Estimates for Elliptic Systems from Composite Material

YANYAN LI
Rutgers, The State University of New Jersey
AND
LOUIS NIRENBERG
Courant Institute

Dedicated to the memory of Jürgen Moser

1 Introduction

1.1 Background

In the closure $\overline{D}$ of a bounded domain in $\mathbb{R}^n$, we consider a composite media whose physical characteristics are smooth in the closures of subdomains D_m but possibly discontinuous across their boundaries. The physical properties of the media are described in terms of a linear second-order elliptic system in divergence form. The coefficients of the system are smooth in each $\overline{D}_m$ but not across their boundaries.

Before stating results we first describe the nature of our subdomains. D is a bounded domain in $\mathbb{R}^n$ that contains L disjoint subdomains $D_1, \dots, D_L$, with $D = (\bigcup \overline{D}_m) \setminus \partial D$. If a point in D lies on some ∂D_m, then we assume for that m, the component of ∂D_m containing the point is smooth. This implies that any point $x \in D$ belongs to the boundaries of at most two of the D_m. Thus if the boundaries of two D_m touch, then they touch on a whole component of such a boundary. However, as will be explained in Remark 1.2, we may include domains as shown in Figure 1.1.

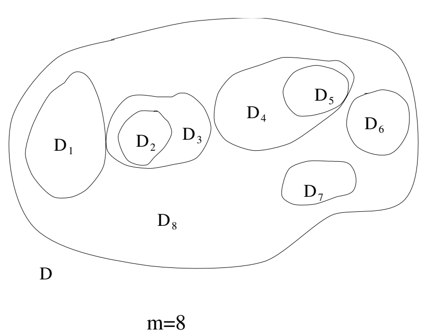

FIGURE 1.1

Communications on Pure and Applied Mathematics, Vol. LVI, 0892–0925 (2003)

We consider a weak solution u in $H^1(D)$; u is vector-valued. In engineering, one is interested in obtaining bounds on the stresses represented by ∇u. For $\varepsilon > 0$ small, we set

$$D_\varepsilon = \left\{x \in D : \mathrm{dist}(x, \partial D) > \varepsilon\right\}.$$

Question. Away from ∂D, is ∇u bounded independently of the distance between the domains? Are higher derivatives also bounded? What about bounds being independent of the number of regions?

Babuška et al. [2] were interested in elliptic systems arising in elasticity. They observed numerically that, for certain homogeneous isotropic linear systems of elasticity, indeed $|\nabla u|$ is bounded independently of the distance between the regions.

This paper is a continuation of a paper by Li and Vogelius [10]. There the case of scalar elliptic equations for a single real function u was considered:

$$\sum_{\alpha,\beta=1}^{n} \partial_\alpha\left(A^{\alpha\beta}(x)\partial_\beta u\right) = \mathrm{RHS}$$

where $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$ and "RHS" denotes the right-hand side. The coefficients $A^{\alpha\beta}$ are measurable and uniformly elliptic,

$$\lambda|\xi|^2 \le A^{\alpha\beta}(x)\xi_\alpha\xi_\beta \le \Lambda|\xi|^2, \quad \lambda, \Lambda > 0,$$

and are C^μ $(0 < \mu < 1)$ in each $\overline{D}_m$. In [10] they obtained uniform estimates for $|\nabla u|$ and $\|u\|_{C^{1,\alpha'}}$ for some $0 < \alpha' \le \frac{1}{4}$ in each $\overline{D}_m \cap D_\varepsilon$, independently of the distance between the regions. Indeed, several regions $\overline{D}_m$ may even touch (of course, then some ∂D_m are not smooth, as in Figure 1.1). The estimates, including α', depend on the number of regions, on the $C^{1,\alpha}$ smoothness of the ∂D_m, on λ and Λ, and on the C^μ norm of A on $\overline{D}_m$ (and of course on ε). Their proof makes use of the De Giorgi–Moser estimates for scalar elliptic equations in divergence form.

Question. What about higher derivatives? They studied a special case in $\mathbb{R}^2$: D is a disk $\{|x| < R\}$, and D_1 and D_2 are unit disks centered at $(0, -1)$ and $(0, 1)$, so their closures touch at the origin, $D_3 = D \setminus (\overline{D}_1 \cup \overline{D}_2)$ (Figure 1.2).

The equation is

$$\partial_i(a(x)\partial_i u) = 0 \text{ in } D, \quad u \in H^1(D),$$

i.e.,

$$\int a(x)\partial_i u \partial_i \zeta = 0 \quad \forall \zeta \in C_0^\infty(D) \tag{1.1}$$

with $a(x) \equiv 1$ in D_3 and $a(x) = a_0 \neq 1$ in D_1 and D_2; here a_0 is a positive constant. Thus the function u is harmonic in each D_i, $i = 1, 2, 3$. It is easy to see

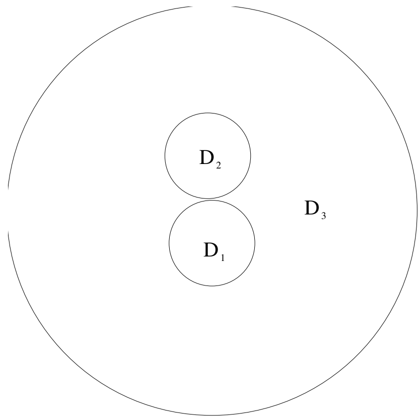

FIGURE 1.2

from (1.1) that the function u is continuous in D and that at any boundary point $x \neq 0$ of D_1 or D_2 with exterior unit normal ν,

$$a_0 u_\nu(x)\big|_{D_m} = u_\nu(x)\big|_{D_3}, \quad m = 1, 2.$$

Here the left-hand side uses the exterior normal derivative from inside D_m, while the RHS uses the interior normal derivative for D_3. This problem was first considered in [4], but in [10] they show that for sufficiently large R,

$$|D^k u| \le C_k \qquad \text{in } D_1 \text{ and } D_2\ \forall k,$$
$$|D^k u| \le C_{k,\varepsilon} \quad \text{in } D_3 \cap D_\varepsilon\ \forall k.$$

Their proof made use of conformal mapping.

Open Problem. For the same problem in higher dimensions, can one estimate derivatives of any order?

1.2 Elliptic Systems and Principal Results

We consider vector-valued functions $u = (u^1, \dots, u^N)$. The systems take the form

$$\partial_\alpha\left(A_{ij}^{\alpha\beta}(x)\partial_\beta u^j\right) = b_i, \quad i = 1, \dots, N. \tag{1.2}$$

(We use the summation convention: α and β are summed from 1 to n, while i and j are summed from 1 to N.)

The coefficients $A_{ij}^{\alpha\beta}$, often denoted by A, are measurable and bounded,

$$\left|A_{ij}^{\alpha\beta}(x)\right| \le \Lambda, \tag{1.3}$$

and they belong to C^μ in $\overline{D}_m$, $m = 1, L$, for some $0 < \mu < 1$. Furthermore, for some $\lambda > 0$, we assume the (rather weak) ellipticity condition

$$\int_D A_{ij}^{\alpha\beta}(x)\partial_\alpha\varphi^i\partial_\beta\varphi^j \ge \lambda \int_D |\nabla\varphi|^2 \quad \forall\varphi \in H_0^1(D, \mathbb{R}^N). \tag{1.4}$$

A consequence of (1.4) is

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta\eta^i\eta^j \geq \lambda|\xi|^2|\eta|^2 \quad \forall \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^N .$$

Hypotheses (1.3) and (1.4) are clearly satisfied if the coefficients $\{A_{ij}^{\alpha\beta}(x)\}$ are strongly elliptic in the sense that

$$\lambda|\xi|^2 \leq A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \leq \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^{nN},\ x \in D .$$

The hypotheses are also satisfied by the linear systems of elasticity. Recall that a system is called a system of elasticity if $N = n$, the coefficients satisfy

$$A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha} = A_{\alpha j}^{i\beta} , \tag{1.5}$$

and, for all $n \times n$ *symmetric* matrices $\{\xi_\alpha^i\}$,

$$\lambda|\xi|^2 \leq A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \leq \Lambda|\xi|^2 , \quad x \in D . \tag{1.6}$$

It is well-known that (1.3)–(1.4), with a smaller λ and larger Λ, follow from (1.5)–(1.6); see, for example, [11, chap. 1].

Concerning the b_i in (1.2), we assume they have the form

$$b_i = h_i + \partial_\beta g_i^\beta \tag{1.7}$$

and that

$$\begin{cases} h = \{h_i\} \in L^\infty(D) \\ g = \{g_i^\beta\} \in C^\mu(\overline{D}_m),\ m = 1, \dots, L. \end{cases}$$

Our principal result yields $C^{1,\alpha'}$ interior estimates for u. First, we formulate more precisely our conditions on the $\partial D_m \subset D$. We assume that each such D_m is of class $C^{1,\alpha}$, $0 < \alpha \leq 1$; that is, in a neighborhood of every point of ∂D_m, ∂D_m is the graph of some $C^{1,\alpha}$ function of $n-1$ variables. For $m > 0$, we define the $C^{1,\alpha}$ norm of a $C^{1,\alpha}$ domain D_m as the largest positive number a such that in the a-neighborhood of every point of ∂D_m, identified as 0 after a possible translation and rotation of the coordinates so that $x_n = 0$ is the tangent to ∂D_m at 0, ∂D is given by the graph of a $C^{1,\alpha}$ function, denoted as f_m, which is defined in $|x'| < 2a$, the $2a$-neighborhood of 0 in the tangent plane. Moreover, $\|f_m\|_{C^{1,\alpha}(|x'|<2a)} \leq \frac{1}{a}$. The principal result gives interior $C^{1,\alpha'}$ estimates of an H^1 solution u of (1.2) (with b_i of the form (1.7)); i.e., u belongs to $H^1(D)$ and satisfies

$$\int_D A_{ij}^{\alpha\beta}(x)\partial_\beta u^j\partial_\alpha\zeta^i + h_i\zeta^i - g_i^\beta\partial_\beta\zeta^i = 0$$

for every vector-valued $\zeta = (\zeta^1, \dots, \zeta^N)$ in $C_0^\infty(D)$, and hence for all $\zeta \in H_0^1(D)$.

