Starshaped Compact Hypersurfaces With Prescribed m-th Mean Curvature in Elliptic Space

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Abstract

We consider the problem of finding a compact starshaped hypresurface in a space form for which the normalized m-th elementary symmetric function of principal curvatures is a prescribed function. In this paper conditions for existence of at least one solution to a nonlinear second order elliptic equation of that problem are established in case of a space form with positive sectional curvature.

1 Introduction

Let $\mathcal{R}^{n+1}(1)$, $n \geq 2$, be a space form of sectional curvature 1 and m an integer, $1 \leq m \leq n$. In this paper we establish conditions for existence of a smooth hypersurface M in $\mathcal{R}^{n+1}(1)$ which is starshaped relative to some point $\mathbf{O}$ and whose m-th mean curvature $H_m = \psi_{|M}$, where ψ is a given function in $\mathcal{R}^{n+1}(1)$. Here, by the m-th mean curvature we understand the normalized elementary symmetric function of order m of principal curvatures $\lambda_1, ..., \lambda_n$ of M, that is,

$$H_m = \frac{1}{\binom{n}{m}} \sum_{i_1 < ... < i_n} \lambda_{i_1} \cdots \lambda_{i_m}.$$

The proof of the main result uses a priori estimates obtained in preceding paper [1] and degree theory for nonlinear elliptic partial differential equations developed by Yan Yan Li [6]. We refer the reader to [1] for the introductory material, including derivation of the required partial diffrential equations, and some history of the problem.

We now state the main result of this paper. First, we describe in a convenient form the Riemannian space $\mathcal{R}^{n+1}(1)$. Let S^{n+1} be a unit sphere in Euclidean space R^{n+2} and h the standard metric on S^{n+1} induced from R^{n+2}. Let $\mathbf{O}$ be a point in S^{n+1}, S^{n+1}_+ the open hemisphere with the pole $\mathbf{O}$, and $T_{\mathbf{O}}$ the hyperplane tangent to S^{n+1} at $\mathbf{O}$. In a natural way $T_{\mathbf{O}}$ can be identified with the usual Euclidean space R^{n+1} with a Cartesian coordinate system $x = (x_1, ..., x_{n+1})$ with origin at $\mathbf{O}$. Using the inverse of the exponential map from $T_{\mathbf{O}}$ to S^{n+1}_+, we may pull the metric h from S^{n+1}_+ to an open ball $x_1^2 + ... + x_{n+1}^2 < \pi/2$ $(= B^{n+1})$

[1]Work partially supported by NSF grant DMS-0100819
[2]Supported by Emory University Research Committee and Emory-TU Berlin Exchange Program.

in $T_{\mathbf{O}}$ with center at $\mathbf{O}$. The space (B^{n+1}, h) is the $\mathcal{R}^{n+1}(1)$. Obviously, it is isometric to S^{n+1}_+.

Introduce in $\mathcal{R}^{n+1}(1)$ polar coordinates (u, ρ), where for a point $x \in \mathcal{R}^{n+1}(1)$ ρ is the geodesic distance from $\mathbf{O}$ to x and u is a point on a standard unit sphere S^n in R^{n+1} centered at $\mathbf{O}$ defining the direction of the geodesic from $\mathbf{O}$ to x.

The metric h in these coordinates is given by

$$h = d\rho^2 + \sin^2 \rho e, \quad 0 \leq \rho < \pi/2, \tag{1}$$

where e is the standard metric on the unit sphere S^n induced from R^{n+1}.

We consider smooth hypersurfaces in $\mathcal{R}^{n+1}(1)$ which are starshaped relative to the origin $\mathbf{O}$ and do not pass through $\mathbf{O}$, that is, such hypersurfaces are radial graphs over the sphere S^n in $\mathcal{R}^{n+1}(1)$ of positive smooth functions $z(u), u \in S^n$.

Theorem 1.1 *Let $1 \leq m \leq n$ and $\psi(x)$ a positive C^∞ function in the annulus $\bar{\Omega} \subset \mathcal{R}^{n+1}(1)$, $\bar{\Omega}:\ u \in S^n,\ \ \rho \in [R_1, R_2],\ 0 < R_1 < R_2 < \pi/2$. Suppose ψ satisfies the conditions:*

$$\psi(u, R_1) \geq \cot^m R_1 \text{ for } u \in S^n, \tag{2}$$

$$\psi(u, R_2) \leq \cot^m R_2 \text{ for } u \in S^n, \tag{3}$$

and

$$\frac{\partial}{\partial \rho}\left[\psi(u, \rho) \cot^{-m} \rho\right] \leq 0 \text{ for all } u \in S^n \text{ and } \rho \in [R_1, R_2]. \tag{4}$$

