An Extension to a Theorem of Jörgens, Calabi, and Pogorelov

L. CAFFARELLI
University of Texas
AND
YANYAN LI
Rutgers—The State University

1 Introduction

A classical theorem of Jörgens ($n = 2$ [17]), Calabi ($n \leq 5$ [7]), and Pogorelov ($n \geq 2$ [18]) states that any classical convex solution of

$$\det(D^2 u) = 1 \quad \text{in } \mathbb{R}^n \tag{1.1}$$

must be a quadratic polynomial. For $n = 2$, a classical solution is either convex or concave; the result holds without the convexity hypothesis.

A simpler and more analytical proof, along the lines of affine geometry, of the theorem was later given by Cheng and Yau [9]. The first author extended the result for classical solutions to viscosity solutions [4]. It was proven by Trudinger and Wang in [19] that the only open convex subset Ω of $\mathbb{R}^n$ which admits a convex C^2 solution of $\det(D^2 u) = 1$ in Ω with $\lim_{x \to \partial\Omega} u(x) = \infty$ is $\Omega = \mathbb{R}^n$. In this paper we give the following extension to the theorem of Jörgens, Calabi, and Pogorelov: Let u be a convex viscosity solution of $\det(D^2 u) = 1$ outside a bounded subset of $\mathbb{R}^n$, $n \geq 3$; then there exist an $n \times n$ real symmetric positive definite matrix A, a vector $b \in \mathbb{R}^n$, and a constant $c \in \mathbb{R}$ such that

$$\limsup_{|x| \to \infty} |x|^{n-2} \left(u - \left[\frac{1}{2} x' A x + b \cdot x + c \right] \right) < \infty \,.$$

Our approach, different from previous ones, is based on the theory of the first author on Monge-Ampère equations [2, 3]. Our above-mentioned results also enable us to establish an existence result for the Dirichlet problem on exterior domains with prescribed asymptotic behavior at infinity. In $\mathbb{R}^2$, similar problems are studied by L. Ferrer, A. Martínez, and F. Milán in [13, 14] using complex variable methods. See also Delanoë [12].

For the reader's convenience, we recall the definition of viscosity solutions (see [5] and the references therein for an extensive study of viscosity solutions to fully nonlinear elliptic equations of second order). Let Ω be an open subset of $\mathbb{R}^n$,

Communications on Pure and Applied Mathematics, Vol. LVI, 0549–0583 (2003)

$g \in C^0(\Omega)$ a positive function, and $u \in C^0(\Omega)$ a locally convex function. We say that u is a viscosity subsolution of

$$\det(D^2 u) = g \quad \text{in } \Omega \tag{1.2}$$

or a viscosity solution of

$$\det(D^2 u) \geq g \quad \text{in } \Omega$$

if for every $\bar{x} \in \Omega$ and every convex $\varphi \in C^2(\Omega)$ satisfying

$$\varphi \geq u \ \text{ on } \Omega \quad \text{and} \quad \varphi(\bar{x}) = u(\bar{x})$$

we have

$$\det(D^2 \varphi(\bar{x})) \geq g(\bar{x})\,.$$

Similarly, u is a viscosity supersolution of (1.2) if for every $\bar{x} \in \Omega$ and every convex $\varphi \in C^2(\Omega)$ satisfying

$$\varphi \leq u \ \text{ on } \Omega \quad \text{and} \quad \varphi(\bar{x}) = u(\bar{x})$$

we have

$$\det(D^2 \varphi(\bar{x})) \leq g(\bar{x})\,.$$

u is a viscosity solution of (1.2) if u is both a viscosity subsolution and a viscosity supersolution of (1.2).

In this paper we study convex viscosity solutions to

$$\det(D^2 u) = f \quad \text{on } \mathbb{R}^n\,, \tag{1.3}$$

where $f \in C^0(\mathbb{R}^n)$ satisfies

$$0 < \inf_{\mathbb{R}^n} f \leq \sup_{\mathbb{R}^n} f < \infty \tag{1.4}$$

and

$$\text{support } (f-1) \text{ is bounded.} \tag{1.5}$$

First is the extension of the classical theorem of Jörgens, Calabi, and Pogorelov for classical solutions to viscosity solutions, due to the first author.

THEOREM 1.1 [4] *For $n \geq 2$, any convex viscosity solution of* (1.1) *must be a quadratic polynomial.*

Let

$$\mathcal{A} = \{A : A \text{ is real } n \times n \text{ symmetric positive definite matrix with } \det(A) = 1\}\,.$$

The following theorem gives asymptotic behavior of solutions of (1.3) under the hypotheses (1.4) and (1.5).

THEOREM 1.2 *Let $f \in C^0(\mathbb{R}^n)$ satisfy* (1.4) *and* (1.5). *Assume that u is a convex viscosity solution of* (1.3). *Then*

(i) *For $n \geq 3$, there exist some $c \in \mathbb{R}$, $b \in \mathbb{R}^n$, and $A \in \mathcal{A}$, such that $E(x) := u(x) - (\frac{1}{2}x'Ax + b \cdot x + c)$ satisfies*

$$\limsup_{|x| \to \infty} |x|^{n-2} |E(x)| < \infty . \tag{1.6}$$

Moreover, u is C^∞ in the complement of the support of $(f - 1)$ and

$$\limsup_{|x| \to \infty} |x|^{n-2+k} |D^k E(x)| < \infty \quad \forall k \geq 1 . \tag{1.7}$$

(ii) *For $n = 2$, there exist some $c \in \mathbb{R}$, $b \in \mathbb{R}^n$, and $A \in \mathcal{A}$ such that*

$$E(x) := u(x) - \left(\frac{1}{2}x'Ax + b \cdot x + d \log \sqrt{x'Ax} + c \right)$$

satisfies

$$\limsup_{|x| \to \infty} |x| |E(x)| < \infty , \tag{1.8}$$

where

$$d = \frac{1}{2\pi} \int_{\mathbb{R}^2} (f - 1) . \tag{1.9}$$

Moreover, u is C^∞ in the complement of the support of $(f - 1)$ and

$$\limsup_{|x| \to \infty} |x|^{k+1} |D^k E(x)| < \infty \quad \forall k \geq 1 . \tag{1.10}$$

COROLLARY 1.3 *Let O be a bounded open convex subset of $\mathbb{R}^n$, and let $u \in C^0(\mathbb{R}^n \setminus \overline{O})$ be a locally convex viscosity solution of*

$$\det(D^2 u(x)) = 1 , \quad x \in \mathbb{R}^n \setminus \overline{O} .$$

Then $u \in C^\infty(\mathbb{R}^n \setminus \overline{O})$, and we have the following:

(i) *For $n \geq 3$, there exist some $c \in \mathbb{R}$, $b \in \mathbb{R}^n$, and $A \in \mathcal{A}$ such that* (1.6) *and* (1.7) *hold.*

(ii) *For $n = 2$, there exist some $c, d \in \mathbb{R}$, $b \in \mathbb{R}^2$, and $A \in \mathcal{A}$ such that* (1.8) *and* (1.10) *hold. Moreover, if $O = \varnothing$, then $d = 0$.*

Remark 1.4. For $n = 2$, Corollary 1.3 is known (see [14]).

The theorem of Jörgens, Calabi, and Pogorelov is an easy consequence of Corollary 1.3. Indeed, let $u \in C^2$ be a convex solution of (1.1). Then, by Corollary 1.3, for some c, b, and $A \in \mathcal{A}$,

$$E(x) := u(x) - \left(\frac{1}{2}x'Ax + b \cdot x + c \right) \to 0 \quad \text{as } |x| \to \infty .$$

Since

$$\det(A + D^2 E) - \det(A) = \det(D^2 u) - 1 = 0$$

and $(A + D^2 E) = (D^2 u)$ is positive definite, it follows from the mean value theorem that for some positive definite matrix function $(a_{ij}(x))$,

$$a_{ij} D_{ij} E = 0 \quad \text{in } \mathbb{R}^n .$$

By the maximum principle, $E(x) \equiv 0$, i.e., $u(x) \equiv \frac{1}{2} x' A x + b \cdot x + c$.

Corollary 1.3 enables us to establish the following existence theorem for the Dirichlet problem on exterior domains with prescribed asymptotic behavior at infinity.

THEOREM 1.5 *Let D be a smooth, bounded, strictly convex open subset of $\mathbb{R}^n$, $n \geq 3$, and let $\varphi \in C^2(\partial D)$. Then for any given $b \in \mathbb{R}^n$ and any given $A \in \mathcal{A}$, there exists some constant c_*, depending only on n, D, φ, b, and A, such that for every $c > c_*$ there exists a unique function $u \in C^\infty(\mathbb{R}^n \setminus \overline{D}) \cap C^0(\overline{\mathbb{R}^n \setminus D})$ that satisfies*

$$\begin{cases} \det(D^2 u) = 1, \ (D^2 u) > 0, & \text{in } \mathbb{R}^n \setminus \overline{D}, \\ u = \varphi & \text{on } \partial D, \end{cases}$$

and

$$\limsup_{|x| \to \infty} \left(|x|^{n-2} \left| u(x) - \left[\frac{1}{2} x' A x + b \cdot x + c \right] \right| \right) < \infty .$$

Remark 1.6. The Dirichlet problem on exterior domains of $\mathbb{R}^2$ was studied by Ferrer, Martínez, and Milán in [13, 14] using complex variable methods.

Our next theorem gives the existence of solutions of (1.3) with given asymptotic behavior at infinity.

THEOREM 1.7 *For $n \geq 3$, let $f \in C^0(\mathbb{R}^n)$ satisfy* (1.4) *and* (1.5). *Then for any $c \in \mathbb{R}$, $b \in \mathbb{R}^n$, and $A \in \mathcal{A}$, there exists a unique convex viscosity solution u of* (1.3) *satisfying*

$$\lim_{|x| \to \infty} E(x) = 0 , \tag{1.11}$$

where $E(x) = u(x) - (\frac{1}{2} x' A x + b \cdot x + c)$. Moreover, u is C^∞ in the complement of the support of $(f - 1)$, and E satisfies

$$(1 + |x|^{n-2}) |E(x)| \leq C , \quad x \in \mathbb{R}^n , \tag{1.12}$$

In addition, for any $k \geq 1$,

$$|x|^{n-2+k} |D^k E(x)| \leq C , \quad |x| \geq r , \tag{1.13}$$

where C and r are some positive constants depending only on n, f, and k.

Remark 1.8. Let u be the convex viscosity solution in Theorem 1.7. Then for $M > C$, $\{x : u(x) < M\}$ contains a ball of radius $\sqrt{2(M - C)}$ and is contained in a ball of radius $\sqrt{2(M + C)}$. By the works of the first author [2, 3], u is strictly convex. Moreover, if $f \in C^{k,\alpha}(O)$ for some $k \geq 0$, $0 < \alpha < 1$, and some open subset of $\mathbb{R}^n$, then $u \in C^{k+2,\alpha}(O)$.

Remark 1.9. For smooth f, Theorem 1.7 is a special case of results that Delanoë [12] obtained by different methods. For any general measurable function f satisfying (1.4), existence of infinitely many entire viscosity solutions was established by Chou and Wang in [10].

2 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1, an extension, due to the first author ([4]), of the theorem of Jörgens, Calabi, and Pogorelov for classical solutions to viscosity solutions. We need some well-known comparison principles. For the reader's convenience, we include the simple proofs.

