is satisfied, i.e. $(1,0,\dots,0)\in\partial\Gamma$, no vector of the form $(\lambda_1,-1,\dots,-1)$ belongs to $\bar\Gamma$. Conversely, if (1.9) fails, i.e.

$$(1,0,\dots,0)\notin\partial\Gamma, \tag{2.1}$$

then there is a unique $(\lambda,-1,\dots,-1)$ on $\partial\Gamma$. We thus define

$$\begin{cases} \mu_\Gamma^- = +\infty & \text{if (1.9) holds,}\\ \mu_\Gamma^-\in[n-1,\infty) & \text{is the number such that } (\mu_\Gamma^-,-1,\dots,-1)\in\\ & \partial\Gamma \text{ if (1.9) does not hold.}\end{cases} \tag{2.2}$$

The following lemma, whose proof can be found in Appendix B, gives some basic properties of $\mu_\Gamma^\pm$.

Lemma 2.1. *Assume that Γ satisfies (1.1) and (1.2). Then*

(a) μ_Γ^+ *and* μ_Γ^- *are monotone in* Γ.
(b) $\mu_\Gamma^+=n-1$ *(or* $\mu_\Gamma^-=n-1$*) if and only if* $\Gamma=\Gamma_1$.
(c) $\mu_\Gamma^+=0$ *if and only if* $\Gamma=\Gamma_n$.
(d) μ_Γ^+ *and* μ_Γ^- *satisfy*

$$(n-2)+\frac{n-1}{\mu_\Gamma^+}\le\mu_\Gamma^-\le\begin{cases}\frac{n-1}{\mu_\Gamma^+-(n-2)} & \text{if } \mu_\Gamma^+>n-2,\\ \infty & \text{otherwise.}\end{cases}$$

(e) if (1.9) holds then $0\le\mu_\Gamma^+\le n-2$.

Theorem 1.6 is a special case of the following result.

Theorem 2.2. *Assume that Γ satisfies (1.1), (1.2) and $0\le a<b\le\infty$. Then every radially symmetric positive viscosity solution u of (1.7) in $\{a<|x|<b\}$ is one of the following smooth solutions:*

(a) $u(x)=C_1|x|^{-C_2}$ *with* $C_1>0$, $0\le C_2\le n-2$ *when* $\mu_\Gamma^+=1$,

(b) $u(x)=(C_3|x|^{-\mu_\Gamma^++1}+C_4)^{\frac{n-2}{\mu_\Gamma^+-1}}$ *with* $C_3\ge0$, $C_4\ge0$, $C_3+C_4>0$ *when* $\mu_\Gamma^+\ne1$,

(c) $u(x)=(C_5|x|^{-\mu_\Gamma^-+1}-C_6)^{\frac{n-2}{\mu_\Gamma^--1}}$ *with* $C_5>0$, $C_6\ge0$, $\lim_{r\to b}C_5r^{-\mu_\Gamma^-+1}-C_6\ge0$ *when* $\mu_\Gamma^-<\infty$,

(d) $u(x)=(-C_7|x|^{-\mu_\Gamma^-+1}+C_8)^{\frac{n-2}{\mu_\Gamma^--1}}$ *with* $C_7\ge0$, $C_8>0$, $-\lim_{r\to a}C_7r^{-\mu_\Gamma^-+1}+C_8\ge0$ *when* $\mu_\Gamma^-<\infty$.

Remark 2.3. Assume that Γ satisfies (1.1), (1.2), (1.9), and $0 < b < \infty$. By the above theorem, the only positive radially symmetric C^2 solutions of (1.7) in the ball $\{|x| < b\}$ are constants. If one has in addition that $\mu_\Gamma^+ \geq 1$, then the only *bounded* positive radially symmetric C^2 solutions of (1.7) in the punctured ball $\{0 < |x| < b\}$ are constants.

We first give the

Proof of Theorem 2.2 for classical solutions. Let $r = |x|$ and

$$\hat{A}^u = \frac{n-2}{2}\, u^{\frac{2n}{n-2}}\, A^u = -u\nabla^2 u + \frac{n}{n-2}\,\nabla u \otimes \nabla u - \frac{1}{n-2}\,|\nabla u|^2 I.$$

Since u is radially symmetric, the eigenvalues of $\hat{A}^u$ are

- $V := -u\,u'' + \frac{n-1}{n-2}\,(u')^2$, which is simple,
- and $v := -\frac{1}{r}\,u\,u' - \frac{1}{n-2}\,(u')^2$, which has multiplicity $n-1$.

Thus, by (1.8) and (2.2), for each $r \in (a,b)$,

$$\begin{aligned}
&\text{either } v(r) = 0,\\
&\text{or } v(r) > 0 \text{ and } V(r) + \mu_\Gamma^+\, v(r) = 0,\\
&\text{or } v(r) < 0, \mu_\Gamma^- < \infty \text{ and } V(r) + \mu_\Gamma^-\, v(r) = 0.
\end{aligned}$$

Case 1: There holds

$$v = -\frac{1}{r}\,u\,u' - \frac{1}{n-2}\,(u')^2 = -\frac{1}{n-2}\,u\,u'\,[\ln(r^{n-2}\,u)]' = 0 \text{ in } (a,b). \tag{2.3}$$

Solutions to (2.3) are $u \equiv C_0$ or $u \equiv \hat{C}_0\, r^{2-n}$. In particular, $V \equiv 0$ and hence $A^u \equiv 0$ in (a,b).

Case 2: v is positive somewhere in (a,b). Let (c,d) be a maximal open subinterval of (a,b) on which v is positive. Then, in the interval (c,d),

$$\begin{cases} v > 0, \\ V + \mu_\Gamma^+\, v = -u\,u'' - \frac{\mu_\Gamma^+}{r}\,u\,u' + \frac{n-1-\mu_\Gamma^+}{n-2}\,(u')^2 = 0. \end{cases} \tag{2.4}$$

If $\mu_\Gamma^+ = 1$, we put $u = e^w$ and obtain $w'' + \frac{1}{r}\,w' = 0$, which gives $w = c_1 + c_2 \ln r$. It follows that

$$u = C_1\, r^{-C_2} \text{ in } (c,d) \text{ for some } C_1 > 0 \tag{2.5}$$

If $\mu_\Gamma^+ \neq 1$, we introduce

$$u = w^{\frac{n-2}{\mu_\Gamma^+ - 1}}.$$

The second line of (2.4) becomes $w'' + \frac{\mu_\Gamma^+}{r}\,w' = 0$, which implies $w = C_3 r^{-\mu_\Gamma^+ + 1} + C_4$ and

$$u = (C_3\, r^{-\mu_\Gamma^+ + 1} + C_4)^{\frac{n-2}{\mu_\Gamma^+ - 1}} \text{ in } (c,d). \tag{2.6}$$

We next show that $(c,d)=(a,b)$. Arguing by contradiction, assume for example that $c\neq a$. By the maximality of (c,d), we must have

$$v(c) = -\frac{1}{c}\,u(c)\,u'(c) - \frac{1}{n-2}\,(u'(c))^2 = 0. \tag{2.7}$$

Since $v\neq 0$ in (c,d), we have $C_2\neq 0$ if $\mu_\Gamma^+=1$ and $C_3\neq 0$ if $\mu_\Gamma^+\neq 1$. From the explicit form of u, it can be seen that $u'(c)\neq 0$. Thus this implies

$$u'(c) = -\frac{n-2}{c}\,u(c). \tag{2.8}$$

If $\mu_\Gamma^+=1$, this implies that $C_2=n-2$ in (2.5) and so v is identically zero in (c,d), contradicting the first line of (2.4). If $\mu_\Gamma^+\neq 1$, this implies that $C_4=0$ in (2.6), and again results a contradiction. We have thus shown that $(c,d)=(a,b)$.

A calculation shows, in view of (2.5) and (2.6), that $\hat{A}^u$ is similar to $\mathrm{diag}(-\mu_\Gamma^+\,v,v,\ldots,v)$ where

$$v = \begin{cases} C_1^2\left(C_2-\frac{C_2^2}{n-2}\right)r^{-2C_2-2} & \text{if } \mu_\Gamma^+=1,\\ (n-2)\,C_3\,C_4\,u^{\frac{2(n-1-\mu_\Gamma^+)}{n-2}}\,r^{-\mu_\Gamma^+-1} & \text{if } \mu_\Gamma^+\neq 1.\end{cases}$$

The restrictions of C_1, C_2, C_3 and C_4 in (a) and (b) follow.

<u>Case 3:</u> v is negative somewhere in (a,b). Let (c,d) be a maximal open subinterval of (a,b) on which v is negative. Then $\mu_\Gamma^-<\infty$ and, in the interval (c,d),

$$\begin{cases} v<0,\\ V+\mu_\Gamma^-\,v = -u\,u'' - \frac{\mu_\Gamma^-}{r}\,u\,u' + \frac{n-1-\mu_\Gamma^-}{n-2}\,(u')^2 = 0.\end{cases}$$

Arguing as in Case 2, we arrive at

$$u = (\hat{C}_5\,r^{-\mu_\Gamma^-+1} + \hat{C}_6)^{\frac{n-2}{\mu_\Gamma^- -1}} \text{ in } (a,b).$$

It follows that $\hat{A}^u$ is similar to $\mathrm{diag}(-\mu_\Gamma^-\,v,v,\ldots,v)$ where

$$v = (n-2)\,\hat{C}_5\,\hat{C}_6\,u^{\frac{2(n-1-\mu_\Gamma^-)}{n-2}}\,r^{-\mu_\Gamma^- -1}$$

The remaining part of the theorem follows easily from the above. □

Here are consequences of what we have just proved:

Corollary 2.4. *Assume that* Γ *satisfies* (1.1), (1.2) *and* (1.9). *For any* $0<a<b<\infty$, $\alpha>0$ *and* $\beta>0$, *there exists a positive radially symmetric function* u *in* $C^2(\{a<|x|<b\})\cap C^0(\{a\le|x|\le b\})$ *satisfying*

$$\begin{cases} \lambda(A^u)\in\partial\Gamma \text{ in } \{a<|x|<b\},\\ u|_{\partial B_a}=\alpha, \qquad u|_{\partial B_b}=\beta\end{cases} \tag{2.9}$$

if any only if

$$0 \le \ln\frac{\alpha}{\beta} \le (n-2)\ln\frac{b}{a}. \tag{2.10}$$

Moreover, the solution is unique.

Corollary 2.5. *Assume that* Γ *satisfies* (1.1), (1.2) *and* (2.1). *For any* $0 < a < b < \infty$, $\alpha > 0$ *and* $\beta > 0$, *there exists a unique positive radially symmetric function solution* $u \in C^2(\{a < |x| < b\}) \cap C^0(\{a \le |x| \le b\})$ *to* (2.9).

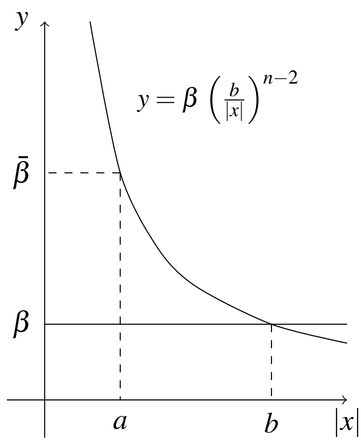

FIGURE 2.1. For (2.9) to have a solution when (1.9) holds, α must satisfy $\beta \le \alpha \le \bar{\beta}$. No such restriction is need when (1.9) does not hold.