Then there exists a closed, C^∞, embedded hypersurface M in $\mathcal{R}^{n+1}(1)$, $M \subset \Omega$, which is a radial graph over S^n of a function z and

$$H_m(\lambda_1(z(u)), ..., \lambda_n(z(u))) = \psi(u, z(u)) \quad \text{for all } u \in S^n. \tag{5}$$

This theorem extends to an arbitrary $m, 1 \leq m \leq n$, the analogous result established by Oliker in [8] for $m = n$. In Euclidean space an analogous result for functions generalizing elementary symmetric functions of principal curvatures was established by Caffarelli, Nirenberg and Spruck [3]. It should be noted that in contrast with the cases studied in [3] and [8] where the usual continuity method was applied to prove existence, we have to use here the special degree theory developed in [6]. The reason for this is that the continuity method requires (among other things) that the corresponding linearized equation be invertible on any admissible solution and this result is not available in the case studied in this paper. However, within the framework of the degree theory it suffices to know invertibility only on spheres and we establish this fact here.

Finally, we note that for hyperbolic space $\mathcal{R}^{n+1}(-1)$ an existence result similar to Theorem 1.1 is not known except for the Gauss curvature case, that is, when $m = n$; see Oliker [8].

2 The equation of the problem

In order to make our presentation reasonably selfcontained, we summarize here the needed facts about starshaped hypersurfaces in $\mathcal{R}^{n+1}(1)$. More details can be found in [1]. Also, in order to make our notation here agree with those in [1], to which we will need to refer often, we put $f(\rho) = \sin^2\rho$ and $q(\rho) = \cot\rho$.

Unless explicitly stated otherwise, all latin indices are in the range $1, ..., n$, the sums are over this range and summation over repeated lower and upper indices is assumed. For a smooth function z on S^n we put $z_i \equiv \nabla'_i z = \partial z/\partial u^i$, where $\nabla'_i z$ denotes the first covariant derivative in the metric e and $u^1, ..., u^n$ are some smooth local coordinates on S^n. The second covariant derivative in e is denoted by $\nabla'_{ij} z$.

Let M be a hypersurface in $\mathcal{R}^{n+1}(1)$ given by $r(u) = (u, z(u)),\ u \in S^n$, where $z \in C^2(S^n)$ and positive function on S^n. The metric ($\equiv$ the first fundsamental form) $g = g_{ij}du^i du^j$ induced on M from $\mathcal{R}^{n+1}(1)$ has coefficients

$$g_{ij} = f e_{ij} + z_i z_j \quad \text{and} \quad \det(g_{ij}) = f^{n-1}(f + |\nabla' z|^2)\det(e_{ij}). \tag{6}$$

The elements of the inverse matrix $(g^{ij}) = (g_{ij})^{-1}$ are

$$g^{ij} = \frac{1}{f}\left[e^{ij} - \frac{z^i z^j}{f + |\nabla' z|^2}\right] \quad \left(z^i = e^{ij} z_j\right). \tag{7}$$

The unit normal vector field on M is given by

$$N = \frac{\nabla' z - fR}{\sqrt{f^2 + f|\nabla' z|^2}}, \tag{8}$$

where $R = \delta/d\rho$ is the tangent vector field on $\mathcal{R}^{n+1}(1)$. The second fundamental form b of M has coefficients:

$$b_{ij} = \frac{f}{\sqrt{f^2 + f|\nabla' z|^2}}\left[-\nabla'_{ij} z + \frac{\partial \ln f}{\partial \rho} z_i z_j + \frac{1}{2}\frac{\partial f}{\partial \rho} e_{ij}\right], \tag{9}$$

With our choice of the normal the second fundamental form of a sphere $z = const > 0$ is positive definite, since for $\mathcal{R}^{n+1}(1)$ $\partial f/\partial \rho > 0$.

The principal curvatures of M are the eigenvalues of the second fundamental form relative to the metric g and are the real roots, $\lambda_1, ..., \lambda_n$, of the equation

$$\det(a^i_j - \lambda \delta^i_j) = 0,$$

where

$$a^i_j = g^{ik} b_{kj}. \tag{10}$$

The elementary symmetric function of order $m,\ 1 \le m \le n$, of $\lambda = (\lambda_1, ..., \lambda_n)$ is

$$S_m(\lambda) = \sum_{i_1 < ... < i_n} \lambda_{i_1} \cdots \lambda_{i_m} \quad \text{and} \quad S_m(\lambda) = \tbinom{n}{m} H_m = F(a^i_j), \tag{11}$$

where F is the sum of the principal minors of (a^i_j) of order m. Evidently,

$$F(a^i_j) \equiv F(u, z, \nabla'_1, ..., \nabla'_n z, \nabla'_{11} z, ..., \nabla'_{nn} z),$$

and the equation (5) assumes the form

$$F(a^i_j) = \bar{\psi}(u, z(u)), \tag{12}$$

where $\bar{\psi} \equiv \binom{n}{m}\psi$.