PROPOSITION 2.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and let $g \in C^0(\Omega)$ be a positive function. Assume that $w \in C^0(\overline{\Omega})$ is a locally convex viscosity subsolution (supersolution) of*

$$\det(D^2 w) = g \quad \textit{in } \Omega\,,$$

and $v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ is a locally convex supersolution (subsolution) of

$$\det(D^2 v) = g \quad \textit{in } \Omega\,.$$

Assume also that

$$w \leq v \ (w \geq v) \quad \textit{on } \partial\Omega\,.$$

Then

$$w \leq v \ (w \geq v) \quad \textit{on } \overline{\Omega}\,.$$

PROOF: Without loss of generality, $\overline{\Omega} \subset B_R$ for some $R > 0$. For $0 < \epsilon < 1$ and $\lambda \geq 0$, let

$$v_\epsilon(x) = v(x) - \epsilon(|x|^2 - R^2) \quad \text{and} \quad v_{\epsilon,\lambda}(x) = v_\epsilon(x) + \lambda\,.$$

For λ sufficiently large,

$$v_{\epsilon,\lambda} \geq w \quad \text{on } \overline{\Omega}\,.$$

Let $\bar{\lambda}(\epsilon)$ be the smallest $\lambda \geq 0$ such that the above holds. We will show that

$$\bar{\lambda} := \limsup_{\epsilon \to 0} \bar{\lambda}(\epsilon) = 0\,. \tag{2.1}$$

Indeed, if $\bar{\lambda} > 0$, then there exist $\epsilon_j \to 0^+$, $\bar{\lambda}_j := \bar{\lambda}_j(\epsilon_j) \to \bar{\lambda}$, and $\bar{x}_j \in \overline{\Omega}\ \bar{x}_j \to \bar{x}$, such that

$$v_{\epsilon_j,\bar{\lambda}_j}(\bar{x}_j) = w(\bar{x}_j)\,,$$

while

$$v_{\epsilon_j,\bar{\lambda}_j} \geq w \quad \text{on } \overline{\Omega}\,.$$

Since $v_{\epsilon,\bar\lambda} > w$ on $\partial\Omega$, we have $\bar x \in \Omega$. Since $\det(D^2 v) > 0$ and v is convex, we have $(D^2 v(\bar x)) > 0$, so, for large j, $v_{\epsilon_j,\bar\lambda_j}$ is strictly convex near $\bar x_j$. By the definition of viscosity solution and taking $v_{\epsilon_j,\bar\lambda_j}$ as a test function, we have

$$\det\left(D^2 v_{\epsilon_j,\bar\lambda_j}(\bar x_j)\right) \le g(\bar x_j)\,.$$

On the other hand, from the explicit expression of $v_{\epsilon,\bar\lambda}$, we have

$$\det\left(D^2 v_{\epsilon_j,\bar\lambda_j}(\bar x_j)\right) = \det\left(D^2 v(\bar x_j) - 2\epsilon_j I\right) < \det\left(D^2 v(\bar x_j)\right) \le g(\bar x_j)\,,$$

a contradiction. So (2.1) holds, and therefore

$$w(x) \le \lim_{\epsilon\to 0} v_{\epsilon,\bar\lambda(\epsilon)}(x) = v(x)\,, \quad x \in \overline{\Omega}\,.$$

The statement concerning viscosity supersolution w can be proven in a similar way. □

There are two immediate consequences of Proposition 2.1:

COROLLARY 2.2 *Let $w \in C^0(\overline{B}_R)$ be a locally convex viscosity subsolution of*

$$\det(D^2 w) = 1 \quad \text{on } B_R\,.$$

Then

$$w(x) \le \frac{1}{2}(|x|^2 - R^2) + \max_{\partial B_R} w \quad \forall x \in B_R\,. \tag{2.2}$$

COROLLARY 2.3 *Let $w \in C^0(\overline{B}_R)$ be a locally convex viscosity supersolution of*

$$\det(D^2 w) = 1 \quad \text{on } B_R\,.$$

Then

$$w(x) \ge \frac{1}{2}(|x|^2 - R^2) + \min_{\partial B_R} w \quad \forall x \in B_R\,.$$

PROOF OF COROLLARY 2.2: The function

$$v(x) := \frac{1}{2}(|x|^2 - R^2) + \max_{\partial B_R} w$$

is a C^2 convex function in $\overline{B}_R$ satisfying

$$\det(D^2 v) = 1 \text{ on } B_R\,, \quad w \le v \text{ on } \partial B_R\,.$$

Estimate (2.2) follows from Proposition 2.1. □

PROOF OF COROLLARY 2.3: Let

$$v(x) := \frac{1}{2}(|x|^2 - R^2) + \min_{\partial B_R} w$$

and apply Proposition 2.1. □

The following result is well-known; see, for example, [1] and [8].

PROPOSITION 2.4 *Let Ω be a bounded open convex subset of $\mathbb{R}^n$, $n \geq 2$, and let $f \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ be a positive function. Then there exists a convex solution $w \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ of*

$$\begin{cases} \det(D^2 w) = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 2.5. The locally convex viscosity solution to the above Dirichlet problem is unique. This follows from Proposition 2.1.

Remark 2.6. If Ω is a C^∞ strictly convex domain and $f \in C^\infty(\overline{\Omega})$ is positive, then, by [6], $w \in C^\infty(\overline{\Omega})$.

To prove Theorem 1.1, we will first show that a convex viscosity solution of (1.1) is a classical solution.

THEOREM 2.7 *Let u be a convex viscosity solution of* (1.1). *Then $u \in C^\infty(\mathbb{R}^n)$.*

For $M > 0$, let $\Omega_M = \{x \in \mathbb{R}^n : u(x) < M\}$.

PROPOSITION 2.8 *Let u be a* nonnegative *convex viscosity subsolution of* (1.1). *Then there exists some constant C, depending only on n, such that*

$$\Omega_M \subset B_{CM^{n/2}} \quad \forall M \geq \max_{\partial B_1} u\,.$$

Proposition 2.8 will be deduced from Corollary 2.2 and the lemma below. Let $e_1 = (0, \dots, 0, 1)$ and, for $\delta > 0$, $B'_\delta = \{(0, x_2, \dots, x_n) : |(0, x_2, \dots, x_n)| < \delta\}$.

LEMMA 2.9 *For $\delta, r, \lambda > 0$, let K be the convex hull of $\overline{B'_\delta} \cup \{re_1\}$, and let $u \in C^0(K)$ be a nonnegative convex viscosity solution of*

$$\det(D^2 u) \geq \lambda \quad \text{in the interior of } K\,.$$

Assume, for some positive constant β, that

$$u \leq \beta \quad \text{on } \overline{B}'_\delta\,.$$

Then, for some constant $C = C(n) \geq 1$,

$$\max\{\beta, u(re_1)\} \geq \frac{\lambda^{1/n} \delta^{\frac{2(n-1)}{n}} r^{2/n}}{C}\,.$$

PROOF: By the convexity of u,

$$\max_K u \leq \alpha := \max\{\beta, u(re_1)\}\,.$$

It is clear that we can put an ellipsoid E in K with

$$|E| \geq \frac{\delta^{n-1} r}{C}$$

for some $C = C(n) \geq 1$. For some $a \in \mathcal{A}$, $b \in \mathbb{R}^n$, and

$$A(x) = ax + b\,,$$

we have, for some $R > 0$,

$$A(E) = B_R\,.$$

Consider

$$w(x) = \frac{1}{\lambda^{1/n}} u(A^{-1}(x))\,, \quad x \in B_R\,.$$

Then

$$\det(D^2 w) \geq 1 \quad \text{in } B_R$$

in the viscosity sense, and

$$w \leq \frac{\alpha}{\lambda^{1/n}} \quad \text{on } \partial B_R\,.$$

It follows from Corollary 2.2 and the fact that $u \geq 0$ that

$$0 \leq w(0) \leq -\frac{1}{2}R^2 + \max_{\partial B_R} w \leq -\frac{1}{2}R^2 + \frac{\alpha}{\lambda^{1/n}}\,.$$

Thus

$$\delta^{n-1} r \leq C(n)|E| \leq C(n)R^n \leq \frac{C(n)\alpha^{n/2}}{\lambda^{1/2}}\,.$$

Lemma 2.9 follows from the above. □

PROOF OF PROPOSITION 2.8: For $M \geq \max_{\partial B_1} u$, without loss of generality, we may assume that

$$u(re_1) = M\,.$$

We need to show that $r \leq C(n)M^{n/2}$. By Corollary 2.2, $\max_{\partial B_1} u \geq \frac{1}{2}$. So we only need to consider $r \geq 2$. Let $\widetilde{K}$ denote the convex hull of $\overline{B}_1$ and re. By the convexity of u and the fact that the values of u on ∂B_1 and at re are bounded by M, we have

$$\max_{\widetilde{K}} u \leq M\,.$$

Proposition 2.8 follows from Lemma 2.9 (with K being the convex hull of $\widetilde{K} \cap \{x_1 = 1\}$ and re_1). □

Let u be a nonnegative convex viscosity subsolution of (1.1). By Proposition 2.8, Ω_M is bounded and convex for every $M > \inf_{\mathbb{R}^n} u$. By a normalization lemma of John-Cordoba and Gallegos (see [11]), there exists some affine transformation

$$A_M(x) = a_M x + b_M\,,$$

where a_M is an $n \times n$ matrix satisfying

$$\det(a_M) = 1 \tag{2.3}$$

and $b_M \in \mathbb{R}^n$ such that

$$B_R \subset A_M(\Omega_M) \subset B_{nR} \quad \text{for some } R > 0\,. \tag{2.4}$$

PROPOSITION 2.10 *Let* $0 < \lambda < \infty$*, and let u be a convex viscosity solution of*

$$\det(D^2 u) \geq \lambda \quad \text{on } \mathbb{R}^n . \tag{2.5}$$

Assume that u is normalized to further satisfy

$$u(0) = 0 , \quad u \geq 0 \ \text{ on } \mathbb{R}^n . \tag{2.6}$$

Then there exists some constant $C \geq 1$ *depending only only n such that*

$$|\Omega_M| \leq C\lambda^{-1/2} M^{n/2} \quad \forall M > 0 .$$

PROOF: Let

$$A_M(x) = a_M x + b_M$$

be an affine transformation satisfying (2.3) and (2.4). Consider

$$w(x) = \lambda^{-1/n} u\big(A_M^{-1}(x)\big) .$$

Then, by (2.5),

$$\det(D^2 w) \geq 1 \ \text{ on } \mathbb{R}^n \quad \text{and} \quad w \leq \lambda^{-1/n} M \ \text{ on } \partial B_R .$$

Applying Corollary 2.2, we have

$$0 \leq w(0) \leq -\frac{1}{2} R^2 + \max_{\partial B_R} w \leq -\frac{1}{2} R^2 + \lambda^{-1/n} M .$$

It follows that

$$|\Omega_M| \leq C(n) R^n \leq C(n) \lambda^{-1/2} M^{n/2} .$$

□

Similarly, we have the following:

PROPOSITION 2.11 *Let* $0 < \Lambda < \infty$*, and let u be a convex viscosity solution of*

$$\det(D^2 u) \leq \Lambda \quad \text{on } \mathbb{R}^n . \tag{2.7}$$

Assume that u is normalized to further satisfy (2.6)*. Then there exists some constant* $C \geq 1$*, depending only only n, such that*

$$|\Omega_M| \geq C^{-1} \Lambda^{-1/2} M^{n/2} \quad \forall M > 0 .$$

PROOF: In the proof of Proposition 2.10, we let instead

$$w(x) = \Lambda^{-1/n} u\big(A_M^{-1}(x)\big) .$$

Then, by (2.7),

$$\det(D^2 w) \leq 1 \ \text{ on } \mathbb{R}^n \quad \text{and} \quad w \geq \Lambda^{-1/n} M \ \text{ on } \partial B_{nR} .$$

Applying Corollary 2.3, we have

$$0 = w(A_M(0)) \geq -\frac{1}{2} (nR)^2 + \min_{\partial B_{nR}} w \geq -\frac{1}{2} (nR)^2 + \Lambda^{-1/n} M .$$

It follows that

$$|\Omega_M| \geq C(n)^{-1} R^n \geq C(n)^{-1} \Lambda^{-1/2} M^{n/2} .$$

□

PROPOSITION 2.12 *For* $0 < \lambda \leq \Lambda < \infty$, *let* u *be a convex viscosity solution of* (2.5) *and* (2.7) *that also satisfies the normalization* (2.6), *and let* $A_M(x) = a_M x + b_M$ *be an affine transformation satisfying* (2.3) *and* (2.4). *Then for some constant* C, *depending only on* n, λ, *and* Λ,

$$2nR \geq \operatorname{dist}\big(A_M(\Omega_{M/2}), \partial A_M(\Omega_M)\big) \geq C^{-1} R \quad \forall M > 0 . \tag{2.8}$$

Consequently,

$$B_{R/C} \subset a_M(\Omega_M) \subset B_{2nR} . \tag{2.9}$$

PROOF: Let w be defined on $O_M := \frac{1}{R} A_M(\Omega_M)$ by

$$w(x) = \frac{\Lambda^{-1/n}}{R^2} u\big(A_M^{-1}(Rx)\big) - \frac{\Lambda^{-1/n} M}{R^2} , \quad x \in O_M .$$

Then

$$B_1 \subset O_M \subset B_n , \quad w = 0 \text{ on } O_M ,$$

and

$$\det(D^2 w) \leq 1 \text{ in } O_M .$$

It follows from Lemma A.1 that

$$w(x) \geq -C(n) \operatorname{dist}(x, \partial O_M))^{\frac{2}{n+1}} , \quad x \in O_M .$$

