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Gradient Estimates for the Perfect Conductivity Problem

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Abstract

We establish both upper and lower bounds of the gradient estimates for the perfect conductivity problem in the case where perfect (stiff) conductors are closely spaced inside an open bounded domain and away from the boundary. These results give the optimal blow-up rates of the stress for conductors with arbitrary shape and in all dimensions.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary, $n \geqq 2$, $0 < \alpha < 1$, D_1 and D_2 be two bounded strictly convex open subsets in Ω with $C^{2,\alpha}$ boundaries which are ε apart and far away from $\partial\Omega$, that is

$$\begin{aligned} &\overline{D}_1, \overline{D}_2 \subset \Omega, \quad \text{the principal curvature of } \partial D_1, \partial D_2 \geqq \kappa_0, \\ &\varepsilon := \operatorname{dist}(D_1, D_2) > 0, \ \ \operatorname{dist}(D_1 \cup D_2, \partial\Omega) > r_0, \ \ \operatorname{diam}(\Omega) < \frac{1}{r_0}, \end{aligned} \tag{1.1}$$

where $\kappa_0, r_0 > 0$ are universal constants independent of ε. We will assume that the $C^{2,\alpha}$ norms of ∂D_i are bounded by some constant independent of ε. This implies $\operatorname{diam}(D_i) \geqq r_0^*$ for some universal constant $r_0^* > 0$ independent of ε. We denote

$$\widetilde{\Omega} := \Omega \backslash \overline{D_1 \cup D_2}.$$

Given $\varphi \in C^2(\partial\Omega)$, consider the following scalar equation with Dirichlet boundary condition:

$$\begin{cases} \operatorname{div}(a_k(x)\nabla u_k) = 0 & \text{in } \Omega, \\ u_k = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where

$$a_k(x) = \begin{cases} k \in (0, \infty) & \text{in } D_1 \cup D_2, \\ 1 \quad \text{in } \Omega \setminus \overline{D_1 \cup D_2}. \end{cases} \tag{1.3}$$

It is well known that there exists a unique solution $u_k \in H^1(\Omega)$ of the above equation, which is also the minimizer of I_k on $H^1_\varphi(\Omega)$, where

$$H^1_\varphi(\Omega) := \{u \in H^1(\Omega) \mid u = \varphi \text{ on } \partial\Omega\}, \quad I_k[v] := \frac{1}{2}\int_\Omega a_k |\nabla v|^2.$$

As explained in the introduction of [10], the above equation in dimension $n = 2$ can be used as a simple model in the study of composite media with closely spaced interfacial boundaries. For this purpose, the domain Ω would model the cross-section of a fiber-reinforced composite, D_1 and D_2 would represent the cross-sections of the fibers, $\widetilde{\Omega}$ would represent the matrix surrounding the fibers, and the shear modulus of the fibers would be k and that of the matrix would be 1. Equation (1.2) is then obtained by using a standard model of anti-plane shear, and the solution u_k represents the out of plane elastic displacement. The most important quantities from an engineering point of view are the stresses, in this case represented by ∇u_k.

It is well known that the solution u_k satisfies $\|u_k\|_{C^{2,\alpha}(D_i)} < \infty$. In fact, if ∂D_1 and ∂D_2 are $C^{m,\alpha}$, we have $\|u_k\|_{C^{m,\alpha}(D_i)} < \infty$. Such results do not require D_i to be convex and hold for general elliptic systems with piecewise smooth coefficients; see for example theorem 9.1 in [10] and proposition 1.6 in [9]. For a fixed $0 < k < \infty$, the $C^{m,\alpha}(D_i)$-norm of the solution might tend to infinity as $\varepsilon \to 0$. Babuska, Anderson, Smith and Levin [4] were interested in linear elliptic systems of elasticity arising from the study of composite material. They observed numerically that, for solution u to certain homogeneous isotropic linear systems of elasticity, $\|\nabla u\|_{L^\infty}$ is bounded independently of the distance ε between D_1 and D_2. Bonnetier and Vogelius [6] proved this in dimension $n = 2$ for the solution u_k of (1.2) when D_1 and D_2 are two unit balls touching at a point. This result was extended by Li and Vogelius in [10] to general second order elliptic equations with piecewise smooth coefficients, where stronger $C^{1,\beta}$ estimates were established. The $C^{1,\beta}$ estimates were further extended by Li and Nirenberg in [9] to general second order elliptic systems including systems of elasticity. For higher derivative estimates, for example an ε-independent L^∞-estimate of second derivatives of u_k in D_1, we draw the attention of readers to the open problem on page 894 of [9]. In [10] and [9], the ellipticity constants are assumed to be away from 0 and ∞. If we allow ellipticity constants to deteriorate, the situation is different. It has been shown in various papers, see for example [7] and [11], that when $k = \infty$ the L^∞-norm of ∇u_k for the solution u_k of equation (1.2) generally becomes unbounded as ε tends to zero. The rate at which the L^∞ norm of the gradient of a special solution has been shown in [7] to be $\varepsilon^{-1/2}$.

In this paper, we consider the perfect conductivity problem, where $k = +\infty$. It was proved by Ammari, Kang and Lim in [1] and Ammari, Kang, H. Lee, J. Lee and Lim in [3] that, when D_1 and D_2 are balls of comparable radii embedded in $\Omega = \mathbb{R}^2$, the blow-up rate of the gradient of the solution to the perfect conductivity problem is $\varepsilon^{-1/2}$ as ε goes to zero; with the lower bound given in [1] and the upper bound given in [3]. Yun in [12] generalized the above mentioned result in [1] by establishing the same lower bound, $\varepsilon^{-1/2}$, for two strictly convex subdomains in $\mathbb{R}^2$. In this paper, we give both lower and upper bounds to blow-up rates of the gradient for the solution to the perfect conductivity problem in a bounded matrix, where two strictly convex subdomains are embedded. Our methods apply to dimension $n \geqq 3$ as well. One might reasonably suspect that the blow-up rate in dimension $n \geqq 3$ should be smaller than that in dimension $n = 2$. However we prove the opposite: As ε goes to zero, the blow-up rate is $\varepsilon^{-1/2}$, $(\varepsilon|\ln\varepsilon|)^{-1}$ and ε^{-1} for $n = 2,\ 3$ and $n \geqq 4$, respectively. We also give a criteria, in terms of a linear functional of the boundary data φ, for the situation where the rate of blow-up is realized. Note that [1] and [3] contain also results for $k < \infty$.

The perfect conductivity problem is described as follows:

$$\begin{cases} \Delta u = 0 & \text{in } \widetilde{\Omega}, \\ u|_+ = u|_- & \text{on } \partial D_1 \cup \partial D_2, \\ \nabla u \equiv 0 & \text{in } D_1 \cup D_2, \\ \int_{\partial D_i} \frac{\partial u}{\partial \nu}\big|_+ = 0 & (i = 1, 2), \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

where

$$\frac{\partial u}{\partial \nu}\bigg|_+ := \lim_{t\to 0^+} \frac{u(x + t\nu) - u(x)}{t}\ .$$

Here and throughout this paper ν is the outward unit normal to the domain and the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. The existence and uniqueness of solutions to equation (1.4) are well known, see the Appendix. Moreover, the solution $u \in H^1(\Omega)$ is the weak limit of the solutions u_k to equations (1.2) as $k \to +\infty$. It can be also described as the unique function which has the *least energy* in appropriate functional space, defined as $I_\infty[u] = \min_{v\in\mathcal{A}} I_\infty[v]$, where

$$I_\infty[v] := \frac{1}{2}\int_{\widetilde{\Omega}} |\nabla v|^2, \quad v \in \mathscr{A},$$

$$\mathscr{A} := \left\{ v \in H^1_\varphi(\Omega)\,|\nabla v \equiv 0 \ \text{ in } D_1 \cup D_2 \right\}.$$

The readers can refer to the Appendix for the proofs of the above statements.

We now state more precisely what it means by saying that the boundary of a domain, say Ω, is $C^{2,\alpha}$ for $0 < \alpha < 1$: In a neighborhood of every point of $\partial\Omega$, $\partial\Omega$ is the graph of some $C^{2,\alpha}$ functions of $n-1$ variables. We define the $C^{2,\alpha}$ norm of $\partial\Omega$, denoted as $\|\partial\Omega\|_{C^{2,\alpha}}$, as the smallest positive number $\frac{1}{a}$ such that in the $2a-$neighborhood of every point of $\partial\Omega$, identified as 0 after a possible translation and rotation of the coordinates so that $x_n = 0$ is the tangent to $\partial\Omega$ at 0, $\partial\Omega$ is given by the graph of a $C^{2,\alpha}$ function, denoted as f, which is defined as $|x'| < a$, the $a-$neighborhood of 0 in the tangent plane. Moreover, $\|f\|_{C^{2,\alpha}(|x'|<a)} \leqq \frac{1}{a}$.

Theorem 1.1. *Let Ω, D_1, $D_2 \subset \mathbb{R}^n$, ε be defined as in (1.1), $\varphi \in C^2(\partial\Omega)$. Let $u \in H^1(\Omega) \cap C^1(\overline{\widetilde{\Omega}})$ be the solution to equation (1.4). For ε sufficiently small, there is a positive constant C which depends only on n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$ and $\|\partial D_2\|_{C^{2,\alpha}}$, but independent of ε such that*

$$\begin{aligned}
\|\nabla u\|_{L^\infty(\widetilde{\Omega})} &\leqq \frac{C}{\sqrt{\varepsilon}}\|\varphi\|_{C^2(\partial\Omega)} && \text{for } n = 2,\\
\|\nabla u\|_{L^\infty(\widetilde{\Omega})} &\leqq \frac{C}{\varepsilon|\ln\varepsilon|}\|\varphi\|_{C^2(\partial\Omega)} && \text{for } n = 3,\\
\|\nabla u\|_{L^\infty(\widetilde{\Omega})} &\leqq \frac{C}{\varepsilon}\|\varphi\|_{C^2(\partial\Omega)} && \text{for } n \geqq 4.
\end{aligned} \tag{1.5}$$

Remark 1.1. We draw the attention of readers to the independent work of Yun [13] where he has also established the upper bound, $\varepsilon^{-1/2}$, in $\mathbb{R}^2$. The methods are very different. Results in this paper and those in [12] and [13] do not really need D_1 and D_2 to be strictly convex, the strict convexity is only needed for the portions in a fixed neighborhood (the size of the neighborhood is independent of ε) of a pair of points on ∂D_1 and ∂D_2 which realize minimal distance ε. In fact, our proofs of Theorem 1.1–1.2 also apply, with minor modification, to more general situations where two inclusions, D_1 and D_2, are not necessarily convex near points on the boundaries where minimal distance ε is realized; see discussions after the proofs of Theorem 1.1–1.2 in Section 1.3.

To prove Theorem 1.1, we first decompose the solution u of equation (1.4) as follows:

$$u = C_1 v_1 + C_2 v_2 + v_3 \tag{1.6}$$

where $C_i := C_i(\varepsilon)$ $(i = 1, 2)$ is the boundary value of u on ∂D_i $(i = 1, 2)$ respectively, and $v_i \in C^2(\overline{\widetilde{\Omega}})$ $(i = 1, 2, 3)$ satisfies

$$\begin{cases} \Delta v_1 = 0 & \text{in } \widetilde{\Omega},\\ v_1 = 1 \quad \text{on } \partial D_1, & v_1 = 0 \quad \text{on } \partial D_2 \cup \partial\Omega, \end{cases} \tag{1.7}$$

$$\begin{cases} \Delta v_2 = 0 & \text{in } \widetilde{\Omega},\\ v_2 = 1 \quad \text{on } \partial D_2, & v_2 = 0 \quad \text{on } \partial D_1 \cup \partial\Omega, \end{cases} \tag{1.8}$$

$$\begin{cases} \Delta v_3 = 0 & \text{in } \widetilde{\Omega},\\ v_3 = 0 \quad \text{on } \partial D_1 \cup \partial D_2, & v_3 = \varphi \quad \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Define

$$Q_\varepsilon[\varphi] := \int_{\partial D_1} \frac{\partial v_3}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} - \int_{\partial D_2} \frac{\partial v_3}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}, \tag{1.10}$$

then $Q_\varepsilon : C^2(\partial\Omega) \to \mathbb{R}$ is a linear functional.

Theorem 1.2. *With the same conditions in Theorem 1.1, let* $u \in H^1(\Omega) \cap C^1(\overline{\widetilde{\Omega}})$ *be the solution to equation (1.4). For* ε *sufficiently small, there exists a positive constant* C *which depends on* n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$, $\|\partial D_2\|_{C^{2,\alpha}}$ *and* $\|\varphi\|_{C^2(\partial\Omega)}$, *but is independent of* ε *such that*

$$\begin{array}{ll} \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \geqq \frac{|Q_\varepsilon[\varphi]|}{C} \cdot \frac{1}{\sqrt{\varepsilon}} & for\ n = 2, \\ \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \geqq \frac{|Q_\varepsilon[\varphi]|}{C} \cdot \frac{1}{\varepsilon|\ln\varepsilon|} & for\ n = 3, \\ \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \geqq \frac{|Q_\varepsilon[\varphi]|}{C} \cdot \frac{1}{\varepsilon} & for\ n \geqq 4. \end{array} \tag{1.11}$$

Remark 1.2. If $\varphi \equiv 0$, then the solution to equation (1.4) is $u \equiv 0$. Theorem 1.1 and Theorem 1.2 are obvious in this case. So we only need to prove them for $\|\varphi\|_{C^2(\partial\Omega)} = 1$, by considering $u/\|\varphi\|_{C^2(\partial\Omega)}$.