Note that on a sphere $z = const = c,\ 0 < c < \pi/2$,

$$F(a^i_j) = \binom{n}{m} q^m(c). \tag{13}$$

Let Γ be the connected component of $\{\lambda \in R^n \mid S_m(\lambda) > 0\}$ containing the positive cone $\{\lambda \in R^n \mid \lambda_1, ..., \lambda_n > 0\}$.

Definition 2.1 *A positive function $z \in C^2(S^n)$ is admissible for the operator F if the corresponding hypersurface $M = (u, z(u)),\quad u \in S^n$, is contained in the annulus $\bar{\Omega}$ defined in Theorem 1.1, and at every point of M with the choice of the normal as in (8), the principal curvatures $(\lambda_1, ..., \lambda_n) \in \Gamma$.*

3 A priori estimates

It will be convenient for ease of reference to recall here the a priori estimates obtained in[1].

Proposition 3.1 *Let $1 \le m \le n$ and let $\psi(x)$ be a positive continuous function in the annulus $\bar{\Omega}:\ u \in S^n,\quad \rho \in [R_1, R_2],\ 0 < R_1 < R_2 < a$. Suppose ψ satisfies the conditions:*

$$\psi(u, R_1) \ge q^m(R_1) \text{ for } u \in S^n, \tag{14}$$

$$\psi(u, R_2) \le q^m(R_2) \text{ for } u \in S^n. \tag{15}$$

Let $z \in C^2(S^n)$ be an admissible solution of equation (12) and $R_1 \le z(u) \le R_2,\quad u \in S^n$. Then either $z \equiv R_1$, or $z \equiv R_2$, or

$$R_1 < z(u) < R_2, \quad u \in S^n. \tag{16}$$

Theorem 3.2 *Let $1 \le m \le n$ and let $\psi(x)$ be a positive C^1 function in the annulus $\bar{\Omega}:\ u \in S^n,\quad \rho \in [R_1, R_2],\ 0 < R_1 < R_2 < a$. Let $z \in C^3(S^n)$ be an admissible solution of equation (12) satisfying the inequalities*

$$R_1 \le z(u) \le R_2, \quad u \in S^n. \tag{17}$$

Suppose, in addition, that for all $u \in S^n$ and $\rho \in [R_1, R_2]$ ψ satisfies the condition

$$\frac{\partial}{\partial \rho}\left[\psi(u, \rho) q^{-m}(\rho)\right] \le 0; \tag{18}$$

Then

$$|grad\ z| \le C_1 \tag{19}$$

where C_1 is a constant depending only on m, n, R_1, R_2, ψ, and $|grad \psi|$.

Theorem 3.3 *Let $1 \leq m \leq n$ and let $\psi(x)$ be a positive C^2 function in the annulus $\bar{\Omega}: \; u \in S^n, \quad \rho \in [R_1, R_2], \; 0 < R_1 < R_2 < a$. Let $z \in C^4(S^n)$ be an admissible solution of equation (12) in $\mathcal{R}^{n+1}(1)$ satisfying the inequalities*

$$R_1 \leq z(u) \leq R_2, \quad u \in S^n, \tag{20}$$

and

$$|grad\ z| \leq C_1 = const \quad on \quad S^n. \tag{21}$$

Then

$$\| z \|_{C^2(S^n)} \leq C_2, \tag{22}$$

where the constant C_2 depends only on $m, n, R_1, R_2, C_1,$ and $\| \psi \|_{C^2(\bar{\Omega})}$.

4 Proof of Theorem 1.1

1. Define a family of functions

$$\psi^t(u, \rho) = t\psi(u, \rho) + (1-t)A^\epsilon q^{m+\epsilon}(\rho), \quad t \in [0, 1],$$

where ϵ is some fixed positive constant and A a positive constant to be specified later. Consider the family of equations

$$\Phi(z, t) \equiv F(a^i_j(z)) - \bar{\psi}^t(u, z(u)) = 0, \qquad u \in S^n, \tag{23}$$

where $\bar{\psi}^t(u, z(u)) \equiv \binom{n}{m}\psi^t(u, z(u))$. For $0 \leq t \leq 1$ we consider the family of operators $\Phi(\cdot, t): C^{4,\alpha}_a(S^n) \longrightarrow C^{2,\alpha}(S^n), \quad 0 < \alpha < 1$, where $C^{4,\alpha}_a(S^n)$ denotes the subset of functions from $C^{4,\alpha}(S^n)$ which are admissible for the operator F.