For $\bar{y} \in A_M(\Omega_{M/2})$, $M > 0$, let $\bar{x} = \frac{1}{R}\bar{y}$; we then have

$$-\frac{\Lambda^{-1/n} M}{2R^2} = \frac{\Lambda^{-1/n}(\frac{M}{2} - M)}{R^2} \geq w(\bar{x}) \geq -C(n) \operatorname{dist}(\bar{x}, \partial O_M)^{\frac{2}{n+1}} .$$

It follows from Proposition 2.10, (2.3), and (2.4) that for some $C = C(n, \Lambda, \lambda) \geq 1$,

$$\operatorname{dist}(\bar{y}, \partial A_M(\Omega_M)) \geq C^{-1} R .$$

Since $A_M(\Omega_M) \subset B_{nR}$,

$$\operatorname{dist}(\bar{y}, \partial A_M(\Omega_M)) \leq 2nR .$$

Estimate (2.8) is established. Estimate (2.9) follows from (2.8) and

$$\operatorname{dist}(A_M(0), \partial A_M(\Omega_M)) = \operatorname{dist}(0, \partial a_M(\Omega_M)) .$$

□

PROOF OF THEOREM 2.7: Without loss of generality, we may assume that u further satisfies (2.6). For $M > 0$, by Proposition 2.8, Ω_M is bounded and convex. Applying Proposition 2.4 to $w = u - M$ on $\Omega = \Omega_M$, we have $u \in C^\infty(\Omega_M)$. Since $\mathbb{R}^n = \cup_{M>0} \Omega_M$, we have $u \in C^\infty(\mathbb{R}^n)$. □

PROPOSITION 2.13 *Let* $u \in C^2(\mathbb{R}^n)$ *be a convex solution of* (2.7) *that further satisfies normalizations* (2.6) *and*

$$D^2u(0) = I \quad \textit{the identity.}$$

For $M > 0$*, let* $A_M(x) = a_M x + b_M$ *be some affine transformation satisfying* (2.3) *and* (2.4)*. Then for some positive constant* $C = C(n)$*,*

$$C^{-1}M^{1/2} \le R \le CM^{1/2} \tag{2.10}$$

and

$$\|a_M\| \le C\,, \quad \left\|a_M^{-1}\right\| \le C\,. \tag{2.11}$$

Moreover,

$$\sup_{\mathbb{R}^n} \|D^2u\| \le C\,. \tag{2.12}$$

PROOF: Estimate (2.10) follows from Propositions 2.10 and 2.11 and (2.4). Consider

$$w(x) = \frac{1}{R^2}u\big(a_M^{-1}(Rx)\big)\,, \quad x \in O := \frac{1}{R}a_M(\Omega_M)\,.$$

By (2.9),

$$B_{1/C} \subset O \subset B_{2n}\,. \tag{2.13}$$

By (1.1),

$$\det(D^2w) = 1 \quad \text{on } O\,.$$

It follows from (2.10) that

$$C^{-1} \le w = \frac{M}{R^2} \le C \quad \text{on } \partial O\,.$$

By (2.13) and the interior second derivative estimates of Pogorelov,

$$\|D^2w(x)\| \le C \quad \forall |x| \le \frac{1}{2C}\,, \tag{2.14}$$

in particular,

$$\|D^2w(0)\| \le C\,.$$

Recall the Pogorelov estimate: Let O be a bounded convex open set of $\mathbb{R}^n$, and let $w \in C^2(O) \cap C^0(\overline{O})$ be a convex function satisfying $\det(D^2w) = 1$ in O and $w = 0$ on ∂O; then for any compact subset K of O, $\|D^2w\| \le C$ on K for some constant C depending only on n, O, and K.

Since $D^2u(0) = I$, we have

$$D^2w(0) = \big(a_M^{-1}\big)'\big(a_M^{-1}\big)\,,$$

where $(a_M^{-1})'$ denotes the transpose of a_M^{-1}. It follows that

$$\left\|a_M^{-1}\right\| \le C\,.$$

Since $\det(a_M^{-1}) = \det(a_M) = 1$, we then have

$$\|a_M\| \le C\,.$$

Estimate (2.11) is established.

By (2.14) and (2.11),

$$\sup_{|y|\le \frac{R}{2C}} |D^2u(y)| \le C \quad \text{where } C = C(n)\,.$$

Since M can be arbitrary large (as can R), estimate (2.12) follows from the above. □

Theorem 1.1 can be deduced from Theorem 2.7, (2.12), and the $C^{2,\alpha}$ interior estimates of Evans and Krylov as follows:

PROOF OF THEOREM 1.1: By Theorem 2.7, $u \in C^\infty(\mathbb{R}^n)$. Then, by (2.12),

$$|u(x)| \le C|x|^2\,, \quad |x| \ge 1\,. \tag{2.15}$$

For $\bar{x} \in \mathbb{R}^n$, we will show that $D^2u(\bar{x}) = D^2u(0)$. For $R > 2(|\bar{x}|+1)$, we consider

$$w(y) = \frac{1}{R^2}u(Ry)\,, \quad y \in B_1\,.$$

By (2.15), (1.1), and (2.12),

$$|w| \le C\,, \quad \det(D^2w) = 1\,, \quad |D^2w| \le C \text{ on } B_1$$

where $C = C(n)$. Let

$$\bar{y} = \frac{\bar{x}}{R}\,.$$

Then

$$|\bar{y}| \le \frac{1}{2}\,.$$

It follows from the above-mentioned estimates of Evans and Krylov that for some $\alpha \in (0, 1)$ and C (independent of R),

$$|D^2w(\bar{y}) - D^2w(0)| \le C|\bar{y}|^\alpha\,,$$

i.e.,

$$|D^2u(\bar{x}) - D^2u(0)| \le C\frac{|\bar{x}|^\alpha}{R^\alpha}\,.$$

Sending $R \to \infty$, we have

$$D^2u(\bar{x}) = D^2u(0)\,.$$

Since $\bar{x}$ is arbitrary, u is a quadratic polynomial. Theorem 1.1 is established. □

3 Proof of Theorem 1.2 and Corollary 1.3

We prove Theorem 1.2 and Corollary 1.3 in this section.

First we have the following:

LEMMA 3.1 *Let u be as in Theorem* 1.2; *then u is C^∞ in the complement of the support of* $(f-1)$.

PROOF: Let $\bar{x}$ be in the complement of the support of $(f-1)$; without loss of generality, $\bar{x}=0$. Subtracting from u a supporting plane to the graph of u at $(0,u(0))$, we may assume without loss of generality that u satisfies (2.6). It follows from Proposition 2.8 that Ω_ϵ is bounded and convex for $\epsilon>0$. Then by [3, theorem 1] we have that $\{x\in\Omega_\epsilon : u(x)=0\}=\{0\}$. Taking $\epsilon>0$ small enough, Ω_ϵ belongs to the complement of the support of $(f-1)$. It follows from Proposition 2.4 that $u\in C^\infty(\Omega_\epsilon)$. □

Let u be as in Theorem 1.2. As explained above, we may assume that u also satisfies (2.6). We know from Lemma 3.1 that for M large, $u\in C^\infty(\mathbb{R}^n\setminus\Omega_M)$ and Ω_M is bounded and strictly convex. Keep u fixed outside Ω_M and redefine u inside Ω_M so that the new u is in $C^\infty(\mathbb{R}^n)$ and $(D^2u)>0$ on $\mathbb{R}^n$. Let $f=\det(D^2u)$ be the new f. So we only need to establish Theorem 1.2 with the additional hypothesis that $u\in C^\infty(\mathbb{R}^n)$ and u satisfies (2.6). These will be assumed in the rest of this section.

Let $A_M(x)=a_Mx+b_M$ be an affine transformation satisfying (2.3) and (2.4), and let

$$\xi(y)=\frac{1}{R^2}u\big(a_M^{-1}(Ry)\big)\,,\quad y\in O:=\frac{1}{R}a_M(\Omega_M)\,.$$

By (2.9),

$$B_{1/C}\subset O\subset B_{2n}\,.$$

Here and in the following, $C\geq 1$ denotes some constant depending only on n and f. Clearly

$$\det(D^2\xi)=f\big(a_M^{-1}(Ry)\big)\,.$$

By Propositions 2.10 and 2.11,

$$\xi=\frac{M}{R^2}\in(C^{-1},C)\quad\text{on }\partial O\,. \tag{3.1}$$

By Proposition 2.4 and Remark 2.5, there exists a unique convex solution $\bar{\xi}:=\bar{\xi}_{O,M/R^2}\in C^0(\overline{O})\cap C^\infty(O)$ of

$$\begin{cases}\det(D^2\bar{\xi})=1 & \text{on } O\\ \bar{\xi}=\frac{M}{R^2} & \text{on }\partial O.\end{cases}$$

Because of (3.1), the interior second derivative estimates of Pogorelov, and the $C^{2,\alpha}$ estimates of Evans and Krylov (also use Schauder estimates), for every $\delta>0$, there exists some positive constant $C=C(\delta)$, independent of M, such that

$$C^{-1}I\leq(D^2\bar{\xi}(x))\leq CI\,,\quad |D^3\bar{\xi}(x)|\leq C\,,\quad x\in O,\ \operatorname{dist}(x,\partial O)\geq\delta\,. \tag{3.2}$$

LEMMA 3.2 *For some positive constant C independent of M,*

$$|\xi - \bar{\xi}| \le C R^{-1} \quad \text{in } O\,.$$

PROOF: By the Alexandrov estimate (see, e.g., [16, lemma 9.2]),

$$-\min_{\overline{O}}(\xi - \bar{\xi}) \le C \left\{ \int\limits_{S^+} \det(D^2(\xi - \bar{\xi}) \right\}^{1/n},$$

where

$$S^+ = \{x \in O : D^2(\xi - \bar{\xi})(x) > 0\}\,.$$

On S^+,

$$\frac{D^2\xi}{2} = \frac{D^2(\xi - \bar{\xi}) + D^2\bar{\xi}}{2}\,,$$

so it follows from the concavity of $(\det)^{1/n}$,

$$\det\left(\frac{D^2\xi}{2}\right)^{1/n} \ge \frac{1}{2}\det(D^2(\xi - \bar{\xi}))^{1/n} + \frac{1}{2}\det(D^2\bar{\xi})^{1/n}\,,$$

i.e.,

$$\det(D^2(\xi - \bar{\xi})(y))^{1/n} \le 1 - f\left(a_M^{-1}(Ry)\right)^{1/n}, \quad y \in S^+\,.$$

It follows that

$$-\min_{\overline{O}}(\xi - \bar{\xi}) \le C\left\{ \int\limits_{S^+} \left[1 - f(a_M^{-1}(Ry))^{1/n}\right]^n \right\}^{1/n} \le \frac{C}{R}\|1 - f^{1/n}\|_{L^n(f<1)} \le \frac{C}{R}\,.$$

Similarly, we can show that

$$-\min_{\overline{O}}(\bar{\xi} - \xi) \le \frac{C}{R}\|f^{1/n} - 1\|_{L^n(f>1)} \le \frac{C}{R}\,.$$

Lemma 3.2 is established. □

Let $\bar{x}$ be the unique minimum point of $\bar{\xi}$ in $\overline{O}$. Recall that

$$A_M(x) = a_M x + b_M$$

is an affine transformation satisfying (2.3) and (2.4). By Proposition 2.10 and Proposition 2.11 ,

$$C^{-1}M^{1/2} \le R \le C M^{1/2}\,.$$

Let

$$E_M = \{x : x' D^2\bar{\xi}(\bar{x})x \le 1\}\,.$$

PROPOSITION 3.3 *There exist $\bar{k}$ and C, depending only on n and f, such that for $\epsilon = \frac{1}{10}$, $M = 2^{(1+\epsilon)k}$, and $2^{k-1} \le M' \le 2^k$, we have*

$$\left(\frac{2M'}{R^2} - C2^{-\frac{3\epsilon k}{2}}\right)^{1/2} E_M \subset \frac{1}{R} a_M(\Omega_{M'}) \subset \left(\frac{2M'}{R^2} + C2^{-\frac{3\epsilon k}{2}}\right)^{1/2} E_M \quad \forall k \ge \bar{k}\,. \tag{3.3}$$

PROOF: In the proof, C and $\bar{k}$ denote various large constants with the specified dependence, and we always assume that $k \ge \bar{k}$. Clearly,

$$C^{-1}2^{-\epsilon k} \le \frac{M'}{R^2} \le C2^{-\epsilon k}\,, \qquad C^{-1}2^{\frac{(1+\epsilon)k}{2}} \le R \le C2^{\frac{(1+\epsilon)k}{2}}\,,$$

and

$$\left\{\xi < \frac{M'}{R^2}\right\} := \left\{z \in O : \xi(z) < \frac{M'}{R^2}\right\} = \frac{1}{R} a_M(\Omega_{M'})\,.$$

By Lemma 3.2,

$$|\xi - \bar{\xi}| \le \frac{C}{R} \le C2^{-\frac{(1+\epsilon)k}{2}} \quad \text{on } \overline{O}\,.$$

Since

$$\frac{M'}{R^2} \ll \frac{C}{R}\,,$$

the level surface of ξ can be well approximated by the level surface of $\bar{\xi}$:

$$\left\{\bar{\xi} < \frac{M'}{R^2} - \frac{C}{R}\right\} \subset \left\{\xi < \frac{M'}{R^2}\right\} \subset \left\{\bar{\xi} < \frac{M'}{R^2} + \frac{C}{R}\right\}. \tag{3.4}$$

By Lemma 3.2 and the fact $\xi \ge 0$, we have

$$-\frac{C}{R} \le \xi(\bar{x}) - \frac{C}{R} \le \bar{\xi}(\bar{x}) \le \bar{\xi}(0) \le \xi(0) + \frac{C}{R} = \frac{C}{R}\,.$$

So, by Lemma A.1, $B_{1/C}(\bar{x}) \subset O$, and therefore by (3.2),

$$\left|\bar{\xi}(x) - \bar{\xi}(\bar{x}) - \frac{1}{2}(x - \bar{x})' D^2\bar{\xi}(\bar{x})(x - \bar{x})\right| \le C|x - \bar{x}|^3 \quad \forall x \in B_{1/C}(\bar{x})$$

and

$$\frac{I}{C} \le (D^2\bar{\xi}(\bar{x})) \le CI\,.$$

Estimate (3.3) follows from (3.4) and the above estimates by elementary consideration. □

Let B denote the unit ball in $\mathbb{R}^n$, and still let $\epsilon = \frac{1}{10}$.