Remark 1.3. It is interesting to know when $|Q_\varepsilon[\varphi]| \geqq \frac{1}{C}$ for some positive constant C independent of ε. Roughly speaking $Q_\varepsilon[\varphi] \to Q^*[\varphi]$ as $\varepsilon \to 0$, and this amounts to $Q^*[\varphi] \neq 0$. For details, see Section 3.

Theorem 1.1–1.2 can be extended to equations with more general coefficients as follows: Let n, Ω, D_1, D_2, ε and φ be the same as in Theorem 1.1, and let

$$A_2(x) := \left(a_2^{ij}(x)\right) \in C^2(\overline{\widetilde{\Omega}})$$

be $n \times n$ symmetric matrix functions in $\widetilde{\Omega}$ satisfying for some constants $0 < \lambda \leqq \Lambda < \infty$,

$$\lambda|\xi|^2 \leqq a_2^{ij}(x)\xi_i\xi_j \leqq \Lambda|\xi|^2, \quad \forall x \in \widetilde{\Omega}, \quad \forall \xi \in \mathbb{R}^n, \tag{1.12}$$

and $a_2^{ij}(x) \in C^2(\overline{\Omega\backslash D_1 \cup D_2})$.

We consider

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i}u\right) = 0 & \text{in } \widetilde{\Omega}, \\ u|_+ = u|_- & \text{on } \partial D_1 \cup \partial D_2, \\ \nabla u = 0 & \text{in } D_1 \cup D_2, \\ \int_{\partial D_i} a_2^{ij}(x)\partial_{x_i}u\nu_j|_+ = 0 & (i = 1, 2), \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.13}$$

where repeated indices denote summations as usual.

Here is an extension of Theorem 1.1:

Theorem 1.3. *With the above assumptions, let* $u \in H^1(\Omega) \cap C^1(\overline{\widetilde{\Omega}})$ *be the solution to equation (1.13). For* ε *sufficient small, there is a positive constant* C *which depends only on* n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$, $\|\partial D_2\|_{C^{2,\alpha}}$, λ, Λ *and* $\|A_2\|_{C^2(\overline{\widetilde{\Omega}})}$, *but independent of* ε *such that estimate (1.5) holds.*

Similar to the decomposition formula (1.6), we decompose the solution u of equation (1.13) as follows:

$$u = C_1V_1 + C_2V_2 + V_3 \tag{1.14}$$

where $C_i := C_i(\varepsilon)$ $(i = 1, 2)$ is the boundary value of u on ∂D_i $(i = 1, 2)$ respectively, and $V_i \in C^2(\overline{\widetilde{\Omega}})$ $(i = 1, 2, 3)$ satisfies

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i} V_1\right) = 0 & \text{in } \widetilde{\Omega}, \\ V_1 = 1 \quad \text{on } \partial D_1, \;\; V_1 = 0 & \text{on } \partial D_2 \cup \partial\Omega, \end{cases} \tag{1.15}$$

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i} V_2\right) = 0 & \text{in } \widetilde{\Omega}, \\ V_2 = 1 \quad \text{on } \partial D_2, \quad V_2 = 0 & \text{on } \partial D_1 \cup \partial\Omega, \end{cases} \tag{1.16}$$

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i} V_3\right) = 0 & \text{in } \widetilde{\Omega}, \\ V_3 = 0 \quad \text{on } \partial D_1 \cup \partial D_2, \quad V_3 = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

Define

$$\begin{aligned} Q_\varepsilon[\varphi] := &\int_{\partial D_1} a_2^{ij}(x)\,\partial_{x_i} V_3\, \nu_j \int_{\partial\Omega} a_2^{ij}(x)\,\partial_{x_i} V_2\, \nu_j \\ &- \int_{\partial D_2} a_2^{ij}(x)\,\partial_{x_i} V_3\, \nu_j \int_{\partial\Omega} a_2^{ij}(x)\,\partial_{x_i} V_1\, \nu_j, \end{aligned} \tag{1.18}$$

then $Q_\varepsilon : C^2(\partial\Omega) \to \mathbb{R}$ is a linear functional.

Theorem 1.4. *With the same conditions in Theorem 1.3, let $u \in H^1(\Omega) \cap C^1(\overline{\widetilde{\Omega}})$ be the solution to equation (1.13). For ε sufficiently small and $Q_\varepsilon[\varphi]$ defined by (1.18), there is a positive constant C which depends only on n, κ_0, r_0, $\|\partial D_1\|_{C^{2,\alpha}}$, $\|\partial D_2\|_{C^{2,\alpha}}$, λ, Λ and $\|A_2\|_{C^2(\overline{\widetilde{\Omega}})}$, but independent of ε such that estimate (1.11) holds.*

Along the approach in this paper, we have extended, in a subsequent paper [5], Theorem 1.1 and 1.3 from two inclusions to multiple inclusions, see Theorem 1.5 below.

The complementary problem to the perfect case is the insulated case when $k = 0$ in (1.2). For $n = 2$, Ammari, Kang, H. Lee, J. Lee and Lim have given in [1] and [3] the optimal bound when D_1 and D_2 are balls of comparable radii embedded in $\Omega = \mathbb{R}^2$. The blow-up rate of the gradient of solutions is $\varepsilon^{-1/2}$ as the distance between D_1 and D_2, ε, goes to zero. They obtained this by converting the insulated case to the perfect case using harmonic conjugators. The situation for $n \geqq 3$ is different since the $k = 0$ case can not be converted to the $k = \infty$ case. In the subsequent paper [5], we have proved that the above mentioned optimal upper bound $\varepsilon^{-1/2}$ is also an upper bound of $\|\nabla u\|_{L^\infty}$ for $n \geqq 3$. In fact, what we have obtained is a local version of the estimates, see Theorem 1.6 below. On the other hand, we do not know yet whether the estimates are sharp for $n \geqq 3$.

Now let, as before, Ω be a bounded open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary, $n \geqq 2$, $0 < \alpha < 1$. Instead of two inclusions, let $\{D_i\}$, $i = 1, 2, \ldots, m$, be m strictly convex open subsets in Ω with $C^{2,\alpha}$ boundaries, $m \geqq 2$, satisfying

$$\begin{aligned} &\overline{D}_i \subset \Omega, \quad \text{the principal curvature of } \partial D_i \geqq \kappa_0, \\ &\varepsilon_{ij} := \text{dist}(D_i, D_j) > 0, \quad (i \neq j) \\ &\text{dist}(D_i, \partial\Omega) > r_0, \quad \text{diam}(\Omega) < \frac{1}{r_0}, \end{aligned} \tag{1.19}$$

where $\kappa_0, r_0 > 0$ are universal constants independent of ε_{ij}. As before we will assume that the $C^{2,\alpha}$ norms of ∂D_i are under control, while ε_{ij} can become very small.

For $i \neq j$, let

$$\text{dist}(x_{ij}^i, x_{ij}^j) = \text{dist}(D_i, D_j) = \varepsilon_{ij} > 0, \ x_{ij}^i \in \partial D_i, \ x_{ij}^j \in \partial D_j,$$

and

$$x_{ij}^0 := \frac{1}{2}(x_{ij}^i + x_{ij}^j).$$

It is easy to see that there exists some positive constant δ which depends only on κ_0, r_0 and $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but is independent of $\{\varepsilon_{ij}\}$ such that any ball of radius 2δ can intersect at most two elements in $\{D_i\}$. We will only be interested in those pair i, j satisfying

$$B_{ij} := B(x_{ij}^0, \delta) \text{ intersects both } D_i \text{ and } D_j. \tag{1.20}$$

Given $\varphi \in C^2(\partial\Omega)$, consider, for $m \geqq 2$,

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i} u\right) = 0 & \text{in } \widetilde{\Omega} := \Omega \backslash \overline{\cup_{i=1}^m D_i}, \\ u|_+ = u|_- & \text{on } \partial D_i \ (i = 1, 2, \ldots, m), \\ \nabla u = 0 & \text{on } D_i \ (i = 1, 2, \ldots, m), \\ \int_{\partial D_i} a_2^{ij}(x)\partial_{x_i} u \nu_j|_+ = 0 & (i = 1, 2, \ldots, m), \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.21}$$

where $a_2^{ij}(x)$ satisfies condition (1.12). Then we have

Theorem 1.5. ([5]) *Let $\Omega, \{D_l\} \subset \mathbb{R}^n$, $\{\varepsilon_{kl}\}$ be defined as in (1.19), $n \geqq 2$, $\varphi \in C^2(\partial\Omega)$, and let $u \in H^1(\Omega) \cap C^1(\overline{\widetilde{\Omega}})$ be the solution to equation (1.21), with $u \equiv C_l$ on D_l. Then for any pair i, j satisfying (1.20),*

$$\begin{aligned} \|\nabla u\|_{L^\infty(\widetilde{\Omega} \cap B_{ij})} &\leqq C\frac{|C_i - C_j|}{\varepsilon_{ij}} + C\|\varphi\|_{C^2(\partial\Omega)} \\ &\leqq \begin{cases} \frac{C}{\sqrt{\varepsilon_{ij}}}\|\varphi\|_{C^2(\partial\Omega)} & for\ n = 2, \\ \frac{C}{\varepsilon_{ij}|\ln \varepsilon_{ij}|}\|\varphi\|_{C^2(\partial\Omega)} & for\ n = 3, \\ \frac{C}{\varepsilon_{ij}}\|\varphi\|_{C^2(\partial\Omega)} & for\ n \geqq 4, \end{cases} \end{aligned} \tag{1.22}$$

where C depends only on n, m, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\{\|\partial D_l\|_{C^{2,\alpha}}\}$, λ, Λ and $\left\{\|a_2^{kl}\|_{C^2(\overline{\widetilde{\Omega}})}\right\}$.

All our previous theorems concern perfect inclusions ($k = \infty$). Now we consider the complementary problem when $k = 0$ (insulated inclusions) in (1.2).

Let $\Omega, D_i \subset \mathbb{R}^n$, ε_{ij} be defined as in (1.19), $\varphi \in C^2(\partial\Omega)$. The insulated conductivity problem can be described as follows, for $i = 1, 2, \ldots, m$,

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i}u\right) = 0 & \text{in } \widetilde{\Omega}, \\ a_2^{ij}(x)\partial_{x_i}u\nu_j|_+ = 0 & \text{on } \partial D_i, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.23}$$

The existence and uniqueness of solutions to equation (1.23) are elementary. By the maximum principle, $\|u\|_{L^\infty(\widetilde{\Omega})} \leqq \|\varphi\|_{L^\infty(\partial\Omega)}$.

To obtain an upper bound of $\|\nabla u\|_{L^\infty(\widetilde{\Omega})}$ for solutions, we only need to consider the following local situation: For any pair i, j satisfying (1.20), consider

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i}u\right) = 0 & \text{in } \widetilde{\Omega} \cap B(x_{ij}^0, 2\delta), \\ a_2^{ij}(x)\partial_{x_i}u\nu_j|_+ = 0 & \text{on } (\partial D_i \cup \partial D_j) \cap B(x_{ij}^0, 2\delta), \\ |u| \leqq 1 & \text{in } \widetilde{\Omega} \cap B(x_{ij}^0, 2\delta). \end{cases} \tag{1.24}$$

Then we have

Theorem 1.6 ([5]). *Let Ω, $\{D_l\} \subset \mathbb{R}^n$, $\{\varepsilon_{kl}\}$ be defined as in (1.19), $n \geqq 2$. For any pair i, j satisfying (1.20), let $u \in C^1(\overline{\widetilde{\Omega} \cap B(x_{ij}^0, 2\delta)})$ be a solution to equation (1.24). Then*

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega}\cap B_{ij})} \leqq \frac{C}{\sqrt{\varepsilon_{ij}}}, \tag{1.25}$$

where the constant C depends only on n, m, κ_0, r_0, $\|\partial D_i\|_{C^{2,\alpha}}$, $\|\partial D_j\|_{C^{2,\alpha}}$, λ, Λ and $\{\|a_2^{kl}\|_{C^2(\overline{\widetilde{\Omega}})}\}$.

The paper is organized as follows. In Section 2 we prove Theorem 1.1−1.2. In Section 3 we give a criteria for $|Q_\varepsilon[\varphi]|$ to be bounded below by a positive constant independent of ε. Theorem 1.3−1.4 are proved in Section 4. In the Appendix we present some elementary results for the conductivity problem.

2. Proof of Theorem 1.1 and 1.2

In the introduction, we write $u = C_1v_1 + C_2v_2 + v_3$ as in (1.6). To prove our main theorems, we first estimate $\|\nabla u\|_{L^\infty(\widetilde{\Omega})}$ in terms of $|C_1 - C_2|$, and then estimate $|C_1 - C_2|$.