In order to apply the degree theory, as in [6] we need to carry out the following three steps.

Step I. It needs to be shown that for $t = 0$ there exists a unique admissible solution $z^0 \in C^{4,\alpha}_a(S^n)$ of the equation $\Phi(z, 0) = 0$ and the derivative $\Phi_z(z^0, 0)$ is an invertible operator from $C^{4,\alpha}(S^n)$ to $C^{2,\alpha}(S^n)$.

Step II. It needs to be shown that for all $t \in [0, 1]$ and any solution $z \in C^{4,\alpha}_a(S^n)$ of equation (23) we have

$$R_1 < z(u) < R_2, \quad u \in S^n, \tag{24}$$

and

$$\| z \|_{C^{4,\alpha}(S^n)} < C \tag{25}$$

for some constant C, depending only on $m, n, R_1, R_2, \| \psi \|_{C^{2,\alpha}(\bar{\Omega})}$.

Step III consists in showing that the degree of the $\Phi(\cdot, 1) \neq 0$.

2. Step I - existence and uniqueness of a solution to $\mathbf{\Phi(z, 0) = 0}$. Set $t = 0$ in (23) and fix some $\bar{R}$ such that $R_1 < \bar{R} < R_2$, where R_1 and R_2 are as in Theorem 1.1. Put

$$A = 1/q(\bar{R}). \tag{26}$$

The selected constant A will remain fixed for the rest of the paper. Taking into account (13), it is clear that $z^0 = \bar{R}$ is a solution of the equation

$$F(a^i_j) = \binom{n}{m} A^\epsilon q^{m+\epsilon}(z), \quad u \in S^n. \tag{27}$$

Let us show that z^0 is the only admissible solution of (27). Suppose that there exists another admissible solution $\tilde{z}$ of (27). Let $\bar{u} \in S^n$ be such that $\tilde{z}(\bar{u}) = \min_{S^n} \tilde{z}(u)$. At $\bar{u}$ $grad\ \tilde{z} = 0$, $Hess(\tilde{z}) \geq 0$ and

$$(g^{ij}) = \frac{1}{f}(e^{ij}),\ (b_{ij}) = -Hess(\tilde{z}) + fq(e_{ij}) \leq fq(e_{ij}), \quad (a^i_j) \leq q(\delta^i_j).$$

Consequently, we have at $\bar{u}$,

$$F(a^i_j(\tilde{z})) = \binom{n}{m} A^\epsilon q^{m+\epsilon}(\tilde{z}) \leq \binom{n}{m} q^m(\tilde{z}).$$

Since q is strictly decreasing, it follows from the choice of A that $\tilde{z(u)} \geq \tilde{z}(\bar{u}) \geq \bar{R}$ on S^n. By a similar argument it is shown that $\tilde{z} \leq \bar{R}$ on S^n. Hence, $\tilde{z} = \bar{R} = z^0$.

3. Step I - Invertiblity of $\Phi_z(z^0, 0)$. By standard results from theory of linear elliptic equations, in order to establish invertibility of $\Phi_z(z^0, 0) : C^{4,\alpha} \to C^{2,\alpha}$, it suffices to show that the $ker \Phi_z(z^0, 0) = \{0\}$. We do this in two stages. First we transform the metric h of $\mathcal{R}^{n+1}(1)$ and the expressions for first and second fundamental forms of an arbitrary hypersurface M so that the operator $F(a^i_j)$ assumes a more convenient form for linearization. Then we compute the corresponding linearized operator on an arbitrary admissible solution and for any t. Finally, determine its kernel at z^0 and $t = 0$.

3.1. The Cayley-Klein model of $\mathcal{R}^{n+1}(1)$. Let

$$\bar{\rho} = \frac{1}{q(\rho)}, \quad \gamma(\bar{\rho}) = \frac{1}{1 + \bar{\rho}^2}. \tag{28}$$

Then

$$f(\rho(\bar{\rho})) = \gamma \bar{\rho}^2 \quad \text{and} \quad h = \gamma(\gamma d\bar{\rho}^2 + \bar{\rho}^2 e_{ij} du^i du^j).$$

Geometrically, this is equivalent to transforming the coordinate ρ in $\mathcal{R}^{n+1}(1)$ so that in coordinates $(u, \bar{\rho})$, $u \in S^n, 0 \leq \bar{\rho} < \infty$, the metric (1) assumes the form corresponding to the Cayley-Klein (projective) model of the elliptic space. In this model the space $\mathcal{R}^{n+1}(1)$ is modeled on the entire R^{n+1} with metric h as above.