PROPOSITION 3.4 *There exist positive constants C and $\bar{k}$ and some real invertible upper-triangular matrices $\{T_k\}_{k\geq\bar{k}}$ such that*

$$\det(T_k) = 1\,, \quad \left\| T_k T_{k-1}^{-1} - I \right\| \leq C 2^{-\epsilon k}\,, \tag{3.5}$$

and

$$(1 - C2^{-\epsilon k})\sqrt{2M'}B \subset T_k(\Omega_{M'}) \subset (1 + C2^{-\epsilon k})\sqrt{2M'}B\,, \qquad 2^{k-1} \leq M' \leq 2^k\,. \tag{3.6}$$

Consequently, for some invertible T,

$$\det(T) = 1\,, \quad \|T_k - T\| \leq C2^{-\epsilon k}\,. \tag{3.7}$$

PROOF: Let $M = 2^{(1+\epsilon)k}$ and let $2^{k-1} \leq M' \leq 2^k$. By (3.3), there exist some constants C and $\bar{k}$ (depending only on n and f) such that

$$(1 - C2^{-\epsilon k})\sqrt{2M'}E_M \subset a_M(\Omega_{M'}) \subset (1 + C2^{-\epsilon k})\sqrt{2M'}E_M\,, \quad k \geq \bar{k}\,.$$

Let Q be the positive definite matrix satisfying $Q^2 = D^2\bar{\xi}(\bar{x})$, and let O be an orthogonal matrix such that

$$T_k := OQa_M \quad \text{is upper-triangular.}$$

Clearly,

$$\det(T_k) = \det(O)\det(Q)\det(a_M) = \sqrt{\det(D^2\bar{\xi}(\bar{x}))}\,\det(a_M) = 1$$

and

$$(1 - C2^{-\epsilon k})\sqrt{2M'}B \subset T_k(\Omega_{M'}) \subset (1 + C2^{-\epsilon k})\sqrt{2M'}B\,. \tag{3.8}$$

Taking some larger $\bar{k}$, we deduce from (3.8), with $M' = 2^k$ and then $M' = 2^{k-1}$, that (with a larger C)

$$(1 - C2^{-\epsilon k})B \subset T_k T_{k-1}^{-1}(B) \subset (1 + C2^{-\epsilon k})B\,, \quad k \geq \bar{k}\,.$$

Since $T_k T_{k-1}^{-1}$ is still upper-triangular, we apply Lemma A.5 (with $U = T_k T_{k-1}^{-1}$) to obtain that

$$\left\| T_k T_{k-1}^{-1} - I \right\| \leq C2^{-\epsilon k}\,, \quad k \geq \bar{k}\,.$$

Estimates (3.5) and (3.6) are established. The existence of T and (3.7) (with a larger C) follow by elementary consideration. Proposition 3.4 is established. □

Let $v = u \circ T$. Then

$$\det(D^2 v) = 1\,, \quad \mathbb{R}^n \setminus T^{-1}(\Omega_{M_0})\,. \tag{3.9}$$

Since $\{x : v(x) < M'\} = T(\Omega_{M'})$, we deduce from (3.6) and (3.7) that

$$(1 - C(M')^{-\epsilon})\sqrt{2M'}B \subset \{x : v(x) < M'\} \subset (1 + C(M')^{-\epsilon})\sqrt{2M'}B \quad \forall M' \geq 2^{\bar{k}}\,.$$

Consequently,

$$\left|v(x) - \frac{1}{2}|x|^2\right| \le C|x|^{2-2\epsilon}, \quad |x| \ge 2^{\bar{k}}. \tag{3.10}$$

LEMMA 3.5 *Let $w \in C^\infty(\mathbb{R}^n \setminus B_1)$ satisfy*

$$(I + D^2 w(x)) > 0, \quad \det(I + D^2 w(x)) = 1 \ \text{on } |x| > 1,$$

and, for some constants $\beta > 0$ and $\gamma > -2$,

$$|w(x)| \le \frac{\beta}{|x|^\gamma} \quad \text{on } |x| > 1.$$

Then there exist some constants $r = r(n, \beta, \gamma) \ge 1$ such that for all $k \ge 1$,

$$|D^k w(x)| \le \frac{C}{|x|^{\gamma+k}} \quad \text{on } |x| \ge r,$$

where C depends only on n, k, β, and γ.

PROOF: Let

$$\eta(x) := \frac{|x|^2}{2} + w(x).$$

For $|x| = R > 8$, let

$$\eta_R(y) := \left(\frac{4}{R}\right)^2 \eta\left(x + \frac{R}{4}y\right), \quad |y| \le 2,$$

and

$$w_R(y) := \left(\frac{4}{R}\right)^2 w\left(x + \frac{R}{4}y\right) = \eta_R(y) - 8\left|\frac{x}{R} + \frac{y}{4}\right|^2, \quad |y| \le 2.$$

By the decay hypothesis on w, there exists some $r = r(n, \beta) \ge 1$ such that for $|x| = R \ge r$,

$$3 \le 4 - \frac{16\beta 2^\gamma}{R^{\gamma+2}} \le \eta_R(y) \le 32 + \frac{16\beta 2^\gamma}{R^{\gamma+2}} \le 33, \quad |y| \le 2.$$

Since η_R satisfies

$$(D^2 \eta_R(y)) > 0, \quad \det(D^2 \eta_R(y)) = 1, \quad |y| < 2,$$

by the estimates of Pogorelov, Evans-Krylov, and Schauder, we have, for every $k \ge 1$,

$$\|\eta_R\|_{C^k(\overline{B})} \le C \quad \text{and} \quad \frac{I}{C} < (D^2 \eta_R) < CI \ \text{ on } B.$$

Here and in the following, $C \ge 1$ denotes some constant depending on n and k unless otherwise stated.

It follows that

$$\|w_R\|_{C^k(\overline{B})} \le C \quad \text{and} \quad \frac{I}{C} < (I + D^2 w_R) < CI \ \text{ on } B. \tag{3.11}$$

Clearly, w_R satisfies

$$\hat{a}_{ij}(y) D_{ij} w_R(y) = 0 \quad \text{on } B_2,$$

where $(\hat{a}_{ij}(y)) = \int_0^1 F_{\xi_{ij}}(I + sD^2 w_R(y))ds$ satisfies, in view of (3.11), that

$$\|\hat{a}_{ij}\|_{C^k(\overline{B})} \le C \quad \text{and} \quad \frac{I}{C} < (\hat{a}_{ij}) < CI \ \text{ on } B\,.$$

Here and throughout the section, we use the notation

$$F(\xi) := \det(\xi_{ij})^{1/n}\,.$$

It is well-known that, in the open set of symmetric positive definite matrices, $(F_{\xi_{ij}})$ is positive definite and F is concave.

By Schauder theory,

$$\left|D^k w_R(0)\right| \le C\|w_R\|_{L^\infty(B)} \le \frac{C(n,k,\beta,\gamma)}{R^{\gamma+2}}\,.$$

It follows that

$$|D^k w(x)| \le \frac{C(n,k,\beta,\gamma)}{|x|^{\gamma+k}}\,.$$

□

LEMMA 3.6 *For $n \ge 3$, there exist $b \in \mathbb{R}^n$, $c \in \mathbb{R}$, and some positive constant C such that*

$$\left|v(x) - \left(\frac{|x|^2}{2} + b\cdot x + c\right)\right| \le \frac{C}{|x|^{n-2}} \quad \forall x \in \mathbb{R}^n \setminus T^{-1}(\Omega_{M_0})\,.$$

PROOF: Let

$$\hat{E} := v(x) - \frac{|x|^2}{2}\,.$$

Then

$$F(I + D^2\hat{E}) = 1 \quad \text{in } \mathbb{R}^n \setminus T^{-1}(\Omega_{M_0})\,. \tag{3.12}$$

By (3.10), we apply Lemma 3.5 to $\hat{E}$ (rather, to $r^{-2}\hat{E}(rx)$, for some harmless r) with $\gamma = 2\epsilon - 2$ to obtain

$$|D^2\hat{E}(x)| \le \frac{C}{|x|^{2\epsilon}}\,.$$

It follows that

$$\tilde{a}_{ij}(x)D_{ij}\hat{E}(x) = F(I + D^2\hat{E}) - F(I) = 0 \quad \text{in } \mathbb{R}^n \setminus T^{-1}(\Omega_{M_0})\,, \tag{3.13}$$

where

$$\tilde{a}_{ij}(x) = \int_0^1 F_{\xi_{ij}}(I + sD^2\hat{E}(x))ds\,.$$

Let $e \in \mathbb{R}^n$ be a unit vector; applying D_e and D_{ee} to (3.12) yields, in view of the concavity of $F(\xi)$, that

$$a_{ij}(x)D_{ij}(D_e\hat{E}(x)) = 0\,, \quad x \in \mathbb{R}^n \setminus T^{-1}(\Omega_{M_0})\,, \tag{3.14}$$

and

$$a_{ij}(x)D_{ij}(D_{ee}\hat{E}(x)) \ge 0\,, \quad x \in \mathbb{R}^n \setminus T^{-1}(\Omega_{M_0})\,, \tag{3.15}$$

where

$$a_{ij}(x) := F_{\xi_{ij}}(I + D^2\hat{E}(x)) .$$

Clearly, $(a_{ij}(x))$ and $(\tilde{a}_{ij}(x))$ are positive definite and satisfy

$$|\tilde{a}_{ij}(x) - \delta_{ij}| + |a_{ij}(x) - \delta_{ij}| \le C|x|^{-2\epsilon} .$$

It is well-known that for such coefficients, there exist positive solutions $G(x)$ of $a_{ij}(x)D_{ij}G(x) = 0$ satisfying $\lim_{|x|\to\infty}(|x|^{n-2}G(x)) = 1$. By (3.15) and the maximum principle,

$$e'D^2\hat{E}(x)e = D_{ee}\hat{E}(x) \le CG(x) \le C|x|^{2-n} , \quad x \in T^{-1}(\Omega_{M_0}) .$$

This means that the largest eigenvalue of $(D^2\hat{E}(x))$ is bounded from above by $C|x|^{2-n}$. By (3.13), the least eigenvalue of $(D^2\hat{E}(x))$ is bounded below by a negative constant multiple (depending only on the ellipticity of $(\tilde{a}_{ij}(x))$) of the largest eigenvalue of $(D^2\hat{E}(x))$. Thus,

$$e'D^2\hat{E}(x)e = D_{ee}\hat{E}(x) \ge -C|x|^{2-n} .$$

It follows that

$$|D^2\hat{E}(x)| \le \frac{C}{|x|^{n-2}} \le \frac{C}{|x|} , \quad \mathbb{R}^n \setminus T^{-1}(\Omega_{M_0}) .$$

For $1 \le m \le n$, $D_m\hat{E}$ satisfies (3.14) with $D_e\hat{E}$ replaced by $D_m\hat{E}$. Applying theorem 4 in [15] (with $u = D_m\hat{E}$ and $r \to \infty$), $D_m\hat{E}(x)$ tends to some constant, denoted as b_m, as $|x|$ tends to infinity, namely,

$$\lim_{|x|\to\infty} D\widetilde{E}(x) = 0 \quad \text{where} \quad \widetilde{E}(x) := \hat{E}(x) - b \cdot x .$$

Since $D_m\widetilde{E}$ satisfies the same equation as $D_m\hat{E}$ (i.e., (3.14)), we deduce from the maximum principle that

$$|D\widetilde{E}(x)| \le C|x|^{2-n} \le \frac{C}{|x|} , \quad \mathbb{R}^n \setminus T^{-1}(\Omega_{M_0}) .$$

By (3.13),

$$\widetilde{a}_{ij}(x)D_{ij}\widetilde{E}(x) = 0 , \quad \mathbb{R}^n \setminus T^{-1}(\Omega_{M_0}) .$$

Applying again [15, theorem 4] to $u = \widetilde{E}$ as $r \to \infty$, we have

$$\lim_{|x|\to\infty} \widetilde{E}(x) = c$$

for some constant c. Applying the maximum principle to $\widetilde{E}(x) - c$, we have

$$|\widetilde{E}(x) - c| \le C|x|^{2-n} ,$$

namely,

$$|\hat{E}(x) - b \cdot x - c| \le C|x|^{2-n} .$$

□

3.1 Proof of Theorem 1.2

In this subsection we complete the proof of Theorem 1.2.