In this section we use, unless otherwise stated, C to denote various positive constants whose values may change from line to line and which depend only on n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$ and $\|\partial D_2\|_{C^{2,\alpha}}$.

Proposition 2.1. *Under the hypotheses of Theorem 1.1, let u be the solution of equation (1.4). There exists a positive constant C, such that, for sufficiently small $\varepsilon > 0$,*

$$\frac{1}{\varepsilon} \mid C_1 - C_2 \mid \leqq \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leqq \frac{C}{\varepsilon} \mid C_1 - C_2 \mid + C\|\varphi\|_{C^2(\partial\Omega)}. \tag{2.1}$$

To prove this proposition, we first estimate the gradients of v_1, v_2 and v_3. Without loss of generality, we may assume throughout the proof of the proposition that $\|\varphi\|_{C^2(\partial\Omega)} = 1$; see Remark 1.2.

Lemma 2.1. *Let v_1, v_2 be defined by equations (1.7) and (1.8), then for $n \geqq 2$, we have*

$$\|\nabla v_1\|_{L^\infty(\widetilde{\Omega})} + \|\nabla v_2\|_{L^\infty(\widetilde{\Omega})} \leqq \frac{C}{\varepsilon}, \quad \left\|\frac{\partial v_1}{\partial \nu}\right\|_{L^\infty(\partial\Omega)} + \left\|\frac{\partial v_2}{\partial \nu}\right\|_{L^\infty(\partial\Omega)} \leqq C.$$

Proof. By the maximum principle, $\|v_1\|_{L^\infty(\widetilde{\Omega})} \leqq 1$, and since v_1 achieves constants on each connected component of $\partial\widetilde{\Omega}$, and each connected component of $\partial\widetilde{\Omega}$ is $C^{2,\alpha}$ then the gradient estimates for harmonic functions implies that

$$\|\nabla v_1\|_{L^\infty(\widetilde{\Omega})} \leqq \frac{C\|v_1\|_{L^\infty}}{\mathrm{dist}(\partial D_1, \partial D_2)} = \frac{C}{\varepsilon}.$$

Similarly, we can prove $\|\nabla v_2\|_{L^\infty(\widetilde{\Omega})} \leqq C/\varepsilon$. The second inequality follows from the boundary estimates for harmonic functions. □

Before estimating $|\nabla v_3|$, we first prove:

Lemma 2.2. *Let $\rho \in C^2(\widetilde{\Omega})$ be the solution to:*

$$\begin{cases} \Delta\rho = 0 & \text{in } \widetilde{\Omega}, \\ \rho = 0 \quad \text{on } \partial D_1 \cup \partial D_2, \;\; \rho = 1 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Then $\|\nabla\rho\|_{L^\infty(\widetilde{\Omega})} \leqq C$.

Proof. Let $\rho_i (i = 1, 2) \in C^2(\Omega\backslash\overline{D}_i) \cap C^1(\overline{\Omega\backslash D_i})$ be the solution to:

$$\begin{cases} \Delta\rho_i = 0 & \text{in } \Omega\backslash\overline{D}_i, \\ \rho_i = 0 \;\text{ on } \partial D_i, & \rho_i = 1 \;\text{ on } \partial\Omega. \end{cases}$$

Again by the maximum principle and the strong maximum principle, we obtain $0 < \rho_1 < 1$ in $\Omega\backslash\overline{D}_1$. Since $\overline{D}_2 \subset \Omega\backslash\overline{D}_1$, we have $\rho_1 > 0 = \rho$ on ∂D_2. And since $\rho_1 = \rho$ on ∂D_1 and $\partial\Omega$, therefore $\rho_1 > \rho$ on $\widetilde{\Omega}$. Now because $\rho_1 = \rho = 0$ on ∂D_1 and $\rho_1 > \rho > 0$ on $\widetilde{\Omega}$, so

$$\|\nabla\rho\|_{L^\infty(\partial D_1)} \leqq \|\nabla\rho_1\|_{L^\infty(\partial D_1)} \leqq C.$$

Similarly,

$$\|\nabla\rho\|_{L^\infty(\partial D_2)} \leqq \|\nabla\rho_2\|_{L^\infty(\partial D_2)} \leqq C.$$

By the boundary estimate of harmonic functions, we know that $\|\nabla\rho\|_{L^\infty(\partial\Omega)} \leqq C$. Since $\Delta\rho = 0$ in $\widetilde{\Omega}$, $\partial_{x_i}\rho$ is also harmonic, by the maximum principle,

$$\|\nabla\rho\|_{L^\infty(\widetilde{\Omega})} \leqq \max\left(\|\nabla\rho\|_{L^\infty(\partial D_1)}, \|\nabla\rho\|_{L^\infty(\partial D_2)}, \|\nabla\rho\|_{L^\infty(\partial\Omega)}\right) \leqq C.$$

□

Now, we estimate $|\nabla v_3|$:

Lemma 2.3. *Let v_3 be defined by equation (1.9), for $n \geqq 2$, we have*

$$\|\nabla v_3\|_{L^\infty(\widetilde{\Omega})} \leqq C.$$

Proof. Since $v_3 = -\rho = \rho = 0$ on $\partial D_i (i = 1, 2)$, and $-\rho \leqq v_3 = \varphi \leqq \rho$ on $\partial\Omega$, we have, by the maximum principle,

$$-\rho \leqq v_3 \leqq \rho \quad \text{in } \widetilde{\Omega}.$$

It follows, for $i = 1, 2$, that

$$\|\nabla v_3\|_{L^\infty(\partial D_i)} \leqq \|\nabla \rho\|_{L^\infty(\partial D_i)} \leqq C.$$

By the boundary estimate,

$$\|\nabla v_3\|_{L^\infty(\partial \Omega)} \leqq C.$$

By the harmonicity of $\partial_{x_i} v_3$ and the maximum principle,

$$\|\nabla v_3\|_{L^\infty(\widetilde{\Omega})} \leqq C.$$

□

Remark 2.1. Without assuming $\|\varphi\|_{C^2(\partial\Omega)} = 1$, we have

$$\|\nabla v_3\|_{L^\infty(\partial D_1 \cup \partial D_2)} \leqq C\|\varphi\|_{L^\infty(\partial\Omega)},$$

where C has the dependence specified at the beginning of this section, except that it does not depend on $\|\partial\Omega\|_{C^{2,\alpha}}$. This is easy to see from the proof of Lemma 2.3.

The above lemma yields the main result of [2].

Corollary 2.1 ([1]). *Let B_1 and B_2 be two spheres with radius R and centered at $(\pm R \pm \frac{\varepsilon}{2}, 0, \dots, 0)$, respectively. Let H be a harmonic function in $\mathbb{R}^3$. Define u to be the solution to*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^3 \backslash \overline{B_1 \cup B_2}, \\ u = 0 & \text{on } \partial B_1 \cup \partial B_2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to +\infty. \end{cases}$$

Then there is a constant C independent of ε such that

$$\|\nabla(u - H)\|_{L^\infty(\mathbb{R}^3 \backslash \overline{B_1 \cup B_2})} \leqq C.$$

Proof. By the maximum principle and interior estimates of harmonic functions, the C^3 norm of $u|_{B_{2R}(0)}$ is bounded by a constant independent of ε. Apply Lemma 2.3 with $\Omega = B_{2R}(0)$ and $\varphi = u|_{B_{2R}(0)}$, we immediately obtain the above corollary. □

With the above lemmas, we give the

Proof of Proposition 2.1. Since $u = C_1$ on ∂D_1, $u = C_2$ on ∂D_2, and dist$(\partial D_1, \partial D_2) = \varepsilon$, by the mean value theorem, $\exists\, \xi \in \widetilde{\Omega}$ such that

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \geqq |\nabla u(\xi)| \geqq \frac{|C_1 - C_2|}{\varepsilon}.$$

By the decomposition formula (1.6),

$$\nabla u = C_1 \nabla v_1 + C_2 \nabla v_2 + \nabla v_3 = (C_1 - C_2)\nabla v_1 + C_2 \nabla(v_1 + v_2) + \nabla v_3.$$

Hence,

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leqq |C_1 - C_2| \|\nabla v_1\|_{L^\infty(\widetilde{\Omega})} + |C_2| \|\nabla(v_1 + v_2)\|_{L^\infty(\widetilde{\Omega})} + \|\nabla v_3\|_{L^\infty(\widetilde{\Omega})}.$$

By Lemma 2.2, since $v_1 + v_2 = 1 - \rho$ in $\widetilde{\Omega}$, we have

$$\|\nabla(v_1 + v_2)\|_{L^\infty(\widetilde{\Omega})} = \|\nabla(1 - \rho)\|_{L^\infty(\widetilde{\Omega})} = \|\nabla \rho\|_{L^\infty(\widetilde{\Omega})} \leqq C.$$

Using the fact we showed in the Appendix, $\|u\|_{H^1(\Omega)} \leqq C$, so $|C_1| + |C_2| \leqq C$. Therefore using also Lemma 2.1 we obtain,

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leqq \frac{C}{\varepsilon} \mid C_1 - C_2 \mid + C.$$

This proof is now completed. □

Later we will give an estimate of $|C_1 - C_2|$, which, together with Proposition 2.1, yields the lower and upper bounds of $\|\nabla u\|_{L^\infty(\widetilde{\Omega})}$ for strictly convex subdomains D_1 and D_2.

2.1. Estimate of $|C_1 - C_2|$

Back to the decomposition formula (1.6), denote

$$a_{ij} = \int_{\partial D_i} \frac{\partial v_j}{\partial \nu} \quad (i, j = 1, 2), \qquad b_i = \int_{\partial D_i} \frac{\partial v_3}{\partial \nu} \quad (i = 1, 2). \tag{2.3}$$

We first give some basic lemmas:

Lemma 2.4. *Let a_{ij} and b_i be defined as in (2.3), then they satisfy the following*:

1. $a_{12} = a_{21} > 0,\ a_{11} < 0,\ a_{22} < 0,$
2. $-C \leqq a_{11} + a_{21} \leqq -\frac{1}{C},\ -C \leqq a_{22} + a_{12} \leqq -\frac{1}{C},$
3. $|b_1| \leqq C,\ |b_2| \leqq C.$

By the fourth line of equation (1.4), C_1 and C_2 satisfy

$$\begin{cases} a_{11} C_1 + a_{12} C_2 + b_1 = 0, \\ a_{21} C_1 + a_{22} C_2 + b_2 = 0. \end{cases} \tag{2.4}$$

By solving the above linear system, using $a_{12} = a_{21}$ and $a_{11}a_{22} - a_{12}a_{21} > 0$ which follows from Lemma 2.4, we obtain

$$C_1 = \frac{-b_1 a_{22} + b_2 a_{12}}{a_{11}a_{22} - a_{12}^2}, \quad C_2 = \frac{-b_2 a_{11} + b_1 a_{12}}{a_{11}a_{22} - a_{12}^2}, \tag{2.5}$$

and therefore,

$$|C_1 - C_2| = \frac{|b_1 - \alpha b_2|}{|a_{11} - \alpha a_{12}|}, \qquad \text{where } \alpha = \frac{a_{11} + a_{12}}{a_{22} + a_{12}} > 0. \tag{2.6}$$

Based on this formula, we will give the estimates for $|a_{11} - \alpha a_{12}|$ and $|b_1 - \alpha b_2|$, then the estimate for $|C_1 - C_2|$ follows immediately.

Proof of Lemma 2.4. (1) By the maximum principle and the strong maximum principle,

$$0 < v_1 < 1 \quad \text{in } \widetilde{\Omega}.$$

By the Hopf Lemma, we know that

$$\frac{\partial v_1}{\partial \nu}\bigg|_{\partial D_1} < 0, \quad \frac{\partial v_1}{\partial \nu}\bigg|_{\partial D_2} > 0, \quad \frac{\partial v_1}{\partial \nu}\bigg|_{\partial \Omega} < 0.$$

Similarly,

$$\frac{\partial v_2}{\partial \nu}\bigg|_{\partial D_1} > 0, \quad \frac{\partial v_2}{\partial \nu}\bigg|_{\partial D_2} < 0, \quad \frac{\partial v_2}{\partial \nu}\bigg|_{\partial \Omega} < 0.$$

Thus $a_{11} < 0$, $a_{12} > 0$, $a_{21} > 0$ and $a_{22} < 0$.

Also, since v_1 and v_2 are the solutions of equations (1.7) and equations (1.8), respectively, we have

$$\begin{aligned} 0 &= \int_{\widetilde{\Omega}} \Delta v_1 \cdot v_2 - \int_{\widetilde{\Omega}} \Delta v_2 \cdot v_1 = -\int_{\partial D_2} \frac{\partial v_1}{\partial \nu} \cdot 1 + \int_{\partial D_1} \frac{\partial v_2}{\partial \nu} \cdot 1 \\ &= -a_{21} + a_{12}, \end{aligned} \tag{2.7}$$

that is $a_{21} = a_{12}$.