It is clear that the proof of Theorem 2.2 for classical solutions can be adapted to give a complete classification for radially symmetric classical solutions of $\sigma_k(A^u) = 0$ (without any ellipticity assumption). In this case, the solutions take the form

$$u(x) = \begin{cases} \hat{C}_1 \, |x|^{-\hat{C}_2} & \text{if } n = 2k, \\ (\hat{C}_3 \, |x|^{-\frac{n-2k}{k}} + \hat{C}_4)^{\frac{(n-2)k}{n-2k}} & \text{if } n \neq 2k, \end{cases}$$

where the only restriction on the constants $\hat{C}_i \in \mathbb{R}$ is such that $u > 0$ in the relevant interval. We omit the details.

In the proof of Theorem 2.2 in the general case, we will use of the following comparison principle which is a consequence of a result in [21] on the first variation of the operator A^u.

Lemma 2.6. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set,* Γ *satisfy* (1.1) *and* (1.2)*,* u *be a positive function in* $USC(\bar{\Omega})$ *(resp.* $LSC(\bar{\Omega})$*),* v *be a positive function in* $C^2(\Omega) \cap LSC(\bar{\Omega})$ *(resp.* $C^2(\Omega) \cap USC(\bar{\Omega})$*) such that* $\lambda(A^u) \in \mathbb{R}^n \setminus \Gamma$ *(resp.* $\lambda(A^u) \in \bar{\Gamma}$*) in* Ω *in the viscosity sense, and* $\lambda(A^v) \in \bar{\Gamma}$ *(resp.* $\lambda(A^v) \in \mathbb{R}^n \setminus \Gamma$*) in* Ω*. Assume that* $u \le v$ *(resp.* $u \ge v$*) on* $\partial\Omega$*. Then* $u \le v$ *(resp.* $u \ge v$*) in* $\bar{\Omega}$*. In particular, if* $\lambda(A^u) \in \partial\Gamma$ *in* Ω *in the viscosity sense,* $\lambda(A^v) \in \partial\Gamma$ *in* Ω *and* $u = v$ *on* $\partial\Omega$*, then* $u \equiv v$ *in* Ω*.*

Proof. To prove the first part, let

$$v_i(x) = \Big(v(x) + \frac{1}{i}e^{\delta\,|x|^2}\Big)^{-\frac{n-2}{2}}, i = 1, 2, \ldots$$

By [21, Lemma 3.7], for some small $\delta > 0$ and for all i,

$$\lambda(A^{v_i}) \in \Gamma \text{ in } \Omega. \tag{2.11}$$

It follows from the assumptions on u and v that $\inf_{\bar\Omega} v > 0$, $\sup_{\bar\Omega} u < \infty$ and $u \le v$ on $\partial\Omega$. Let β_i be the smallest number such that $\beta_i v_i \ge u$ on $\bar\Omega$. If $\limsup_{i\to\infty}\beta_i \le 1$, then, since $v_i \to v$ uniformly on $\bar\Omega$, $v \ge u$ in $\bar\Omega$ as desired. Otherwise, along a subsequence, $\beta_i \to \bar\beta > 1$. We have $\beta_i v_i(x_i) = u(x_i)$ for some $x_i \in \bar\Omega$. Since $v \ge u$ on $\partial\Omega$ and $v_i \to v$ on $\bar\Omega$, we know that $x_i \in \Omega$. It follows, taking $\beta_i v_i$ as a test function, $\lambda(A^{\beta_i v_i}(x_i)) \in \mathbb{R}^n \setminus \Gamma$, i.e. $\lambda(A^{v_i}(x_i) \in \mathbb{R}^n \setminus \Gamma$, violating (2.11). This completes the proof of the first part of Lemma 2.6. The proof of the second part is the same. $\square$

The following estimate for viscosity super-solutions of (1.7) can be viewed as a generalization of (2.10).

Lemma 2.7. *Assume that* Γ *satisfies* (1.1), (1.2) *and* (1.9). *For* $0 \le a < b \le \infty$, *let* $u \in LSC(\{a < |x| < b\})$ *be positive, radially symmetric and satisfy* $\lambda(A^u) \in \bar\Gamma$ *in* $\{a < |x| < b\}$ *in the viscosity sense. Then* u *is non-increasing and* $|x|^{n-2}u$ *is non-decreasing in* $|x|$, *i.e. for* $a < c < d < b$,

$$0 \le \ln\frac{u(c)}{u(d)} \le (n-2)\ln\frac{d}{c}. \tag{2.12}$$

In particular, u *is locally Lipschitz in* $\{a < |x| < b\}$.

Proof. Let

$$m := \frac{\ln u(c) - \ln u(d)}{\ln d - \ln c}.$$

We first show the second half of the estimate: $m \le n-2$. Assume otherwise that $m = (n-2) + \varepsilon$ for some $\varepsilon > 0$. Define for $\mu > 1$,

$$\xi_\mu(x) = \xi_\mu(r) = u(c)\,\frac{c^{n-2}}{r^{n-2}}\,\exp\Big[\frac{-\varepsilon(\ln r - \ln c)^\mu}{(\ln d - \ln c)^{\mu-1}}\Big].$$

It is easy to see that $\xi_\mu(c) = u(c)$ and $\xi_\mu(d) = u(d)$. Note that A^{ξ_μ} has two eigenvalues λ_1 of multiplicity one and λ_2 has multiplicity $(n-1)$. A direct computation using the explicit formula for ξ_μ shows that

$$\lambda_2 = -\frac{1}{r}\xi_\mu\,\xi_\mu' - \frac{1}{n-2}(\xi_\mu')^2 < 0 \text{ in } (a,b).$$

In view of (1.10), this implies $\lambda(A^{\xi_\mu}) \in \mathbb{R}^n \setminus \bar\Gamma$.

Now, u is a super-solution while ξ_μ is a sub-solution of (1.7) and both have the same boundary values. By Lemma 2.6, $u \ge \xi_\mu$. Sending $\mu \to \infty$ results in

$$u \ge u(c)\,\frac{c^{n-2}}{r^{n-2}} \text{ in } (c,d),$$

which contradicts the assumption that $m > n-2$.

The first half of the conclusion that $m \geq 0$ can be shown similarly. Assume otherwise that this was wrong. Then the function

$$\hat{\xi}_\mu = u(d)\,\exp\left[\frac{m(\ln d - \ln r)^\mu}{(\ln d - \ln c)^{\mu-1}}\right]$$

is a sub-solution of (1.7) which has the same boundary values as u. Thus, by Lemma 2.6, $u \geq \hat{\xi}_\mu$ in (c,d) which leads to a contradiction when we send $\mu \to \infty$. □

We are now in a position to give the

Proof of Theorem 2.2. It suffices to show that u is a classical solution. For any $a < c < d < b$ there exists a smooth positive radially symmetric solution $\hat{u}$ of (1.7) such that $\hat{u}(c) = u(c)$ and $\hat{u}(d) = u(d)$: This is a consequence of Lemma 2.7 and Corollary 2.4 in case (1.9) holds, or of Lemma 2.5 in case (2.1) holds. By Lemma 2.6, $u \equiv \hat{u}$ in (c,d). Since $\hat{u}$ is smooth, so is u. The conclusion follows. □

As mentioned in the introduction, the strong maximum principle fails for solutions of (1.7) when (1.9) holds. The next result recovers a strong maximum principle statement in the radially symmetric setting.

Lemma 2.8. *Assume that* Γ *satisfies* (1.1) *and* (1.2). *For* $0 \leq a < b \leq \infty$, *let* $u \in C^0(\{a < |x| < b\})$ *and* $\bar{u} \in LSC(\{a < |x| < b\})$ *be positive, radially symmetric and satisfy respectively* $\lambda(A^u) \in \partial\Gamma$ *and* $\lambda(A^{\bar{u}}) \in \bar{\Gamma}$ *in* $\{a < |x| < b\}$ *in the viscosity sense. Assume that* $u \leq \bar{u}$ *in* $\{a < |x| < b\}$. *Then*

$$\text{either } u < \bar{u} \text{ in } \{a < |x| < b\} \text{ or } u \equiv u \text{ in } \{a < |x| < b\}.$$

Proof. Suppose the contrary, then for some $c, d \in (a,b)$, $u(c) < \bar{u}(c)$ and $u(d) = \bar{u}(d)$. We may assume that $c < d$; the other case can be proved similarly. According to Theorem 2.2, u is smooth (and takes some specific form).

We first observe that

$$u \equiv \bar{u} \text{ in } \{d \leq |x| < b\}.$$

The reason is that if $u(\bar{r}) < \bar{u}(\bar{r})$ for some $d < \bar{r} < b$, we can apply Lemma 2.6 on $\{c < |x| < \bar{r}\}$ to obtain, for small $\varepsilon > 0$, $(1+\varepsilon)u \leq \bar{u}$ in $\{c < |x| < \bar{r}\}$, violating $u(d) = \bar{u}(d)$.

Fix a $\bar{d} \in (d,b)$ and let $\alpha = \frac{1}{2}[u(c) + \bar{u}(c)]$. If (1.9) holds, an application of Lemma 2.7 to both u and $\bar{u}$ gives

$$0 < \ln\frac{u(c)}{u(\bar{d})} < \ln\frac{\alpha}{u(\bar{d})} < \ln\frac{\bar{u}(c)}{\bar{u}(\bar{d})} \leq (n-2)\ln\frac{\bar{d}}{c},$$

and hence, by Corollary 2.4, there exists a unique smooth radially symmetric solution v of (1.7) in $\{c < |x| < \bar{d}\}$ satisfying $v(c) = \alpha$ and $v(\bar{d}) = u(\bar{d}) = \bar{u}(\bar{d})$. If (2.1) holds, the existence of v is assured by Lemma 2.5. By Lemma 2.6, $v \leq \bar{u}$ on $\{c < |x| < \bar{d}\}$. On the other, since $u(c) < v(c)$ and $u(\bar{d}) = v(\bar{d})$, we have, in view of the explicit form of radial solutions given by Theorem 2.2, $u < v$ in $\{c < |x| < \bar{d}\}$. Thus, $u(d) < v(d) \le \bar{u}(d)$, a contradiction. □

A consequence is the following comparison type result, which will be used later.