For $n \geq 3$, inequality (1.6) follows from Lemma 3.6, and estimate (1.7) follows from (1.6) and Lemma 3.5 with $\gamma = n-2$. This completes the proof of Theorem 1.2 in the case $n \geq 3$.

In the rest of this subsection we prove Theorem 1.2 when $n = 2$. First, instead of Lemma 3.6, we have the following:

LEMMA 3.7 *For $n = 2$, there exist $b \in \mathbb{R}^n$, $c, d \in \mathbb{R}$, and some positive constant C such that*

$$\left| v(x) - \left(\frac{|x|^2}{2} + b \cdot x + d \log |x| + c \right) \right| \leq \frac{C}{|x|} \quad \forall x \in \mathbb{R}^n \setminus T^{-1}(\Omega_{M_0}) .$$

PROOF: By (3.10) and Lemma 3.5, we have

$$|x|^{2\epsilon-1} |\nabla \hat{E}(x)| + |x|^{2\epsilon} |\nabla^2 \hat{E}(x)| + |x|^{1+2\epsilon} |\nabla^3 \hat{E}(x)| \leq C , \quad |x| \geq 2^{\bar{k}} ,$$

where

$$\hat{E}(x) := v(x) - \frac{|x|^2}{2} .$$

Differentiating (3.9) and using the above decay estimates on $D^2 \hat{E}$, we have

$$\left(\delta_{ij} + O(|D^2 \hat{E}|\right) D_{ij}(\hat{E}_m) = 0 , \quad \text{where} \quad \hat{E}_m = \frac{\partial \hat{E}}{\partial x_m} , \quad m = 1, 2.$$

It follows that

$$\Delta \hat{E}_m = O(|D^2 \hat{E}||D^3 \hat{E}|) = O\left(\frac{1}{|x|^{1+4\epsilon}} \right) .$$

Let

$$\hat{\psi}_m(x) = -\frac{1}{2\pi} \int\limits_{|y| \geq 2^{\bar{k}}} \Delta \hat{E}_m(y) (\log |x-y| - \log |y|) dy .$$

Then

$$\Delta(\hat{E}_m - \hat{\psi}_m) = 0 , \quad |x| \geq 2^{\bar{k}} ,$$

and, for any $\epsilon' < 2\epsilon$,

$$|\hat{\psi}_m(x)| \leq C(\epsilon') |x|^{1-\epsilon'} , \quad |x| \geq 2^{\bar{k}} .$$

Since $\hat{E}_m - \hat{\psi}_m$ is harmonic in $|x| > 2^{\bar{k}}$, and its growth is at most of order $|x|^{1-2\epsilon}$,

$$\hat{E}_m(x) - \hat{\psi}_m(x) = O(\log |x|) ,$$

and therefore, for any $\epsilon' < 4\epsilon$,

$$|\nabla \hat{E}(x)| \leq C(\epsilon') (\log |x| + |x|^{1-\epsilon'}) .$$

Integrating the above, we have, for any $\epsilon' < 4\epsilon$,

$$|\hat{E}(x)| \leq C(\epsilon') (|x| \log |x| + |x|^{2-\epsilon'}) .$$

We have improved estimate (3.10) of $|\hat{E}(x)|$. Applying Lemma 3.5 with this improved estimate and arguing as above, we have, for any $\epsilon' < 8\epsilon$,

$$|\hat{E}(x)| \le C(\epsilon')(|x|\log|x| + |x|^{2-\epsilon'}) .$$

By induction, we have, for any $\epsilon' > 0$,

$$|\hat{E}(x)| \le C(\epsilon')|x|^{1+\epsilon'} , \quad |x| \ge 2^{\bar{k}} .$$

Then by Lemma 3.5, we have, for any $\epsilon' > 0$,

$$|x|^{-\epsilon'}|\nabla \hat{E}(x)| + |x|^{1-\epsilon'}|\nabla^2 \hat{E}(x)| + |x|^{2-\epsilon'}|\nabla^3 \hat{E}(x)| \le C , \quad |x| \ge 2^{\bar{k}} .$$

Since $\hat{E}$ satisfies

$$\delta_{ij} + O(|D^2\hat{E}|)D_{ij}\hat{E} = 0 ,$$

we have, for any $\epsilon' > 0$,

$$\Delta\hat{E} = O(|D^2\hat{E}|^2) = O\left(\frac{1}{|x|^{4-\epsilon'}}\right) .$$

Let

$$\hat{\psi}(x) = -\frac{1}{2\pi}\int\limits_{|y|\ge 2^{\bar{k}}} \Delta\hat{E}(y)(\log|x-y| - \log|y|)dy .$$

Then, for any $\epsilon' > 0$,

$$|\hat{\psi}(x)| \le \frac{C(\epsilon')}{|x|^{2-\epsilon'}} , \quad |x| \ge 2^{\bar{k}} .$$

Since $\hat{E} - \hat{\psi}$ is harmonic in $|x| > 2^{\bar{k}}$, and since its growth is at most of order $|x|^{1+\epsilon'}$, we have, for some $b \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$, that

$$\hat{E}(x) - \psi(x) = b\cdot x + d\log|x| + c + O\left(\frac{1}{|x|}\right) .$$

By the decay of ψ,

$$\hat{E}(x) = b\cdot x + d\log|x| + c + O\left(\frac{1}{|x|}\right) .$$

□

Next we have the following proof:

PROOF OF THEOREM 1.2 FOR $n = 2$: Relation (1.8) follows from Lemma 3.7, and estimate (1.10), for some $d \in \mathbb{R}$, follows from (1.8) and Lemma 3.5 with $\gamma = 1$. We only need to establish (1.9). In fact, by making an affine transformation, we only need to establish (1.9) for $A = I$. We first prove it under an additional hypothesis that $f \in C^\infty(\mathbb{R}^2)$. In this case $u \in C^\infty(\mathbb{R}^2)$. Write $w = |x|^2/2$, $\eta = d\log|x|$, $\bar{u} = w + \eta + E$, and

$$\det(D^2\bar{u}) = \bar{u}_{11}\bar{u}_{22} - \bar{u}_{12}^2 = \partial_1(\bar{u}_1\bar{u}_{22}) - \partial_2(\bar{u}_1\bar{u}_{12}) .$$

By (1.3) and $u = \bar{u} + b \cdot x + c$,

$$\det(D^2\bar{u}) = \det(D^2 u) = f\,.$$

By (1.10), as $|x| \to \infty$,

$$|DE(x)| = O\left(\frac{1}{|x|^2}\right) \quad \text{and} \quad |D^2E(x)| = O\left(\frac{1}{|x|^3}\right).$$

Integrating the equation of $\bar{u}$ on B_r and integrating by parts, we have, as $r \to \infty$,

$$\begin{aligned}
\int_{B_r} f &= \int_{|x|=r} \left[(\bar{u}_1\bar{u}_{22})\frac{x_1}{|x|} - (\bar{u}_1\bar{u}_{12})\frac{x_2}{|x|}\right] \\
&= \int_{|x|=r} \left[(w_1 w_{22})\frac{x_1}{|x|} - (w_1 w_{12})\frac{x_2}{|x|}\right] + \int_{|x|=r} \left[(\eta_1 w_{22})\frac{x_1}{|x|} - (\eta_1 w_{12})\frac{x_2}{|x|}\right] \\
&\quad + \int_{|x|=r} \left[(w_1 \eta_{22})\frac{x_1}{|x|} - (w_1 \eta_{12})\frac{x_2}{|x|}\right] + O\left(\frac{1}{r}\right) \\
&= \int_{B_r} \det(D^2 w) + \int_{|x|=r} \left([\eta_1 w_{22} + w_1\eta_{22}]\frac{x_1}{|x|} - (\eta_1 w_{12})\frac{x_2}{|x|}\right) + O\left(\frac{1}{r}\right).
\end{aligned}$$

We know that

$$\int_{B_r} \det(D^2 w) = \pi r^2$$

and

$$\begin{aligned}
\int_{|x|=r} \left([\eta_1 w_{22} + w_1\eta_{22}]\frac{x_1}{|x|} - (\eta_1 w_{12})\frac{x_2}{|x|}\right) &= \frac{2d}{r^3}\int_{|x|=r} (x_1^2) \\
&= \frac{d}{r^3}\int_{|x|=r} |x|^2 = 2\pi d\,.
\end{aligned}$$

It follows that

$$\int_{B_r} (f-1) = 2\pi d + O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty\,.$$

Formula (1.9) for $A = I$ follows after sending r to infinity. As pointed out earlier, formula (1.9) in general follows by applying the special case to $\tilde{u}(x) = u(A^{-1/2}x)$.

If $f \in C^0$ only, let $\bar{u}_\epsilon = \bar{u} * \rho_\epsilon$, the convolution of $\bar{u}$ and ρ_ϵ with $\rho_\epsilon(x) = \epsilon^{-2}\rho(\epsilon^{-1}x)$, where ρ is some nonnegative smooth function of compact support satisfying $\int \rho = 1$. For large r, B_r contains the support of $(f-1)$, and therefore $\bar{u}$ is C^∞ near ∂B_r. We also know that

$$\lim_{\epsilon\to 0}\int_{B_r} \det(D^2\bar{u}_\epsilon) = \int_{B_r} f\,.$$

As shown above,

$$\int_{B_r} \det(D^2\bar{u}_\epsilon) = \int_{|x|=r} \left[(\bar{u}_{\epsilon 1}\bar{u}_{\epsilon 22})\frac{x_1}{|x|} - (\bar{u}_{\epsilon 1}\bar{u}_{\epsilon 12})\frac{x_2}{|x|} \right].$$

Sending ϵ to zero, we have

$$\int_{B_r} f = \int_{|x|=r} \left[(\bar{u}_1\bar{u}_{22})\frac{x_1}{|x|} - (\bar{u}_1\bar{u}_{12})\frac{x_2}{|x|} \right].$$

Following previous arguments, we obtain (1.9) (for continuous f). □

3.2 Proof of Corollary 1.3

Finally, we can prove Corollary 1.3.

By enlarging O slightly, we may assume that $u \in C^0(\mathbb{R}^n \setminus O)$. We divide the proof into three steps.

Step 1. First we prove Corollary 1.3 under the additional hypothesis that u can be extended as a convex function on $\mathbb{R}^n$.

In this case, we first show that

$$u \in C^\infty(\mathbb{R}^n \setminus \overline{O}) . \tag{3.16}$$

For $\bar{x} \in \mathbb{R}^n \setminus \overline{O}$, by subtracting from u a supporting plane to the graph at $(\bar{x}, u(\bar{x}))$, we may assume that $u \geq 0$ on $\mathbb{R}^n \setminus \overline{O}$ and $u(\bar{x}) = 0$. An application of Lemma 2.9 yields that $u(x) \to \infty$ as $|x| \to \infty$. For large M, $\{u < M\}$ contains $\bar{x}$ and $\overline{O}$. Applying theorem 1 in [3] on $\{u < M\} \setminus \overline{O}$, we know that $\bar{x}$ is the only point in $\{u < M\} \setminus \overline{O}$ where $u = 0$. So for $\epsilon > 0$ small, $\{u < \epsilon\}$ is a convex open subset of $\mathbb{R}^n \setminus \overline{O}$. By Proposition 2.4 and Remark 2.5, u is C^∞ in $\{u < \epsilon\}$. In particular, u is C^∞ near $\bar{x}$. (3.16) is established.