THEOREM 1.1 *Assume the conditions above, even if some D_m touch as in Figure* 1.1 (*see Remark* 1.2 *below*). *For any $\varepsilon > 0$, there exists a constant C such that for any α' satisfying*

$$0 < \alpha' \leq \min\left\{\mu, \frac{\alpha}{2(\alpha+1)}\right\} ,$$

we have for $m = 1, \dots, L$,

$$\sum_{m=1}^{L} \|u\|_{C^{1,\alpha'}(\overline{D}_m \cap D_\varepsilon)} \le C\left(\|u\|_{L^2(D)} + \|h\|_{L^\infty(D)} + \sum_{m=1}^{L} \|g\|_{C^{\alpha'}(\overline{D}_m)} \right). \tag{1.8}$$

Here C depends only on n, N, L, μ, α, ε, λ, Λ, $\|A\|_{C^{\alpha'}(\overline{D}_m)}$, and the $C^{1,\alpha'}$ norms of the D_m; in particular,

$$\|\nabla u\|_{L^\infty(D_\varepsilon)} \le C\left(\|u\|_{L^2(D)} + \|h\|_{L^\infty(D)} + \sum_{m=1}^{L} \|g\|_{C^{\alpha'}(\overline{D}_m)} \right).$$

Remark 1.2. The solution u is unique if $u|_{\partial D}$ is a given function in $H^{1/2}(\partial D)$. It follows that, by approximation, we may assume that the coefficients A and f belong to $C^\infty(\overline{D}_m)$ $\forall m$. Furthermore, it suffices to prove estimate (1.8) in case no more than two of the $\overline{D}_m$ touch, for we may move or change them slightly to achieve that. In addition, by approximation, we may suppose that ∂D_m is in C^∞ for $m > 0$. From now on, we assume all these conditions.

Remark 1.3. Theorem 1.1 for scalar equations was established in [10] for slightly more restrictive α': $0 < \alpha' \le \mu$ and $\alpha' < \frac{\alpha}{n(\alpha+1)}$.

1.3 Outline of Proof and C^∞ Property of u in Each $\overline{D}_m \cap D$

In Section 2, using Remark 1.2, for $\overline{D}_m \subset D$, we first prove the following:

PROPOSITION 1.4 *For each m, the solution u belongs to $C^\infty(\overline{D}_m \cap D)$.*

Remark 1.5. Proposition 1.4 still holds for the more general operator

$$\partial_\alpha\left(A_{ij}^{\alpha\beta}\partial_\beta u^j + B_{ij}^{\alpha} u^j\right) + C_{ij}^{\beta}\partial_\beta u^j + D_{ij}u^j$$

provided that B_{ij}^α, C_{ij}^β, and D_{ij} are also in $C^\infty(\overline{D}_m)$ for each m.

However, the proof of the proposition does not yield the kind of uniform bounds that we desire. The proof of Proposition 1.4 is based on a result of Chipot, Kinderlehrer, and Vergara-Caffarelli [8] for solutions of laminar systems. We consider D to be the cube Ω,

$$\Omega = \{x : |x_i| < 1\} \quad \text{with } x = (x', x_n)$$

divided into Ω_m. However, the Ω_m are different; they are "strips":

$$\Omega_m = \{x \in \Omega : c_{m-1} < x_n < c_m\},$$

where the c_m are increasing constants lying between -1 and 1. There may be infinitely many strips; if so, we set $c_{-\infty} = -1$ and $c_\infty = 1$. In Ω we consider system (1.2) for a vector-valued function v,

$$\partial_\alpha\left(A_{ij}^{\alpha\beta}\partial_\beta v^j\right) = H_i + \partial_\alpha\left(G_i^\alpha\right), \quad i = 1, \dots, N. \tag{1.9}$$

The coefficients A are uniformly smooth in each $\overline{\Omega}_m$ and satisfy (1.3) and (1.4). The H_i and the G_i^α are also assumed to be smooth in each $\overline{\Omega}_m$.

PROPOSITION 1.6 *Assume the above. Let $v \in H^1(\Omega, \mathbb{R}^n)$ be a weak solution of* (1.9). *Then for all γ', $D_{x'}^{\gamma'} v \in C^0(\Omega)$, and for each m, $v \in C^\infty(\overline{\Omega}_m \cap \Omega)$. Moreover, for any $0 < \varepsilon < 1$, any nonnegative k, and any m,*

$$\|v\|_{C^k(\overline{\Omega}_m \cap (1-\varepsilon)\Omega)} \leq C\|v\|_{L^2(\Omega)} + C \sum_{|\gamma'| \leq \tilde{k}-1} \left\| D_{x'}^{\gamma'} H \right\|_{L^2(\Omega)} + C \sum_{|\gamma'| \leq \tilde{k}} \left\| D_{x'}^{\gamma'} G \right\|_{L^2(\Omega)},$$

where $\tilde{k} = k + [\frac{n-1}{2}] + 2$ and C depends on ε, k, n, N, λ, Λ, and the $L^\infty(\Omega)$ norm of $D_{x'}^{\gamma'} A$ for $|\gamma'| \leq \tilde{k}$.

COROLLARY 1.7 *If we further assume in Proposition* 1.6 *that $A = \overline{A}$, $G = \overline{G}$, and $H = \overline{H}$ are constants in each $\overline{\Omega}_m$, then for any $\varepsilon > 0$, any nonnegative integer k, and any m,*

$$\|v\|_{C^k(\overline{\Omega}_m \cap (1-\varepsilon)\Omega)} \leq C\left(\|v\|_{L^2(\Omega)} + \|\overline{H}\|_{L^\infty(\Omega)} + \|\overline{G}\|_{L^\infty(\Omega)}\right),$$

where $C = C(\varepsilon, k, n, N, \lambda, \Lambda)$.

Remark 1.8. Both Proposition 1.6 and Corollary 1.7 hold for more general systems as described in Remark 1.5. Naturally, the constants C in Proposition 1.6 and Corollary 1.7 also depend on appropriate bounds of the coefficients B_{ij}^α, C_{ij}^β, and D_{ij}.

Proposition 1.6 can be deduced from Proposition 2.1, a result in [8]. In Section 2 we present a proof of Proposition 1.6 that is a bit different from that in [8]. In particular, our proof does not use the reverse Hölder inequality. Proposition 1.4 follows from Proposition 1.6 and Remark 1.8 by straightening boundaries using a smooth local diffeomorphism.

1.4 Outline of Proof of $C^{1,\alpha'}$ Estimates

Most of the paper is devoted to these estimates. We make use of ideas of L. Caffarelli of [5, 6].

To estimate $|\nabla u(x)|$ at a point x in D_ε, we need only consider the case that x is close to some ∂D_m; otherwise, standard interior estimates yield the result. In that case we approximate the problem by a laminar one as in the preceding section, with a finite number of strips. To this end, in Section 2 we present a general perturbation result, Lemma 3.1. It asserts, roughly, the following: Suppose u is a solution of system

$$\partial(A\partial u) = \partial g$$

in (for convenience) a cube Ω. Suppose that B are the coefficients of a similar system also satisfying (1.3) and (1.4) with the L^1 norm of $(A - B) \leq \varepsilon$ small.

Then in $\frac{3}{4}\Omega$, there is an H^1 solution of the "B system"

$$\partial(B\partial v) = 0 \quad \text{in } \frac{3}{4}\Omega \quad \text{with } \|u - v\|_{H^1(\frac{1}{2}\Omega)} \leq C\big(\|g\|_{L^2(\Omega)} + \varepsilon^\gamma \|u\|_{L^2(\Omega)}\big)$$

for some constant $\gamma > 0$ and some C.

This is used only in the case that the system B is a laminar one, with piecewise constant coefficients, which we rename $\overline{A}$. Because of the geometry (here we take x as the origin), we have for r small

$$\left(\fint_{r\Omega} |A - \overline{A}|^2\right)^{1/2} \leq E r^{\alpha'} .$$

We will describe below the ideas of the proof of Theorem 1.1 when the system is homogeneous. Applying Lemma 3.1 on perturbation in a suitable cube Ω, we obtain a solution w_0 of the $\overline{A}$ system

$$\partial\big(\overline{A}\partial w_0\big) = 0 \quad \text{with } \|u - w_0\|_{L^2(\frac{1}{2}\Omega)} \leq \left(\frac{1}{4}\right)^{\frac{n}{2}+1+\alpha'} .$$

In addition, using Proposition 1.6, we show that

$$\|\nabla w_0\|_{L^\infty(\frac{1}{4}\Omega)} \leq C .$$

By repeated use of Lemma 3.1, applied first to $u - w_0$ in smaller and smaller cubes and by scaling, we obtain a sequence of functions $w_1, w_2, \dots,$ satisfying, with C a fixed constant,

$$\|\nabla w_k\|_{L^\infty(4^{-(k+1)}\Omega)} \leq C 4^{-k\alpha'} , \quad |w_k(0)| \leq C 4^{-k(1+\alpha')} , \tag{1.10}$$

and

$$\left\| u - \sum_{j=0}^{k} w_j \right\|_{L^2(4^{-k}\Omega)} \leq C 4^{-(k+1)(\frac{n}{2}+1+\alpha')} . \tag{1.11}$$

Using (1.10) and (1.11) finally yields

$$\left\| u - \sum_{j=0}^{\infty} w_j(0) \right\|_{L^2(4^{-(k+1)}\Omega)} \leq C 4^{-(k+1)\frac{n+2}{2}} ,$$

which yields

$$|\nabla u(0)| \leq C .$$

The procedure is unfortunately rather long. It is carried out in Sections 2 and 3. Sections 4 and 5, also technical, treat the Hölder-continuity of ∇u. Take two points in some D_{m_0}; one we take as the origin while the other we call x. We wish to show that for $|x|$ small,

$$|\nabla u(0) - \nabla u(x)| \leq C|x|^{\alpha'} . \tag{1.12}$$

Pick a point on $\bigcup_m \partial D_m$ such that the distance from the origin to this point is the shortest distance from the origin to $\bigcup_m \partial D_m$. Let the line going through this point and the origin be the x_n-axis. This is illustrated in Figure 1.3.