We now re-calculate the first and second fundamental forms of M using (28). Put

$$\sigma = \frac{1}{q(z)}.$$

Thus, in coordinates $(u, \bar{\rho})$ M is the graph of the function $\bar{\rho} = \sigma(u)$, $u \in S^n$.

Differentiating σ, we obtain $z_i = \gamma \sigma_i$. It follows from (6) that

$$g_{ij} = \gamma \sigma^2 e_{ij} + \gamma^2 \sigma_i \sigma_j = \gamma \hat{g}_{ij} - \sigma^2 \gamma^2 \sigma_i \sigma_j, \quad \text{where} \quad \hat{g}_{ij} = \sigma^2 e_{ij} + \sigma_i \sigma_j.$$

Note that $\hat{g} = \hat{g}_{ij} du^i du^j$ is the metric of M in euclidean sense, that is, $\hat{g}$ is induced from R^{n+1}; see [7]. Put

$$W = \sqrt{\sigma^2 + |\nabla' \sigma|^2}, \quad V = \sqrt{W^2 + \sigma^4}.$$

Then for the inverse matrices we have

$$g^{ij} = \frac{W^2}{\gamma V^2}\left(\hat{g}^{ij} + \frac{\sigma^2 e^{ij}}{W^2}\right),$$

where

$$\hat{g}^{ij} = \frac{1}{\sigma^2}\left(e^{ij} - \frac{\sigma^i\sigma^j}{W^2}\right) \quad (\sigma^i = e^{ij}\sigma_j). \tag{29}$$

Next, we re-calculate the second fundamental form b of M. For the factor in front of the square brackets in (9) we have

$$\frac{f}{\sqrt{f^2 + f|\nabla' z|^2}} = \frac{\sigma}{\sqrt{\sigma^2 + \gamma|\nabla'\sigma|^2}} = \frac{\sigma}{V\sqrt{\gamma}}.$$

For the expression in the square brackets in (9) we need

$$\frac{\partial \ln f}{\partial \rho}|_{\rho=z} = \frac{2}{\sigma}, \quad \frac{1}{2}\frac{\partial f}{\partial \rho}|_{\rho=z} = \gamma\sigma.$$

Then

$$-\nabla'_{ij}z + \frac{\partial \ln f}{\partial \rho} z_i z_j + \frac{1}{2}\frac{\partial f}{\partial p} e_{ij} = -\gamma\nabla'_{ij}\sigma - \gamma_i\sigma_j + 2\frac{\gamma^2}{\sigma}\sigma_i\sigma_j + \gamma\sigma e_{ij}.$$

Since $\gamma_i = -2\gamma^2\sigma\sigma_i$, we obtain

$$b_{ij} = \frac{\sqrt{\gamma}}{V}\left(-\sigma\nabla'_{ij}\sigma + 2\sigma_i\sigma_j + \sigma^2 e_{ij}\right) = \frac{W\sqrt{\gamma}}{V}\hat{b}_{ij}, \tag{30}$$

where

$$\hat{b}_{ij} = \frac{-\sigma\nabla'_{ij}\sigma + 2\sigma_i\sigma_j + \sigma^2 e_{ij}}{W}, \tag{31}$$

which is the second fundamental form of M in euclidean sense [7]. Thus

$$a^i_j = g^{ik}b_{kj} = \frac{W^3}{V^3\sqrt{\gamma}}\left(\hat{g}^{ik}\hat{b}_{kj} + \frac{\sigma^2}{W^2}e^{ik}\hat{b}_{kj}\right). \tag{32}$$

3.2. Linearization of (23) and completion of step I. Upon replacing z by σ in (23) we get

$$\Phi(\sigma, t) \equiv F(a^i_j(\sigma)) - \bar{\psi}^t(u, \sigma(u)) = 0, \qquad u \in S^n, \tag{33}$$

with a^i_j given by (32) and

$$F(a^i_j) \equiv F(u, \sigma, \nabla'_1\sigma, ...\nabla'_n, \nabla'_{11}\sigma, ..., \nabla'_{nn}\sigma).$$

Lemma 4.1 *Let $z \in C^2(S^n)$ be an admissible solution of (23) for some $t \in [0,1]$ and σ the corresponding solution of (33). Let $\Phi_\sigma(\sigma, t)$ be the operator obtained by linearizing $\Phi(\sigma, t)$ on σ. Then the $ker\Phi_\sigma(\sigma, t)$ consists of functions $v \in C^2(S^n)$ satisfying the equation*

$$v\left[\left(-\frac{3\sigma^4}{V^2} + \gamma\sigma^2 - 1\right)m\bar{\psi}^t(\cdot, \sigma) + 2Q\frac{\sigma^2}{W^2}F^j_i e^{ik}\hat{b}_{kj} - \bar{\psi}^t_\sigma(\cdot, \sigma)\sigma\right] + T(v) = 0 \quad on \quad S^n, \tag{34}$$

where

$$Q(\sigma) = \frac{W^3(\sigma)}{V^3(\sigma)\sqrt{\gamma(\sigma)}}$$

and T is a linear second order negatively elliptic operator with coefficients depending on σ and with no zero order term in v.