For large M, $\{u < M\}$ is a strictly convex, bounded open set containing $\overline{O}$. As explained at the beginning of Section 3, we can keep u fixed outside $\{u < M\}$ while redefining u inside $\{u < M\}$ so that the new u is in $C^\infty(\mathbb{R}^n)$ and $(D^2u) > 0$ on $\mathbb{R}^n$. Let $f = \det(D^2u)$; Corollary 1.3 follows from Theorem 1.2 applied to the new u.

Step 2. We show that there exists some affine function $l(x)$ such that

$$u(x) - l(x) \geq 0 \quad \text{in } \mathbb{R}^n \setminus \overline{O} . \tag{3.17}$$

Without loss of generality, we may assume that $\overline{O} \subset B_1$. Fix some $\lambda \gg 1$; by subtracting from u a supporting plane to the graph at $(-\lambda e_1, u(-\lambda e_1))$, we may assume that $u \geq 0$ on $\mathbb{R}^n \setminus \Gamma$, where Γ denotes the cone generated by $-\lambda e_1$ and B_1. Fix some $M > 0$ such that

$$u(x) < M , \quad x \in B_3 \setminus O .$$

By Lemma 2.9, $\lim_{\alpha\to\infty} u(-\alpha e_2) = \infty$. Let α be the largest value such that $u(-\alpha e_2) \leq M$. Then $3 \leq \alpha < \infty$ and $u(-\alpha e_2) = M$. Let Γ_i denote the closed cone generated by $-\alpha e_2$ and B_i, $i = 1, 2$, and let $l_1(x)$ denote a supporting plane of the graph of u at $(-\alpha e_2, u(-\alpha e_2))$. Then $l_1(-\alpha e_2) = M$ and $u(x) \geq l_1(x)$ for $x \in \mathbb{R}^n \setminus \Gamma_1$. Since $u \leq M$ on $B_2 \setminus O$, we know that $l_1(x) \leq M$ on Γ_2. Since λ is large, $\Gamma_2 \cap \Gamma$ is compact. By Lemma 2.9,

$$\lim_{x \in \Gamma_1, |x| \to \infty} u(x) = \infty .$$

It follows that

$$u(x) \geq l_1(x) \text{ on } \mathbb{R}^n \setminus \{\text{a compact set}\}.$$

Therefore (3.17) holds with $l(x) = l_1(x) - c_2$ for a suitably large c_2.

Step 3. Now we can complete the proof of Corollary 1.3.

By step 2, we may assume without loss of generality that $u \geq 0$ on $\mathbb{R}^n \setminus O$. By Lemma 2.9, $\lim_{|x| \to \infty} u(x) = \infty$. It follows that, for large M, the function

$$\tilde{u}(x) = \begin{cases} M, & x \in \overline{O}, \\ \max\{M, u(x)\}, & x \in \mathbb{R}^n \setminus \overline{O}, \end{cases}$$

is convex on $\mathbb{R}^n$. Corollary 1.3(i) and Corollary 1.3(ii) follow from step 2 applied to $\tilde{u}$. In the following we show that $u \in C^\infty(\mathbb{R}^n \setminus \overline{O})$. For $\bar{x} \in \mathbb{R}^n \setminus \overline{O}$, let $l(x)$ be a supporting plane to the graph of u at $(\bar{x}, u(\bar{x}))$. We already know that u grows quadratically at infinity. So, by applying [3, theorem 1], we know that $\bar{x}$ is an isolated local minimum point of $u(x) - l(x)$. By the same argument as in step 1, u is C^∞ near $\bar{x}$.

4 Proof of Theorem 1.7

The uniqueness part in the proof of 1.7 can be deduced easily from the maximum principle (see, e.g., Lemma A.2). By the affine invariance of the problem, we may assume $A = I$, $b = 0$, and $c = 0$. We may also assume that

$$\text{support of } (f - 1) \subset B_1 .$$

First, we explore the proof under the additional hypothesis that $f \in C^\infty(\mathbb{R}^n)$. For $R > 1$, let $u_R \in C^\infty(\overline{B}_R)$ be the unique convex solution of

$$\begin{cases} \det(D^2 u_R) = f & \text{on } B_R \\ u_R = \frac{R^2}{2} & \text{on } \partial B_R . \end{cases} \tag{4.1}$$

We will show that as R tends to infinity, u_R tends to some u that satisfies (1.3) and, for some constant depending only on n, $\min_{\mathbb{R}^n} f$ and $\max_{\mathbb{R}^n} f$.

$$\sup_{\mathbb{R}^n} |u(x) - \frac{|x|^2}{2}| \leq C . \tag{4.2}$$

To prove the above, we need some barrier functions. Let $h(r)$ be defined on $r > 0$, and let

$$u(x) = h(|x|) .$$

Then

$$D^2u(x) = \begin{pmatrix} h''(r) & & & \\ & \frac{h'(r)}{r} & & \\ & & \ddots & \\ & & & \frac{h'(r)}{r} \end{pmatrix}.$$

So,

$$\det(D^2u(x)) = h''(r)\left(\frac{h'(r)}{r}\right)^{n-1}.$$

For $a = 2^n(\max_{\mathbb{R}^n} f)$, let

$$h_-(r) = \begin{cases} \int_1^r (s^n + a)^{1/n}\,ds, & r \geq 1, \\ (\max_{\mathbb{R}^n} f)^{1/n}(r^2 - 1), & 0 \leq r < 1, \end{cases}$$

and

$$u_-(x) = h_-(|x|)\,, \quad x \in \mathbb{R}^n\,.$$

Then $u_- \in C^0(\mathbb{R}^n) \cap C^\infty(\overline{B}_1) \cap C^\infty(\overline{\mathbb{R}^n \setminus B_1})$, u_- is locally convex in $\mathbb{R}^n \setminus B_1$,

$$\det(D^2u_-) = 1 \quad \text{on } \mathbb{R}^n \setminus \overline{B}_1\,,$$

$$\det(D^2u_-) > f \quad \text{on } B_1\,,$$

and

$$\lim_{r \to 1^-} h_-'(r) < \lim_{r \to 1^+} h_-'(r)\,. \tag{4.3}$$

It is easy to see that (recall that $n \geq 3$)

$$\sup_{x \in \mathbb{R}^n} \left| u_-(x) - \frac{|x|^2}{2} \right| < \infty\,. \tag{4.4}$$

Next we define

$$h_+(r) = \int_1^r (s^n - 1)^{1/n}\,ds\,, \quad r \geq 1\,,$$

and

$$u_+(x) = \begin{cases} h_+(|x|), & |x| \geq 1, \\ 0, & |x| < 1. \end{cases}$$

Then $u_+ \in C^0(\mathbb{R}^n) \cap C^1(\overline{B}_1) \cap C^\infty(\mathbb{R}^n \setminus \overline{B}_1)$, u_+ is locally convex in $\mathbb{R}^n \setminus B_1$, and

$$\det(D^2u_+) = 1 \quad \text{on } \mathbb{R}^n \setminus \overline{B}_1\,.$$

We also know that

$$h_+'(1) = 0 \tag{4.5}$$

and

$$\sup_{x \in \mathbb{R}^n} \left| u_+(x) - \frac{|x|^2}{2} \right| < \infty\,. \tag{4.6}$$

By (4.4) and (4.6),

$$\beta_+ := \sup_{x\in\mathbb{R}^n} \left(\frac{|x|^2}{2} - u_+(x) \right) < \infty$$

and

$$\beta_- := \inf_{x\in\mathbb{R}^n} \left(\frac{|x|^2}{2} - u_-(x) \right) > -\infty\,.$$

We will use $(u_+ + \beta_+)$ and $(u_- + \beta_-)$ as barrier functions to establish the following:

LEMMA 4.1 *Let u_R be defined in* (4.1) *for $R > 1$; then*

$$u_-(x) + \beta_- \le u_R(x) \le u_+(x) + \beta_+ \quad \forall x \in B_R\,. \tag{4.7}$$

PROOF: Let $R > 1$; for β sufficiently large, we have

$$u_+(x) + \beta \ge u_R(x)\,, \quad x \in \overline{B}_R\,.$$

Let $\bar\beta$ be the smallest number for which the above holds with $\beta = \bar\beta$. If $\bar\beta \le \beta_+$, then the second inequality in (4.7) holds. Otherwise, $\bar\beta > \beta_+$, and for some $\bar x \in \overline{B}_R$,

$$u_R(\bar x) = u_+(\bar x) + \bar\beta\,.$$

In view of the boundary data of u_R and the definition of β_+ (recall that $\bar\beta > \beta_+$), we must have

$$|\bar x| < R\,.$$

We also know that

$$|\bar x| > 1\,.$$

Indeed, if $|\bar x| \le 1$, the supporting plane of the graph of u_R at $(\bar x, u_R(\bar x))$ must be horizontal ((4.5) is used when $|\bar x| = 1$), and therefore $u_R(x) = \bar\beta$ for all $|x| \le 1$, which is impossible.

On the other hand, by the strong maximum principle, $1 < |\bar x| < R$ cannot occur either. We have established the second inequality in (4.7).

To establish the first inequality, we argue similarly, using $(u_- + \beta_-)$ as a barrier function. First, we know that for β very negative, we have

$$u_-(x) + \beta \le u_R(x)\,, \quad x \in \overline{B}_R\,.$$

Let $\bar\beta$ be the largest number for which the above holds with $\beta = \bar\beta$. If $\bar\beta \ge \beta_-$, then the first inequality in (4.7) holds. Otherwise, $\bar\beta < \beta_-$, and for some $\bar x \in \overline{B}_R$,

$$u_R(\bar x) = u_-(\bar x) + \bar\beta\,.$$

Since

$$\det(D^2 u_-) \ge \det(D^2 u_R) \quad \text{on } B_R \setminus \overline{B}_1$$

and

$$\det(D^2u_-) > \det(D^2u_R) \quad \text{on } B_1 ,$$

we have, by the maximum principle, $|\bar{x}| = 1$. But this is impossible in view of (4.3) and the smoothness of u_R. The first inequality in (4.7) is established. □

By Lemma 4.1, we can apply Pogorelov estimates and then Evans-Krylov estimates and Schauder theory to u_R to obtain that on every compact subset K of $\mathbb{R}^n$,

$$\|u_R\|_{C^k(K)} \le C(K,k) \quad \forall k .$$

So along a subsequence $R_i \to \infty$,

$$u_{R_i} \to u \quad \text{in } C^k_{\text{loc}}(\mathbb{R}^n) \quad \forall k .$$

It follows that u is convex and satisfies (1.3) and

$$u_-(x) + \beta_- \le u(x) \le u_+(x) + \beta_+ \quad \text{on } \mathbb{R}^n . \tag{4.8}$$

In particular, $u \in C^\infty(\mathbb{R}^n)$ and satisfies (1.3) and (4.2).

Without the additional smoothness hypothesis on f, let $f_\epsilon = f * \rho_\epsilon$, where ρ_ϵ is the usual mollifier. Let u_ϵ be the solution found above for f_ϵ. We know from the proof that $|u_\epsilon(x) - \frac{1}{2}|x|^2| \le C$ on $\mathbb{R}^n$ for some C depending only on n and f. Then, by the convexity of u_ϵ and the above estimates, $\{|u_\epsilon| + |Du_\epsilon|\}$ is uniformly bounded on any compact subset of $\mathbb{R}^n$. Passing to a subsequence (still denoted as $\{u_\epsilon\}$), $u_\epsilon \to u$ in $C^0_{\text{loc}}(\mathbb{R}^n)$ for some convex function u. So u is a viscosity solution of (1.3) and satisfies (4.2).

By Theorem 1.2, there exist $b \in \mathbb{R}^n$, $c \in \mathbb{R}$, and $A \in \mathcal{A}$ such that (1.6) holds. In view of (4.2), $A = I$ and $b = 0$. Replacing u by $u - c$ (but still calling it u), clearly $E(x) := u(x) - [|x|^2/2 + c]$ satisfies (1.11). To see (1.12), we first deduce from (4.8) that

$$|E(x)| \le C , \quad x \in \mathbb{R}^n .$$

Here and in the following, C and r denote various large positive constants depending only on n and f. Applying Lemma 3.5, with $\gamma = 0$, we have

$$|D^2E(x)| \le \frac{C}{|x|^2} , \quad |x| \ge r .$$

Since E satisfies

$$\tilde{a}_{ij}(x) D_{ij} E = 0 \ \text{ on } \mathbb{R}^n \setminus \overline{B}_1 \quad \text{with} \quad \tilde{a}_{ij}(x) = \int_0^1 F_{\xi_{ij}}(I + sD^2E(x))ds .$$

In view of the above estimates on D^2E, the Green's function for $(\tilde{a}_{ij}(x))$ is bounded by $C|x|^{2-n}$ for $|x| \ge r$. Estimate (1.12) then follows from (1.11) and the maximum principle. Estimate (1.13) follows from (1.12) and Lemma 3.5 with $\gamma = n - 2$. Theorem 1.7 is established.