Corollary 2.9. *Assume that* Γ *satisfies* (1.1) *and* (1.2). *For* $0 \le a < b < \infty$, *let* $u \in C^0(\{a \le |x| \le b\}), \bar{u} \in LSC(\{a \le |x| \le b\})$ *be positive, radially symmetric and satisfy respectively* $\lambda(A^u) \in \partial\Gamma$ *and* $\lambda(A^{\bar{u}}) \in \bar{\Gamma}$ *in* $\{a < |x| < b\}$ *in the viscosity sense. Assume that* $u|_{\partial B_b} \le \bar{u}|_{\partial B_b}$ *and* $u|_{\partial B_d} \ge \bar{u}|_{\partial B_d}$ *for some* $a < d < b$, *then*

$$\bar{u} \le u \text{ in } \{a < |x| < d\}.$$

Proof. Assume the contrary that $u(c) < \bar{u}(c)$ for some $c \in (c,d)$. According to Theorem 2.2, u is a smooth function. An application of Lemma 2.6 yields

$$\bar{u} \ge u \text{ on } \bar{B}_b \setminus B_c.$$

In particular, $\bar{u}(d) \ge u(d)$. We also know from the assumption that $\bar{u}(d) \le u(d)$. So we have $\bar{u}(d) = u(d)$. By Lemma 2.8, we obtain $\bar{u} \equiv u$ on $\bar{B}_b \setminus B_c$, violating $u(c) < \bar{u}(c)$. □

Lemma 2.10. *Assume that* Γ *satisfies* (1.1), (1.2) *and* (1.9). *For* $0 \le a < b \le \infty$, *let* $u \in LSC(\{a < |x| < b\})$ *be a positive, radially symmetric solution of* $\lambda(A^u) \in \bar{\Gamma}$ *in the viscosity sense in* $\{a < |x| < b\}$. *Then, for any* $a < R_0 < b$, *the function*

$$\Psi_{R_0}(r) = \begin{cases} \frac{\ln u(r) - \ln u(R_0)}{\ln R_0 - \ln r} & \text{if } \mu_\Gamma^+ = 1, \\ \frac{u(r)^{\frac{\mu_\Gamma^+ - 1}{n-2}} - u^{\frac{\mu_\Gamma^+ - 1}{n-2}}(R_0)}{r^{-\mu_\Gamma^+ + 1} - R_0^{-\mu_\Gamma^+ + 1}} & \text{if } \mu_\Gamma^+ \ne 1 \end{cases}$$

is non-decreasing in r *for* $r \in (a, R_0)$.

Proof. Fix $a < R_2 < R_1 < R_0$. Using estimate (2.12) in Lemma 2.7 and Corollary 2.4, we can find uniquely two smooth radial functions $v_i \in C^\infty(B_{R_0} \setminus \{0\})$, $i = 1,2$ such that

$$\begin{cases} \lambda(A^{v_i}) \in \partial\Gamma \text{ in } B_{R_0} \setminus \{0\}, \\ v_i(R_0) = u(R_0), v_i(R_i) = u(R_i). \end{cases}$$

By Corollary 2.9, $u(R_2) \le v_1(R_2)$. It then follows from the explicit formula for v_1 and v_2 in Theorem 2.2 that

$$v_2 \le v_1 \text{ in } B_{R_0} \setminus \{0\}.$$

To proceed, consider first the case where $\mu_\Gamma^+ \ne 1$. By Theorem 2.2, there exist non-negative constants μ_i and ν_i such that

$$v_i(r) = \left(\mu_i r^{-\mu_\Gamma^+ + 1} + \nu_i\right)^{\frac{n-2}{\mu_\Gamma^+ - 1}}.$$

As $v_1(R_0) = v_2(R_0) = u(R_0)$, we have

$$\nu_i = u(R_0)^{\frac{\mu_\Gamma^+ - 1}{n-2}} - \mu_i R_0^{-\mu_\Gamma^+ + 1}.$$

We thus have

$$v_i(r) = \left[\mu_i\left(r^{-\mu_\Gamma^+ + 1} - R_0^{-\mu_\Gamma^+ + 1}\right) + u(R_0)^{\frac{\mu_\Gamma^+ - 1}{n-2}}\right]^{\frac{n-2}{\mu_\Gamma^+ - 1}}.$$

Recalling $v_2 \le v_1$ we thus get

$$\mu_2 \le \mu_1.$$

On the other hand, as $v_i(R_i) = u(R_i)$, we have $\mu_i = \Psi_{R_0}(R_i)$ and so $\Psi_{R_0}(R_2) \le \Psi_{R_0}(R_1)$.

Let's turn to the case where $\mu_\Gamma^+ = 1$. The argument is similar. By Theorem 2.2, there exist constants $\mu_i \in [0, n-2]$ and ν_i such that

$$\ln v_i(r) = -\mu_i \ln r + \nu_i.$$

As before, this leads

$$\ln v_i(r) = \Psi_i(R_i)\Big(\ln R_0 - \ln r) + \ln u(R_0).$$

Recalling $v_2 \le v_1$, we have $\Psi_{R_0}(R_2) \le \Psi_{R_0}(R_1)$, which finishes the proof. □

3 Key gradient estimates

In this section, we prove Theorem 1.5, a local gradient estimate for locally Lipschitz viscosity solutions of (1.7).

For a locally Lipschitz function v in B_1, $0 < \alpha < 1$, $x \in B_1$ and $0 < \delta < 1 - |x|$, define

$$[v]_{\alpha,\delta}(x) = \sup_{0<|y-x|<\delta} \frac{|v(y) - v(x)|}{|y-x|^\alpha}.$$

Note that $[v]_{\alpha,\delta}(x)$ is continuous and non-decreasing in δ. Thus we can define

$$\delta(v,x,\alpha) = \begin{cases} \infty & \text{if } (1-|x|)^\alpha\,[v]_{\alpha,1-|x|}(x) < 1, \\ \mu & \text{where } 0 < \mu \le 1-|x| \text{ and } \mu^\alpha\,[v]_{\alpha,\mu}(x) = 1 \\ & \text{if } (1-|x|)^\alpha\,[v]_{\alpha,1-|x|}(x) \ge 1. \end{cases}$$

The above function $\delta(v,x,\alpha)$ was introduced in [22]. Its inverse $\delta(v,x,\alpha)^{-1}$ plays a similar role to $|\nabla v(x)|$ in performing a rescaling argument for a sequence of functions blowing up in C^α-norms. In particular, if $\delta = \delta(v,x,\alpha) < \infty$, then the rescaled function $w(y) := v(x+\delta y) - v(x)$ satisfies

$$w(0) = 0 \text{ and } [w]_{\alpha,1}(0) = \delta^\alpha [v]_{\alpha,\delta}(x) = 1.$$

We start the proof in a special case.

Lemma 3.1. *Let u be as in Theorem 1.5. There exists $C = C(n)$ such that*

$$|\nabla \ln u| \le C(n)\left[\frac{\sup_{B_{3/4}} u}{\inf_{B_{3/4}} u}\right]^{\frac{1}{n-2}} \qquad a.e.\ \text{ in } B_{1/2}$$

Proof. For $x \in B_{1/2}$, $0 < \lambda \le R := \frac{1}{4}\left[\frac{\sup_{B_{3/4}} u}{\inf_{B_{3/4}} u}\right]^{-\frac{1}{n-2}}$ and $|y| = 3/4$, we have

$$u_{x,\lambda}(y) := \frac{\lambda^{n-2}}{|y-x|^{n-2}}\, u\Big(x + \frac{\lambda^2(y-x)}{|y-x|^2}\Big) \le (4R)^{n-2} \sup_{B_{3/4}} u = \inf_{B_{3/4}} u \le u(y).$$

Also, we know that $u_{x,\lambda}$ satisfies $\lambda(A^{u_{x,\lambda}}) \in \partial\Gamma$ in $B_1 \setminus B_\lambda(x)$ in the viscosity sense. Since $u_{x,\lambda} = u$ on $\partial B_\lambda(x)$, we can apply [22, Proposition 1.14] to obtain

$$u_{x,\lambda} \le u \text{ in } B_{3/4} \setminus B_\lambda(x) \text{ for all } 0 < \lambda \le R, |x| \le 1/2. \tag{3.1}$$

By [23, Lemma 2], (3.1) implies the gradient estimate

$$|\nabla \ln u| \le \frac{C(n)}{R} \qquad a.e. \text{ in } B_{1/2}.$$

This concludes the proof. □

We now give the

Proof of Theorem 1.5. We follow the proof of Theorem 1.10 in [22]. Since the equation $\lambda(A^u) \in \partial\Gamma$ is invariant under scaling, it suffices to consider $\varepsilon = 15/16$. We first claim that

$$\sup_{x \ne y \in B_{1/8}} \frac{|\ln u(x) - \ln u(y)|}{|x-y|^\alpha} \le C(\Gamma, \alpha) \text{ for any } 0 < \alpha < 1. \tag{3.2}$$

Assume otherwise that (3.2) fails. Then, for some $0 < \alpha < 1$, we can find a sequence of positive $C^{0,1}$ functions u_i in B_1 such that $\lambda(A^{u_i}) \in \partial\Gamma$ there but

$$\sup_{x \ne y \in B_{1/8}} \frac{|\ln u_i(x) - \ln u_i(y)|}{|x-y|^\alpha} \to \infty.$$

This implies that, for any fixed $0 < r < 3/4$,

$$\sup_{x \in B_{1/8}} [\ln u_i]_{\alpha,r}(x) \to \infty,$$

which consequently implies

$$\inf_{x \in B_{1/8}} \delta(\ln u_i, x, \alpha) \to 0.$$

It follows that for some $x_i \in B_{3/4}$,

$$\frac{3/4 - |x_i|}{\delta(\ln u_i, x_i, \alpha)} > \frac{1}{2} \sup_{x \in B_{3/4}} \frac{3/4 - |x|}{\delta(\ln u_i, x, \alpha)} \to \infty.$$

Let $\sigma_i = \frac{3/4 - |x_i|}{2}$ and $\varepsilon_i = \delta(\ln u_i, x_i, \alpha)$. Then

$$\frac{\sigma_i}{\varepsilon_i} \to \infty, \varepsilon_i \to 0, \text{ and } \varepsilon_i \le 4\,\delta(\ln u_i, z, \alpha) \text{ for any } |z - x_i| \le \sigma_i. \tag{3.3}$$

We now define

$$v_i(y) = \frac{1}{u_i(x_i)}\, u_i(x_i + \varepsilon_i\, y) \text{ for } |y| \le \frac{\sigma_i}{\varepsilon_i}.$$

Then

$$[\ln v_i]_{\alpha,1}(0) = \varepsilon_i^{\alpha}\, [\ln u_i]_{\alpha,\varepsilon_i}(x_i) = 1. \tag{3.4}$$

Also, by (3.3), for any fixed $\beta > 1$ and $|y| < \beta$, there holds

$$\begin{aligned}
[\ln v_i]_{\alpha,1}(y) &= \varepsilon_i^{\alpha}\, [\ln u_i]_{\alpha,\varepsilon_i}(x_i + \varepsilon_i\, y)\\
&\le 4^{-\alpha} \Big\{ 3 \sup_{|z-(x_i+\varepsilon_i y)|\le\varepsilon_i} \varepsilon_i^{\alpha}\, [\ln u_i]_{\alpha,\varepsilon_i/4}(z) + \varepsilon_i^{\alpha}\, [\ln u_i]_{\alpha,\varepsilon_i/4}(x_i + \varepsilon_i\, y) \Big\}\\
&\le 3 \sup_{|z-(x_i+\varepsilon_i y)|\le\varepsilon_i} \delta(\ln u_i, z, \alpha)^{\alpha}\, [\ln u_i]_{\alpha,\delta(\ln u_i,z,\alpha)}(z)\\
&\qquad + \delta(\ln u_i, x_i + \varepsilon_i\, y, \alpha)^{\alpha}\, [\ln u_i]_{\alpha,\delta(\ln u_i,x_i+\varepsilon_i y,\alpha)}(x_i + \varepsilon_i\, y)\\
&= 4
\end{aligned} \tag{3.5}$$

for all sufficiently large i. Since $v_i(0) = 1$ by definition, we deduce from (3.4) and (3.5) that

$$\frac{1}{C(\beta)} \le v_i(y) \le C(\beta) \text{ for } |y| \le \beta \text{ and all sufficiently large } i. \tag{3.6}$$

Thanks to (3.6), we can apply Lemma 3.1 to obtain

$$|\nabla \ln v_i| \le C(\beta) \text{ in } B_{\beta/2} \text{ for all sufficiently large } i. \tag{3.7}$$

Passing to a subsequence and recalling (3.3) and (3.6), we see that v_i converges in $C^{0,\alpha'}$ ($\alpha < \alpha' < 1$) on compact subsets of $\mathbb{R}^n$ to some positive, locally Lipschitz function v_* which satisfies $\lambda(A^{v_*}) \in \partial\Gamma$ in the viscosity sense. By the Liouville-type theorem [22, Theorem 1.4],

$$v_* \equiv v_*(0) = \lim_{i\to\infty} v_i(0) = 1.$$

This contradicts (3.4), in view of (3.7) and the convergence of v_i to v_*. We have proved (3.2).