Proof. Consider a deformation of M given by $\sigma_s = \sigma + s\xi$, where s is a small real parameter and $\xi \in C^2(S^n)$. The equation obtained by linearizing (33) is given by

$$\Phi_\sigma(\sigma,t)(\xi) \equiv \sum_{i,j} F_{\nabla'_{ij}\sigma}\nabla'_{ij}\xi + \sum_i F_{\nabla'_i\sigma}\nabla'_i\xi + F_\sigma\xi - \bar{\psi}^t_\sigma\xi = 0, \tag{35}$$

where the subscripts at F and $\bar{\psi}$ indicate differentiaton with respect to the corresponding variables and s is set to zero after differentiation.

We claim that the second order linear operator $\Phi_\sigma(\sigma,t)$ is negatively elliptic on σ. Put

$$F_i^j = \frac{\partial F}{\partial a^i_j}.$$

Fix an arbitrary point on S^n and diagonalize there the matrices (g_{ij}) and (b_{ij}) using an orthonormal set of principal directions as the basis. At that point

$$F_i^j = 0 \text{ when } i \neq j \text{ and } F_i^i = \frac{\partial S_m}{\partial \lambda_i} > 0,$$

where the inequality on the right follows from a well known property of elementary symmetric functions [2]. Then

$$\sum_{i,j} F_{\nabla'_{ij}\sigma}\nabla'_{ij}\xi = -\sigma\frac{\sqrt{\gamma}}{V}F_i^i\delta^{ii}\nabla'_{ii}\xi$$

and we conclude that $\Phi_\sigma(\sigma,t)$ is negatively elliptic.

Next, we derive (34). We make in (35) a substitution $\xi = \sigma v$. Then, the equation satisfied by v is given by

$$v\left[\sum_{i,j} F_{\nabla'_{ij}\sigma}\nabla'_{ij}\sigma + \sum_i F_{\nabla'_i\sigma}\nabla'_i\sigma + F_\sigma\sigma - \bar{\psi}^t_\sigma\sigma\right] + T(v) = 0 \text{ at } s = 0, \tag{36}$$

where T is a linear second order negatively elliptic operator with no zero order terms in v. We compute now the coefficient at v.

Note that

$$\left[\sum_{i,j} F_{\nabla'_{ij}\sigma}\nabla'_{ij}\sigma + \sum_i F_{\nabla'_i\sigma}\nabla'_i\sigma + F_\sigma\sigma - \bar{\psi}^t_\sigma\sigma\right]_{s=0} = \frac{d[F(a^i_j(s\sigma) - \bar{\psi}^t(u,s\sigma)]}{ds}|_{s=1} \tag{37}$$

and

$$\frac{dF(a^i_j(s\sigma)}{ds}|_{s=1} = F_i^j\frac{da^i_j(s\sigma)}{ds}|_{s=1}. \tag{38}$$

By (32) we have

$$a^i_j(s\sigma) = Q(s\sigma)\left(\hat{a}^i_j(s\sigma) + \frac{s^2\sigma^2}{W^2(s\sigma)}e^{ik}\hat{b}_{kj}(s\sigma)\right),$$

where we put $\hat{a}^i_j \equiv \hat{g}^{ik}\hat{b}_{kj}$.

It follows from (29) and (31) that

$$\hat{a}^i_j(s\sigma) = (1/s)\hat{a}^i_j(\sigma) \quad \text{and} \quad \hat{b}_{kj}(s\sigma) = s\hat{b}_{kj}(\sigma).$$

The quotient $s^2\sigma^2/W^2(s\sigma)$ is homogeneous in s of order zero. Thus we have

$$\frac{da^i_j(s\sigma)}{ds}|_{s=1} = \frac{Q'}{Q}a^i_j + Q\left(-\hat{a}^i_j + \frac{\sigma^2}{W^2}e^{ik}\hat{b}_{kj}\right) \quad \text{at} \quad s = 1,$$

where Q' denotes the derivative with respect to s. Using (32) we obtain

$$\frac{da^i_j(s\sigma)}{ds}|_{s=1} = \left(\frac{Q'}{Q} - 1\right)a^i_j + 2Q\frac{\sigma^2}{W^2}e^{ik}\hat{b}_{kj} \quad \text{at} \quad s = 1. \tag{39}$$

We have

$$\frac{Q'}{Q} = -\frac{3\sigma^4}{V^2} + \gamma\sigma^2 \quad \text{at} \quad s = 1.$$

Substituting it in (39) and noting that

$$F^j_i a^i_j = mF(a^i_j) = m\bar{\psi}^t(u,\sigma) \quad \text{at} \quad s = 1,$$

we obtain

$$\left(\frac{Q'}{Q} - 1\right)F^j_i a^i_j = \left(-\frac{3\sigma^4}{V^2} + \gamma\sigma^2 - 1\right)m\bar{\psi}^t.$$

This, together with (39), (38), (37) and (36) imply that v satisfies (34).