5 Proof of Theorem 1.5

By an affine transformation and by subtracting a linear function from u, we only need to prove the theorem for $A = I$, $b = 0$, and $B_2(0) \subset D$. These will be assumed below.

LEMMA 5.1 *There exists some constant C, depending only on n, φ, and D, such that, for every $\xi \in \partial D$, there exists $\bar{x}(\xi) \in \mathbb{R}^n$ satisfying*

$$|\bar{x}(\xi)| \le C \quad \text{and} \quad w_\xi < \varphi \ \text{ on } \partial D \setminus \{\xi\},$$

where

$$w_\xi(x) := \varphi(\xi) + \frac{1}{2}\big(|x - \bar{x}(\xi)|^2 - |\xi - \bar{x}(\xi)|^2\big), \quad x \in \mathbb{R}^n.$$

PROOF: Let $\xi \in \partial D$. By a translation and a rotation, we may assume without loss of generality that $\xi = 0$ and ∂D is locally represented by the graph of

$$x_n = \rho(x') := \frac{1}{2} \sum_{1 \le \alpha, \beta \le n-1} B_{\alpha\beta} x_\alpha x_\beta + \circ(|x'|^2),$$

and φ locally has the expansion

$$\varphi(x', \rho(x')) = \varphi(0) + \varphi_{x_1}(0) x_1 + \frac{1}{2} \sum_{1 \le \alpha, \beta \le n-1} A_{\alpha\beta} x_\alpha x_\beta + \circ(|x'|^2),$$

where $x' = (x_1, \dots, x_{n-1})$ and $(B_{\alpha\beta})$ is positive definite.

Let

$$\bar{x} = (-\varphi_{x_1}(0), 0, \dots, 0, \bar{x}_n)$$

and

$$w(x) = \varphi(0) + \frac{1}{2}(|x - \bar{x}|^2 - |\bar{x}|^2), \quad x \in \mathbb{R}^n.$$

Then

$$w(x', \rho(x')) = \varphi(0) + \varphi_{x_1}(0) x_1 + \frac{1}{2}\big[|x'|^2 + \rho(x')^2\big] - \bar{x}_n \rho(x').$$

It follows that

$$(w - \varphi)(x', \rho(x')) = \\ \frac{1}{2}\big[|x'|^2 + \rho(x')^2\big] - \frac{1}{2} \sum_{1 \le \alpha, \beta \le n-1} A_{\alpha\beta} x_\alpha x_\beta - \bar{x}_n \rho(x') + \circ(|x'|^2).$$

By the strict convexity of ∂D, there exists some constant $\delta > 0$ depending only on D such that

$$\rho(x') \ge \delta |x'|^2 \quad \forall |x'| < \delta. \tag{5.1}$$

Clearly, for large $\bar{x}_n$, we have

$$(w - \varphi)(x', \rho(x')) < 0 \quad \forall 0 < |x'| < \delta.$$

The largeness of $\bar{x}_n$ depends only on δ and φ.

On the other hand,

$$w(x) = \varphi(0) + \frac{1}{2}|x|^2 + \varphi_{x_1}(0)x_1 - x_n\bar{x}_n .$$

By the strict convexity of ∂D and (5.1),

$$x_n \geq \delta^3 \quad \forall x \in \partial D \setminus \{(x', \rho(x')) : |x'| < \delta\} .$$

It follows that

$$w(x) \leq \varphi(0) + \frac{1}{2}|x|^2 + \varphi_{x_1}(0)x_1 - \delta^3\bar{x}_n \quad \forall x \in \partial D \setminus \{(x', \rho(x')) : |x'| < \delta\} .$$

By making $\bar{x}_n$ larger (still under control), we have

$$w(x) - \varphi(x) < 0 \quad \forall x \in \partial D \setminus \{(x', \rho(x')) : |x'| < \delta\} .$$

Lemma 5.1 is established. □

Fix some $c_1 \in \mathbb{R}^n$ such that

$$w_\xi(x) \leq \frac{1}{2}|x|^2 + c_1 \quad \forall \xi \in \partial D,\ x \in \mathbb{R}^n \setminus D,\ \operatorname{dist}(x, \partial D) \leq 1 .$$

For $x \in \mathbb{R}^n \setminus \overline{D}$, let $\mathcal{S}_{c,x}$ denote the set of functions in $C^0(\mathbb{R}^n \setminus D)$ that are locally convex viscosity subsolutions of

$$\det(D^2 w) \geq 1 \quad \text{in } \mathbb{R}^n \setminus \overline{D}$$

satisfying

$$w \leq \varphi \quad \text{on } \partial D$$

and

$$w(y) \leq \frac{1}{2}|y|^2 + c \quad \forall y \in \mathbb{R}^n \setminus D,\ |y - x| \leq 2\operatorname{diam}(D) .$$

Clearly $\mathcal{S}_{c,x}$ is nonempty for all x and c.

Define

$$u_c(x) = \sup\{w(x) : w \in \mathcal{S}_{c,x}\}, \quad x \in \mathbb{R}^n \setminus \overline{D} .$$

LEMMA 5.2 *We have*

(i) $u_c(x) \leq \frac{1}{2}|x|^2 + c$, $x \in \mathbb{R}^n \setminus \overline{D}$;
(ii) u_c *is a locally convex viscosity subsolution of* $\det(D^2u_c) = 1$ *in* $\mathbb{R}^n \setminus \overline{D}$;
(iii) u_c *can be extended to a continuous function on* $\mathbb{R}^n \setminus D$ *with* $u_c = \varphi$ *on* ∂D; *and*
(iv) u_c *is a viscosity solution of* $\det(D^2u_c) = 1$ *on* $\mathbb{R}^n \setminus \overline{D}$.

PROOF: (i) follows from the definition since $w(x) \leq \frac{1}{2}|x|^2 + c$ for all $w \in \mathcal{S}_{c,x}$. (ii) holds since u_c locally is the sup over a family of convex viscosity subsolutions.

For $\bar{\xi} \in \partial D$ and x close to $\bar{\xi}$, since $w_\xi \in \mathcal{S}_{c,x}$ for all $\xi \in \partial D$, we have $u_c(x) \geq w_{\bar{\xi}}(x)$ for x close to $\bar{\xi}$. It follows that $\liminf_{x\to\bar{\xi}} u_c(x) \geq \varphi(\bar{\xi})$. On the other hand, $\limsup_{x\to\bar{\xi}} u_c(x) \leq \varphi(\bar{\xi})$. Indeed, if along a sequence $x_i \to \bar{\xi}$,

$\lim_{i\to\infty} u_c(x_i) \geq \varphi(\bar{\xi}) + 3\delta$ for some $\delta > 0$. Then by the definition of u_c, there exists $w_i \in \mathcal{S}_{c,x_i}$ such that $w_i(x_i) \geq \varphi(\bar{\xi}) + 2\delta$ for large i. But w_i is locally convex and, for ξ close to $\bar{\xi}$, $w_i(\xi) \leq \varphi(\xi) < \varphi(\bar{\xi}) + \delta$. This forces w_i to be unbounded near $\bar{\xi}$, contradicting the fact that $w_i \in \mathcal{S}_{c,x_i}$. (iii) is established.

For $\bar{x} \in \mathbb{R}^n \setminus \overline{D}$, fix some ϵ satisfying $0 < \epsilon < 2\,\mathrm{diam}(D)$ and $B_\epsilon(\bar{x}) \subset \mathbb{R}^n \setminus \overline{D}$. By the definition of u_c,

$$u_c(y) \leq \frac{1}{2}|y|^2 + c \quad \forall |y - \bar{x}| \leq \epsilon\,.$$

It is well-known (see, e.g., the appendix) that there is a unique convex viscosity solution $\widetilde{u} \in C^0(\overline{B_\epsilon(\bar{x})})$ to

$$\begin{cases} \det(D^2 \widetilde{u}) = 1 & \text{in } B_\epsilon(\bar{x}) \\ \widetilde{u} = u_c & \text{on } \partial B_\epsilon(\bar{x}). \end{cases}$$

By the maximum principle, $\widetilde{u} \geq u_c$ on $B_\epsilon(\bar{x})$. Define

$$w(y) = \begin{cases} \widetilde{u}(y) & \text{if } y \in B_\epsilon(\bar{x}) \\ u_c(y) & \text{if } y \in \mathbb{R}^n \setminus (D \cup B_\epsilon(\bar{x}))\,. \end{cases}$$

Clearly, $w \in \mathcal{S}_{c,x}$. So, by the definition of u_c, $u_c \geq w$ on $B_\epsilon(\bar{x})$. It follows that $u_c \equiv \widetilde{u}$ on $B_\epsilon(\bar{x})$. (iv) is established. □

For $b \in \mathbb{R}$ and $a > -1$, let

$$w_{a,b}(x) = b + \int_1^{|x|} (s^n + a)^{1/n}\, ds\,.$$

Then $w_{a,b}$ is a locally convex smooth solution of

$$\begin{cases} \det(D^2 w_{a,b}) = 1\,, & \mathbb{R}^n \setminus B_1\,, \\ w_{a,b} = b\,, & \partial B_1\,. \end{cases}$$

Let $\bar{r} = 2\,\mathrm{diam}(D)$ and $b(a) := \min_{\partial D} \varphi - \int_1^{\bar{r}} (s^n + a)^{1/n}\, ds$; clearly $w_{a,b(a)} \leq \varphi$ on ∂D. It is easy to see that

$$\begin{aligned}
&\lim_{|x|\to\infty} \left(w_{a,b(a)}(x) - \frac{|x|^2}{2} \right) \\
&= b(a) - \frac{1}{2} + \int_1^\infty s\left(\left(1 + \frac{a}{s^n}\right)^{1/n} - 1\right) ds \\
&= \mu(a) := \min_{\partial D} \varphi - \frac{1}{2} - \int_1^{\bar{r}} (s^n + a)^{1/n}\, ds + \int_1^\infty s\left(\left(1 + \frac{a}{s^n}\right)^{1/n} - 1\right) dsr \\
&= \min_{\partial D} \varphi - \frac{1}{2} - \frac{1}{2}(\bar{r}^2 - 1) + \int_{\bar{r}}^\infty s\left(\left(1 + \frac{a}{s^n}\right)^{1/n} - 1\right) ds\,.
\end{aligned}$$

Clearly $\mu(a)$ is smooth, strictly monotonically increasing, and $\mu(a) \to \infty$ as $a \to \infty$.

Fix some $a_* > -1$ such that $c_* := \mu(a_*) \ge c_1$.

LEMMA 5.3 *For $c > c_*$,*

$$\liminf_{|x|\to\infty} \left(u_c(x) - \frac{1}{2}|x|^2 \right) \ge c\,.$$

PROOF: For all $\mu^{-1}(c_*) < a < \mu^{-1}(c)$,

$$\lim_{|x|\to\infty} \left(w_{a,b(a)}(x) - \frac{|x|^2}{2} \right) = \mu(a) < c\,.$$

It follows that $w_{a,b(a)} \in \mathcal{S}_{c,x}$ for sufficiently large $|x|$. It follows that $u_c(x) \ge w_{a,b(a)}(x)$ and therefore

$$\liminf_{|x|\to\infty} \left(u_c(x) - \frac{|x|^2}{2} \right) \ge \mu(a)\,.$$

Lemma 5.3 follows after sending a to $\mu^{-1}(c)$. □

PROOF OF THEOREM 1.5: It follows from Lemma 5.2, Lemma 5.3, and Theorem 1.2. □

Appendix

The following lemma and its proof can be found in [3]. For the reader's convenience, we include them here.