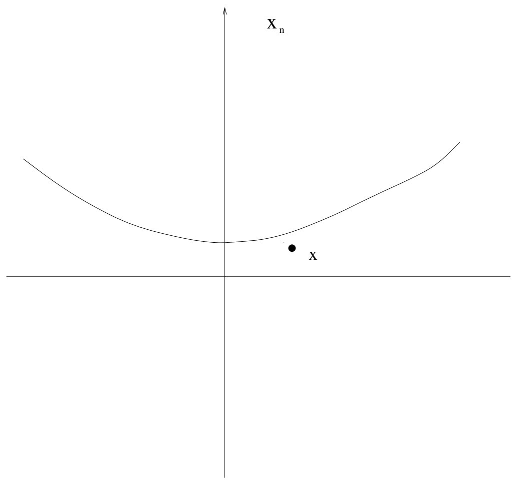

FIGURE 1.3

To prove (1.12), we compare ∇u at 0 and x with ∇u at two other points $\bar{x}$ and $\bar{z}$, as in [10]. Since the number of regions D_m is finite, we may find $\bar{x}$ on the x_n-axis such that $|\bar{x}| \sim |x|$ and $\bar{x} + 8|x|\Omega$ lies entirely in some D_m. We prove that

$$|\nabla u(\bar{x}) - T\nabla u(0)| \leq C|x|^{\alpha'},$$

where T is some invertible linear transformation with $\|T\|$ and $\|T^{-1}\|$ bounded from above by some universal constant. Similarly, we can find $\bar{z}$ with $|\bar{z}-\bar{x}| \leq 2|x|$ and

$$|\nabla u(\bar{z}) - T\nabla u(x)| \leq C|x|^{\alpha'};$$

see Figure 1.4.

Finally, we show that

$$|\nabla u(\bar{x}) - \nabla u(y)| \leq C|x|^{\alpha'} \quad \forall y \in \bar{x} + 6|x|\Omega;$$

in particular,

$$|\nabla u(\bar{x}) - \nabla u(\bar{z})| \leq C|x|^{\alpha'}.$$

The desired estimate (1.12) follows from the above.

Let $\{D_m\}$ be domains of a flat torus $\mathbb{T}^n$ as described above. Here $\mathbb{T}^n$ is the quotient of $\mathbb{R}^n$ with respect to the equivalence relation $x \sim y$ if and only if $x^\alpha - y^\alpha$ are integers. Based on Theorem 1.1 and the method in [1], we have the following extension of a result of Avellaneda and Lin [1].

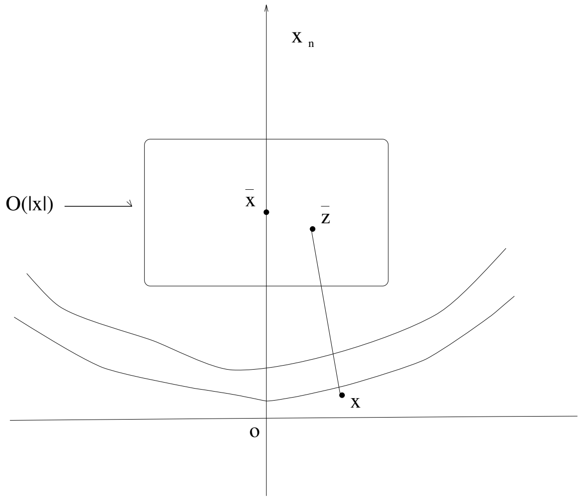

FIGURE 1.4

THEOREM 1.9 *Let $\{D_m\}$ be as above and let A be "piecewise Hölder" as described earlier. Assume that A is* 1*-periodic in each x^α and, for a unit ball B_1 of $\mathbb{R}^n$, that $u \in H^1(B_1, \mathbb{R}^N)$ is a solution of*

$$\partial_\alpha \left(A_{ij}^{\alpha\beta} \left(\frac{x}{\varepsilon} \right) \partial_\beta u^j \right) = 0 \,, \quad B_1 \,.$$

Then

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C \|u\|_{L^2(B_1)} \,,$$

where C is independent of ε and the distances between the $\{\partial D_m\}$.

Remark 1.10. A $W^{1,\infty}$ estimate is given in the above theorem, while a $W^{1,p}$ estimate for $p < \infty$ is due to Caffarelli and Peral [7]. Under the additional hypothesis that A is Hölder on $\mathbb{T}^n$, the $W^{1,\infty}$ estimate is due to Avellaneda and Lin [1].

2 Proofs of Propositions 1.4 and 1.6

Let Ω be the unit cube and Ω_m be the strips defined in the introduction. We assume that coefficients A are uniformly smooth in each $\overline{\Omega}_m$ and satisfy (1.3) and (1.4). H and G are also assumed to be smooth in each $\overline{\Omega}_m$.

We first prove the following:

PROPOSITION 2.1 [8] *Assume the above. Let $v \in H^1(\Omega, \mathbb{R}^n)$ be a weak solution of* (1.9). *Then for any $0 < \varepsilon < 1$ and for any positive k, $D_{x'}^{\gamma'} v \in H^1_{\mathrm{loc}}(\Omega)$ for all $|\gamma'| \leq k$, and, for some constant C depending only on n, N, λ, Λ, ε, and*

$\sum_{|\gamma'|\le k}\|D_{x'}^{\gamma'}A\|_{L^\infty(\Omega)}$, *we have*

$$\sum_{|\gamma'|\le k}\int\limits_{(1-\varepsilon)\Omega}\left|DD_{x'}^{\gamma'}v\right|^2\le C\|v\|^2_{L^2(\Omega)}+C\sum_{|\gamma'|\le k-1}\left\|D_{x'}^{\gamma'}H\right\|^2_{L^2(\Omega)}+C\sum_{|\gamma'|\le k}\left\|D_{x'}^{\gamma'}G\right\|^2_{L^2(\Omega)}. \tag{2.1}$$

Moreover, for

$$w=(w_i)=\left(A_{ij}^{n\beta}\partial_\beta v^j-G_n^i\right),$$

we have $D_{x'}^{\gamma'}w$, $D_{x'}^{\gamma'}\partial_n w\in L^2_{\mathrm{loc}}(\Omega)$ *for all* $|\gamma'|\le k-1$, *and*

$$\sum_{|\gamma'|\le k-1}\left(\left\|D_{x'}^{\gamma'}w\right\|^2_{L^2((1-\varepsilon)\Omega)}+\left\|D_{x'}^{\gamma'}\partial_n w\right\|^2_{L^2((1-\varepsilon)\Omega)}\right)\le C\|v\|^2_{L^2(\Omega)}+C\sum_{|\gamma'|\le k-1}\left\|D_{x'}^{\gamma'}H\right\|^2_{L^2(\Omega)}+C\sum_{|\gamma'|\le k}\left\|D_{x'}^{\gamma'}G\right\|^2_{L^2(\Omega)}. \tag{2.2}$$

The proof of Proposition 2.1 relies on a convenient form of Sobolev's inequality, which is fairly well known:

LEMMA 2.2 *Let* f *be a real function in* Ω *with* $D_{x'}^{\gamma'}f$ *and* $D_{x'}^{\gamma'}\partial_n f\in L^2(\Omega)$ *for all* $0\le|\gamma'|\le[\frac{n-1}{2}]+1=:k$. *Then* $f\in C^0(\overline{\Omega})$ *and*

$$\|f\|_{L^\infty(\Omega)}\le C(n)\sum_{|\gamma'|\le k}\left(\|D_{x'}^{\gamma'}\partial_n f\|_{L^2(\Omega)}+\|D_{x'}^{\gamma'}f\|_{L^2(\Omega)}\right).$$

PROOF: Our conditions on f assert that f belongs to H^1 on $[-1,1]$ with values in $H^k([-1,1]^{n-1})$. By the usual form of Sobolev's inequality,

$$H^k([-1,1]^{n-1})\subset C^0([-1,1]^{n-1}).$$

Thus f is in $H^1((-1,1),C^0([-1,1]^{n-1}))$ and hence in $C^0(\overline{\Omega})$. In fact, f is Hölder-continuous in $\overline{\Omega}$. □

PROOF OF PROPOSITION 2.1: First we establish (2.1). We sketch the argument without giving every detail since the steps are all rather familiar ones. If we multiply (1.9) by v and a suitable cutoff function, we find, on integrating by parts and using (1.3) and (1.4),

$$\int\limits_{(1-\varepsilon)\Omega}|Dv|^2\le C\left(\|v\|^2_{L^2(\Omega)}+\|H\|^2_{L^2(\Omega)}+\|G\|^2_{L^2(\Omega)}\right). \tag{2.3}$$

To estimate higher derivatives, it is customary to differentiate the equation, multiply by a suitable derivative of v and by a cutoff function, and integrate by parts. Clearly, we are not allowed to apply ∂_n since the coefficients are smooth only in $x'=(x_1,\dots,x_{n-1})$ derivatives. Furthermore, we do not yet know that v

has additional derivatives in the x'-directions. So in place of taking derivatives, it is standard to use difference quotients in these directions. To save space and the reader's patience, we shall simply differentiate. Applying $D_{x'}^{\gamma'}$ for $|\gamma'| = 1$ to (1.9), we obtain

$$\partial_\alpha\big(A_{ij}^{\alpha\beta}\partial_\beta(D_{x'}^{\gamma'}v^j)\big) = D_{x'}^{\gamma'}H_i + \partial_\alpha\big(D_{x'}^{\gamma'}G_i^\alpha - (D_{x'}^{\gamma'}A_{ij}^{\alpha\beta})\partial_\beta v^j\big)\,,$$

and, consequently, as above,

$$\int\limits_{(1-\varepsilon)\Omega} \big|DD_{x'}^{\gamma'}v\big|^2 \le C\big(\|H\|^2_{L^2((1-\frac{\varepsilon}{2})\Omega)} + \|D_{x'}^{\gamma'}G\|^2_{L^2((1-\frac{\varepsilon}{2})\Omega)} + \|Dv\|^2_{L^2((1-\frac{\varepsilon}{2})\Omega)}\big)\,.$$

It follows, in view of (2.3), that

$$\int\limits_{(1-\varepsilon)\Omega} \big|DD_{x'}^{\gamma'}v\big|^2 \le C\big(\|v\|^2_{L^2(\Omega)} + \|H\|^2_{L^2(\Omega)} + \|G\|^2_{L^2(\Omega)} + \|D_{x'}^{\gamma'}G\|^2_{L^2(\Omega)}\big)\,.$$

We have established (2.1) for $k = 1$. Estimate (2.1) for general k follows by induction through further differentiation in horizontal directions in a standard way.