From (3.2), we can find some universal constant $C > 1$ such that

$$\frac{u(0)}{C} \le u \le C\,u(0) \text{ in } B_{1/8}.$$

Applying Lemma 3.1 again we obtain the required gradient estimate in $B_{1/16}$. □

4 Bôcher-type theorems

In this section we prove the Bôcher-type theorems stated in the introduction. We start in Subsection 4.1 by proving the regularity assertion across isolated singularities with mild growth in our Bôcher-type theorems. Subsection 4.1 contains the proof of Theorem 1.8. Theorems 1.2, 1.3 and 1.4 are proved in the next three subsections. In Subsection 4.6 we consider a case where (1.9) does not hold. Subsection 4.1 and Subsections 4.2-4.6 can be read independently.

4.1 Isolated singularities with mild growth

We will need the following removable singularity result for super-solutions of (1.7).

Lemma 4.1. *Let Γ satisfy* (1.1) *and* (1.2)*, $u \in LSC(B_1 \setminus \{0\})$ be a positive solution of $\lambda(A^u) \in \bar{\Gamma}$ in $B_1 \setminus \{0\}$ in the viscosity sense. Then u, with $u(0) = \liminf_{x\to 0} u(x)$, is a positive function in $LSC(B_1)$ satisfying $\lambda(A^u) \in \bar{\Gamma}$ in B_1 in the viscosity sense.*

Proof. It is easy to see that u, with $u(0) = \liminf_{x\to 0} u(x)$, is in $LSC(B_1)$. We know from $\lambda(A^u) \in \bar{\Gamma}$ and (1.2) that $\Delta u \le 0$ in $B_1 \setminus \{0\}$ in the viscosity sense. Since $\{0\}$ has zero Newtonian capacity, $\Delta u \le 0$ in B_1 in the viscosity sense. Consequently,

$$\inf_{B_{1/2}\setminus\{0\}} u \ge \min_{\partial B_{1/2}} u > 0.$$

In particular, $u(0) > 0$.

We have shown that u is a positive function in $LSC(B_1)$ and satisfies $\Delta u \le 0$ in B_1 in the viscosity sense. It follows that u is lower-conical at $\{0\}$ (as defined in [4]) : For any $\eta \in C^\infty(B_{1/2})$ and for any $\varepsilon > 0$,

$$\inf_{x\in B_{1/2}} \Big[(u+\eta)(x) - (u+\eta)(0) - \varepsilon|x|\Big] < 0.$$

The proof of [4, Theorem 1.1] gives that $\lambda(A^u) \in \bar{\Gamma}$ in B_1 in the viscosity sense. $\square$

Proposition 4.2. *Assume that Γ satisfies* (1.1)*,* (1.2)*, and $0 \le \mu_\Gamma^+ < 1$. Let $u \in LSC(B_1 \setminus \{0\}) \cap L^\infty_{\rm loc}(B_1 \setminus \{0\})$ be a positive function satisfying $\lambda(A^u) \in \bar{\Gamma}$ in $B_1 \setminus \{0\}$ in the viscosity sense and*

$$\liminf_{|x|\to 0} |x|^{n-2} u(x) = 0.$$

Then, the function u with $u(0) = \liminf_{|x|\to 0} u(x)$ is in $C^{0,1-\mu_\Gamma^+}_{\rm loc}(B_1)$. Moreover,

$$\|u^{\frac{\mu_\Gamma^+ - 1}{n-2}}\|_{C^{0,\alpha}(B_{1/2})} \le C(\Gamma) \max_{\partial B_{3/4}} u^{\frac{\mu_\Gamma^+ - 1}{n-2}}.$$

Proof. By Lemma 4.1, $\lambda(A^u) \in \bar{\Gamma}$ in the viscosity sense. Let $v(x) = v(|x|) = \min_{\partial B_{|x|}} u$. Then $\lambda(A^v) \in \bar{\Gamma}$ in B_1 in the viscosity sense, hence v is super-harmonic.

It follows that v is non-increasing. Also, by the hypothesis, $\liminf_{r\to 0} r^{n-2}\, v(r) = 0$, hence there exists $0 < r_1 < 3/4$ such that

$$r_1^{n-2} v(r_1) < (3/4)^{n-2}\, v(3/4). \tag{4.1}$$

Thus, since $v(r_1) \geq v(3/4)$, there exists $C_1 \geq 0$ and $C_2 > 0$ such that the function

$$\hat{v}(r) = (C_1\,|x|^{-\mu_\Gamma^+ +1} + C_2)^{\frac{n-2}{\mu_\Gamma^+ -1}}$$

satisfies $\hat{v}(r_1) = v(r_1)$ and $\hat{v}(3/4) = v(3/4)$. By Theorem 2.2, $\lambda(A^{\hat{v}}) \in \partial\Gamma$ in $B_1 \setminus \{0\}$. By Corollary 2.9, we have $v \leq \hat{v}$ in $(0, r_1)$. In particular, v is bounded at the origin and

$$u(0) = \liminf_{|x|\to 0} u = \liminf_{r\to 0} v(r) < \infty.$$

By Lemma 4.1, $\lambda(A^u) \in \bar{\Gamma}$ in the viscosity sense. By the super-harmonicity of u,

$$c := \inf_{B_{3/4}} u = \min_{\partial B_{3/4}} u > 0.$$

For $\bar{x} \in B_{1/2}$, consider

$$\xi_{\bar{x}}(x) := c\Big(\frac{|x-\bar{x}|^{-\mu_\Gamma^+ +1}}{4^{\mu_\Gamma^+ -1}} + b\Big)^{\frac{n-2}{\mu_\Gamma^+ -1}}$$

where $b > 0$ satisfies

$$\xi_{\bar{x}}(\bar{x}) = c\, b^{\frac{n-2}{\mu_\Gamma^+ -1}} = u(\bar{x}). \tag{4.2}$$

We will show that

$$u \geq \xi_{\bar{x}} \text{ in } B_{3/4}. \tag{4.3}$$

It is easy to see that

$$\xi_{\bar{x}}(x) \leq \frac{c}{4^{n-2}}\,|x-\bar{x}|^{2-n} \leq c \text{ for all } x \in \partial B_{3/4}.$$

Also, by (4.2), for any $0 < \varepsilon < 1$, there exists $0 < \delta < \frac{1}{8}$ such that

$$(1-\varepsilon)\,\xi_{\bar{x}} \leq u \text{ in } B_\delta(\bar{x}). \tag{4.4}$$

Since $\lambda(A^{(1-\varepsilon)\xi_{\bar{x}}}) \in \partial\Gamma$ in $B_{3/4} \setminus \{\bar{x}\}$ according to Theorem 2.2 and $(1-\varepsilon)\xi_{\bar{x}} \leq u$ on $\partial(B_{3/4} \setminus B_\delta(\bar{x}))$, we can apply Lemma 2.6 to obtain

$$(1-\varepsilon)\xi_{\bar{x}} \leq u \text{ in } B_{3/4} \setminus B_\delta(\bar{x}).$$

Thus, in view of (4.4),

$$(1-\varepsilon)\xi_{\bar{x}} \leq u \text{ in } B_{3/4}.$$

Sending $\varepsilon \to 0$, we obtain (4.3).

Set

$$w = u^{\frac{\mu_\Gamma^+ -1}{n-2}}.$$

We deduce from (4.3), in view of (4.2), that

$$w(x)-w(\bar{x}) \le \frac{|x-\bar{x}|^{-\mu_\Gamma^+ +1}}{4^{\mu_\Gamma^+ -1}} \max_{\partial B_{3/4}} w \text{ for all } x,\bar{x} \in B_{1/2}.$$

Switching the role of x and $\bar{x}$ we obtain

$$|w(x)-w(\bar{x})| \le \frac{|x-\bar{x}|^{-\mu_\Gamma^+ +1}}{4^{\mu_\Gamma^+ -1}} \max_{\partial B_{3/4}} w \text{ for all } x,\bar{x} \in B_{1/2},$$

which proves the result. □

Proposition 4.3. *Assume that* Γ *satisfies* (1.1), (1.2), *and* $1 \le \mu_\Gamma^+ \le n-1$. *Let* $u \in C^{0,1}_{\mathrm{loc}}(B_1 \setminus \{0\})$ *be a positive viscosity solution to* (1.7) *in* $B_1 \setminus \{0\}$ *satisfying*

$$\liminf_{|x|\to 0} |x|^{n-2} u(x) = 0.$$

Then, for all $0<\alpha<1$, *the function* u *with* $u(0)=\liminf_{|x|\to 0} u(x)$ *is in* $C^{0,\alpha}_{\mathrm{loc}}(B_1)$. *Moreover,*

$$\|u\|_{C^{0,\alpha}(B_{1/2})} \le C(\Gamma,\alpha) \inf_{B_{1/2}} u.$$

Proof. Let u be the extended function. We first prove that

$$\max_{\partial B_r} u = \sup_{B_r} u, \qquad 0<r<1. \tag{4.5}$$

By Theorem 2.2, the function, with $0<\varepsilon<1$ and $0<r<1$,

$$v_{\varepsilon,r}(x) = \begin{cases} \left[\varepsilon\,|x|^{-\mu_\Gamma^+ +1} + \sup_{\partial B_r} u^{\frac{\mu_\Gamma^+ -1}{n-2}}\right]^{\frac{n-2}{\mu_\Gamma^+ -1}} & \text{if } \mu_\Gamma^+ > 1, \\ \sup_{\partial B_r} u\, r^{\varepsilon}\,|x|^{-\varepsilon} & \text{if } \mu_\Gamma^+ = 1, \end{cases}$$

satisfies $\lambda(A^{v_{\varepsilon,r}}) \in \partial\Gamma$ in $B_r \setminus \{0\}$, $v_{\varepsilon,r} \ge u$ on ∂B_r. Clearly, there exists $\delta_i \to 0^+$ such that

$$\min_{\partial B_{\delta_i}} [v_{\varepsilon,r} - u] \to \infty \text{ as } i \to \infty.$$

Here we have used $\mu_\Gamma^+ \ge 1$. An application of Lemma 2.6 on $B_r \setminus B_{\delta_i}$ gives

$$u \le v_{\varepsilon,r} \text{ in } B_r \setminus B_{\delta_i}.$$

Sending $i\to\infty$ and then $\varepsilon \to 0$, we obtain (4.5).