(2) We will prove the first inequality, the second one stands for the same reason. By the harmonicity of v_1 in $\widetilde{\Omega}$,

$$a_{11} + a_{21} = -\int_{\widetilde{\Omega}} \Delta v_1 + \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} = \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} < 0.$$

By Lemma 2.1,

$$a_{11} + a_{21} = \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} \geqq -C.$$

On the other hand, since $0 < v_1 < 1$ in $\widetilde{\Omega}$ and $v_1 = 1$ on ∂D_1, by the boundary gradient estimates of a harmonic function, $\exists\, B(\bar{x}, 2\bar{r}) \subset \widetilde{\Omega}$, such that $v_1 > 1/2$ in

$B(\bar{x}, \bar{r})$, where $\bar{r}$ is independent of ε. Let $\rho \in C^2(\Omega \backslash \overline{D_2 \cup B(\bar{x}, \bar{r})}) \cup C^1(\partial\Omega \cup \partial D_2 \cup \partial B(\bar{x}, \bar{r}))$ be the solution of the following equation:

$$\begin{cases} \Delta\rho = 0 & \text{in } \Omega \backslash \overline{D_2 \cup B(\bar{x}, \bar{r})}, \\ \rho = 1/2 \ \text{ on } \partial B(\bar{x}, \bar{r}) & \rho = 0 \ \text{ on } \partial D_2 \cup \partial\Omega. \end{cases}$$

By the maximum principle and the strong maximum principle, $0 < \rho < 1/2$ in $\Omega \backslash \overline{D_2 \cup B(\bar{x}, \bar{r})}$. A contradiction argument based on the Hopf Lemma yields,

$$-\frac{\partial\rho}{\partial\nu} \geqq \frac{1}{C} \qquad \text{on } \partial\Omega.$$

On the other hand, since $\rho \leqq v_1$ on the boundary of $\Omega \backslash \overline{D_1 \cup D_2 \cup B(\bar{x}, \bar{r})}$, we obtain, via the maximum principle, $0 < \rho \leqq v_1$ in $\Omega \backslash \overline{D_1 \cup D_2 \cup B(\bar{x}, \bar{r})}$. It follows, using $\rho = v_1 = 0$ on $\partial\Omega$, that

$$\frac{\partial v_1}{\partial\nu} \leqq \frac{\partial\rho}{\partial\nu} \qquad \text{on } \partial\Omega.$$

Thus,

$$a_{11} + a_{21} = \int_{\partial\Omega} \frac{\partial v_1}{\partial\nu} \leqq \int_{\partial\Omega} \frac{\partial\rho}{\partial\nu} \leqq -\frac{1}{C}.$$

(3) Clearly,

$$0 = \int_{\tilde{\Omega}} \Delta v_1 \cdot v_3 - \int_{\tilde{\Omega}} \Delta v_3 \cdot v_1 = \int_{\partial\Omega} \frac{\partial v_1}{\partial\nu} \cdot \varphi + \int_{\partial D_1} \frac{\partial v_3}{\partial\nu} \cdot 1 = \int_{\partial\Omega} \frac{\partial v_1}{\partial\nu} \cdot \varphi + b_1.$$

Thus,

$$|b_1| = \left| \int_{\partial\Omega} \frac{\partial v_1}{\partial\nu} \cdot \varphi \right| \leqq \int_{\partial\Omega} \left| \frac{\partial v_1}{\partial\nu} \right| \leqq C.$$

Therefore, we finished the proof. □

2.2. *Estimate of* $|a_{11} - \alpha a_{12}|$

By a translation and rotation of the axis, we may assume without loss of generality that D_1, D_2 are two strictly convex subdomains in $\Omega \subset \mathbb{R}^n$ which satisfy the following:

$$(-\varepsilon/2, 0') \in \partial D_1, \ (\varepsilon/2, 0') \in \partial D_2, \ \varepsilon = \text{dist}(D_1, D_2). \tag{2.8}$$

Near the origin, we can find a ball $B(0, r)$ such that the portion of ∂D_i $(i = 1, 2)$ in $B(0, r)$ is strictly convex, where $r > 0$ is independent of ε. Then $\partial D_1 \cap B(0, r)$ and $\partial D_2 \cap B(0, r)$ can be represented by the graph of $x_1 = f(x') - \varepsilon/2$ and $x_1 = g(x') + \varepsilon/2$ respectively, where $x' = (x_2, \dots, x_n)$. Thus $f(0') = g(0') = 0$, $\nabla f(0') = \nabla g(0') = 0$, and $-CI \leqq \left(D^2 f(0')\right) \leqq -\frac{1}{C}I$, $\frac{1}{C}I \leqq \left(D^2 g(0')\right) \leqq CI$. With these notations, we first estimate a_{ii} for $i = 1, 2$.

Lemma 2.5. *Let a_{ii} be defined by (2.3), then*

$$\frac{1}{C\sqrt{\varepsilon}} \leqq -a_{ii} \leqq \frac{C}{\sqrt{\varepsilon}}, \quad \text{for } n = 2,\ i = 1, 2.$$

Proof. It suffices to prove it for a_{11}. By the harmonicity of v_1, we have

$$0 = \int_{\widetilde{\Omega}} \Delta v_1 \cdot v_1 = -\int_{\widetilde{\Omega}} |\nabla v_1|^2 - \int_{\partial D_1} \frac{\partial v_1}{\partial \nu} = -\int_{\widetilde{\Omega}} |\nabla v_1|^2 - a_{11},$$

that is

$$a_{11} = -\int_{\widetilde{\Omega}} |\nabla v_1|^2.$$

Now we construct a function (here in $\mathbb{R}^2$, we let $x = x_1,\ y = x_2$)

$$\overline{w}(x, y) = -\frac{x - g(y) - \frac{\varepsilon}{2}}{g(y) - f(y) + \varepsilon} \tag{2.9}$$

on $O_r := \widetilde{\Omega} \cap \{(x, y) \mid |y| < r\}$. It is clear that $\overline{w}(x, y)$ is linear in x for fixed y and

$$\overline{w}\ |_{B(0,r)\cap\partial D_1} = 1; \quad \overline{w}\ |_{B(0,r)\cap\partial D_2} = 0,$$

so we have

$$\int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_x \overline{w}(x, y)|^2 \mathrm{d}x \leqq \int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_x v_1(x, y)|^2 \mathrm{d}x,$$

that is

$$\frac{1}{g(y) - f(y) + \varepsilon} \leqq \int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_x v_1(x, y)|^2.$$

Integrating on y we get

$$\begin{aligned}\int_0^{r/2} \int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_x v_1(x, y)|^2 \mathrm{d}x\mathrm{d}y &\geqq \int_0^{r/2} \frac{1}{g(y) - f(y) + \varepsilon} \mathrm{d}y \\ &\geqq \frac{1}{C} \int_0^{r/2} \frac{1}{y^2 + \varepsilon} \mathrm{d}y = \frac{1}{C\sqrt{\varepsilon}}.\end{aligned} \tag{2.10}$$

Thus

$$-a_{11} \geqq \int_0^{r/2} \int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_x v_1(x, y)|^2 \mathrm{d}x\mathrm{d}y \geqq \frac{1}{C\sqrt{\varepsilon}}.$$

On the other hand, we can find $\psi \in C^2(\overline{\Omega})$ such that

$$\begin{aligned}&\psi = 0 \text{ on } \overline{O}_{r/8}, \quad \psi = 1 \text{ on } \partial D_1 \backslash (\overline{O_{r/4}}), \quad \psi = 0 \text{ on } \partial D_2 \backslash (\overline{O_{r/4}}), \\ &\psi = 0 \text{ on } \partial\Omega, \quad \text{and} \quad \|\nabla\psi\|_{L^\infty(\Omega)} \leqq C.\end{aligned}$$

We can also find $\rho \in C^2(\overline{\Omega})$ such that

$$0 \leqq \rho \leqq 1,\ \rho = 1 \text{ on } \overline{O}_{r/2},\ \rho = 0 \text{ on } \overline{\Omega} \backslash O_r \quad \text{and} \quad |\nabla \rho| \leqq C.$$

Let $w = \rho \overline{w} + (1-\rho)\psi$, then $w = 1 = v_1$ on ∂D_1; $w = 0 = v_1$ on ∂D_2; $w = 0 = v_1$ on $\partial \Omega$ and $w = \overline{w}$ on $\overline{O}_{r/2}$. Then by the properties of ψ, ρ and the harmonicity of v_1, we have

$$\int_{\widetilde{\Omega}} |\nabla v_1|^2 \leqq \int_{\widetilde{\Omega}} |\nabla w|^2 \leqq \int_{\widetilde{\Omega} \cap O_{r/2}} |\nabla \overline{w}|^2 + C. \tag{2.11}$$

A calculation gives

$$\partial_y \overline{w} = \frac{g'(y)(g(y) - f(y) + \varepsilon) - (g(y) - x + \frac{\varepsilon}{2})(g'(y) - f'(y))}{(g(y) - f(y) + \varepsilon)^2}.$$

We will show $\int_{\widetilde{\Omega} \cap O_{r/2}} |\partial_y \overline{w}|^2 \leqq C$. Indeed,

$$\begin{aligned}
&\int_0^{r/2} \int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |\partial_y \overline{w}(x, y)|^2 \mathrm{d}x\mathrm{d}y \\
&\leqq 2 \int_0^{r/2} \int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} \\
&\quad \times \left(\frac{g'(y)^2}{(g(y) - f(y) + \varepsilon)^2} + \frac{(g(y) - x + \frac{\varepsilon}{2})^2 (g'(y) - f'(y))^2}{(g(y) - f(y) + \varepsilon)^4} \right) \mathrm{d}x\mathrm{d}y \\
&= 2 \int_0^{r/2} \frac{g'(y)^2}{g(y) - f(y) + \varepsilon} \mathrm{d}y + 2 \int_0^{r/2} \frac{(g'(y) - f'(y))^2}{g(y) - f(y) + \varepsilon} \mathrm{d}y \\
&\leqq C \int_0^{r/2} \frac{y^2}{y^2 + \varepsilon} \mathrm{d}y + C \int_0^{r/2} \frac{y^2}{y^2 + \varepsilon} \mathrm{d}y \\
&\leqq C.
\end{aligned} \tag{2.12}$$

Then by (2.11) and (2.12)

$$\begin{aligned}
|a_{11}| = \int_{\widetilde{\Omega}} |\nabla v_1|^2 &\leqq \int_{\widetilde{\Omega} \cap O_{r/2}} |\nabla \overline{w}|^2 + C \\
&\leqq C \int_0^{r/2} \int_{f(y)-\frac{\varepsilon}{2}}^{g(y)+\frac{\varepsilon}{2}} |D_x \overline{w}(x, y)|^2 \mathrm{d}x\mathrm{d}y + C \\
&= C \int_0^{r/2} \frac{1}{g(y) - f(y) + \varepsilon} \mathrm{d}y + C \leqq C \int_0^{r/2} \frac{1}{y^2 + \varepsilon} \mathrm{d}y + C \\
&\leqq \frac{C}{\sqrt{\varepsilon}}.
\end{aligned} \tag{2.13}$$

The proof is completed. □

Similarly, we have

Lemma 2.6. *Let a_{ii} be defined by* (2.3),

$$\frac{1}{C}|\ln\varepsilon| \leqq -a_{ii} \leqq C|\ln\varepsilon|, \quad for\ n=3,\ i=1,2.$$

Proof. We consider

$$\overline{w}(x_1,x') = -\frac{x-g(x')-\frac{\varepsilon}{2}}{g(x')-f(x')+\varepsilon} \tag{2.14}$$

on $O_{r/2} := \widetilde{\Omega} \cap \{(x_1,x')|\ |x'| < \frac{r}{2}\}$. Use the same proof in Lemma 2.5, we have

$$\int_0^{r/2}\int_{f(x')-\frac{\varepsilon}{2}}^{g(x')+\frac{\varepsilon}{2}} |\partial_{x'}\overline{w}(x_1,x')|^2 \mathrm{d}x_1\mathrm{d}x' \leqq C.$$

Therefore, it suffices to verify that

$$\int_{\widetilde{\Omega}\cap O_{r/2}} |\partial_{x_1}\overline{w}(x_1,x')|^2 \sim |\ln\varepsilon|.$$

Indeed,

$$\begin{aligned}\int_{\widetilde{\Omega}\cap O_{r/2}} |\partial_{x_1}\overline{w}(x_1,x')|^2 &= \int_{|x'|<r/2}\frac{1}{g(x')-f(x')+\varepsilon}\mathrm{d}x' \\ &\sim \int_0^{r/2}\frac{t}{Ct^2+\varepsilon}\mathrm{d}t \sim |\ln\varepsilon|.\end{aligned}$$

This completes the proof. □

Lemma 2.7. *Let a_{ii} be defined by* (2.3),

$$\frac{1}{C} \leqq -a_{ii} \leqq C \quad for\ n \geqq 4,\ i=1,2.$$

Proof. We only need

$$\int_{O_{r/2}} |\partial_{x_1}\overline{w}(x_1,x')|^2 = \int_{|x'|<r/2}\frac{1}{g(x')-f(x')+\varepsilon}\mathrm{d}x' \sim \int_0^{r/2}\frac{t^{n-2}}{Ct^2+\varepsilon}\mathrm{d}t \sim C.$$

The proof is completed. □

Lemma 2.8. *Let α be defined by* (2.6), *we have*

$$\frac{1}{C} \leqq \alpha \leqq C.$$

Proof. By the definition of α and using the second statement in Lemma 2.4, we are done. □

To summarize, we have

Proposition 2.2. *Let* a_{ij} *and* α *be defined by* (2.3) *and* (2.6), *we have*

1. $\frac{1}{C\sqrt{\varepsilon}} \leqq |a_{11} - \alpha a_{12}| \leqq \frac{C}{\sqrt{\varepsilon}}$ $\quad$ *for* $n = 2$,
2. $\frac{1}{C}|\ln \varepsilon| \leqq |a_{11} - \alpha a_{12}| \leqq C|\ln \varepsilon|$ $\quad$ *for* $n = 3$,
3. $\frac{1}{C} \leqq |a_{11} - \alpha a_{12}| \leqq C$ $\quad$ *for* $n \geqq 4$.