Because of (2.1), $D_{x'}^{\gamma'}w \in L^2_{\text{loc}}(\Omega)$ for $|\gamma'| \le k-1$, and the estimate of $\sum_{|\gamma'|\le k-1}\|D_{x'}^{\gamma'}w\|^2_{L^2((1-\varepsilon)\Omega)}$ in (2.2) also follows from (2.1) and (2.3). Rewriting equation (1.9) as

$$\partial_n w = H_i + \sum_{\alpha\le n-1}\partial_\alpha\big(G_i^\alpha - A_{ij}^{\alpha\beta}\partial_\beta v^j\big)$$

and applying horizontal differentiation to it, we obtain, in view of (2.1), $D_{x'}^{\gamma'}\partial_n w \in L^2_{\text{loc}}(\Omega)$ ($|\gamma'| \le k-1$) and the estimate of $\sum_{|\gamma'|\le k-1}\|D_{x'}^{\gamma'}\partial_n w\|^2_{L^2((1-\varepsilon)\Omega)}$ in (2.2). Proposition 2.1 is established. □

PROOF OF PROPOSITION 1.6: It is well known that for each m, $v \in C^\infty(\Omega_m)$. For $k \ge [\frac{n-1}{2}]+1$ and $|\gamma'| \le k - [\frac{n-1}{2}] - 1$, by Proposition 2.1 and an application of Lemma 2.2 with $f = D_{x'}^{\gamma'}v$, we have $D_{x'}^{\gamma'}v \in C^0(\Omega)$ and

$$\sum_{|\gamma'|\le k-[\frac{n-1}{2}]-1}\big\|D_{x'}^{\gamma'}v\big\|^2_{L^\infty((1-\varepsilon)\Omega)} \le C\|v\|^2_{L^2(\Omega)} + C\sum_{|\beta'|\le k-1}\big\|D_{x'}^{\beta'}H\big\|^2_{L^2(\Omega)} + C\sum_{|\beta'|\le k}\big\|D_{x'}^{\beta'}G\big\|^2_{L^2(\Omega)}\,, \tag{2.4}$$

where C has the same dependence as in Proposition 2.1. Similarly, for $k \ge [\frac{n-1}{2}]+2$ and $|\gamma'| \le k - [\frac{n-1}{2}] - 2$, by Proposition 2.1 and an application of Lemma 2.2

with $f = D_{x'}^{\gamma'} w$, we have $D_{x'}^{\gamma'} w \in C^0(\Omega)$, and

$$\sum_{|\gamma'|\le k-[\frac{n-1}{2}]-2} \left\| D_{x'}^{\gamma'} w \right\|_{L^\infty((1-\varepsilon)\Omega)}^2 \le C\|v\|_{L^2(\Omega)}^2 + C \sum_{|\gamma'|\le k-1} \left\| D_{x'}^{\gamma'} H \right\|_{L^2(\Omega)}^2 + C \sum_{|\gamma'|\le k} \left\| D_{x'}^{\gamma'} G \right\|_{L^2(\Omega)}^2 , \tag{2.5}$$

where C has the same dependence as in Proposition 2.1. Consequently, $D_{x'}^{\gamma'} v \in W_{\mathrm{loc}}^{1,\infty}(\Omega)$, and

$$\sum_{|\gamma'|\le k-[\frac{n-1}{2}]-2} \left\| D D_{x'}^{\gamma'} v \right\|_{L^\infty((1-\varepsilon)\Omega)}^2 \le C\|v\|_{L^2(\Omega)}^2 + C \sum_{|\beta'|\le k-1} \left\| D_{x'}^{\beta'} H \right\|_{L^2(\Omega)}^2 + C \sum_{|\beta'|\le k} \left\| D_{x'}^{\beta'} G \right\|_{L^2(\Omega)}^2 . \tag{2.6}$$

Indeed, by (2.4), we only need to show that $\partial_n D_{x'}^{\gamma'} v \in L_{\mathrm{loc}}^\infty(\Omega)$ and establish (2.6) for $\|\partial_n D_{x'}^{\gamma'} v\|_{L^\infty((1-\varepsilon)\Omega)}^2$. By (2.4) and (2.5), $A_{ij}^{nn} \partial_n D_{x'}^{\gamma'} v^j \in L_{\mathrm{loc}}^\infty(\Omega)$ and

$$\left\| A_{ij}^{nn} \partial_n D_{x'}^{\gamma'} v^j \right\|_{L^\infty((1-\varepsilon)\Omega)}^2 \le C\|v\|_{L^2(\Omega)}^2 + C \sum_{|\beta'|\le k-1} \left\| D_{x'}^{\beta'} H \right\|_{L^2(\Omega)}^2 + C \sum_{|\beta'|\le k} \left\| D_{x'}^{\beta'} G \right\|_{L^2(\Omega)}^2 .$$

Because of (1.3) and (1.4), (A_{ij}^{nn}) is a positive definite $N \times N$ matrix with eigenvalues in $[\lambda, \Lambda]$. Consequently, $D_{x'}^{\gamma'} v \in W_{\mathrm{loc}}^{1,\infty}(\Omega)$ and

$$\left\| \partial_n D_{x'}^{\gamma'} v^j \right\|_{L^\infty((1-\varepsilon)\Omega)}^2 \le C\|v\|_{L^2(\Omega)}^2 + C \sum_{|\beta'|\le k-1} \left\| D_{x'}^{\beta'} H \right\|_{L^2(\Omega)}^2 + C \sum_{|\beta'|\le k} \left\| D_{x'}^{\beta'} G \right\|_{L^2(\Omega)} .$$

Inequality (2.6) gives us the desired bounds for tangential (i.e., x') derivatives of v and of $\partial_n v$. To estimate derivatives involving $\partial_n^j v$ for $j > 1$, we simply observe that these may be derived recursively from those already established. Indeed, according to (1.9),

$$A_{ij}^{nn} \partial_n^2 v^j = -\partial_n\big(A_{ij}^{nn}\big)\partial_n v^j + f_i - \sum_{\alpha+\beta<2n} \partial_\alpha\big(A_{ij}^{\alpha\beta} \partial_\beta v^j\big) , \tag{2.7}$$

where $f_i = H_i + \partial_\alpha(G_i^\alpha)$.

Since the matrix A_{ij}^{nn} has a bounded inverse, we can estimate $D_{x'}^{\gamma'} \partial_n^2 v$ pointwise for each open strip. Applying ∂_n to (2.7), we can then estimate tangential derivatives of $\partial_n^3 v$ and so on. We thus obtain

$$\sum_{|\gamma|\le k} \|D^\gamma v\|_{L^\infty(\Omega_m\cap(1-\varepsilon)\Omega)} \le C\|v\|_{L^2(\Omega)} + C \sum_{|\gamma'|\le \tilde{k}-1} \big\|D^{\gamma'}_{x'} H\big\|_{L^2(\Omega)} + C \sum_{|\gamma'|\le \tilde{k}} \big\|D^{\gamma'}_{x'} G\big\|_{L^2(\Omega)} .$$

Hence, $v \in C^\infty(\overline{\Omega}_m \cap \Omega)$. Proposition 1.6 is proven. □

Remark 2.3. The use of Proposition 1.6 shows that in some situations in Theorem 1.1 we may allow infinitely many D_m. Here is an example. Suppose D contains a closed ball centered, say, at the origin, of radius R, and suppose the region D_m for $m = (-\infty, \infty)$ are infinitely many disjoint concentric shells lying in $R/2 < |x| < R$ with $\bigcup \overline{D}_m = \{R/2 \le |x| \le R\}$. Then the conclusion of Theorem 1.1 holds. This is because about any point x with $|x| = 3R/4$ we may make a smooth transformation of variable mapping $\{R/4 \le |x| \le R\} \cap$ a cone centered at the origin into a cube in which the images of ∂D_m for all m lie on parallel hyperplanes. This reduces the problem to that of Proposition 1.6.

3 A General Perturbation Lemma

In this section we present some perturbation lemmas in, for simplicity, the unit cube Ω. Such perturbation lemmas will be used in our proof of Theorem 1.1 at all scales. For $0 < \lambda \le \Lambda < \infty$, we denote by $\mathcal{A}(\lambda, \Lambda)$ the class of measurable vector-valued functions $\{A^{\alpha\beta}_{ij}(x)\}$ satisfying (1.3) and (1.4).

LEMMA 3.1 *For* $0 < \varepsilon < 1$, *let* $A, B \in \mathcal{A}(\lambda, \Lambda)$ *satisfy*

$$\int_\Omega |A - B| < \varepsilon . \tag{3.1}$$

Then for any $g = (g^\beta_i) \in L^2(\Omega, \mathbb{R}^{nN})$ *and any solution* $u \in H^1(\Omega)$ *of*

$$\partial_\alpha\big(A^{\alpha\beta}_{ij}(x)\partial_\beta u^j\big) = \partial_\beta g^\beta_i , \quad 1 \le i \le N , \quad in\ \Omega ,$$

there exists some solution $v \in H^1(\frac{3}{4}\Omega)$ *of*

$$\partial_\alpha\big(B^{\alpha\beta}_{ij}(x)\partial_\beta v^j\big) = 0 , \quad 1 \le i \le N , \quad in\ \frac{3}{4}\Omega ,$$

such that

$$\|u - v\|_{H^1(\frac{1}{2}\Omega)} \le C\big(\|g\|_{L^2(\Omega)} + \varepsilon^\gamma \|u\|_{L^2(\Omega)}\big) ,$$

where C *and* γ *are some positive constants depending only on* n, N, λ, *and* Λ.