Since u is a positive super-harmonic function in $B_1 \setminus \{0\}$ and the Newtonian capacity of $\{0\}$ is zero, we have

$$\min_{\partial B_r} u = \inf_{B_r} u. \tag{4.6}$$

For $0<|x|<\frac{7}{8}$, applying Theorem 1.5 to $u(x+\frac{|x|}{8}\cdot)$ leads to

$$|\nabla \ln u(x)| \le \frac{C(\Gamma)}{|x|} \text{ for all } x \in B_{7/8}\setminus\{0\}. \tag{4.7}$$

In particular,

$$R := \frac{1}{4}\left[\frac{\max_{\partial B_{3/4}} u}{\min_{\partial B_{3/4}} u}\right]^{-\frac{1}{n-2}} \geq C(\Gamma)^{-1} > 0. \tag{4.8}$$

For $0 < \lambda < |x| < R$ and $|y| = \frac{3}{4}$, we have, in view of (4.5),

$$\begin{aligned} u_{x,\lambda}(y) &:= \frac{\lambda^{n-2}}{|y-x|^{n-2}} u\Big(x + \frac{\lambda^2(y-x)}{|y-x|^2}\Big) \\ &\leq (2R)^{n-2} \sup_{B_{3/4}} u = (2R)^{n-2} \max_{\partial B_{3/4}} u \leq \min_{\partial B_{3/4}} u \leq u(y). \end{aligned}$$

Since $u_{x,\lambda} = u$ on $\partial B_\lambda(x)$ and $\lambda(A^{u_{x,\lambda}}) \in \partial\Gamma$ in $B_{3/4} \setminus B_\lambda(x)$, we can apply the comparison principle [22, Proposition 1.14] to obtain

$$u_{x,\lambda} \leq u \text{ in } B_{3/4} \setminus (B_\lambda(x) \cup \{0\}) \text{ for all } 0 < \lambda < |x| < R.$$

By Lemma A.1, we have

$$|\max_{\partial B_r} \ln u - \min_{\partial B_r} \ln u| \leq \frac{C(n)\, r}{R} \text{ for all } 0 < r < R/2. \tag{4.9}$$

We deduce from (4.5), (4.6) and (4.9) that

$$\sup_{B_r} |\ln u - \ln u(0)| \leq \max_{\partial B_r} \ln u - \min_{\partial B_r} \ln u \leq \frac{C(n)\, r}{R} \text{ for all } 0 < r < R/2. \tag{4.10}$$

From (4.10) and (4.7), we can use interpolation to show that $\ln u \in C^{0,\frac{1}{2}}(B_{R/2})$.

To obtain better regularity, we refine our usage of Lemma A.1 and the super-harmonicity of u. Fix $\alpha \in (0,1)$, $x_0 \in B_{R/8}$ and let $r_0 = |x_0|$. By Lemma A.1, we have

$$\ln u(x) - \ln u(x_0) \leq \frac{C(n)}{R} |x - x_0| \text{ for any } x \in B_{r_0/2}(x_0) \setminus B_{r_0}(0). \tag{4.11}$$

Also by the same lemma,

$$\ln u(x) - \ln u(\tilde{x}) \leq \frac{C(n)}{R} |x - \tilde{x}| \text{ for any } x, \tilde{x} \in \partial B_{r_0}(0). \tag{4.12}$$

It remains to bound $\ln u(x) - \ln u(x_0)$ from below for $x \in B_{r_0/2}(x_0) \setminus B_{r_0}(0)$.

Let $y_0 = \frac{x_0}{|x_0|}$ and define

$$v(y) = \frac{1}{r_0}[\ln u(x_0 + \frac{r_0}{2} y) - \ln u(x_0)] \text{ for } y \in B_1(0) \setminus B_2(-2y_0).$$

As u is super-harmonic, so is v. In addition, by (4.10) and (4.12),

$$|v(y)| \leq \frac{C(n)}{R} |y| \text{ for any } y \in B_1(0) \setminus B_2(-2y_0), \tag{4.13}$$

$$v(y) - v(\tilde{y}) \leq \frac{C(n)}{R} |y - \tilde{y}| \text{ for any } y, \tilde{y} \in \partial B_2(-2y_0) \cap B_1(0). \tag{4.14}$$

Define w as the harmonic function in $B_1(0)\setminus B_2(-2y_0)$ such that $w=v$ on $\partial(B_1(0)\setminus B_2(-2y_0))$. Then (4.13), (4.14) and elliptic regularity imply that

$$\|w\|_{C^\alpha(B_{1/2}(0)\setminus B_2(-2y_0))} \le C\big(\|v\|_{C^{0,1}(\partial B_2(-2y_0)\cap B_1(0))} + \|v\|_{L^\infty(B_2(-2y_0)\cap B_1(0))}\big) \le \frac{C(n,\alpha)}{R}.$$

Thus, by the maximum principle,

$$v(y)-v(0) \ge w(y)-w(0) \ge -\frac{C(n,\alpha)}{R}\,|y|^\alpha \text{ for any } y\in B_{1/2}(0)\setminus B_2(-2y_0)$$

Recalling back we obtain that

$$\ln u(x) - \ln u(x_0) \ge -\frac{C(n,\alpha)}{R}\,|x-x_0|^\alpha \text{ for any } x\in B_{r_0/2}(x_0)\setminus B_{r_0}(0). \tag{4.15}$$

From (4.11) and (4.15), we get

$$|\ln u(x) - \ln u(x_0)| \le \frac{C(n,\alpha)}{R}\,|x-x_0|^\alpha \text{ for any } x\in B_{r_0/2}(x_0)\setminus B_{r_0}(0).$$

This implies that

$$|u(x)-u(y)| \le \frac{C(n,\alpha)}{R}\,|x-y|^\alpha \text{ for any } x\in B_{R/8}(0) \text{ and } y\in B_{|x|/4}(x). \tag{4.16}$$

(Here x_0 could be either x or y, whoever that has smaller norm.)

To complete the proof, we show that

$$|\ln u(x) - \ln u(y)| \le \frac{C(n,\alpha)}{R}\,|x-y|^\alpha \text{ for any } x,y\in B_{R/8}(0)\setminus\{0\}. \tag{4.17}$$

The assertion is readily seen from (4.7), (4.8) and (4.17). To prove (4.16), we may assume without loss of generality that $|x|\ge|y|$. If $|x-y|<|x|/4$, (4.17) follows from (4.16). Otherwise, $|x-y|\ge|x|/4$ and so by (4.10),

$$|\ln u(x)-\ln u(y)| \le |\ln u(x)-\ln u(0)| + |\ln u(y)-\ln u(0)| \le \frac{C(n)}{R}\,|x| \le \frac{C(n)}{R}\,|x-y|,$$

which also implies (4.17). □

4.2 Leading term at an isolated singularity

Proof of Theorem 1.8. Define

$$v(r) = \min_{\partial B_r} u.$$

Then v is positive and super-harmonic in $B_1\setminus\{0\}$. Since $\{0\}$ has zero Newtonian capacity, v is super-harmonic in B_1. In particular, v is non-increasing.

We claim that $\lim_{r\to 0} r^{n-2}\,v(r)$ exists and is finite. Fix some $0<\rho_1<1$ and for $0<\rho<\rho_1$, let w_ρ be the radially symmetric function which is harmonic in $B_1\setminus\{0\}$ such that $w_\rho(\rho)=v(\rho)+1$ and $w_\rho(\rho_1)=v(\rho_1)$. In fact, $w_\rho(r)=a_{1,\rho}\,r^{2-n}+a_{2,\rho}$ where

$$a_{1,\rho} = \frac{v(\rho)+1-v(\rho_1)}{\rho^{2-n}-\rho_1^{2-n}} > 0 \text{ and } a_{2,\rho} = v(\rho_1) - a_{1,\rho}\,\rho_1^{2-n}.$$

Note that $w_\rho(r) \geq v(r)$ for all $0 < r < \rho$. (Because if $w(s) < v(s)$ for some $s < \rho$, the maximum principle implies that $w_\rho(r) \leq v(r)$ for $s < r < \rho_1$, which implies in particular that $w_\rho(\rho) \leq v(\rho)$ contradicting our choice of $w_\rho(\rho)$.) It follows that

$$\limsup_{r\to 0} r^{n-2}\, v(r) \leq \limsup_{r\to 0} r^{n-2}\, w_\rho(r) = a_{1,\rho} \text{ for all } 0 < \rho < \rho_1.$$

In particular, $\limsup_{r\to 0} r^{n-2}\, v(r)$ is finite. Also, we obtain from the above that

$$\limsup_{r\to 0} r^{n-2}\, v(r) \leq \liminf_{\rho\to 0} a_{1,\rho} = \liminf_{r\to 0} r^{n-2}\, v(r),$$

which proves the claim. We thus have

$$a := \liminf_{|x|\to 0} |x|^{n-2}\, u(x) = \lim_{r\to 0} r^{n-2}\, v(r) < \infty.$$

We next claim that

$$A := \limsup_{|x|\to 0} |x|^{n-2}\, u(x) \text{ is finite.}$$

To prove the claim, let, for $0 < r < 1/4$,

$$u_r(y) = u(ry), \qquad \frac{1}{2} < |y| < 2.$$

Then v_r satisfies $\lambda(A^{u_r}) \in \partial\Gamma$ in $\{1/2 < |y| < 2\}$. Thus, by Theorem 1.5,

$$\max_{\partial B_1} u_r \leq C \min_{\partial B_1} u_r$$

where C depends only on n. Equivalently,

$$\max_{\partial B_r} u \leq C \min_{\partial B_r} u.$$

It follows that $A \leq C\, a < \infty$.

Next, we show that $A = a$. Assume by contradiction that $A > a$. Then, for some $\varepsilon > 0$, we can find a sequence $x_j \to 0$ such that

$$|x_j|^{n-2}\, u(x_j) \geq a + 2\varepsilon. \tag{4.18}$$

Furthermore, we can assume that

$$|x_j|^{n-2} \min_{\partial B_{|x_j|}} u = |x_j|^{n-2}\, v(|x_j|) \leq a + \varepsilon. \tag{4.19}$$

Define

$$u_j(y) = \frac{1}{R_j^{n-2}}\, u\Big(\frac{y}{R_j}\Big), \qquad |y| < R_j = |x_j|^{-1}.$$

Then, by (4.18) and (4.19),

$$\begin{cases} \lambda(A^{u_j}) \in \partial\Gamma \text{ in } B_{R_j} \setminus \{0\}, \\ \min_{\partial B_1} u_j \leq a + \varepsilon \text{ and } \max_{\partial B_1} u_j \geq a + 2\varepsilon. \end{cases} \tag{4.20}$$

Since $\min_{\partial B_1} u_j$ is bounded, we can apply Theorem 1.5 to obtain the boundedness of u_j and $|\nabla u_j|$ on every compact subset of $\mathbb{R}^n \setminus \{0\}$. By the Ascoli-Arzela theorem,

u_j, after passing to a subsequence, converges uniformly on compact subset of $\mathbb{R}^n \setminus \{0\}$ to some locally Lipschitz function u_*. Furthermore, by (4.20), u_* satisfies (1.7) in $\mathbb{R}^n \setminus \{0\}$ in the viscosity sense. By [22, Theorem 1.18], u_* is radially symmetric about the origin, i.e. $u_*(y) = u_*(|y|)$. This results in a contradiction as the second line in (4.20) and the convergence of u_j to u_* imply that

$$\max_{\partial B_1} u_* \geq a + 2\varepsilon > a + \varepsilon \geq \min_{\partial B_1} u_*.$$

We conclude that $A = a$ and thereby finish the proof. □

When $\mu_\Gamma^+ = 1$, the leading term for a singular solution of (1.7) might not be $|x|^{-(n-2)}$; see Theorem 2.2. A more precise picture is given by the following lemma.