Proof. Since $a_{11} < 0$, $a_{12} > 0$, $a_{11} + a_{12} < 0$ and $\alpha > 0$, we have

$$|a_{11}| < |a_{11} - \alpha a_{12}| < (1 + \alpha)|a_{11}|.$$

Combining the results of Lemma 2.5, Lemma 2.6, Lemma 2.7 and Lemma 2.8, the proof is completed. □

2.3. Estimate of $|b_1 - \alpha b_2|$

Proposition 2.3. *Let* $b_1,\ b_2$, α *and* $Q_\varepsilon[\varphi]$ *be defined by* (2.3), (2.6) *and* (1.10), *we have*

$$\frac{|Q_\varepsilon[\varphi]|}{C} \leqq |b_1 - \alpha b_2| \leqq C\|\varphi\|_{C^2(\partial\Omega)}.$$

Proof. Combining the third result in Lemma 2.4 and Lemma 2.8, we have

$$|b_1 - \alpha b_2| \leqq |b_1| + |\alpha||b_2| \leqq C\|\varphi\|_{C^2(\partial\Omega)}.$$

On the other hand, by the definition and the harmonicity of v_1 and v_2 and using Lemma 2.4, we obtain

$$\begin{aligned} |b_1 - \alpha b_2| &= \frac{|b_1(a_{22} + a_{12}) - b_2(a_{11} + a_{12})|}{|a_{22} + a_{12}|} \\ &\geqq \frac{1}{C} \cdot \left| \int_{\partial D_1} \frac{\partial v_3}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} - \int_{\partial D_2} \frac{\partial v_3}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} \right| = \frac{|Q_\varepsilon[\varphi]|}{C}. \end{aligned}$$

This completes the proof. □

Now we are ready to prove our two main theorems:

Proof of Theorem 1.1–1.2. By Proposition 2.1 and (2.6), then using Proposition 2.2, 2.3, we are done. □

As we mentioned in Remark 1.1, the strict convexity assumption of the two inclusions can be weakened. In fact, our proofs of Theorem 1.1–1.2 apply, with minor modification, to more general situations:

In $\mathbb{R}^n$, $n \geqq 2$, under the same assumptions in the beginning of Section 1.2 except for the strict convexity condition, $\partial D_1 \cap B(0, r)$ and $\partial D_2 \cap B(0, r)$ can be represented by the graph of $x_1 = f(x') - \frac{\varepsilon}{2}$ and $x_1 = g(x') + \frac{\varepsilon}{2}$, then $f(0') = g(0') = 0$, $\nabla(g - f)(0') = 0$. Assume further that

$$\lambda_0 |x'|^{2m} \leqq g(x') - f(x') \leqq \lambda_1 |x'|^{2m}, \quad \forall |x'| \leqq r/2, \tag{2.15}$$

for some ε-independent $\lambda_0, \lambda_1 > 0$, $m \geqq 1 \in \mathbb{Z}$.

Under the above assumption, let $u \in H^1(\Omega)\cap C^1(\overline{\widetilde{\Omega}})$ be the solution to equation (1.4). For ε sufficiently small, there exist positive constants C and C', such that

$$\begin{aligned}
\frac{|Q_\varepsilon[\varphi]|}{C'}\cdot\varepsilon^{-\frac{n-1}{2m}} &\leqq \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leqq C\|\varphi\|_{C^2(\partial\Omega)}\cdot\varepsilon^{-\frac{n-1}{2m}}, && \text{if } n-1<2m,\\
\frac{|Q_\varepsilon[\varphi]|}{C'}\cdot\frac{1}{\varepsilon|\ln\varepsilon|} &\leqq \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leqq C\|\varphi\|_{C^2(\partial\Omega)}\cdot\frac{1}{\varepsilon|\ln\varepsilon|}, && \text{if } n-1=2m,\\
\frac{|Q_\varepsilon[\varphi]|}{C'}\cdot\frac{1}{\varepsilon} &\leqq \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leqq C\|\varphi\|_{C^2(\partial\Omega)}\cdot\frac{1}{\varepsilon}, && \text{if } n-1>2m,
\end{aligned} \tag{2.16}$$

where $Q_\varepsilon[\varphi]$ is defined by (1.10), and C depends on n, m, λ_0, λ_1, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$ and $\|\partial D_2\|_{C^{2,\alpha}}$, C' depends on the same as C and also $\|\varphi\|_{C^2(\partial\Omega)}$, but both are independent of ε.

The proof is essentially the same except for the computation of $\int_{\widetilde{\Omega}}|\nabla v_1|^2$. In fact,

$$\int_0^{r/2}\int_{f(x')-\frac{\varepsilon}{2}}^{g(x')+\frac{\varepsilon}{2}}|\partial_{x'}\overline{w}(x_1,x')|^2\mathrm{d}x_1\mathrm{d}x' \leqq C,$$

still holds. Then by (2.10) and (2.13) we only need to calculate

$$\int_{|x'|<r/2}\frac{1}{g(x')-f(x')+\varepsilon}\mathrm{d}x' \sim \int_0^{r/2}\frac{\rho^{n-2}}{\rho^{2m}+\varepsilon}d\rho.$$

Indeed, if $n-1<2m$,

$$\int_0^{r/2}\frac{\rho^{n-2}}{\rho^{2m}+\varepsilon}d\rho = \varepsilon^{\frac{n-1}{2m}-1}\int_0^{r/2\varepsilon^{\frac{1}{2m}}}\frac{s^{n-2}}{s^{2m}+1}\mathrm{d}s \sim C\varepsilon^{\frac{n-1}{2m}-1},$$

if $n-1=2m$,

$$\int_0^{r/2}\frac{\rho^{n-2}}{\rho^{2m}+\varepsilon}d\rho = \frac{1}{2m}\int_0^{r/2}\frac{1}{\rho^{2m}+\varepsilon}d\rho^{2m} \sim C|\ln\varepsilon|,$$

if $n-1>2m$,

$$\int_0^{r/2}\frac{\rho^{n-2}}{\rho^{2m}+\varepsilon}d\rho \sim C.$$

Therefore, we obtain (2.16) by using the same arguments in the proofs of Theorem 1.1 and Theorem 1.2.

Actually, we can replace $2m$ by any real number $\beta>0$, the results still hold.

3. Estimate of $|Q_\varepsilon[\varphi]|$

In order to identify situations when $\|\nabla u\|_{L^\infty}$ behaves exactly as the upper bound established in Theorem 1.1, we estimate in this section $|Q_\varepsilon[\varphi]|$. To emphasize the dependence on ε, we denote D_1, D_2 by $D_{1\varepsilon}$, $D_{2\varepsilon}$, denote φ by φ_ε, and denote v_1, v_2, v_3 defined by equation (1.7), (1.8), (1.9) as $v_{1\varepsilon}$, $v_{2\varepsilon}$, $v_{3\varepsilon}$. In this section we assume, in addition to the hypotheses in Theorem 1.1, that along a sequence $\varepsilon \to 0$ (we still denote it as ε), $D_{1\varepsilon} \to D_1^*$, $D_{2\varepsilon} \to D_2^*$ in $C^{2,\alpha}$ norm, $\varphi_\varepsilon \to \varphi^*$ in $C^{1,\alpha}(\partial\Omega)$. We use notation $\widetilde{\Omega}^* = \Omega\backslash\overline{D_1^* \cup D_2^*}$, and assume, without loss of generality, that $D_1^* \cap D_2^* = \{0\}$. We will show that as $\varepsilon \to 0$, $v_{i\varepsilon}$ converges, in an appropriate sense, to v_i^* which satisfies

$$\begin{cases} \Delta v_1^* = 0 & \text{in } \widetilde{\Omega}^*, \\ v_1^* = 1 \ \text{ on } \partial D_1^*\backslash\{0\}, & v_1^* = 0 \ \text{ on } \partial\Omega \cup \partial D_2^*\backslash\{0\}, \end{cases} \tag{3.1}$$

$$\begin{cases} \Delta v_2^* = 0 & \text{in } \widetilde{\Omega}^*, \\ v_2^* = 1 \ \text{ on } \partial D_2^*\backslash\{0\}, & v_2^* = 0 \ \text{ on } \partial\Omega \cup \partial D_1^*\backslash\{0\}, \end{cases} \tag{3.2}$$

$$\begin{cases} \Delta v_3^* = 0 & \text{in } \widetilde{\Omega}^*, \\ v_3^* = 0 \ \text{ on } \partial D_1^* \cup \partial D_2^*, & v_3^* = \varphi^* \ \text{ on } \partial\Omega. \end{cases} \tag{3.3}$$

First we prove

Lemma 3.1. *There exist unique* $v_i^* \in L^\infty(\widetilde{\Omega}^*) \cap C^0(\overline{\widetilde{\Omega}^*} \setminus \{0\}) \cap C^2(\widetilde{\Omega}^*)$, $i = 1, 2, 3$, *which solve equations (3.1), (3.2) and (3.3) respectively. Moreover,* $v_i^* \in C^1(\overline{\widetilde{\Omega}^*}\backslash\{0\})$.

Proof. The existence of solutions to the above equations can easily be obtained by Perron's method, see theorem 2.12 and lemma 2.13 in [8]. For the reader's convenience, we give below a simple proof of the uniqueness. We only need to prove that 0 is the only solution in $L^\infty(\widetilde{\Omega}^*) \cap C^0(\overline{\widetilde{\Omega}^*} \setminus \{0\}) \cap C^2(\widetilde{\Omega}^*)$ to the following equation:

$$\begin{cases} \Delta w = 0 & \text{in } \widetilde{\Omega}^*, \\ w = 0 & \text{on } \partial\widetilde{\Omega}^*\backslash\{0\}. \end{cases} \tag{3.4}$$

Indeed, $\forall\, \varepsilon > 0$, we have

$$|w(x)| \leqq \frac{\varepsilon^{n-2}\|w\|_{L^\infty(\widetilde{\Omega}^*)}}{|x|^{n-2}}, \quad \text{on } \partial(\widetilde{\Omega}^*\backslash B_\varepsilon(0)).$$

By the maximum principle,

$$|w(x)| \leqq \frac{\varepsilon^{n-2}\|w\|_{L^\infty(\widetilde{\Omega}^*)}}{|x|^{n-2}}, \quad \forall\, x \in \widetilde{\Omega}^*\backslash B_\varepsilon(0).$$

Thus $w \equiv 0$ in $\widetilde{\Omega}^*$. The additional regularity $v_i^* \in C^1(\overline{\widetilde{\Omega}^*} \setminus \{0\})$ follows from standard elliptic estimates and the regularity of the ∂D_i and $\partial\Omega$. $\square$

Lemma 3.2. *For $i = 1, 2, 3$,*

$$v_{i\varepsilon} \longrightarrow v_i^* \quad in \quad C^2_{loc}(\widetilde{\Omega}^*), \quad as \quad \varepsilon \to 0, \tag{3.5}$$

$$\int_{\partial\Omega} \frac{\partial v_{i\varepsilon}}{\partial \nu} \longrightarrow \int_{\partial\Omega} \frac{\partial v_i^*}{\partial \nu}, \quad as \quad \varepsilon \to 0, \quad i = 1, 2, \tag{3.6}$$

$$\int_{\partial D_{i\varepsilon}} \frac{\partial v_{3\varepsilon}}{\partial \nu} \longrightarrow \int_{\partial D_i^*} \frac{\partial v_3^*}{\partial \nu}, \quad as \quad \varepsilon \to 0. \tag{3.7}$$

Proof. By the maximum principle, $\{\|v_{i\varepsilon}\|_{L^\infty}\}$ is bounded by a constant independent of ε. By the uniqueness part of Lemma 3.1, we obtain (3.5) using standard elliptic estimates. By Lemma 2.3, $\{\|\nabla v_{3\varepsilon}\|_{L^\infty}\}$ is bounded by some constant independent of ε, so $\|\nabla v_3^*\|_{L^\infty} < \infty$. Estimate (3.6) and (3.7) follow from standard elliptic estimates. The proof is completed. □

Similar to $Q_\varepsilon[\varphi_\varepsilon]$, we define

$$Q^*[\varphi^*] := \int_{\partial D_1^*} \frac{\partial v_3^*}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_2^*}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial v_3^*}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu}, \tag{3.8}$$

then $Q^* : C^2(\partial\Omega) \mapsto \mathbb{R}$ is a linear functional. Let $Q_\varepsilon[\varphi_\varepsilon]$ and $Q^*[\varphi^*]$ be defined by equation (1.10), (3.8), then, by the above lemmas,

$$Q_\varepsilon[\varphi_\varepsilon] \longrightarrow Q^*[\varphi^*], \quad \text{as} \quad \varepsilon \to 0.$$

Corollary 3.1. *If $\varphi^* \in C^2(\partial\Omega)$ satisfies $Q^*[\varphi^*] \neq 0$, then $|Q_\varepsilon[\varphi_\varepsilon]| \geqq \frac{1}{C}$, for some positive constant C which is independent of ε.*

In the following we give some examples to show that, in general, the rates of the lower bounds established in Theorem 1.2 are optimal. Let $\Omega \subset \mathbb{R}^n$, $n \geqq 2$, be a bounded open set with $C^{2,\alpha}$ boundary, $0 < \alpha < 1$, which is symmetric with respect to x_1-variable, that is, $(x_1, x') \in \Omega$ if and only if $(-x_1, x') \in \Omega$, where $x' = (x_2, \cdots, x_n)$.