PROOF: By the ellipticity,

$$\|u\|_{H^1(\frac{4}{5}\Omega)} \le C\big(\|g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big) .$$

Then, by the Fubini theorem, there exists $\frac{3}{4} < \sigma < 1$ such that

$$\|u\|_{H^1(\partial(\sigma\Omega))} \le C\big(\|g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big) .$$

Let $v \in H^1(\sigma\Omega)$ be the solution of

$$\begin{cases} \partial_\alpha(B_{ij}^{\alpha\beta}(x)\partial_\beta v^j) = 0\,, \ 1 \le i \le N\,, & \text{in } \sigma\Omega\,, \\ v = u & \text{on } \partial(\sigma\Omega)\,. \end{cases}$$

Fixing some $0 < \delta < \frac{1}{2}$, let $U \in H^{3/2-\delta}(\sigma\Omega)$ be an extension of u on $\partial(\sigma\Omega)$ satisfying

$$\|\nabla U\|_{L^{\bar p}(\sigma\Omega)} \le C\|U\|_{H^{3/2-\delta}(\sigma\Omega)} \le C\|u\|_{H^{1-\delta}(\partial(\sigma\Omega))} \le C\big(\|g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)\,,$$

where $\bar p = 2n/(n-1+2\delta) \in (2, 2n/(n-1))$. Since $v - U \in H_0^1(\sigma\Omega)$ satisfies

$$\partial_\alpha\big(B_{ij}^{\alpha\beta}(x)\partial_\beta(v^j - U^j)\big) = -\partial_\alpha\big(B_{ij}^{\alpha\beta}(x)\partial_\beta U^j\big) \quad \text{in } \sigma\Omega\,,$$

it follows from the reverse Hölder inequalities (see, e.g., [9, pp. 151–154], as outlined in the appendix) that for some $2 < p \le \bar p$, depending only on n, N, λ, and Λ,

$$\|\nabla(v - U)\|_{L^p(\sigma\Omega)} \le C\|\nabla U\|_{L^p(\sigma\Omega)} \le C\big(\|g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)\,.$$

Consequently,

$$\|\nabla v\|_{L^p(\sigma\Omega)} \le C\big(\|g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)\,.$$

A combination of the equations of u and v leads to

$$\partial_\alpha\big(A_{ij}^{\alpha\beta}(x)\partial_\beta(u^j - v^j)\big) = \partial_\beta g_\beta^i + \partial_\alpha\big((B_{ij}^{\alpha\beta} - A_{ij}^{\alpha\beta})\partial_\beta v^j\big)\,, \qquad 1 \le i \le N,\ \text{in } \sigma\Omega\,.$$

Multiplying the above equations by $u - v$ and integrating by parts, we find, by using the Hölder inequality and (3.1), that

$$\begin{aligned} \|\nabla(u - v)\|_{L^2(\sigma\Omega)} &\le C\big(\|g\|_{L^2(\sigma\Omega)} + \|B - A\|_{L^{2p/(p-2)}(\sigma\Omega)}\|\nabla v\|_{L^p(\sigma\Omega)}\big) \\ &\le C\big(\|g\|_{L^2(\Omega)} + \varepsilon^{(p-2)/(2p)}\|u\|_{L^2(\sigma\Omega)}\big)\,. \end{aligned}$$

Lemma 3.1 follows from the above with $\gamma = (p-2)/(2p)$. □

Essentially the same proof yields the following more general lemma.

LEMMA 3.2 *For $0 < \varepsilon < 1$, let $A, B \in \mathcal{A}(\lambda, \Lambda)$ satisfy* (3.1). *Then for any $g = (g_i^\beta) \in L^2(\Omega, \mathbb{R}^{nN})$, $h = (h_i) \in L^2(\Omega, \mathbb{R}^N)$, $G = (G_i^\beta) \in L^\infty(\Omega, \mathbb{R}^{nN})$, and $H = (H_i) \in L^\infty(\Omega, \mathbb{R}^N)$, and for any solution $u \in H^1(\Omega)$ of*

$$\partial_\alpha\big(A_{ij}^{\alpha\beta}(x)\partial_\beta u^j\big) = h_i + \partial_\beta g_i^\beta\,, \quad 1 \le i \le N\,, \quad \text{in } \Omega\,,$$

there exists some solution $v \in H^1(\frac{3}{4}\Omega)$ of

$$\partial_\alpha\big(B_{ij}^{\alpha\beta}(x)\partial_\beta v^j\big) = H_i + \partial_\beta G_i^\beta\,, \quad 1 \le i \le N\,, \quad \text{in } \frac{3}{4}\Omega\,,$$

such that

$$\begin{aligned} \|u - v\|_{H^1(\frac{1}{2}\Omega)} \le C\big(&\|h - H\|_{L^2(\Omega)} + \|g - G\|_{L^2(\Omega)} \\ &+ \varepsilon^\gamma\big[\|H\|_{L^\infty(\Omega)} + \|G\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega)}\big]\big)\,, \end{aligned}$$

where C and γ are some positive constants depending only on n, N, λ, and Λ.

PROOF: By the ellipticity and the Fubini theorem, we can find $\frac{3}{4} < \sigma < 1$ such that

$$\|u\|_{H^1(\partial(\sigma\Omega))} \le C\big(\|h\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)\,.$$

Let $v \in H^1(\sigma\Omega)$ be the solution of

$$\begin{cases} \partial_\alpha(B_{ij}^{\alpha\beta}(x)\partial_\beta v^j) = H_i + \partial_\beta G_i^\beta\,,\ 1 \le i \le N\,, & \text{in } \sigma\Omega\,,\\ v = u & \text{on } \partial(\sigma\Omega)\,. \end{cases}$$

Fixing some $0 < \delta < \frac{1}{2}$, let $U \in H^{3/2-\delta}(\sigma\Omega)$ be an extension of u on $\partial(\sigma\Omega)$ satisfying

$$\begin{aligned}\|\nabla U\|_{L^{\bar p}(\sigma\Omega)} \le C\|U\|_{H^{3/2-\delta}(\sigma\Omega)} &\le C\|u\|_{H^{1-\delta}(\partial(\sigma\Omega))}\\ &\le C\big(\|h\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)\,,\end{aligned}$$

where $\bar p = 2n/(n-1+2\delta) \in (2, 2n/(n-1))$. Since $v - U \in H_0^1(\sigma\Omega)$ satisfies

$$\partial_\alpha\big(B_{ij}^{\alpha\beta}(x)\partial_\beta(v^j - U^j)\big) = H_i + \partial_\beta G_i^\beta - \partial_\alpha\big(B_{ij}^{\alpha\beta}(x)\partial_\beta U^j\big) \quad \text{in } \sigma\Omega\,,$$

it follows that for some $2 < p \le \bar p$, depending only on n, N, λ, and Λ,

$$\|\nabla(v - U)\|_{L^p(\sigma\Omega)} \le C\big(\|H\|_{L^\infty(\Omega)} + \|G\|_{L^\infty(\Omega)} + \|\nabla U\|_{L^p(\sigma\Omega)}\big)\,,$$

so

$$\|\nabla v\|_{L^p(\sigma\Omega)} \le C\big(\|H\|_{L^\infty(\Omega)} + \|G\|_{L^\infty(\Omega)} + \|\nabla U\|_{L^p(\sigma\Omega)}\big)\,.$$

Combining the equations of u and v leads to

$$\begin{aligned}&\partial_\alpha\big(A_{ij}^{\alpha\beta}(x)\partial_\beta(u^j - v^j)\big) =\\ &\qquad h_i - H_i + \partial_\beta(g_i^\beta - G_i^\beta) + \partial_\alpha\big((B_{ij}^{\alpha\beta} - A_{ij}^{\alpha\beta})\partial_\beta v^j\big) \quad \text{in } \sigma\Omega\,.\end{aligned}$$

Multiplying the above equations by $u - v$ and integrating by parts, we obtain

$$\begin{aligned}&\|\nabla(u - v)\|_{L^2(\sigma\Omega)}\\ &\quad\le C\big(\|h - H\|_{L^2(\sigma\Omega)} + \|g - G\|_{L^2(\sigma\Omega)} + \|B - A\|_{L^{2p/(p-2)}(\sigma\Omega)}\|\nabla v\|_{L^p(\sigma\Omega)}\big)\\ &\quad\le C\big(\|h - H\|_{L^2(\sigma\Omega)} + \|g - G\|_{L^2(\sigma\Omega)}\\ &\qquad\quad + \varepsilon^{(p-2)/(2p)}\big[\|H\|_{L^\infty(\Omega)} + \|G\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega)}\big]\big)\,.\end{aligned}$$

Lemma 3.2 follows immediately. □

4 Preliminaries for Estimating $|\nabla u|$

As mentioned in Section 1.4, to estimate $|\nabla u|$ at a point x in some $\overline{D}_m$, we need only consider the case that for some m_0, x is in D_{m_0} and close to ∂D_{m_0}. We take x as the origin. By suitable rotation and scaling, we may suppose that a finite number of the ∂D_m lie in the usual cube Ω and that these take the form

$$x_n = f_j(x') \quad \forall x' \in [-1, 1]^{n-1}, \; j = 1, \dots, l,$$

with

$$-1 < f_1(x') < \cdots < f_l(x') < 1$$

and with the f_j in $C^{1,\alpha}([-1, 1]^{n-1})$. We set $f_0(x') = -1$ and $f_{l+1} = 1$, and have $l + 1$ regions

$$D_m = \{x \in \Omega : f_{m-1}(x') < x_n < f_m(x')\}, \quad 1 \le m \le l + 1\,.$$

We may suppose that $f_{m_0+1}(0') < 0 < f_{m_0}(0')$, and the closest point on ∂D_{m_0} to the origin is $(0', f_{m_0+1}(0'))$. Thus

$$\nabla' f_{m_0+1}(0') = 0\,;$$

see Figure 4.1.