Finally, we note that since $\sigma > 0$ on S^n, any solution ξ of (35) can be obtained from v in obvious way. The lemma is proved.

Lemma 4.2

$$ker\Phi_\sigma(\sigma^0, 0) = 0.$$

Proof. Set in (34) $\sigma = \sigma^0$ and $t = 0$. Then,

$$-\frac{3\sigma^4}{V^2} + \gamma\sigma^2 - 1 = -(1 + 2\gamma\sigma^2) \quad \text{with} \quad \sigma = \sigma^0.$$

It follows from (31) that $\hat{b}_{ij}(\sigma^0) = \sigma^0 e_{ij}$. Using (32) and (29), we get

$$a^i_j(\sigma^0) = (\sigma^0)^{-1}\delta^i_j = (\sigma^0)^{-2}e^{ik}\hat{b}_{kj}(\sigma^0).$$

Therefore,

$$2Q\frac{\sigma^2}{W^2}F_i^j e^{ik}\hat{b}_{kj} = 2\gamma\sigma^2 m\bar{\psi}^0 \quad \text{at} \quad \sigma = \sigma^0.$$

It follows now from (34) that the coefficient at v is equal to

$$-m\bar{\psi}^0(\sigma^0)\sigma^0 - \bar{\psi}^0_\sigma(\sigma^0)\sigma^0 = \epsilon A^\epsilon(\sigma^0)^{-m-\epsilon}$$

which is positive by our choice of ϵ. Now the standard maximum principle implies that $\ker \Phi_\sigma(\sigma^0, 0) = 0$. The lemma is proved.

The Lemma 4.2 and the theory of linear second order elliptic equations imply that $\Phi_\sigma(\sigma^0, 0) : C^{4,\alpha}(S^n) \longrightarrow C^{2,\alpha}(S^n)$ is invertible. The **step I** is now complete.

4. Step II - a priori bounds. Let $t \in [0,1]$ and $z \in C_a^{4,\alpha}(S^n)$ is a solution of (23) such that

$$R_1 \leq z(u) \leq R_2. \tag{40}$$

We may assume that $z(u) \equiv R_1$ and $z(u) \equiv R_2$ are not solutions of (5), - otherwise we are done.

Now we will show that $z(u) \equiv R_1$ and $z(u) \equiv R_2$ are not solutions of (23) for any $t \in [0,1]$. In order to see that, note first that since $z(u) \equiv R_1$ does not satisfy (5), there exists some $\bar{u} \in S^n$ such that $q^m(R_1) \neq \psi(\bar{u}, R_1)$. Then, because of (2),

$$q^m(R_1) < \psi(\bar{u}, R_1).$$

Since $\bar{R} > R_1$, we have

$$A^\epsilon q^{m+\epsilon}(R_1) > A^\epsilon q^\epsilon(\bar{R}) q^m(R_1) = q^m(R_1).$$

It follows that

$$\psi^t(\bar{u}, R_1) = t\psi(\bar{u}, R_1) + (1-t)A^\epsilon q^{m+\epsilon}(R_1) > q^m(R_1) \quad \text{for any} \quad t \in [0,1]. \tag{41}$$

Therefore, $z(u) \equiv R_1$ is not a solution of (23) for any $t \in [0,1]$.

Similarly, it is shown that for some $\hat{u} \in S^n$

$$\psi^t(\hat{u}, R_2) = t\psi(\hat{u}, R_2) + (1-t)A^\epsilon q^{m+\epsilon}(R_2) < q^m(R_2) \quad \text{for any} \quad t \in [0,1], \tag{42}$$

and $z(u) \equiv R_2$ is not a solution of (23) for any $t \in [0,1]$. It also follows from (41) and (42) that $\psi^t(u, \rho)$ satisfies the conditions of Proposition 3.1. Consequently, any solution $z \in C_a^{4,\alpha}(S^n)$ of (23) must satisfy the inequalities