Let D_1^* be a strictly convex bounded open set in $\{(x_1, x') \in \mathbb{R}^n | x_1 < 0\}$ with $C^{2,\alpha}$ boundary, $0 < \alpha < 1$, satisfying $0 \in \partial D_1^*$ and $\overline{D_1^*} \subset \Omega$. Set $D_2^* = \{(x_1, x') \in \mathbb{R}^n | (-x_1, x') \in D_1^*\}$.

Let $\varphi \in C^2(\partial\Omega) \backslash \{0\}$ satisfy

$$\varphi_{\text{odd}}(x_1, x') := \frac{1}{2}\left[\varphi(x_1, x') - \varphi(-x_1, x')\right] \leqq 0 \text{ (or} \geqq 0), \tag{3.9}$$

on $(\partial\Omega)^+ := \{(x_1, x') \in \partial\Omega | x_1 > 0\}$.

For $\varepsilon > 0$ sufficiently small, let

$$D_{1\varepsilon} := \left\{(x_1, x') \in \Omega \,\middle|\, (x_1 + \frac{\varepsilon}{2}, x') \in D_1^*\right\},$$
$$D_{2\varepsilon} := \left\{(x_1, x') \in \Omega \,\middle|\, (x_1 - \frac{\varepsilon}{2}, x') \in D_2^*\right\},$$
$$\varphi_\varepsilon := \varphi.$$

Proposition 3.1. *Under the above assumptions, we have* $|Q_\varepsilon[\varphi]| \geqq \frac{1}{C}$, *for some positive constant* C *independent of* ε. *Consequently*,

$$\begin{array}{ll} \|\nabla u_\varepsilon\|_{L^\infty(\widetilde{\Omega})} \geqq \frac{1}{C\sqrt{\varepsilon}} & \text{for } n = 2, \\ \|\nabla u_\varepsilon\|_{L^\infty(\widetilde{\Omega})} \geqq \frac{1}{C\varepsilon|\ln\varepsilon|} & \text{for } n = 3, \\ \|\nabla u_\varepsilon\|_{L^\infty(\widetilde{\Omega})} \geqq \frac{1}{C\varepsilon} & \text{for } n \geqq 4, \end{array} \tag{3.10}$$

where u_ε *is the solution to equation* (1.4).

The above proposition can be easily obtained by the following lemma which gives a necessary and sufficient condition instead of condition (3.9) on φ for the lower bounds (3.10) to hold.

Let

$$(v_3^*)_{\text{odd}}(x_1, x') := \frac{1}{2}\left[v_3^*(x_1, x') - v_3^*(-x_1, x')\right], \tag{3.11}$$

we have

Lemma 3.3. *Under the same hypotheses in Proposition* 3.1 *except for the condition* (3.9), *let* $Q_\varepsilon[\varphi]$ *and* $(v_3^*)_{\text{odd}}(x)$ *be defined by equation* (1.10) *and* (3.11), *then the following statements are equivalent*:

1. *For some positive constant* C *independent of* ε, *we have* $|Q_\varepsilon[\varphi]| \geqq \frac{1}{C}$,
2. $\int_{\partial D_2^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu} \neq 0.$

Proof. By symmetry, the strong maximum principle and the Hopf Lemma, we can easily obtain

$$\int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu} = \int_{\partial\Omega} \frac{\partial v_2^*}{\partial \nu} < 0.$$

Then

$$\begin{aligned} Q^*[\varphi] &= \int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu}\left(\int_{\partial D_1^*} \frac{\partial v_3^*}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial v_3^*}{\partial \nu}\right) \\ &= \int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu}\left(\int_{\partial D_1^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu}\right) \\ &= -2\int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu} \int_{\partial D_2^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu}. \end{aligned}$$

Hence, $Q^*[\varphi] \neq 0$ if and only if $\int_{\partial D_2^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu} \neq 0$. Then by Corollary 3.1, we complete the proof. □

Proof of Proposition 3.1. Note that $(v_3^*)_{\text{odd}}(0, x') = 0$ by symmetry, and $(v_3^*)_{\text{odd}}$ is harmonic with $(v_3^*)_{\text{odd}} = \varphi_{\text{odd}} \leqq 0$ (or $\geqq 0$) but not identically zero on $(\partial\Omega)^+$. Now by using the strong maximum principle and the Hopf Lemma, it is clear that $\int_{\partial D_2^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu} \neq 0$, Hence, by Lemma 3.3 and Theorem 1.2, we are done. □

Remark 3.1. If $\varphi = \sum_{i=1}^{n} b_i x_i$ with $b_i \in \mathbb{R}$ and $b_1 \neq 0$, then by Proposition 3.1 we have $|Q_\varepsilon[\varphi]| \geqq \frac{1}{C}$. Therefore, by Theorem 1.1 and 1.2, the blow-up rates of $\|\nabla u\|_{L^\infty(\widetilde{\Omega})}$ are $\varepsilon^{-1/2}$ in dimension $n = 2$, $(\varepsilon|\ln\varepsilon|)^{-1}$ in dimension $n = 3$ and ε^{-1} in dimension $n \geqq 4$.

Now instead of a bounded set Ω, we consider $\mathbb{R}^n$:

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } \mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}}, \\ u_\varepsilon|_+ = u_\varepsilon|_- & \text{on } \partial D_{1\varepsilon} \cup \partial D_{2\varepsilon}, \\ \nabla u_\varepsilon \equiv 0 & \text{in } D_{1\varepsilon} \cup D_{2\varepsilon}, \\ \int_{\partial D_{i\varepsilon}} \frac{\partial u_\varepsilon}{\partial \nu}\Big|_+ = 0 & (i = 1, 2), \\ \limsup_{|x|\to\infty} \ |x|^{n-1}|u_\varepsilon(x) - H(x)| < \infty, \end{cases} \tag{3.12}$$

where $H(x)$ is a given entire harmonic function in $\mathbb{R}^n$.

we have the following result regarding the lower bound for $|\nabla u_\varepsilon|$:

Proposition 3.2. *With the same assumptions on $D_{1\varepsilon}$ and $D_{2\varepsilon}$ as in Proposition 3.1, and let $H(x)$ be an entire harmonic function in $\mathbb{R}^n$ satisfying $H_{\text{odd}}(x_1, x') := \frac{1}{2}\left[H(x_1, x') - H(-x_1, x')\right] < 0$ (or > 0) on $\mathbb{R}^n_+ := \{(x_1, x') \in \mathbb{R}^n | x_1 > 0\}$, then for some positive constant C independent of ε, we have*

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} &\geqq \frac{1}{C\sqrt{\varepsilon}} && for\ n = 2, \\ \|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} &\geqq \frac{1}{C\varepsilon|\ln\varepsilon|} && for\ n = 3, \\ \|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} &\geqq \frac{1}{C\varepsilon} && for\ n \geqq 4, \end{aligned} \tag{3.13}$$

where u_ε is the solution to equation (3.12).

Proof. Step 1. First, we show that there exists a positive constant C independent of ε, such that for any small $\varepsilon > 0$,

$$|x|^{n-1}|u_\varepsilon(x) - H(x)| \leqq C, \qquad \forall\, x \in \mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}}. \tag{3.14}$$

(i) For any bounded open set $U \subset \mathbb{R}^n$ with C^1 boundary ∂U satisfying $\partial U \cap \overline{D_{1\varepsilon} \cup D_{2\varepsilon}} = \emptyset$, we have, in view of the first and the fourth lines in (3.12),

$$\int_{\partial U} \frac{\partial u_\varepsilon}{\partial \nu} = \int_{U \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}}} \Delta u_\varepsilon = 0. \tag{3.15}$$

(ii) We show that there exists a positive constant M independent of ε, such that

$$\|u_\varepsilon - H\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} \leqq M, \qquad \forall \text{ small } \varepsilon > 0.$$

We only need to prove

$$\|u_\varepsilon - H\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} \leqq \sum_{i=1}^{2} \left(\max_{\overline{D}_{i\varepsilon}} H - \min_{\overline{D}_{i\varepsilon}} H \right). \tag{3.16}$$

Since $\nabla u_\varepsilon = 0$ in $D_{1\varepsilon} \cup D_{2\varepsilon}$, u_ε is constant on each $D_{i\varepsilon}$, denoted as $C_i(\varepsilon)$. We know that

$$\lim_{|x|\to\infty} (u_\varepsilon(x) - H(x)) = 0, \tag{3.17}$$

and

$$C_i(\varepsilon) - \max_{\overline{D}_{i\varepsilon}} H \leqq u_\varepsilon - H \leqq C_i(\varepsilon) - \min_{\overline{D}_{i\varepsilon}} H, \quad \text{on } D_{i\varepsilon}, \quad i = 1, 2. \tag{3.18}$$

If (3.16) did not hold, say,

$$\sup_{\mathbb{R}^n}(u_\varepsilon - H) > \sum_{i=1}^{2}\left(\max_{\overline{D}_{i\varepsilon}} H - \min_{\overline{D}_{i\varepsilon}} H\right),$$

then, because of (3.17) and (3.18), there would exist $0 < a < \sup_{\mathbb{R}^n}(u_\varepsilon - H)$ such that $U := \{x \in \mathbb{R}^n \mid (u_\varepsilon - H)(x) > a\} \neq \emptyset$ satisfies $\partial U \cap \overline{D_{1\varepsilon} \cup D_{2\varepsilon}} = \emptyset$. We may assume, by the Sard theorem, that a is a regular value of $u_\varepsilon - H$, and therefore ∂U is C^1. By the Hopf lemma, $\frac{\partial(u_\varepsilon - H)}{\partial \nu} < 0$ on ∂U, and therefore

$$\int_{\partial U} \frac{\partial(u_\varepsilon - H)}{\partial \nu} < 0.$$

On the other hand, using (3.15) and the harmonicity of H in U, we have

$$\int_{\partial U} \frac{\partial(u_\varepsilon - H)}{\partial \nu} = -\int_{\partial U} \frac{\partial H}{\partial \nu} = -\int_U \Delta H = 0.$$

A contradiction.

(iii) Consider $w_\varepsilon(x) := u_\varepsilon(x) - H(x)$. Fix a constant $R_0 > 0$, independent of ε, such that $D_1^* \cup D_2^* \subset B_{R_0/2}(0)$, and let

$$\widetilde{w_\varepsilon}(y) := \frac{1}{|y|^{n-2}} w_\varepsilon\left(\frac{y}{|y|^2}\right), \quad 0 < |y| < \frac{1}{R_0}.$$

Then $\widetilde{w_\varepsilon}$ is harmonic in $B_{1/R_0}\setminus\{0\}$. By the last line of (3.12), there exists a positive constant $C(\varepsilon)$ such that

$$|\widetilde{w_\varepsilon}(y)| \leqq C(\varepsilon)|y|, \quad 0 < |y| < \frac{1}{R_0}.$$

Therefore, $\Delta\widetilde{w_\varepsilon} = 0$ in B_{1/R_0} and $\widetilde{w_\varepsilon}(0) = 0$. By (ii), we have $|\widetilde{w_\varepsilon}| \leqq C$, on $\partial B_{1/R_0}$, for some positive constant C independent of ε. Hence, $|\widetilde{w_\varepsilon}| \leqq C$, $|\nabla\widetilde{w_\varepsilon}| \leqq C$ in $B_{1/(2R_0)}$, then

$$|\widetilde{w_\varepsilon}(y)| \leqq C|y|, \quad |y| < \frac{1}{2R_0}.$$

Therefore, also using (ii), (3.14) holds.