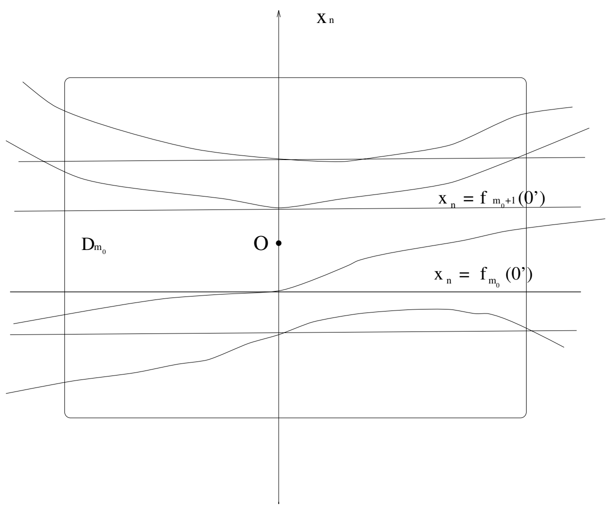

FIGURE 4.1

Our system (1.2) still takes the same form, with (1.3) and (1.4) still holding. As before, the coefficients A, h_i, and g_i^α are smooth in $\overline{D}_m \cap \Omega \ \forall m$. Our desired estimate for $\nabla u(0)$ is given by the following:

PROPOSITION 4.1 *Let $u \in H^1(\Omega)$ be a solution of* (1.2) *in* Ω *with D_m as above. Then, for any ε in* $(0, 1)$,

$$|\nabla u(0)| \le C\big(\|u\|_{L^2(\Omega)} + \|h\|_{L^\infty(\Omega)} + \max_{1\le m\le l+1} \|g\|_{C^\mu(\overline{D}_m)}\big)\,,$$

where C depends only on n, N, l, α, μ, λ, Λ, ε, $\max_{1\le m\le l+1}\|A\|_{C^\mu(\overline{D}_m)}$*, and* $\max_{1\le m\le l+1}\|f_m\|_{C^{1+\alpha}}$.

Proposition 4.1 will be proven using the perturbation lemma of Section 2 in Ω. We approximate the "A system" by a laminar system with coefficients $\overline{A}$ that are *piecewise constant*. Namely, we introduce strips in Ω,

$$\Omega_m = \{x \in \Omega : f_{m-1}(0') < x_n < f_m(0')\}\,,$$

and define the coefficients $\overline{A}$ as

$$\overline{A}(x) = \begin{cases} \lim_{y\in D_m, y\to(0,f_{m-1}(0'))} A(y)\,, & x \in \Omega_m, m > m_0\,, \\ A(0)\,, & x\in\Omega_{m_0}\,, \\ \lim_{y\in D_m, y\to(0,f_m(0'))} A(y)\,, & x\in\Omega_m, m < m_0\,. \end{cases}$$

Using h and g, we similarly define piecewise constant vectors $\overline{H}$ and $\overline{G}$.

We will measure $A - \overline{A}$ in terms of a norm $Y^{s,p}$ defined below.

Definition 4.2. For $s > 0$, $1 \le p < \infty$, and any vector- or matrix-valued function F, we introduce the norm

$$\|F\|_{Y^{s,p}} = \sup_{0<r<1} r^{1-s}\left(\fint_{r\Omega} |F|^p\right)^{1/p}.$$

We have the following lemma; it is proven in the same way as [10, lemma 5.2].

LEMMA 4.3 *Let*

$$0 < \alpha' \le \min\left\{\mu, \frac{\alpha}{2(\alpha+1)}\right\}.$$

With A, $\overline{A}$, g, and $\overline{G}$ as above, there exists a positive constant E, depending only on n, l, α, α′, λ, and Λ, as well as $\max_{1\le m\le l+1}\|A\|_{C^{\alpha'}(\overline{D}_m)}$, $\max_{1\le m\le l+1}\|g\|_{C^{\alpha'}(\overline{D}_m)}$, *and* $\max_{1\le m\le l+1}\|f_m\|_{C^{1,\alpha}(\overline{D}_m)}$*, such that*

$$\|A-\overline{A}\|_{Y^{1+\alpha',2}} + \|h-\overline{H}\|_{Y^{1+\alpha',2}} + \|g-\overline{G}\|_{Y^{1+\alpha',2}} \le E\,.$$

We turn now to the proof of Proposition 4.1; here we use ideas of Caffarelli [5].

PROOF OF PROPOSITION 4.1: For simplicity, we treat the case $b_i \equiv 0$. We will show that

$$|\nabla u(0)| \le C\|u\|_{L^2(\Omega)}\,. \tag{4.1}$$

By Lemma 4.3,

$$\|A-\overline{A}\|_{Y^{1+\alpha',2}} \le E\,.$$

In fact, we can further assume that

$$\|A-\overline{A}\|_{Y^{1+\alpha',2}} \le \varepsilon_0 \tag{4.2}$$

for some small enough $\varepsilon_0 > 0$ (depending only on n, N, λ, Λ, α', and E). Indeed, we pick r_0 satisfying $r_0^{\alpha'}(1+E) = \varepsilon_0$ and let

$$\widetilde{A}(x) = A(r_0 x)\,, \quad \widetilde{\overline{A}}(x) = \overline{A}(r_0 x)\,, \quad \text{and} \quad \widetilde{u}(x) = u(r_0 x)\,.$$

A simple calculation yields

$$\|\widetilde{A} - \widetilde{\overline{A}}\|_{Y^{1+\alpha',2}} \le r_0^{\alpha'} \|A - \overline{A}\|_{Y^{1+\alpha',2}} \le \varepsilon_0\,,$$

and, since $b_i \equiv 0$,

$$\partial(\widetilde{A}\partial\widetilde{u}) = 0 \quad \text{in } \Omega\,.$$

In the following we will always assume the additional hypothesis (4.2) for sufficiently small ε_0. We also assume that u is normalized to satisfy

$$\|u\|_{L^2(\Omega)} = 1\,.$$

We will find $w_k \in H^1(\frac{3}{4^{k+1}}\Omega, \mathbb{R}^N)$, $k \ge 0$, such that for all k,

$$\partial(\overline{A}\partial w_k) = 0\,, \quad \frac{3}{4^{k+1}}\Omega\,, \tag{4.3}$$

$$\|w_k\|_{L^2(\frac{2}{4^{k+1}}\Omega)} \le C' 4^{-\frac{k(n+2+2\alpha')}{2}}\,, \quad \|\nabla w_k\|_{L^\infty(\frac{1}{4^{k+1}}\Omega)} \le C' 4^{-k\alpha'}\,, \tag{4.4}$$

$$\left\| u - \sum_{j=0}^{k} w_j \right\|_{L^2((\frac{1}{4})^{k+1}\Omega)} \le 4^{-\frac{(k+1)(n+2+2\alpha')}{2}}\,. \tag{4.5}$$

An easy consequence of (4.4) is

$$\|w_k\|_{L^\infty(4^{-(k+1)}\Omega)} \le C 4^{-(k+1)(1+\alpha')}\,. \tag{4.6}$$

In the following, C, C', and ε_0 denote various constants that depend only on parameters specified in the proposition. In particular, they are independent of k. C will be chosen first and will be large, then C' (much larger than C), and finally $\varepsilon_0 \in (0, 1)$ (much smaller than $1/C'$).

By Lemma 3.1, we can find $w_0 \in H^1(\frac{3}{4}\Omega, \mathbb{R}^N)$ such that

$$\partial(\overline{A}\partial w_0) = 0 \quad \text{in } \frac{3}{4}\Omega \quad \text{and} \quad \|u - w_0\|_{L^2(\frac{1}{2}\Omega)} \le C\varepsilon_0^{\gamma} \le 4^{-\frac{n+2+2\alpha'}{2}}\,,$$

so

$$\|w_0\|_{L^2(\frac{1}{2}\Omega)} \le C \le C'$$

and, by Corollary 1.7,

$$\|\nabla w_0\|_{L^\infty(\frac{1}{4}\Omega)} \le C \le C'\,.$$

We have verified (4.3)–(4.5) for $k = 0$. Suppose that (4.3)–(4.5) hold up to k ($k \ge 0$); we will prove them for $k+1$. Let

$$W(x) = \left(u - \sum_{j=0}^{k} w_j\right)(4^{-(k+1)}x)\,,$$

$$A_{k+1}(x) = A(4^{-(k+1)}x)\,, \quad \overline{A}_{k+1}(x) = \overline{A}(4^{-(k+1)}x)\,,$$

$$g_{k+1}(x) = 4^{-(k+1)}\Big([A_{k+1} - \overline{A}_{k+1}](x)\sum_{j=0}^{k}(\partial w_j)(4^{-(k+1)}x)\Big).$$

Then W satisfies

$$\partial(A_{k+1}\partial W) = \partial(g_{k+1}) \quad \text{in } \Omega .$$

A simple calculation, using (4.2), yields

$$\begin{aligned}\|A_{k+1} - \overline{A}_{k+1}\|_{L^2(\Omega)} &= \Big(\fint_{4^{-(k+1)}\Omega} |A - \overline{A}|^2\Big)^{1/2} \le 4^{-(k+1)\alpha'}\|A - \overline{A}\|_{Y^{1+\alpha',2}} \\ &\le 4^{-(k+1)\alpha'}\varepsilon_0 .\end{aligned}$$

By the induction hypothesis (see (4.4) and (4.5)), we have

$$\sum_{j=0}^{k}\big|(\partial_\beta w_j)(4^{-(k+1)}x)\big| \le C'\sum_{j=0}^{k}4^{-j\alpha'} \le C' , \quad x \in \Omega ,$$

and

$$\|W\|_{L^2(\Omega)} \le 4^{-(k+1)(1+\alpha')} ,$$

so

$$\|g_{k+1}\|_{L^2(\Omega)} \le C' 4^{-(k+1)(1+\alpha')}\varepsilon_0 .$$

By Lemma 3.1, there exists $v_{k+1} \in H^1(\frac{3}{4}\Omega, \mathbb{R}^N)$ such that

$$\partial(\overline{A}_{k+1}\partial v_{k+1}) = 0 \quad \text{in } \frac{3}{4}\Omega$$

and

$$\begin{aligned}\|W - v_{k+1}\|_{L^2(\frac{1}{2}\Omega)} &\le C'\big(\|g_{k+1}\|_{L^2(\Omega)} + 4^{-(k+1)(1+\alpha')\gamma}\varepsilon_0^\gamma\|W\|_{L^2(\Omega)}\big) \\ &\le C'\big(\varepsilon_0 + \varepsilon_0^\gamma\big)4^{-(k+1)(1+\alpha')} .\end{aligned} \tag{4.7}$$

Let

$$w_{k+1}(x) = v_{k+1}(4^{k+1}x) , \quad x \in \frac{3}{4^{k+2}}\Omega .$$

A change of variables in (4.7) and in the equation of v_{k+1} yields (4.3) and (4.5) for $k+1$.