Lemma 4.4. *Assume that* Γ *satisfies* (1.1), (1.2), (1.9) *and* $\mu_\Gamma^+ = 1$. *Let* $u \in C^{0,1}_{\rm loc}(B_1 \setminus \{0\})$ *be a positive viscosity solution of* (1.7) *in* $B_1 \setminus \{0\}$. *Then there exists* $0 \leq \alpha \leq n-2$ *such that*

$$\lim_{|x|\to 0} \frac{\ln u(x)}{\ln |x|} = -\alpha.$$

Proof. Let

$$v(x) = v(|x|) = \min_{\partial B_{|x|}} u. \tag{4.21}$$

Then $v \in C^0(B_1 \setminus \{0\})$ and satisfies $\lambda(A^v) \in \bar{\Gamma}$ in the viscosity in $B_1 \setminus \{0\}$. By Lemma 2.10, the function

$$\frac{\ln v(r) - \ln v(1/2)}{|\ln r|}$$

is non-decreasing for $r \in (0, 1/2)$. This implies in particular that

$$\alpha := \liminf_{|x|\to 0} \frac{\ln u(x)}{|\ln |x||} = \lim_{r\to 0} \frac{\ln v(r)}{|\ln r|} \text{ exists and is in } [0,\infty).$$

Here we have used the fact that $u \geq \min_{\partial B_{1/2}} u > 0$, a consequence of the superharmonicity of u in $B_1 \setminus \{0\}$. Also, by Lemma 2.7 (or Theorem 1.8), $\alpha \in [0, n-2]$.

Next, by Theorem 1.5,

$$|\nabla \ln u(x)| \leq \frac{C}{|x|} \text{ in } B_1 \setminus \{0\}, \text{ and so } \operatorname*{osc}_{\partial B_{|x|}} \ln u \leq C \text{ for } 0 < |x| < 1.$$

It follows that

$$\frac{\ln u(x)}{|\ln |x||} \leq \frac{\ln v(x)}{|\ln |x||} + \frac{C}{|\ln |x||}.$$

The conclusion easily follows. □

4.3 Proof of Theorem 1.2

We start by showing that

$$\min_{\partial B_r} \mathring{w} \leq \mathring{w} \leq \max_{\partial B_r} \mathring{w} \text{ in } B_r \setminus \{0\} \quad \forall\, 0 < r < 1. \tag{4.22}$$

Fix $0 < r < 1$. We first consider the case where $a > 0$. For $0 < \varepsilon < a$, set

$$v^+_{\varepsilon,r}(x) = \Big[(a+\varepsilon)^{\frac{\mu_\Gamma^+ - 1}{n-2}} |x|^{-\mu_\Gamma^+ + 1} + \max_{\partial B_r} \mathring{w}\Big]^{\frac{n-2}{\mu_\Gamma^+ - 1}},$$

$$v^-_{\varepsilon,r}(x) = \Big[(a-\varepsilon)^{\frac{\mu_\Gamma^+ - 1}{n-2}} |x|^{-\mu_\Gamma^+ + 1} + \min_{\partial B_r} \mathring{w}\Big]^{\frac{n-2}{\mu_\Gamma^+ - 1}}.$$

Then, by Theorem 2.2, we have $\lambda(A^{v^+_{\varepsilon,r}}) \in \partial\Gamma$ and $\lambda(A^{v^-_{\varepsilon,r}}) \in \partial\Gamma$ in $B_r \setminus \{0\}$, and $v^-_{\varepsilon,r} < u < v^+_{\varepsilon,r}$ on ∂B_r. Furthermore, by Theorem 1.8, there exists $\delta = \delta(\varepsilon, r) > 0$ such that

$$v^-_{\varepsilon,r} < u < v^+_{\varepsilon,r} \text{ in } B_\delta \setminus \{0\}.$$

Thus, by Lemma 2.6,

$$v^-_{\varepsilon,r} \leq u \leq v^+_{\varepsilon,r} \text{ in } B_r \setminus \{0\}.$$

Sending $\varepsilon \to 0$ we obtain (4.22).

Next, consider the case where $a = 0$. The argument above establishes the first part of (4.22). The second part follows from the super-harmonicity of $u = \mathring{w}^{\frac{(n-2)k}{n-2k}}$.

We turn to the proof of the dichotomy (1.11)-(1.12). Assume that (1.12) does not hold. Then by (4.22)

$$\min_{\partial B_r} \mathring{w} = \inf_{B_r \setminus \{0\}} \mathring{w}(x) = 0.$$

We thus have $\Delta u \leq 0 = \Delta(a|x|^{-(n-2)})$ in $B_1 \setminus \{0\}$, $u \geq a\,|x|^{-(n-2)}$ in $B_1 \setminus \{0\}$ and the set $\{x \in B_1 \setminus \{0\} : u(x) = a\,|x|^{-(n-2)}\}$ is non-empty. The strong maximum principle for the Laplacian implies that $u \equiv a\,|x|^{-(n-2)}$ in $B_1 \setminus \{0\}$. The last assertion follows from Proposition 4.3. □

Corollary 4.5. *Assume that* Γ *satisfies* (1.1), (1.2), (1.9) *and* $\mu_\Gamma^+ > 1$. *Let* Ω *be an open subset of* $\mathbb{R}^n$ *containing* $\cup_{i=1}^2 B_{|p_1 - p_2|}(p_i)$ *for two distinct points* p_1 *and* p_2. *Assume that* $u \in C^0(\Omega)$ *is a positive solution of* (1.7) *in the viscosity sense in* $\Omega \setminus \{p_1, p_2\}$ *and*

$$\lim_{|x-p_i| \to 0} |x - p_i|^{n-2} u(x) = a_i > 0.$$

Then, for any $r < |p_1 - p_2|$,

$$\inf_{B_r(p_i) \setminus \{p_i\}} \Big(u(x)^{\frac{\mu_\Gamma^+ - 1}{n-2}} - a_i^{\frac{\mu_\Gamma^+ - 1}{n-2}} |x|^{-\mu_\Gamma^+ - 1} \Big) > 0.$$

Proof. Assume otherwise that, for some $0 < r < |p_1 - p_2|$,

$$\inf_{B_r(p_1)\setminus\{p_1\}} \left(u(x)^{\frac{\mu_\Gamma^+ - 1}{n-2}} - a_1^{\frac{\mu_\Gamma^+ - 1}{n-2}} |x|^{-\mu_\Gamma^+ - 1} \right) = 0.$$

By (1.11) in Theorem 1.2,

$$u(x) \equiv a_1 |x - p_1|^{-(n-2)} \text{ in } B_{|p_1 - p_2|}(p_1).$$

This implies that

$$a_2 = \lim_{|x - p_2| \to 0} |x - p_2|^{n-2} u(x) = 0,$$

a contradiction. □

4.4 Proof of Theorem 1.3

By Lemma 4.4,

$$\lim_{|x| \to 0} \frac{\ln u(x)}{|\ln |x||} = \alpha \in [0, n-2].$$

To proceed, consider first the case $\alpha = n-2$. The function v given by (4.21) satisfies $\lambda(A^v) \in \bar{\Gamma}$ in the viscosity sense in $B_1 \setminus \{0\}$. By Lemma 2.10, the function

$$\frac{\ln v(r) - \ln v(s)}{\ln s - \ln r}$$

is non-decreasing in r for $r \in (0, s)$. It follows that

$$\frac{\ln v(r) - \ln v(s)}{\ln s - \ln r} \geq \lim_{r \to 0} \frac{\ln v(r) - \ln v(s)}{\ln s - \ln r} = \alpha = n-2.$$

On the other hand, by estimate (2.12) in Lemma 2.7,

$$\frac{\ln v(r) - \ln v(s)}{\ln s - \ln r} \leq n-2.$$

Combining the last two estimate we immediately get

$$v(r) = \frac{C}{|x|^{n-2}} \text{ for some positive constant } C.$$

In particular, v is harmonic in $B_1 \setminus \{0\}$. As u is super-harmonic in $B_1 \setminus \{0\}$, $u \geq v$ in $B_1 \setminus \{0\}$ and u touches v in the interior, the strong maximum principle implies that $u \equiv v$. This establishes the result for $\alpha = n-2$.

Next, consider (1.13) for $0 < \alpha < n-2$. For $0 < \varepsilon < \min(\alpha, n-2-\alpha)$, let

$$v_{\varepsilon,r}^+(x) = \exp\Big[-(\alpha + \varepsilon) \ln |x| + \max_{\partial B_r} \mathring{w} \Big],$$

$$v_{\varepsilon,r}^-(x) = \exp\Big[-(\alpha - \varepsilon) \ln |x| + \min_{\partial B_r} \mathring{w} \Big].$$

As in the proof of Theorem 1.2, an application of Lemma 2.6 gives $v_{\varepsilon,r}^- \leq u \leq v_{\varepsilon,r}^+$ in $B_r \setminus \{0\}$, which implies (1.13).

Finally, consider $\alpha = 0$. The argument above shows the first part of (1.13). The second part of (1.13) follows from the super-harmonicity of $u = e^{\mathring{w}}$. The remaining assertion on the regularity of $\mathring{w}$ follows from Proposition 4.3. □

4.5 Proof of Theorem 1.4

The function v defined by (4.21) belongs to $LSC(B_1 \setminus \{0\}) \cap L^\infty_{\rm loc}(B_1 \setminus \{0\})$ and satisfies $\lambda(A^v) \in \bar\Gamma$ in $B_1 \setminus \{0\}$ in the viscosity sense. We claim that

$$\text{either } v(x) = \frac{C}{|x|^{n-2}} \text{ for some } C > 0 \text{ or } \sup_{B_{\frac12}\setminus\{0\}} v < \infty. \tag{4.23}$$

Indeed, if the first alternative in (4.23) does not hold, we can find $0 < r_1 < r_2 < 1$ such that

$$v(r_1) \neq \frac{v(r_2)\, r_2^{n-2}}{r_1^{n-2}}.$$

Using (2.12) in Lemma 2.7, we thus have

$$v(r_2) \leq v(r_1) < \frac{v(r_2)\, r_2^{n-2}}{r_1^{n-2}}.$$

As in the proof of Proposition 4.2 (see the argument following (4.1)), this implies that v is bounded near the origin. This proves (4.23).