$$R_1 < z(u) < R_2 \ \text{ for all } u \in S^n. \tag{43}$$

Next, because of condition (4), Theorems 3.2 and 3.3 imply that any solution $z \in C_a^{4,\alpha}(S^n)$ of (23) satisfies

$$\| z \|_{C^2(S^n)} \leq C_2 \tag{44}$$

where the constant C_2 depends only on $m, n, R_1, R_2,$ *and* $\| \psi \|_{C^2(\bar{\Omega})}$. Then, the $C^{2,\alpha}$ estimates with any $\alpha \in (0,1)$ follow from (44) and the results of L. C. Evans [4] and N. V. Krylov [5]. In these circumstances the $C^{4,\alpha}$ estimate (25) follow from $C^{2,\alpha}$ estimates and the regularity theory for second order uniformly elliptic equations. This completes the **step II** of the proof of Theorem 1.1.

5. Step III - computing the degree of the map $\Phi(\cdot, 1)$.

Lemma 4.3 *For any $\delta > 0$ and $C > 0$, there exists a bounded open set $V \subset \Gamma$ (depending on n, m, R_1, R_2, δ and C) with $\bar{V} \subset \Gamma$, such that for any $z \in C_a^{4,\alpha}(S^n)$ satisfying inequalities (40), estimate (25) and*

$$\delta \leq S_m(\lambda_1(z(u)), ..., \lambda_n(z(u)) \leq 1/\delta \ \textit{ for all } u \in S^n, \tag{45}$$

we have

$$(\lambda_1(z(u)), ..., \lambda_n(z(u)) \in V \ \textit{ for all } u \in S^n.$$

Lemma 4.3 is evident and its proof is omitted.

It is also easy to see that there exists $\delta > 0$ (depending on n and constant C in estimate (25)) such that

$$\delta \leq \bar{\psi}^t(u, z(u)) \leq 1/\delta \quad \text{for all } u \in S^n, \tag{46}$$

where $0 \leq t \leq 1$ and $z \in C^{4,\alpha}(S^n)$ satisfies (40) and (25).

For $C > 0$ and $\delta > 0$ above, we define an open bounded subset O^* of $C^{4,\alpha}(S^n)$ as follows: $z \in O^*$ if $z \in C^{4,\alpha}(S^n)$, satisfies (43), (25) and

$$(\lambda_1(z(u)), ..., \lambda_n(z(u)) \in V \quad \text{for all } u \in S^n. \tag{47}$$

Lemma 4.4 *If for some* $0 \leq t \leq 1$ *and* $z \in \bar{O}^*$

$$\Phi(z, t) = 0, \tag{48}$$

then $z \in O^*$, *that is,*

$$\partial O^* \bigcap \Phi(\cdot, t)^{-1}(0) = \emptyset, \quad 0 \leq t \leq 1, \tag{49}$$

where $\Phi(\cdot, t)$ *is viewed as map from* $\bar{O}^* \subset C^{4,\alpha}(S^n)$ *to* $C^{2,\alpha}(S^n)$.

Proof. It follows from Step II that z satisfies (43), (25). Hence, (46) is satisfied. With the δ as in (46), the equation (48) and the explicit form of $\Phi(z, t)$ imply (45). By Lemma 4.3 the condition (47) is satisfied. Therefore $z \in O^*$.

Consequently, by definition 2.2 in [6], the degree $\deg(\Phi(\cdot, t), O^*, 0)$ is defined for all $0 \leq t \leq 1$. By (49) and Proposition 2.2 in [6] $\deg(\Phi(\cdot, t), O^*, 0)$ is independent of $0 \leq t \leq 1$. In particular,

$$\deg(\Phi(\cdot, 1), O^*, 0) = \deg(\Phi(\cdot, 0), O^*, 0).$$

By Step I the equation

$$\Phi(z, 0) = 0, \quad z \in O^*$$

has a unique solution $z = z^0 \in O^*$ and the operator $\Phi_z(z^0, 0) : C^{4,\alpha}(S^n) \to C^{2,\alpha}(S^n)$ is invertible.

By Proposition 2.3 in [6],

$$\deg(\Phi(\cdot, 0), O^*, 0) = \deg(\Phi_z(z^0, 0), B_1, 0),$$

where

$$B_1 = \{z \in C^{4,\alpha}(S^n) | \parallel z \parallel_{C^{4,\alpha}(S^n)} < 1\}.$$

By Proposition 2.4 in [6],

$$\deg(\Phi_z(z^0, 0), B_1, 0) = \pm 1 \neq 0.$$

It follows that

$$\deg(\Phi(\cdot, 1), O^*, 0) \neq 0.$$

Therefore, the equation

$$\Phi(z, 1) = 0, \quad z \in O^*$$

has at least one solution. This completes the proof of Theorem 1.1.

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