It follows from the above and Corollary 1.7 that

$$\|\nabla v_{k+1}\|_{L^\infty(\frac{1}{4}\Omega)} \le C\|v_{k+1}\|_{L^2(\frac{1}{2}\Omega)} \le C4^{-(k+1)(1+\alpha')} .$$

Estimates (4.4) for $k+1$ follow from the above estimates for v_{k+1}. We have thus established (4.3)–(4.5) for all k.

For $|x| \le 4^{-(k+1)}$, using (4.4) and (4.6), it follows that

$$\begin{aligned}\Big|\sum_{j=0}^{k} w_j(x) - \sum_{j=0}^{\infty} w_j(0)\Big| &\le C\sum_{j=0}^{k}4^{-j\alpha'}|x| + C\sum_{j=k+1}^{\infty}4^{-j(1+\alpha')} \\ &\le C|x| + C4^{-k(1+\alpha')} .\end{aligned}$$

So we derive from (4.5) that

$$\left\| u - \sum_{j=0}^{\infty} w_j(0) \right\|_{L^2(4^{-(k+1)}\Omega)} \le C 4^{-\frac{(k+1)(n+2)}{2}} . \tag{4.8}$$

Consequently,

$$u(0) = \sum_{j=0}^{\infty} w_j(0) \quad \text{and} \quad |\nabla u(0)| \le C . \tag{4.9}$$

Estimate (4.1) is established. We have completed the proof of Proposition 4.1 when $b_i \equiv 0$. The general case can be established by similar arguments (using Lemma 3.2 in the proof instead of Lemma 3.1). We leave the details to the interested reader. □

Remark 4.4. By Corollary 1.7 (applied to v_{k+1}), we also know

$$\|\nabla^2 w_k\|_{L^\infty(\frac{1}{4^{k+1}}\Omega\cap\Omega_m)} \le C 4^{k(1-\alpha')} . \tag{4.10}$$

This estimate will be used in our proof of (1.8), the Hölder estimates of the gradients of u.

5 Hölder Estimates of the Gradient

We use the notation of Section 3.

PROPOSITION 5.1 *Let A be as in Section* 3, *and let $u \in H^1(\Omega, \mathbb{R}^N)$ be a solution of*

$$\partial(A\partial u) = 0 \quad in\ \Omega .$$

Then for all $x \in D_{m_0} \cap \frac{1}{2}\Omega$,

$$|\nabla u(x) - \nabla u(0)| \le C\|u\|_{L^2(\Omega)} |x|^{\alpha'} ,$$

where $\alpha' = \min\{\mu, \frac{\alpha}{2(\alpha+1)}\}$ and C depends only on $n, N, l, \alpha, \mu, \lambda$, and Λ, as well as $\max_{1\le m\le l} \|f_m\|_{C^{1,\alpha}([-1,1]^{n-1})}$ *and* $\max_{1\le m\le l} \|A^{(m)}\|_{C^\mu(\overline{D}_m)}$.

The proof is rather technical.

5.1 Beginning of the Proof of Proposition 5.1

As explained in Section 3 we may assume without loss of generality that

$$\|u\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \|A - \overline{A}\|_{Y^{1+\alpha',2}} \le \varepsilon_0 ,$$

where ε_0 is the small constant in Section 3.

As in the proof of Proposition 4.1, we can find $\{w_k\}_{k=0}^{\infty}$ in $H^1(\frac{3}{4^{k+1}}\Omega, \mathbb{R}^N)$ such that for $k \ge 0$, w_k satisfies (4.3), (4.4), (4.5), (4.6), (4.9), and (4.10).

Associated with $\overline{A}^{(m)} := \overline{A}|_{\Omega_m}$, we introduce a linear transformation $N^{(m)}$: $\mathbb{R}^{nN} \to \mathbb{R}^{nN}$ as follows: For $b = (b^i_\alpha) \in \mathbb{R}^{nN}$ $(1 \le \alpha \le n, 1 \le i \le N)$,

$$(N^{(m)}b)^i_\alpha = b^i_\alpha\,, \qquad 1 \le i \le N,\ 1 \le \alpha \le n-1\,,$$
$$(N^{(m)}b)^i_n = \overline{A}^{(m)n\beta}_{ij} b^j_\beta\,, \quad 1 \le i \le N\,.$$

Since $(\overline{A}^{(m)nn}_{ij})$ is a positive definite $N \times N$ matrix with eigenvalues in $[\lambda, \Lambda]$, it is clear that $N^{(m)}$ is invertible and

$$\|N^{(m)}\|, \|(N^{(m)})^{-1}\| \le C(n, N, \lambda, \Lambda)\,. \tag{5.1}$$

We also define linear transformations $T^{(m)} : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ by setting

$$T^{(m)} = (N^{(m)})^{-1} N^{(m_0)}\,.$$

LEMMA 5.2

$$\nabla u(0) = \sum_{j=0}^{\infty} \nabla w_j(0)\,, \tag{5.2}$$

and, for $x \in \frac{1}{4^{k+1}}\Omega \cap \Omega_m \setminus \frac{1}{4^{k+2}}\Omega$,

$$\left| \sum_{j=0}^{k} \nabla w_j(x) - \sum_{j=0}^{k} T^{(m)} \nabla w_j(0) \right| \le C|x|^{\alpha'}\,. \tag{5.3}$$

PROOF: We first prove (5.2). For $4^{-(k+1)}\Omega \subset \Omega_{m_0}$, it follows from (4.10) that

$$\left| w_j(x) - [w_j(0) + \nabla w_j(0)x] \right| \le 4^{j(1-\alpha')}|x|^2\,, \quad j \le k,\ x \in 4^{-(k+1)}\Omega\,.$$

This, and (4.5), yield

$$\left\| u - \left[\sum_{j=0}^{k} w_j(0) + \nabla w_j(0)x \right] \right\|_{L^2(4^{-(k+1)}\Omega)} \le \tag{5.4}$$
$$C4^{-k(n+2+2\alpha')/2} + C\sum_{j=0}^{k} 4^{j(1-\alpha')} \| |x|^2 \|_{L^2(4^{-(k+1)}\Omega)} \le C4^{-k(n+2+2\alpha')/2}\,.$$

From (4.6) and (4.4), we know that $\sum_{j=0}^{\infty} w_j(0)$ and $\sum_{j=0}^{\infty} \nabla w_j(0)$ are convergent and

$$\left| \sum_{j=0}^{\infty} w_j(0) - \sum_{j=0}^{k} w_j(0) \right| \le C4^{-k(1+\alpha')}\,,$$
$$\left| \sum_{j=0}^{\infty} \nabla w_j(0) - \sum_{j=0}^{k} \nabla w_j(0) \right| \le C4^{-k\alpha'}\,. \tag{5.5}$$

Combining (5.4) and (5.5), we have

$$\Big\| u - \Big[\sum_{j=0}^{\infty} w_j(0) + \sum_{j=0}^{\infty} \nabla w_j(0)x \Big] \Big\|_{L^2(4^{-(k+1)}\Omega)} \le C 4^{-k(n+2+2\alpha')/2} .$$

Equality (5.2) follows from the above.

Next we prove (5.3). The matching condition of w_j at $x_n = c_{m-1}$ is, for all $x' \in (-1,1)^{n-1}$,

$$N^{(m)} \nabla w_j^{(m)}(x', c_{m-1}) = N^{(m-1)} \nabla w_j^{(m-1)}(x', c_{m-1}) , \tag{5.6}$$

where $w_j^{(m)} = w_j|_{\Omega_m}$.