If the first alternative in (4.23) holds, we have $u \geq v$ in $B_1 \setminus \{0\}$, $\Delta u \leq 0 = \Delta v$ in $B_1 \setminus \{0\}$ and the set $\{x \in B_1 \setminus \{0\} : u = v\}$ is non-empty. By the strong maximum principle for the Laplacian, $u \equiv v$ and the conclusion follows. If the second alternative in (4.23) holds, the conclusion follows from Proposition 4.2. □

4.6 An analogue of Theorem 1.4 when (1.9) fails

In contrast to Theorem 1.4, when (1.9) does not hold, there are unbounded solutions in a punctured ball of (1.7) which are not of the form $\frac{C}{|x|^{n-2}}$. See the remark below Theorem 1.4. In any event, we have:

Theorem 4.6. *Assume that* Γ *satisfies* (1.1), (1.2), (2.1) *and* $0 \leq \mu_\Gamma^+ < 1$. *Let* $u \in LSC(B_1 \setminus \{0\}) \cap L^\infty_{\rm loc}(B_1 \setminus \{0\})$ *be a positive viscosity solution of* $\lambda(A^u) \in \bar\Gamma$ *in* $B_1 \setminus \{0\}$. *Then* $u \in C^{0,1-\mu_\Gamma^+}_{\rm loc}(B_1 \setminus \{0\})$ *and*

$$\lim_{|x|\to 0} |x|^{n-2} u(x) = a \in [0, +\infty).$$

In addition, if $a = 0$ *then* $u \in C^{0,1-\mu_\Gamma^+}_{\rm loc}(B_1)$ *and* (1.14) *holds.*

Proof. Extend u by $u(0) = \liminf_{|x|\to 0} u(x)$. By Proposition 4.1, $\lambda(A^u) \in \bar\Gamma$ in B_1 in the viscosity sense. Also, by Proposition 4.2, $u \in C^{0,1-\mu_\Gamma^+}_{\rm loc}(B_1 \setminus \{0\})$.

As before, the proof evolves around function v defined by (4.21), which belongs to $C(B_1 \setminus \{0\})$ and satisfies $\lambda(A^v) \in \bar\Gamma$ in B_1 in the viscosity sense. By super-harmonicity, v is non-increasing.

Case 1: There exists $0 < r_1 < r_2 < 1$ such that

$$v(r_1) < \frac{v(r_2)\, r_2^{n-2}}{r_1^{n-2}}.$$

The proof of Proposition 4.2 (see the argument following (4.1)) shows that v is bounded at the origin, $u \in C^{0,1-\mu_\Gamma^+}_{\rm loc}(B_1)$ and (1.14) holds.

Case 2 For all $0 < r_1 < r_2 < 1$,

$$v(r_1) \geq \frac{v(r_2)\, r_2^{n-2}}{r_1^{n-2}}. \tag{4.24}$$

In other words, $r^{n-2}\, v$ is non-increasing.

We claim that

$$a := \liminf_{|x|\to 0} |x|^{n-2}\, u(x) = \liminf_{|x|\to 0} r^{n-2}\, v(r) \text{ is finite.} \tag{4.25}$$

Indeed, by (4.24), we can choose $C_5 > 0$ and $C_6 \geq 0$ such that the function

$$\tilde v(r) = (C_5\, r^{-\mu_\Gamma^- +1} - C_6)^{\frac{n-2}{\mu_\Gamma^- -1}}$$

satisfies $\tilde v(1/2) = v(1/2)$ and $\tilde v(2/3) = v(2/3)$. By Theorem 2.2, $\tilde v$ satisfies $\lambda(A^{\tilde v}) \in \partial\Gamma$ in $B_{2/3} \setminus \{0\}$. By Corollary 2.9, we have $v(r) \leq \tilde v(r)$ for $0 < r < 1/2$, which proves the claim.

Recalling (4.24), we see that

$$v(r) \leq \frac{a}{r^{n-2}} \text{ for all } 0 < r < 1. \tag{4.26}$$

Since v is positive, a is non-zero.

Next, we prove that

$$u(x) \geq \Big[a^{\frac{\mu_\Gamma^- -1}{n-2}}\, |x|^{-\mu_\Gamma^- +1} - \max_{\partial B_{3/4}} \mathring{w}\Big]^{\frac{n-2}{\mu_\Gamma^- -1}}, \tag{4.27}$$

where $\mathring{w}(x) = u(x)^{\frac{\mu_\Gamma^- -1}{n-2}} - a^{\frac{\mu_\Gamma^- -1}{n-2}}\, |x|^{-\mu_\Gamma^- +1}$. For sufficiently small $\varepsilon > 0$, define

$$v_{\varepsilon,r}(x) = \Big[(a-\varepsilon)^{\frac{\mu_\Gamma^- -1}{n-2}}\, |x|^{-\mu_\Gamma^- +1} - \max_{\partial B_r} \mathring{w}\Big]^{\frac{n-2}{\mu_\Gamma^- -1}}.$$

Clearly, $v_{\varepsilon,r} \leq u$ on ∂B_r and, by (4.26), for some $\delta_i \to 0$, $v_{\varepsilon,r} \leq u$ on ∂B_{δ_i}. Also, by Theorem 2.2, $\lambda(A^{v_{\varepsilon,r}}) \in \partial\Gamma$ in $B_r \setminus \{0\}$. Hence, by Lemma 2.6, $v_{\varepsilon,r} \leq u$ in $B_r \setminus B_{\delta_i}$. Sending $\delta_i \to 0$ and then $\varepsilon \to 0$, we obtain (4.27).

Note that the argument leading to (4.3) is applicable in the present situation and leads to

$$\Big[\frac{u(x)}{\inf_{\partial B_r} u}\Big]^{\frac{\mu_\Gamma^+ -1}{n-2}} \geq \Big[\frac{|x-\bar x|}{(1-A)r}\Big]^{-\mu_\Gamma^+ +1} + \Big[\frac{u(\bar x)}{\inf_{\partial B_r} u}\Big]^{\frac{\mu_\Gamma^+ -1}{n-2}}$$

for all $x \in B_r \setminus \{0\}, \bar{x} \in B_{Ar} \setminus \{0\}, 0 < A < 1, 0 < r < 1$. In particular, this implies that the function $w := u^{\frac{\mu_\Gamma^+ - 1}{n-2}}$ extends to a $C^{0,1-\mu_\Gamma^+}$ function in B_1 (with $w(0) = 0$ in view of (4.25)).

For $j > 1$, define

$$w_j(x) = j^{1-\mu_\Gamma^-} w\Big(\frac{x}{j}\Big) \text{ for } |x| < j.$$

The Hölder continuity of w implies that the w_j is bounded in $C^{0,1-\mu_\Gamma^+}(B_R)$ for any fixed $R > 0$. Thus, up to a subsequence, w_j converges uniformly to some $w_\infty \in C^{0,1-\mu_\Gamma^+}_{\rm loc}(\mathbb{R}^n)$. Furthermore, if we define $u_\infty = w_\infty^{\frac{n-2}{\mu_\Gamma^+ - 1}}$, then $\lambda(A^{u_\infty}) \in \bar{\Gamma}$ in $\mathbb{R}^n$.

By (4.26) and (4.27),

$$\max_{\partial B_r} w_\infty = a^{\frac{\mu_\Gamma^+ - 1}{n-2}} r^{1-\mu_\Gamma^+} \text{ and } \min_{\partial B_r} u_\infty = a\, r^{-(n-2)} \text{ for all } 0 < r < \infty.$$

In particular, $u_\infty(x) \geq a\,|x|^{-(n-2)}$. As u_∞ is super-harmonic and $u_\infty(x) = a\,|x|^{-(n-2)}$ for some x, the strong maximum principle implies that $u_\infty(x) = a\,|x|^{-(n-2)}$ and $w_\infty(x) = a^{\frac{\mu_\Gamma^+ - 1}{n-2}} |x|^{1-\mu_\Gamma^+}$. Recalling the convergence of w_j to w_∞, we see that

$$a = \lim_{|x| \to 0} |x|^{n-2} u(x),$$

which finishes the proof. □

Appendix: A calculus lemma

For a continuous function w, let $w_{y,\lambda}$ denote the Kelvin transformation of w about the sphere $\partial B_\lambda(y)$, i.e.

$$w_{y,\lambda}(x) = \frac{\lambda^{n-2}}{|x-y|^{n-2}} w\Big(y + \frac{\lambda^2(x-y)}{|x-y|^2}\Big) \text{ wherever the expression make sense.}$$

In [23, Lemma 2], we show, as an extension of [19, Lemma A.2], that if w is a positive continuous function in $B_1(0)$ and

$$w_{y,\lambda}(x) \leq w(x) \text{ for any } B_\lambda(y) \subset B_1(0) \text{ and } x \in B_1(0) \setminus B_\lambda(y),$$

then $\ln w$ is locally Lipschitz in $B_1(0)$ and

$$|\nabla \ln w(x)| \leq \frac{n-2}{1-|x|} \text{ a.e. in } B_1(0).$$

We present a generalization which is needed in the body of the paper.

Lemma A.1. *Assume that w is a positive continuous function in $B_1(0) \setminus \{0\}$ and*

$$w_{y,\lambda}(x) \leq w(x) \textit{ for any } B_\lambda(y) \subset B_1(0) \setminus \{0\} \textit{ and } x \in B_1(0) \setminus (B_\lambda(y) \cup \{0\}),$$

then $\ln w$ *is locally Lipschitz in* $B_1(0)\setminus\{0\}$. *Furthermore, for all* $x\in B_{1/2}(0)\setminus\{0\}$ *and all* $y\in B_{1/2}(0)\setminus B_{|x|/2}(0)$, *there holds*

$$\ln w(y)-\ln w(x)\le C(n)\max\Big(|y-x|,\frac{|x|^2-|y|^2}{|x|^2}\Big).$$

In particular,

$$\sup_{\partial B_R}\ln w-\inf_{\partial B_R}\ln w\le C(n)R \textit{ for any } 0<R<1/2.$$

Proof. By [23, Lemma 2], $\ln w$ is locally Lipschitz in $B_1(0)\setminus\{0\}$ and

$$|\nabla\ln w(x)|\le\frac{C(n)}{|x|}\text{ a.e. in }B_{1/2}(0)\setminus\{0\}.$$

Thus it suffices to consider $x\in B_{1/16}(0)\setminus\{0\}$ and all $y\in B_{1/16}(0)\setminus B_{|x|/2}(0)$. Let

$$e=\frac{y-x}{|y-x|}\text{ and }t=|y-x|\le\frac{1}{8}.$$

Consider first the case $|y|\ge|x|$, i.e. $2x\cdot e+t\ge0$. Then, for $z_1=x+\frac{1}{4}e$ and $\lambda_1=\frac{1}{2}(\frac{1}{4}-t)^{1/2}$, we have

$$\lambda_1^2=\frac{1}{16}-\frac{t}{4}\le\frac{1}{16}+\frac{1}{2}x\cdot e\le|z_1|^2,$$

and thus

$$w(x)\ge w_{z_1,\lambda_1}(x)=(4\lambda_1)^{n-2}w(y)=(1-4t)^{\frac{n-2}{2}}w(y).$$