LEMMA A.1 *Let Ω be a convex open set with* $\operatorname{diam}(\Omega) \le 1$*, and let $u \in C^0(\overline{\Omega})$ be a convex viscosity solution of*

$$\begin{cases} \det(D^2u) \le 1 & \text{in } \Omega \\ u \ge 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$u(x) \ge \begin{cases} -C(n)\operatorname{dist}(x,\partial\Omega)^{2/n} & \forall x \in \Omega,\ n \ge 3, \\ -C(\alpha)\operatorname{dist}(x,\partial\Omega)^{\alpha} & \forall x \in \Omega,\ n = 2,\ 0 < \alpha < 1. \end{cases}$$

PROOF: Pick a point on $\partial\Omega$, call it the origin 0, and then let the x_n-axis point in the inward normal direction of $\partial\Omega$. Let

$$h(x) = \begin{cases} (|x'|^2 - C)x_n^{2/n}\,, & n \ge 3\,, \\ (x_1^2 - C)x_2^{\alpha}\,, & n = 2\,, \end{cases}$$

where $0 < \alpha < 1$. As in [3], for C suitably large (depending only on n when $n \ge 3$, while depending only on α when $n = 2$), h satisfies

$$(D^2h) > 0\,, \quad \det(D^2h) \ge 1 \ \text{ on } \Omega\,, \quad \text{and} \quad h \le 0 \ \text{ on } \partial\Omega\,.$$

By the maximum principle,

$$h \leq u \quad \text{on } \Omega\,,$$

in particular,

$$u(x', x_n) \geq h(x', 0)\,, \quad (x', x_n) \in \Omega\,.$$

Lemma A.1 follows. □

LEMMA A.2 *Let B be a ball in $\mathbb{R}^n$, $n \geq 2$, and let f be a positive continuous function on $\overline{B}$. Assume that u and v are convex continuous functions on $\overline{B}$ that satisfy, in the viscosity sense,*

$$\det(D^2 u) \leq f \;\; \textit{in } B \quad \textit{and} \quad \det(D^2 u) \geq f \;\; \textit{in } B\,.$$

Assume also that

$$u \geq v \quad \textit{on } \partial B\,.$$

Then

$$u \geq v \quad \textit{on } \overline{B}\,.$$

PROOF: We may assume without loss of generality that $u > v$ on ∂B. Indeed, we may consider $u + \epsilon$ for $\epsilon > 0$ and then let ϵ tend to zero. We prove it by contradiction. Suppose the contrary; for some $\bar{x} \in B$,

$$(u - v)(\bar{x}) = \min_{\overline{B}}(u - v) < 0\,.$$

Let $\{f_j^-\}$, $\{f_j^+\} \subset C^\infty(\overline{B})$, satisfy

$$f_j^- > f > f_j^+ \;\; \text{on } \overline{B} \quad \text{and} \quad f_j^\pm \to f \;\; \text{in } C^0(\overline{B})\,.$$

Let $\varphi \in C^\infty(\partial B)$ satisfy $v + \epsilon < \varphi < u - \epsilon$ on ∂B where $3\epsilon = \min_{\partial B}(u - v) > 0$. By Proposition 2.4, let $w_j^\pm \in C^0(\overline{B}) \cap C^\infty(B)$ be strictly convex solutions of

$$\begin{cases} \det(D^2 w_j^\pm) = f_j^\pm & \text{in } B \\ w_j^\pm = \varphi & \text{on } \partial B. \end{cases}$$

Using $w_j^\pm$ as test functions in the definition of viscosity solutions, we have

$$u - \epsilon \geq w_j^- \;\; \text{on } \overline{B} \quad \text{and} \quad w_j^+ \geq v + \epsilon \;\; \text{on } \overline{B}\,;$$

in particular,

$$u(\bar{x}) - \epsilon \geq w_j^-(\bar{x})\,, \quad w_j^+(\bar{x}) \geq v(\bar{x}) + \epsilon\,.$$

Since $u(\bar{x}) < v(\bar{x})$, we have

$$w_j^+(\bar{x}) \geq w_j^-(\bar{x}) + 2\epsilon\,. \tag{A.1}$$

By the Alexandrov estimate and (A.1), we have

$$2\epsilon \leq -\min_{\overline{B}}\left(w_j^- - w_j^+\right) \leq C\Bigg(\int_{S_j^+} \det\left(D^2(w_j^- - w_j^+)\right)\Bigg)^{1/n}, \tag{A.2}$$

where $S_j^+ = \{x \in B : D^2(w_j^- - w_j^+)(x) > 0\}$.

Write

$$\frac{D^2 w_j^-(x)}{2} = \frac{[D^2 w_j^-(x) - D^2 w_j^+(x)] + [D^2 w_j^+(x)]}{2}.$$

For $x \in S_j^+$, $D^2 w_j^-(x) - D^2 w_j^+(x)$ is positive definite, so by the concavity of $(\det)^{1/n}$, we have

$$\left(\det\left[\frac{D^2 w_j^-(x)}{2}\right]\right)^{1/n} \geq \frac{1}{2}\left(\det\left[D^2 w_j^-(x) - D^2 w_j^+(x)\right]\right)^{1/n} + \frac{1}{2}\left(\det\left[D^2 w_j^+(x)\right]\right)^{1/n},$$

i.e.,

$$\frac{1}{2} f_j^-(x)^{1/n} \geq \frac{1}{2}\left(\det\left[D^2 w_j^-(x) - D^2 w_j^+(x)\right]\right)^{1/n} + \frac{1}{2} f_j^+(x)^{1/n}.$$

Since $f_j^\pm \to f$ in $C^0(\overline{B})$, we have

$$\sup_{x \in S_j^+} \det\left[D^2 w_j^-(x) - D^2 w_j^+(x)\right] \to 0.$$

Sending j to ∞ in (A.2), we have $2\epsilon \leq 0$, which is a contradiction. □

LEMMA A.3 *Let B be a ball of $\mathbb{R}^n$ and $\varphi \in C^0(\partial B)$. Assume that $\underline{u} \in C^0(\overline{B})$ is a convex viscosity subsolution to* $\det(D^2\underline{u}) \geq 1$. *Then*

$$\begin{cases} \det(D^2 u) = 1 & \text{in } B \\ u = \underline{u} & \text{on } \partial B \end{cases}$$

has a unique convex viscosity solution $u \in C^0(\overline{B})$.

Remark A.4. The same conclusion holds when replacing B by any bounded convex open set of $\mathbb{R}^n$.

PROOF: Uniqueness follows from the maximum principle. Let $\varphi_i \in C^\infty(\partial B)$ satisfy

$$\underline{u} < \varphi_i \leq \underline{u} + \frac{1}{i} \text{ on } \partial B \quad \text{and} \quad \varphi_i \to \underline{u} \text{ in } C^0(\partial B). \tag{A.3}$$

It follows from [6] that there exists a unique, strictly convex solution $u_i \in C^\infty(\overline{B})$ of $\det(D^2 u_i) = 1$ in B and $u_i = \varphi_i$ on ∂B. By the maximum principle, $\underline{u} \leq u_i \leq h_i$ on $\overline{B}$, where h_i is the harmonic function on B with boundary value φ_i. By the convexity and the uniform bound of $\{u_i\}$, $|\nabla u_i|$ is bounded on compact subsets of B. So, after passing to a subsequence, u_i uniformly converges on compact subsets of B to some convex function $u \in C^0(B)$. Consequently, u is a viscosity solution to $\det(D^2 u) = 1$. Clearly, $\underline{u} \leq u \leq h$ on B, where h is the harmonic function on B with boundary value $h = \underline{u}$. It follows that u can be extended as a continuous function on $\overline{B}$ with $u = \underline{u}$ on ∂B. Lemma A.3 is established. □

The following is a linear algebra lemma.

LEMMA A.5 *Let U be an $n \times n$ real upper-triangular matrix. Assume that the diagonals of U are nonnegative and, for some $0 < \epsilon < 1$,*

$$(1-\epsilon)B \subset U(B) \subset (1+\epsilon)B\,, \tag{A.4}$$

where $B \subset \mathbb{R}^n$ is the unit ball centered at the origin. Then for some constant $C = C(n)$,

$$\|U - I\| \le C\epsilon\,. \tag{A.5}$$

PROOF: Letting $U = (U_{ij})$, we know that $U_{ij} = 0$ for $i < j$. Since $U(B)$ contains an open neighborhood of $\mathbb{R}^n$, U is invertible, and therefore $U_{ii} > 0\ \forall i$. Write $U^{-1} = (U^{ij})$; then U^{-1} is also upper-triangular, $U^{ii} = 1/U_{ii}$ for every i, and, by (A.4),

$$\frac{1}{1+\epsilon}B \subset U^{-1}(B) \subset \frac{1}{1-\epsilon}B\,.$$

For $1 \le k \le n$, let e_k denote the unit vector with the k^{th} component equal to 1 and all the other components equal to zero. By (A.4)

$$\|Ue_k\| = \sqrt{\sum_{j=1}^{n} U_{jk}^2} \le 1+\epsilon\,; \tag{A.6}$$

in particular, $U_{kk} \le 1+\epsilon$, $1 \le k \le n$. The same argument can be applied to U^{-1}, so, in particular,

$$\frac{1}{U_{kk}} = U^{kk} \le \frac{1}{1-\epsilon}\,, \quad 1 \le k \le n\,.$$

We deduce from the above two estimates that

$$1-\epsilon \le U_{kk} \le 1+\epsilon\,, \quad 1 \le k \le n\,. \tag{A.7}$$

It follows from (A.6) and (A.7) that

$$\sum_{j \ne k} U_{jk}^2 \le (1+\epsilon)^2 - (1-\epsilon)^2 = 4\epsilon\,, \quad 1 \le k \le n\,. \tag{A.8}$$

Estimate (A.5) follows from (A.7) and (A.8). □

Bibliography

[1] Aubin, T. Équations de Monge-Ampère réelles. *J. Funct. Anal.* **41** (1981), no. 3, 354–377.

[2] Caffarelli, L. A. Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math. (2)* **131** (1990), no. 1, 135–150.

[3] Caffarelli, L. A. A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. *Ann. of Math. (2)* **131** (1990), no. 1, 129–134.

[4] Caffarelli, L. A. Topics in PDEs: The Monge-Ampère equation. Graduate course. Courant Institute, New York University, 1995.

[5] Caffarelli, L. A.; Cabré, X. *Fully nonlinear elliptic equations.* American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence, R.I., 1995.

[6] Caffarelli, L.; Nirenberg, L.; Spruck, J. The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation. *Comm. Pure Appl. Math.* **37** (1984), no. 3, 369–402.

[7] Calabi, E. Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. *Michigan Math. J.* **5** (1958), 105–126.

[8] Cheng, S. Y.; Yau, S. T. On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x, u)$. *Comm. Pure Appl. Math.* **30** (1977), no. 1, 41–68.

[9] Cheng, S. Y.; Yau, S. T. Complete affine hypersurfaces. I. The completeness of affine metrics. *Comm. Pure Appl. Math.* **39** (1986), no. 6, 839–866.

[10] Chou, K.-S.; Wang, X.-J. Entire solutions of the Monge-Ampère equation. *Comm. Pure Appl. Math.* **49** (1996), no. 5, 529–539.

[11] de Guzmán, M. *Differentiation of integrals in* R^n. Lecture Notes in Mathematics, 481. Springer, Berlin–New York, 1975.

[12] Delanoë, P. Partial decay on simple manifolds. *Ann. Global Anal. Geom.* **10** (1992), no. 1, 3–61.

[13] Ferrer, L.; Martínez, A.; Milán, F. An extension of a theorem by K. Jörgens and a maximum principle at infinity for parabolic affine spheres. *Math. Z.* **230** (1999), no. 3, 471–486.

[14] Ferrer, L.; Martínez, A.; Milán, F. The space of parabolic affine spheres with fixed compact boundary. *Monatsh. Math.* **130** (2000), no. 1, 19–27.

[15] Gilbarg, D.; Serrin, J. On isolated singularities of solutions of second order elliptic differential equations. *J. Analyse Math.* **4** (1955/56), 309–340.

[16] Gilbarg, D.; Trudinger, N. S. *Elliptic partial differential equations of second order.* Second edition. Grundlehren der Mathematischen Wissenschaften, 224. Springer, Berlin, 1983.

[17] Jörgens, K. Über die Lösungen der Differentialgleichung $rt - s^2 = 1$. *Math. Ann.* **127** (1954), 130–134.

[18] Pogorelov, A. V. On the improper convex affine hyperspheres. *Geometriae Dedicata* **1** (1972), no. 1, 33–46.

[19] Trudinger, N. S.; Wang, X.-J. The Bernstein problem for affine maximal hypersurfaces. *Invent. Math.* **140** (2000), no. 2, 399–422.

L. CAFFARELLI
University of Texas
Department of Mathematics
Austin, TX 78712
E-mail: caffarel@math.utexas.edu

YANYAN LI
Rutgers–The State University
Department of Mathematics
New Brunswick, NJ 08903
E-mail: yyli@math.rutgers.edu

Received November 2001.