Step 2. For $R > R_0$, let $\Omega = B_R(0)$. Let $\varphi_\varepsilon := u_\varepsilon|_{\partial\Omega}$, then by Corollary 3.1 and Theorem 1.2 it is enough to show, for some R, that $Q^*[\varphi^*] \neq 0$, where φ^* is defined at the beginning of this section. By symmetry, we have

$$Q^*[\varphi^*] = \int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu}\left(\int_{\partial D_1^*} \frac{\partial v_3^*}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial v_3^*}{\partial \nu}\right).$$

Without loss of generality, we may assume $H_{\text{odd}}(x) > 0$ on $\mathbb{R}^n_+$. Recall that v_3^* is the solution of (3.3) with boundary data φ^*. In the following we use notation $(v_3^*)_h$ to denote the solution of (3.3) with boundary data h. Since $Q^*[\varphi^*]$ is linear on φ^* and by symmetry $Q^*[H_{\text{even}}] = H[\varphi^*_{\text{even}}] = 0$, where $H_{\text{even}}(x) := H(x) - H_{\text{odd}}(x) = \frac{1}{2}\left[H(x_1, x') + H(-x_1, x')\right]$ and similar for φ^*_{even}, we may assume $H(x) = H_{\text{odd}}(x)$.

Now consider $w(x) = H(x) - (v_3^*)_H(x)$. Then $w(x)$ is harmonic in $\widetilde{\Omega}^*$ which is defined at the beginning of this section. By symmetry, $w(-x_1, x') = -w(x_1, x')$, $w(x) = H(x)$ on $\partial D_1^* \cup \partial D_2^*$ and $w(x) = 0$ on $\partial\Omega$. Therefore,

$$-2\int_{\partial D_2^*} H\frac{\partial w}{\partial \nu} = \int_{\widetilde{\Omega}^*} w(x)\Delta w(x) + \int_{\widetilde{\Omega}^*} |\nabla w|^2 = \int_{\widetilde{\Omega}^*} |\nabla w|^2 \geqq 0.$$

On the other hand, $(v_3^*)_H = 0$ on ∂D_2^*, $(v_3^*)_H > 0$ on $(\partial\Omega)^+$ and, by the oddness of $(v_3^*)_H$, $(v_3^*)_H = 0$ on $\{(x_1, x') \mid x_1 = 0\}$. Thus, by the maximum principle and the strong maximum principle, $(v_3^*)_H > 0$ in $\widetilde{\Omega}^*$ and in turn, using the Hopf lemma, $\dfrac{\partial (v_3^*)_H}{\partial \nu} > 0$ on ∂D_2^*. Hence, using the harmonicity of H,

$$\begin{aligned}\max_{\partial D_2^*} H \int_{\partial D_2^*} \frac{\partial (v_3^*)_H}{\partial \nu} &\geqq \int_{\partial D_2^*} H\frac{\partial (v_3^*)_H}{\partial \nu} \geqq \int_{\partial D_2^*} H\frac{\partial H}{\partial \nu} - \int_{\partial D_2^*} H\frac{\partial w}{\partial \nu}\\ &\geqq \int_{D_2^*} |\nabla H|^2 \geqq \frac{1}{C},\end{aligned}$$

Therefore,

$$\int_{\partial D_2^*} \frac{\partial (v_3^*)_H}{\partial \nu} \geqq \frac{1}{C},$$

for positive constant C independent of R.

For $s_\varepsilon := \varphi_\varepsilon - H$ on $\partial\Omega$, by step 1, there exists a constant $C > 0$ which is independent of ε and R, such that $\|s_\varepsilon\|_{L^\infty(\partial\Omega)} \leqq CR^{1-n}$. By Remark 2.1, we have $\|\nabla (v_3^*)_{s^*}\|_{L^\infty(\partial D_1^* \cup \partial D_2^*)} \leqq C\|s^*\|_{L^\infty(\partial\Omega)}$, thus,

$$\left|\int_{\partial D_i^*} \frac{\partial (v_3^*)_{s^*}}{\partial \nu}\right| \leqq C\int_{\partial D_i^*} \|s^*\|_{L^\infty(\partial\Omega)} \leqq CR^{1-n},$$

for some positive constant C independent of ε and R.

Therefore, for large enough R,

$$\int_{\partial D_2^*} \frac{\partial (v_3^*)_{\varphi^*}}{\partial \nu} = \int_{\partial D_2^*} \frac{\partial (v_3^*)_H}{\partial \nu} + \int_{\partial D_2^*} \frac{\partial (v_3^*)_{s^*}}{\partial \nu} \geqq \frac{1}{C} \neq 0.$$

It is also clear that $\int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu} < 0$, Thus,

$$Q^*[\varphi^*] = -2\int_{\partial\Omega} \frac{\partial v_1^*}{\partial \nu} \int_{\partial D_2^*} \frac{\partial (v_3^*)_{\varphi^*}}{\partial \nu} \neq 0.$$

This proof is completed. □

Remark 3.2. In $\mathbb{R}^2$, when $D_{1\varepsilon}$ and $D_{2\varepsilon}$ are identical balls of radius 1, the estimate (3.13) was established in [2] under a weaker assumption $\partial_{x_1} H(0) \neq 0$.

4. Proof of Theorem 1.3 and 1.4

In the introduction, similar to the harmonic case, we still decompose $u = C_1 V_1 + C_2 V_2 + V_3$ as in (1.14). Proposition 2.1 holds since Lemma 2.1–2.3 hold for V_1, V_2, V_3 defined by (1.15)–(1.17) and $\rho \in C^2(\widetilde{\Omega})$ which is the solution to:

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i}\rho\right) = 0 \quad \text{in } \widetilde{\Omega}, \\ \rho = 0 \ \text{ on } \partial D_1 \cup \partial D_2, \quad \rho = 1 \ \text{ on } \partial\Omega. \end{cases}$$

The proofs are essentially the same. Now we start to estimate $|C_1 - C_2|$. By the decomposition formula (1.14), instead of (2.3), we denote

$$\begin{aligned} a_{lm} &= \int_{\partial D_l} a_2^{ij}(x)\,\partial_{x_i} V_m\,\nu_j \quad (l, m = 1, 2), \\ b_l &= \int_{\partial D_l} a_2^{ij}(x)\,\partial_{x_i} V_3\,\nu_j \quad (l = 1, 2). \end{aligned} \tag{4.1}$$

Then Lemma 2.4 and (2.4)–(2.6) still hold for a_{lm} and b_l defined above. In fact, to prove Lemma 2.4 with general coefficients, we only need to change $\frac{\partial *}{\partial \nu}$ to $a_2^{ij}(x)\,\partial_{x_i} * \nu_j$, change $\Delta *$ in $\partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i} *\right)$ and change v_1, v_2, v_3 in V_1, V_2, V_3, respectively, in the original proof of Lemma 2.4. For instance, (2.7) is changed to

$$\begin{aligned} 0 &= \int_{\widetilde{\Omega}} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i} V_1\right)\cdot V_2 - \int_{\widetilde{\Omega}} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i} V_2\right)\cdot V_1 \\ &= -\int_{\partial D_2} a_2^{ij}(x)\,\partial_{x_i} V_1\,\nu_j \cdot 1 + \int_{\partial D_1} a_2^{ij}(x)\,\partial_{x_i} V_2\,\nu_j \cdot 1 \\ &= -a_{21} + a_{12}. \end{aligned} \tag{4.2}$$

Therefore, to estimate $|C_1 - C_2|$, it is equivalent to estimating $|a_{11} - \alpha a_{12}|$ and $|b_1 - \alpha b_2|$. For $|a_{11} - \alpha a_{12}|$, Lemma 2.5–2.7 still hold for $a_{ll}(l = 1, 2)$ defined by (4.1). The proof is quite similar and the only thing which needs to be shown is the following:

$$\begin{aligned} 0 &= \int_{\widetilde{\Omega}} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i} V_1\right)\cdot V_1 \\ &= -\int_{\widetilde{\Omega}} a_2^{ij}(x)\,\partial_{x_i} V_1 \partial_{x_j} V_1 - \int_{\partial D_1} a_2^{ij}(x)\,\partial_{x_i} V_1\,\nu_j \cdot 1 \\ &= -\int_{\widetilde{\Omega}} a_2^{ij}(x)\,\partial_{x_i} V_1 \partial_{x_j} V_1 - a_{11}, \end{aligned}$$

that is

$$a_{11} = -\int_{\widetilde{\Omega}} a_2^{ij}(x)\,\partial_{x_i} V_1 \partial_{x_j} V_1.$$

Then by the uniform ellipticity of $a_2^{ij}(x)$ and the harmonicity of v_1,

$$|a_{11}| \geqq \lambda \int_{\widetilde{\Omega}} |\nabla V_1|^2 \geqq \lambda \int_{\widetilde{\Omega}} |\nabla v_1|^2,$$

and

$$|a_{11}| \leqq \int_{\widetilde{\Omega}} a_2^{ij}(x)\, \partial_{x_i} w \partial_{x_j} w \leqq \Lambda \int_{\widetilde{\Omega}} |\nabla w|^2 \leqq \Lambda \int_{\widetilde{\Omega} \cap O_{r/2}} |\nabla \overline{w}|^2 + C,$$

where w is defined in the proof of Lemma 2.5 with the same boundary data of V_1 and $\overline{w}$ is defined by (2.9) and (2.14). Thus, Lemma 2.5–2.7 follow by the same computations. Then Lemma 2.8 and Proposition 2.2 hold with the same proofs. For $|b_1 - \alpha b_2|$, Proposition 2.3 also holds for $b_l (l = 1, 2)$ defined by (4.1) and $Q_\varepsilon[\varphi]$ defined by (1.18). The proof is the same after changing $\frac{\partial *}{\partial \nu}$ to $a_2^{ij}(x)\, \partial_{x_i} * \nu_j$. Combining the above propositions, we obtain our theorems.

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5. Appendix: Some elementary results for the conductivity problem

Assume that in $\mathbb{R}^n$, Ω and ω are bounded open sets with $C^{2,\alpha}$ boundaries, $0 < \alpha < 1$, satisfying

$$\overline{\omega} = \bigcup_{s=1}^{m} \overline{\omega}_s \subset \Omega,$$

where $\{\omega_s\}$ are connected components of ω. Clearly, $m < \infty$ and ω_s is open for all $1 \leqq s \leqq \omega$. Given $\varphi \in C^2(\partial\Omega)$, the conductivity problem we consider is the following transmission problem with Dirichlet boundary condition:

$$\begin{cases} \partial_{x_j} \left\{ \left[\left(k a_1^{ij}(x) - a_2^{ij}(x) \right) \chi_\omega + a_2^{ij}(x) \right] \partial_{x_i} u_k \right\} = 0 & \text{in } \Omega, \\ u_k = \varphi & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $k = 1, 2, 3, \ldots$, and χ_ω is the characteristic function of ω.

The $n \times n$ matrices $A_1(x) := \left(a_1^{ij}(x) \right)$ in ω, $A_2(x) := \left(a_2^{ij}(x) \right)$ in $\Omega \backslash \overline{\omega}$ are symmetric and $\exists$ a constant $\Lambda \geqq \lambda > 0$ such that

$$\lambda |\xi|^2 \leqq a_1^{ij}(x) \xi_i \xi_j \leqq \Lambda |\xi|^2 \ (\forall x \in \omega), \quad \lambda |\xi|^2 \leqq a_2^{ij}(x) \xi_i \xi_j \leqq \Lambda |\xi|^2 \ (\forall x \in \Omega \backslash \omega)$$

for all $\xi \in \mathbb{R}^n$ and $a_1^{ij}(x) \in C^2(\overline{\omega})$, $a_2^{ij}(x) \in C^2(\overline{\Omega} \backslash \omega)$.

Equation (5.1) can be rewritten in the following form to emphasize the transmission condition on $\partial\omega$:

$$\begin{cases} \partial_{x_j} \left(a_1^{ij}(x)\, \partial_{x_i} u_k \right) = 0 & \text{in } \omega, \\ \partial_{x_j} \left(a_2^{ij}(x)\, \partial_{x_i} u_k \right) = 0 & \text{in } \Omega \backslash \overline{\omega}, \\ u_k|_+ = u_k|_-, & \text{on } \partial\omega, \\ a_2^{ij}(x) \partial_{x_i} u_k \nu_j \Big|_+ = k a_1^{ij}(x) \partial_{x_i} u_k \nu_j \Big|_- & \text{on } \partial\omega, \\ u_k = \varphi & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Here and throughout this paper ν is the outward unit normal and the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. We list the following results which are well known and omit the proofs.

Theorem 5.1. *If $u_k \in H^1(\Omega)$ is a solution of equation (5.1), then $u_k \in C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$ and satisfies equation (5.2).*

If $u_k \in C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$ is a solution of equation (5.2), then $u_k \in H^1(\Omega)$ and satisfies equation (5.1).