It follows that

$$\ln w(y)-\ln w(x)\le-\frac{n-2}{2}\ln(1-4t)\le C(n)t.$$

Next, assume that $|y|>|x|$. Let

$$s=\frac{|x|^2}{-(2x\cdot e+t)}=\frac{|x|^2}{|x|^2-|y|^2}t>\frac{4}{3}t>0.$$

If $s\ge\frac{1}{4}$, then

$$\lambda_1^2=\frac{1}{16}-\frac{t}{4}\le\frac{1}{16}+\frac{1}{2}x\cdot e+|x|^2=|z_1|^2,$$

and so we continue to have $\ln w(y)-\ln w(x)\le C(n)t$ as desired. If $s<\frac{1}{4}$, we consider $z_2=x+s\,e$ and $\lambda_2=\sqrt{s(s-t)}$. We have

$$\lambda_2^2=s^2-st=s^2+2s\,x\cdot e+|x|^2=|z_2|^2.$$

This leads to

$$w(x)\ge w_{z_2,\lambda_2}(x)=\frac{\lambda_2^{n-2}}{s^{n-2}}w(y)=\frac{(s-t)^{(n-2)/2}}{s^{(n-2)/2}}w(y)$$

and so

$$\ln w(y)-\ln w(x)\le-\frac{n-2}{2}\ln\Big(1-\frac{t}{s}\Big)\le C(n)\frac{t}{s}.$$

The assertion follows. □

Appendix: Proof of Lemma 2.1 and more on μ_Γ^+ and μ_Γ^-

Associated with a Γ satisfying (1.1) and (1.2), we have introduced $\mu_\Gamma^+ \in [0, n-1]$ and $\mu_\Gamma^- \in [n-1,\infty]$ in (1.8) and (2.2) respectively. In this appendix we provide more properties of Γ in connection with μ_Γ^+ and μ_Γ^- and prove Lemma 2.1.

It is convenient to extend the definition of $\mu_\Gamma^\pm$ for cones Γ satisfying (1.1) and $\Gamma \subset \Gamma_1$ (instead of the stronger condition (1.2)):

$$\mu_\Gamma^+ = \sup\Big\{t : (-t,1,1,\ldots,1) \in \Gamma\Big\},$$
$$\mu_\Gamma^- = \inf\Big\{t : (t,-1,-1,\ldots,-1) \in \Gamma\Big\}.$$

In the definition of μ_Γ^-, if the set of such t is empty, the corresponding infimum is taken to be $+\infty$. Evidently, $\mu_\Gamma^- \in [n-1,\infty]$ and $\mu_\Gamma^+ \leq n-1$.

We claim that $\mu_\Gamma^+ > -1$. To see this, pick an arbitrary $\lambda = (\lambda_1,\ldots,\lambda_n) \in \Gamma$ and consider the set of all permutations of λ. This is a subset of Γ and so its center of mass $(\frac{\lambda_1+\ldots+\lambda_n}{n},\ldots,\frac{\lambda_1+\ldots+\lambda_n}{n})$ belongs to Γ. Since $\lambda_1+\ldots+\lambda_n > 0$, this implies that $(1,\ldots,1) \in \Gamma$. As Γ is open, we can thus find some $\varepsilon = \varepsilon(\Gamma) > 0$ such that $(1-\varepsilon,1,\ldots,1) \in \Gamma$, which implies $\mu_\Gamma^+ \geq -(1-\varepsilon)$, as claimed.

Define

$$\mathscr{C}^+(\mu) = \Big\{\Gamma :\ \Gamma \text{ satisfying (1.1)}, \Gamma \subset \Gamma_1 \text{ and } \mu_\Gamma^+ = \mu\Big\}, \mu \in (-1,n-1],$$
$$\mathscr{C}^-(\mu) = \Big\{\Gamma :\ \Gamma \text{ satisfying (1.1)}, \Gamma \subset \Gamma_1 \text{ and } \mu_\Gamma^- = \mu\Big\}, \mu \in [n-1,\infty],$$

and

$$L\Gamma^\pm(\mu) = \cap\mathscr{C}^\pm(\mu), \text{ and } U\Gamma^\pm(\mu) = \cup\mathscr{C}^\pm(\mu).$$

In what to follow, we show that $L\Gamma^\pm(\mu)$ and $U\Gamma^\pm(\mu)$ belong to $\mathscr{C}^\pm(\mu)$ and give an explicit description for these cones. More specifically, we have

Proposition B.1. *There hold*

$$\mathscr{C}^\pm(n-1) = \{\Gamma_1\}, L\Gamma^\pm(n-1) = U\Gamma^\pm(n-1) = \Gamma_1, \tag{B.1}$$

$$L\Gamma^+(\mu) = \Big\{\lambda : \lambda_i - \frac{1}{n-1-\mu}\sum_{j=1}^n \lambda_j < 0 \textit{ for all } i\Big\}, \forall\, \mu \in (-1,n-1), \tag{B.2}$$

$$U\Gamma^+(\mu) = \Big\{\lambda : \lambda_i + \frac{\mu}{n-1-\mu}\sum_{j=1}^n \lambda_j > 0 \textit{ for all } i\Big\}, \forall\, \mu \in (-1,n-1), \tag{B.3}$$

$$L\Gamma^-(\mu) = \Big\{\lambda : \lambda_i + \frac{1}{\mu-(n-1)}\sum_{j=1}^n \lambda_j > 0 \textit{ for all } i\Big\}, \forall\, \mu \in (n-1,\infty], \tag{B.4}$$

(B.5) $$U\Gamma^-(\mu) = \Big\{\lambda : \lambda_i - \frac{\mu}{\mu-(n-1)}\sum_{j=1}^n \lambda_j < 0 \textit{ for all } i\Big\}, \forall\, \mu \in (n-1,\infty],$$

(B.6) $$U\Gamma^+(\mu) = \cup\Big\{\Gamma :\ (1.1) \textit{ and } (1.2) \textit{ hold and } \mu_\Gamma^+ = \mu\Big\}, \forall\, \mu \in [0,n-1].$$

Proof. We will only prove the statements about $\mathscr{C}^+(n-1)$, $L\Gamma^+(\mu)$ and $U\Gamma^+(\mu)$. The ones about $\mathscr{C}^-(n-1)$, $L\Gamma^-(\mu)$ and $U\Gamma^-(\mu)$ can be proved analogously. Let S be the set consisting of $(-\mu,1,\ldots,1)$ and its permutations, and conv(S) the open convex hull of S. For convenience we denote $S = \{v_1, v_2, ..., v_n\}$ with $v_1 = (-\mu,1,\ldots,1)$.

Assume that $\mu = n-1$. If $\Gamma \in \mathscr{C}^+(\mu)$, then conv$(S) \subset \bar{\Gamma}$. On the other hand, since $\mu \neq -1$, $\{v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n\}$ is linearly independent. Also, as $\mu = n-1$, $S \subset \partial\Gamma_1$. Note that 0 is the center of mass of S and hence is in conv(S). Thus conv(S), and therefore $\bar{\Gamma}$, contains a neighborhood of the origin relative to the plane $\partial\Gamma_1$. By homothety $\bar{\Gamma} \supset \partial\Gamma_1$, which implies that $\Gamma = \Gamma_1$. We have shown that $\mathscr{C}^+(n-1) = \{\Gamma_1\}$, and so $L\Gamma^+(n-1) = U\Gamma^+(n-1) = \Gamma_1$.

Assume that $\mu < n-1$. Observe that $L\Gamma^+(\mu)$ is the cone consisting of points of the form $t\lambda$ for some $t > 0$ and some $\lambda \in$ conv(S). This is because the latter cone is a member of $\mathscr{C}^+(\mu)$. Consider a face, say F, of $L\Gamma^+(\mu)$. F is a plane going through the origin and $n-1$ other points in S. Clearly, there is a unique i such that the i-th coordinate of those $n-1$ points is 1. It follows that the equation of F is

$$\lambda_i - \frac{1}{n-1-\mu}\sum_{j=1}^n \lambda_j = 0,$$

whence (B.2).

We turn to (B.3). Let A denote the cone on the right hand side of (B.3). It is easy to check that $A \in \mathscr{C}^+(\mu)$ and hence $A \subset U\Gamma^+(\mu)$. Arguing by contradiction, assume that $U\Gamma^+(\mu) \setminus A \neq \emptyset$. Then we can find a cone $\Gamma \in \mathscr{C}^+(\mu)$ and a vector $\lambda \in \Gamma$ such that $\lambda_1 + \ldots + \lambda_n = n-1-\mu$ and $\lambda_i + \mu \leq 0$ for some i. By symmetry, we can assume that $i = 1$, i.e. $\lambda_1 \leq -\mu$. Note that this implies $x := \lambda_2 + \ldots + \lambda_n \geq n-1$. Now, by convexity, $(\lambda_1, \frac{x}{n-1}, \ldots, \frac{x}{n-1}) \in \Gamma$, which implies that

$$\Big(\frac{(n-1)\lambda_1}{x}, 1, \ldots, 1\Big) \in \Gamma.$$

It follows that $\mu_\Gamma^+ \geq \frac{(n-1)|\lambda_1|}{x} > \mu$, contradicting the definition of $\mathscr{C}^+(\mu)$. (B.3) is proved.

Finally, (B.6) follows from (B.3) as $U\Gamma^+(\mu) \supset \Gamma_n$ for $\mu \geq 0$. □

Proof of Lemma 2.1. (a) is clear. (b) follows from $\mathscr{C}^\pm(n-1) = \{\Gamma_1\}$. (c) follows from (1.2) and $U\Gamma^+(0) = \Gamma_n$. (e) is a consequence of (d).

For (d), note first that $(-\mu_\Gamma^+, 1, \ldots, 1) \in \bar{\Gamma} \subset \overline{U\Gamma^-(\mu_\Gamma^-)}$, which implies that

$$1 - \frac{\mu_\Gamma^-}{\mu_\Gamma^- - (n-1)}(-\mu_\Gamma^+ + (n-1)) \leq 0.$$

Likewise, $(\mu_\Gamma^-, -1, \dots, -1) \in \bar\Gamma \subset \overline{U\Gamma^+(\mu_\Gamma^+)}$ and so

$$-1 + \frac{\mu_\Gamma^+}{n-1-\mu_\Gamma^+}(\mu_\Gamma^- - (n-1)) \geq 0.$$

(d) follows. □

Note that the cone $U\Gamma^+(\mu)$ was used in Li and Li [18], Gursky and Viaclovsky [15] and Trudinger and Wang [25]. A family of cones connecting Γ_1 and Γ was used in [18]:

$$\Gamma_t = \Big\{\lambda \;:\; t\lambda + (1-t)\sigma_1(\lambda)e \in \Gamma\Big\}, \qquad 0 \leq t \leq 1,$$

where $e = (1,1,...,1)$. The so-called $\theta-$convex cone

$$\Sigma_\theta = \Big\{\lambda \;:\; \lambda_i + \theta\sum_{j=1}^n \lambda_j > 0 \text{ for all } i\Big\}$$

was used in [15, 25]. It is clear that

$$\Sigma_\theta = (\Gamma_n)_{\frac{1}{1+\theta}} = U\Gamma^+\left(\frac{(n-1)\theta}{1+\theta}\right), \qquad \text{for all } \theta \geq 0.$$

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