Theorem 5.2. *There exists at most one solution $u_k \in H^1(\Omega)$ to equation (5.1).*

The existence of the solution can be obtained by using the variational method. For every k, we define the energy functional

$$I_k[v] := \frac{k}{2}\int_\omega a_1^{ij}(x)\partial_{x_i}v\partial_{x_j}v + \frac{1}{2}\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}v\partial_{x_j}v, \tag{5.3}$$

where v belongs to the set

$$H^1_\varphi(\Omega) := \{v \in H^1(\Omega) |\ v = \varphi \ \text{ on } \partial\Omega\}.$$

Theorem 5.3. *For every k, there exists a minimizer $u_k \in H^1(\Omega)$ satisfying*

$$I_k[u_k] = \min_{v\in H^1_\varphi(\Omega)} I_k[v].$$

Moreover, $u_k \in H^1(\Omega)$ is a solution of equation (5.1).

Comparing equation (5.2), when $k = +\infty$, the perfectly conducting problem turns out to be:

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}u\right) = 0 & \text{in } \Omega\backslash\overline{\omega}, \\ u|_+ = u|_- & \text{on } \partial\omega, \\ \nabla u = 0 & \text{in } \omega, \\ \displaystyle\int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\ |_+ = 0 & (s = 1, 2, \cdots, m), \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

We also have similar results:

Theorem 5.4. *If $u \in H^1(\Omega)$ satisfies equation (5.4) except for the fourth line, then $u \in C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$.*

Proof. By the third line of equation (5.4), we have $u \equiv$ const on each component of ω, so $u \equiv$ const on each component of $\partial\omega$. Thus $u \equiv$ const on each component of $\partial(\Omega\backslash\overline{\omega})$.

Since $u \in H^1(\Omega)$ satisfies $\partial_{x_i}\left(a_2^{ij}(x)\ \partial_{x_i}u_k\right) = 0$ in $\Omega\backslash\overline{\omega}$, $u|_{\partial\Omega} = \varphi \in C^2(\partial\Omega)$ and $u \equiv$ const on each component of $\partial(\Omega\backslash\overline{\omega})$, by the elliptic regularity theory, we have $u \in C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$. □

Theorem 5.5. *There exists at most one solution $u \in H^1(\Omega) \cap C^1(\overline{\Omega\backslash\omega}) \cap C^1(\overline{\omega})$ of equation (5.4).*

Proof. It is equivalent to showing that if $\varphi = 0$, equation (5.4) only has the solution $u \equiv 0$. Integrating by parts in the first line of equation (5.4), we have

$$\begin{aligned}
0 &= -\int_{\Omega\setminus\overline{\omega}} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i}u_k\right)\cdot u \\
&= \int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\partial_{x_j}u - \int_{\partial\Omega} u\cdot a_2^{ij}(x)\partial_{x_i}u\nu_j\Big|_- + \int_{\partial\omega} u\cdot a_2^{ij}(x)\partial_{x_i}u\nu_j\Big|_+ \\
&\geqq \lambda\int_{\Omega\setminus\overline{\omega}} |\nabla u|^2 - \int_{\partial\Omega} \varphi\cdot a_2^{ij}(x)\partial_{x_i}u\nu_j\Big|_- + C_s\int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\Big|_+ \\
&= \lambda\int_{\Omega\setminus\overline{\omega}} |\nabla u|^2.
\end{aligned}$$

Thus $\nabla u = 0$ in $\Omega\setminus\overline{\omega}$. And since $u = \varphi = 0$ on $\partial\Omega$, we have $u \equiv 0$ in $\Omega\setminus\overline{\omega}$. Since $u|_+ = u|_-$ on $\partial\omega$ and $u \equiv C$ on $\overline{\omega}$, we get $u = 0$ on $\overline{\omega}$. Hence $u \equiv 0$ in Ω, that is $u \equiv 0$ is the only solution of (5.4) when $\varphi = 0$. □

Define the energy functional

$$I_\infty[v] := \frac{1}{2}\int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i}v\partial_{x_j}v, \tag{5.5}$$

where v belongs to the set

$$\mathscr{A} := \left\{v \in H_\varphi^1(\Omega)|\nabla v \equiv 0 \text{ in } \omega\right\}.$$

Theorem 5.6. *There exists a minimizer $u \in \mathscr{A}$ satisfying*

$$I_\infty[u] = \min_{v\in\mathscr{A}} I_\infty[v].$$

Moreover, $u \in H^1(\Omega)\cap C^1(\overline{\Omega\setminus\omega})\cap C^1(\overline{\omega})$ is a solution of equation (5.4).

Proof. By the lower-semi continuity of I_∞ and the weakly closed property of $\mathscr{A}$, it is easy to see that the minimizer $u \in \mathcal{A}$ exists and satisfies $\partial_{x_j}\left(a_2^{ij}(x)\partial_{x_i}u\right) = 0$ in $\Omega\setminus\overline{\omega}$. The only thing which needs to be shown is the fourth line in equation (5.4), that is

$$\int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\,|_+ = 0, \quad s = 1, 2, \cdots, m\ .$$

In fact, since u is a minimizer, for any $\phi \in C_c^\infty(\Omega)$ satisfying $\phi \equiv 1$ on $\overline{\omega}_s$ and $\phi \equiv 0$ on $\overline{\omega}_t (t \neq s)$, let

$$i(t) := I_\infty[u + t\phi] \qquad (t \in \mathbb{R}),$$

we have

$$i'(0) := \frac{di}{dt}\Big|_{t=0} = \int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\phi_{x_j} = 0\ .$$

Therefore

$$\begin{aligned}0 &= -\int_{\Omega\setminus\overline{\omega}} \partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i}u_k\right)\phi\\ &= \int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\phi_{x_j} + \int_{\partial\omega_s}\phi\cdot a_2^{ij}(x)\partial_{x_i}u\nu_j\,|_+\\ &= \int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\,|_+,\end{aligned}$$

for all $s = 1, 2, \cdots, m$. □

Finally, we give the relationship between u_k and u.

Theorem 5.7. *Let u_k and u in $H^1(\Omega)$ be the solutions of equations* (5.2) *and* (5.4), *respectively. Then*

$$u_k \rightharpoonup u \;\; in \; H^1(\Omega), \quad as\; k \to +\infty,$$

and

$$\lim_{k\to+\infty} I_k[u_k] = I_\infty[u],$$

where I_k and I_∞ are defined as (5.3) *and* (5.5).

Proof. Step 1. By the uniqueness of the solution to equation (5.4), we only need to show that there exists a weak limit u of a subsequence of $\{u_k\}$ in $H^1(\Omega)$ and u is the solution of equation (5.4).

(1) To show that after passing to a subsequence, u_k weakly converges in $H^1(\Omega)$ to some u.

Let $\eta \in H^1_\varphi(\Omega)$ be fixed and satisfy $\eta \equiv 0$ on $\overline{\omega}$, then since u_k is the minimizer of I_k in $H^1_\varphi(\Omega)$, we have

$$\frac{\lambda}{2}\|\nabla u_k\|^2_{L^2(\Omega)} \leqq I_k[u_k] \leqq I_k[\eta] = \frac{1}{2}\int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\eta_{x_i}\eta_{x_j} \leqq \frac{\Lambda}{2}\|\eta\|^2_{H^1(\Omega)},$$

that is

$$\|\nabla u_k\|_{L^2(\Omega)} \leqq \|\eta\|_{H^1(\Omega)} \doteq \overline{M},$$

where $\overline{M}$ is independent of k.
Since $u_k = \varphi$ on $\partial\Omega$ and $\sup_k \|u_k\|_{H^1(\Omega)} < \infty$, we have $u_k \rightharpoonup u \;\; in \;\; H^1_\varphi(\Omega)$.

(2) To show that u is a solution of equation (5.4).

In fact, we only need to prove the following three conditions:

$$\partial_{x_j}\left(a_2^{ij}(x)\,\partial_{x_i}u\right) = 0 \qquad \text{in } \Omega\setminus\overline{\omega}, \tag{5.6}$$

$$\nabla u = 0 \qquad \text{in } \omega, \tag{5.7}$$

$$\int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\,|_+ = 0, \qquad s = 1, 2, \ldots, m. \tag{5.8}$$

(i) For every k, since $u_k \in H^1(\Omega)$ is the solution of equation (5.1), then $\forall\, \phi \in C_c^\infty(\Omega)$, we have

$$k\int_\omega a_1^{ij}(x)\partial_{x_i}u_k\phi_{x_j} + \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\phi_{x_j} = 0.$$

Thus, $\forall\, \phi \in C_c^\infty(\Omega\backslash\overline{\omega}) \subset C_c^\infty(\Omega)$,

$$0 = \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\phi_{x_j} \longrightarrow \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\phi_{x_j},$$

since $u_k \rightharpoonup u \;\; in \;\; H_\varphi^1(\Omega) \subset H^1(\Omega)$.
Therefore,

$$\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\phi_{x_j} = 0, \quad \forall\, \phi \in C_c^\infty(\Omega\backslash\overline{\omega}),$$

that is (5.6).

(ii) Let $\eta \in H_\varphi^1(\Omega)$ be fixed and satisfy $\eta \equiv 0$ on $\overline{\omega}$, then since u_k is the minimizer of I_k in $H_\varphi^1(\Omega)$, we have

$$\frac{k\lambda}{2}\|\nabla u_k\|_{L^2(\omega)}^2 \leqq I_k[u_k] \leqq I_k[\eta] = \frac{1}{2}\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}\eta\partial_{x_j}\eta \leqq \frac{\Lambda}{2}\|\eta\|_{H^1(\Omega)}^2,$$

which implies

$$\|\nabla u_k\|_{L^2(\omega)}^2 \to 0, \quad \text{as } k \to \infty.$$

By (1), since $u_k \rightharpoonup u$ in $H^1(\Omega)$, then $u_k \rightharpoonup u$ in $H^1(\omega)$. Therefore, by the lower-semi continuity, we get

$$\begin{aligned} 0 \leqq \lambda\int_\omega |\nabla u|^2 &\leqq \int_\omega a_1^{ij}(x)\partial_{x_i}u\partial_{x_j}u \leqq \int_\omega a_1^{ij}(x)\partial_{x_i}u_k\partial_{x_j}u_k \\ &\leqq \Lambda\|\nabla u_k\|_{L^2(\omega)}^2 \longrightarrow 0, \qquad as \;\; k \longrightarrow \infty. \end{aligned}$$

Hence, $\int_\omega |\nabla u|^2 = 0 \implies \nabla u \equiv 0$ in ω, which is just (5.7).

(iii) By (i) and (ii), u satisfies (5.6) and is either constant or φ on each component of $\partial(\Omega\backslash\overline{\omega})$. Thus, $u \in C^2(\overline{\Omega\backslash\omega})$. For each $s = 1, 2, \ldots, m$, we construct a function $\rho \in C^2(\overline{\Omega\backslash\omega})$, such that $\rho = 1$ on $\partial\omega_s$, $\rho = 0$ on $\partial\omega_t (t \neq s)$, and $\rho = 0$ on $\partial\Omega$. By Green's Identity, we have the following:

$$\begin{aligned} 0 &= -\int_{\Omega\backslash\overline{\omega}} \partial_{x_j}\left(a_2^{ij}(x)\;\partial_{x_i}u_k\right)\rho \\ &= \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\partial_{x_j}\rho - \int_{\partial\Omega} \rho a_2^{ij}(x)\partial_{x_i}u_k\nu_j\Big|_- + \int_{\partial\omega} \rho a_2^{ij}(x)\partial_{x_i}u_k\nu_j\Big|_+ \\ &= \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\partial_{x_j}\rho + k\int_{\partial\omega_s} a_1^{ij}(x)\partial_{x_i}u_k\nu_j\,|_- \\ &= \int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\partial_{x_j}\rho. \end{aligned}$$

Similarly,

$$0 = -\int_{\Omega\setminus\overline{\omega}} \partial_{x_j}\left(a_2^{ij}(x)\partial_{x_i}u\right)\rho = \int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\partial_{x_j}\rho + \int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\,|_+.$$

Since $u_k \rightharpoonup u$ in $H^1(\Omega)$, it follows

$$0 = \int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u_k\partial_{x_j}\rho \longrightarrow \int_{\Omega\setminus\overline{\omega}} a_2^{ij}(x)\partial_{x_i}u\partial_{x_j}\rho.$$

Thus,

$$\int_{\partial\omega_s} a_2^{ij}(x)\partial_{x_i}u\nu_j\,|_+ = 0,$$

for any $s = 1, 2, \ldots, m$. Therefore, we finish the proof of the first part.

Step 2. Since u_k is a minimizer of I_k and $\nabla u = 0$ in ω, for any $k \in \mathbb{N}$,

$$I_k[u_k] \leqq I_k[u] = I_\infty[u].$$

Then $\limsup_{k\to+\infty} I_k[u_k] \leqq I_\infty[u]$.

On the other hand, by Theorem 5.7, since u is the weak limit of $\{u_k\}$ in $H^1(\Omega)$, we obtain

$$I_\infty[u] = \int_\Omega a_2^{ij}(x)\partial_{x_i}u\partial_{x_j}u \leqq \liminf_{k\to+\infty}\int_\Omega a_2^{ij}(x)\partial_{x_i}u_k\partial_{x_j}u_k \leqq \liminf_{k\to+\infty} I_k[u_k].$$

Therefore,

$$\lim_{k\to+\infty} I_k[u_k] = I_\infty[u].$$

The proof is completed. □

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