For $m = m_0$, (5.3) follows from (4.10). We will only show (5.3) for $m \ge m_0+1$ since the proof is the same for $m \le m_0 - 1$. For $x = (x', x_n) \in \frac{1}{4^{k+1}}\Omega \cap \Omega_m \setminus \frac{1}{4^{k+2}}\Omega$, $m \ge m_0 + 1$, we have

$$\sum_{j=0}^{k} \big| \nabla w_j^{(m)}(x) - T^{(m)} \nabla w_j(0) \big| \le$$

$$\sum_{j=0}^{k} \Big(\big| \nabla w_j^{(m)}(x) - \nabla w_j^{(m)}(0', c_{m-1}) \big| + \big| \nabla w_j^{(m)}(0', c_{m-1}) - T^{(m)} \nabla w_j(0) \big| \Big) .$$

By (4.10),

$$\big| \nabla w_j^{(m)}(x) - \nabla w_j^{(m)}(0', c_{m-1}) \big| \le C 4^{j(1-\alpha')} (|x'| + x_n - c_{m-1}) \le C 4^{j(1-\alpha')} |x| .$$

By (5.1), (5.6), and (4.10),

$$\begin{aligned}
&\big| \nabla w_j^{(m)}(0', c_{m-1}) - T^{(m)} \nabla w_j(0) \big| \\
&\quad \le C \big| N^{(m)} \nabla w_j^{(m)}(0', c_{m-1}) - N^{(m_0)} \nabla w_j^{(m_0)}(0) \big| \\
&\quad \le C \sum_{i=m_0+2}^{m} \big| N^{(i)} \nabla w_j^{(i)}(0', c_{i-1}) - N^{(i-1)} \nabla w_j^{(i-1)}(0', c_{i-2}) \big| \\
&\qquad + C \big| N^{(m_0+1)} \nabla w_j^{(m_0+1)}(0', c_{m_0}) - N^{(m_0)} \nabla w_j^{(m_0)}(0) \big| r \\
&\quad \le C \sum_{i=m_0+2}^{m} \big| N^{(i-1)} \nabla w_j^{(i-1)}(0', c_{i-1}) - N^{(i-1)} \nabla w_j^{(i-1)}(0', c_{i-2}) \big| \\
&\qquad + C \big| N^{(m_0)} \nabla w_j^{(m_0)}(0', c_{m_0}) - N^{(m_0)} \nabla w_j^{(m_0)}(0) \big| \\
&\quad \le C \sum_{i=m_0+2}^{m} 4^{j(1-\alpha')} (c_{i-1} - c_{i-2}) + 4^{j(1-\alpha')} (c_{m_0} - 0) \\
&\quad = C 4^{j(1-\alpha')} c_{m-1} \le C 4^{j(1-\alpha')} |x| .
\end{aligned}$$

It follows that

$$\sum_{j=0}^{k} \left|\nabla w_j^{(m)}(x) - T^{(m)}\nabla w_j(0)\right| \le C4^{k(1-\alpha')}|x| \le C4|x|^{\alpha'} .$$

Estimate (5.3) is established; so is Lemma 5.2. □

LEMMA 5.3 *Let $\bar{x}$ be on the x_n-axis and $\bar{x} + a|\bar{x}|\Omega \subset D_{m+1} \cap \Omega_{m+1}$ for some $a > 0$. Then*

$$\left|\nabla u(y) - \sum_{j=0}^{k} \nabla w_j(y)\right| \le C(a)|\bar{x}|^{\alpha'} , \quad y \in \bar{x} + \frac{a}{2}|\bar{x}|\Omega , \tag{5.7}$$

where k satisfies $4^{-(k+2)} \le |\bar{x}| < 4^{-(k+1)}$; consequently,

$$|\nabla u(y) - \nabla u(z)| \le C(a)|\bar{x}|^{\alpha'} , \quad y, z \in \bar{x} + \frac{a}{2}|\bar{x}|\Omega . \tag{5.8}$$

PROOF: Let

$$\hat{w}(y) = u(\bar{x} + a|\bar{x}|y) - \sum_{j=0}^{k} w_j(\bar{x} + a|\bar{x}|y) , \quad y \in \Omega .$$

By the equations of u and w_j,

$$\partial\big(A(\bar{x} + a|\bar{x}|\cdot)\partial\hat{w}\big) = \partial\hat{g} \quad \text{in } \Omega ,$$

where

$$\hat{g} = -a|\bar{x}| \sum_{j=0}^{k} \big(A^{(m+1)}(\bar{x} + a|\bar{x}|y) - A^{(m+1)}(0', c_m)\big)\partial w_j(\bar{x} + a|\bar{x}|y) ,$$

with $A^{(m+1)} := A|_{D_{m+1}}$.

Since $\bar{x} + a|\bar{x}|\Omega \in D_{m+1} \cap \Omega_{m+1}$, the $C^\mu(\Omega)$-seminorm of $A^{(m+1)}(\bar{x} + a|\bar{x}|\cdot)$ is bounded by $C(a)|\bar{x}|^\mu$. Thus, by (4.4) and (4.10),

$$\|\hat{g}\|_{C^\mu(\Omega)} \le C(a)|\bar{x}|^{1+\mu} .$$

We also deduce from (4.5) that

$$\|\hat{w}\|_{L^2(\Omega)} \le C(a)|\bar{x}|^{1+\mu} .$$

By the Schauder theory,

$$\|\nabla\hat{w}\|_{L^\infty(\frac{1}{2}\Omega)} \le C(a)|\bar{x}|^{1+\alpha'} .$$

Estimate (5.7) follows from the above. Estimate (5.8) follows from (5.7) and (4.10). □

5.2 Completion of the Proof of Proposition 5.1

For some small r_1, depending only on the parameters specified in Proposition 5.1, if x satisfies $|x| \geq r_1$, the desired estimate in Proposition 5.1 follows from the gradient estimate in Proposition 4.1. So we always assume that $x \in D_{m_0} \setminus \{0\}$ and $|x| < r_1$. In the following we repeatedly use the smallness of x (i.e., r_1). We select an $\bar{x}$ as follows. If $c_{m_0} > 80|x|$, set $\bar{x} = (0', 10|x|)$ (and $m = m_0 - 1$), otherwise let $m \geq m_0$ be the smallest index for which $c_{m+1} - c_m > 80|x|$, and set $\bar{x} = (0', c_m + 10|x|)$. Clearly, $10|x| \leq |\bar{x}| \leq 100(l+1)|x|$ and $\bar{x} + a|x|\Omega \subset D_{m+1} \cap \Omega_{m+1}$, with $a = 8$. With this choice of $\bar{x}$, let k satisfy $4^{-(k+2)} \leq |\bar{x}| < 4^{-(k+1)}$. Then by (5.2), (5.3), and (5.7), we have

$$
\begin{aligned}
\left|\nabla u(\bar{x}) - T^{(m)}\nabla u(0)\right| &\leq \left|\nabla u(\bar{x}) - \sum_{j=0}^{k} \nabla w_j(\bar{x})\right| \\
&\quad + \left|\sum_{j=0}^{k} \nabla w_j(\bar{x}) - \sum_{j=0}^{k} T^{(m)}\nabla w_j(0)\right| \\
&\leq C|\bar{x}|^{\alpha'} \leq C|x|^{\alpha'} .
\end{aligned} \tag{5.9}
$$

Let z be on either the graph of f_{m_0} or f_{m_0-1}, so that the distance of x to z is the least distance of x to the union of graphs of $\{f_j\}$. Let L be the line passing through z that is normal to this graph. Clearly $x \in L$. Let $z^{(j)}$ denote the intersection of L with the graph of f_j for $m_0 - 1 \leq j \leq m + 1$. Using the smallness of $|x|$ and the $C^{1,\alpha}$ property of $\{f_j\}$, it is not difficult to see that

$$
\left|z^{(j)} - (0', f_j(0'))\right| \leq 4|x| , \quad m_0 \leq j \leq m , \tag{5.10}
$$

and

$$
|z^{(m+1)} - z^{(m)}| \geq 40|x| .
$$

Here m is as defined before, and we have used the fact that the point $(0', f_{m_0-1}(0'))$ is the projection of the origin onto the graph of the function f_{m_0-1}. The same argument shows that we can find $\bar{z}$ on the segment determined by $z^{(m)}$ and $z^{(m+1)}$ with $|\bar{z} - z^{(m)}| = 10|x|$ such that

$$
\left|\nabla u(\bar{z}) - \widetilde{T}^{(m)}\nabla u(x)\right| \leq C|x|^{\alpha'} ,
$$

where the $\{\widetilde{T}^{(m)}\}$ are defined in the natural way. Due to (5.10) and the Hölder continuity of $A^{(j)}$, we have

$$
\left|T^{(m)} - \widetilde{T}^{(m)}\right| \leq C|x|^{\mu} ,
$$

so

$$
\left|\nabla u(\bar{z}) - T^{(m)}\nabla u(x)\right| \leq C|x|^{\alpha'} . \tag{5.11}
$$

It is easy to see, by the smallness of r_1 and Hölder-continuity of $\{\nabla f_j\}$, that

$$
|\bar{x} - \bar{z}| \leq 2|x| .
$$

By (5.8),

$$|\nabla u(\bar{x}) - \nabla u(\bar{z})| \le C|\bar{x}|^{\alpha'} \le C|x|^{\alpha'} . \tag{5.12}$$

A combination of (5.9), (5.11)–(5.12), and (5.1) yields

$$|\nabla u(x) - \nabla u(0)| \le C\left|T^{(m)}[\nabla u(x) - \nabla u(0)]\right| \le C|x|^{\alpha'} .$$

Proposition 5.1 is established.

Similarly, we can prove the following more general proposition; we leave the details to the interested reader.

PROPOSITION 5.4 *Let A and g be as in Section 3, $h \in L^\infty(\Omega, \mathbb{R}^N)$, and $u \in H^1(\Omega, \mathbb{R}^N)$ be a solution of*

$$\partial(A\partial u) = h + \partial g \ \ in\ \Omega , \quad 1 \le i \le N .$$

Then for all $x \in \Omega_{m_0} \cap \frac{1}{2}\Omega$,

$$|\nabla u(x) - \nabla u(0)| \le C\big(\|u\|_{L^2(\Omega)} + \|h\|_{L^\infty(\Omega)} + \max_{1\le m\le l} \|g\|_{C^\mu(\overline{\widetilde{D}}_m)}\big)|x|^{\alpha'} ,$$

where $\alpha' = \min\{\mu, \frac{\alpha}{2(\alpha+1)}\}$ and C depends only on $n, N, l, \alpha, \mu, \lambda$, and Λ, as well as $\max_{1\le m\le l} \|f_m\|_{C^{1,\alpha}([-1,1]^{n-1})}$ and $\max_{1\le m\le l} \|A\|_{C^\mu(\overline{\widetilde{D}}_m)}$.

6 Proof of Theorem 1.9

In this section we prove Theorem 1.9. Our proof is based on Theorem 1.1 and the arguments of Avellaneda and Lin in [1], which we follow closely. They assume Hölder-continuity of the coefficients and make use of classical gradient estimates while we rely on our Theorem 1.1.

Let $\widetilde{\mathcal{A}}$ denote our class of coefficients (with control on the ellipticity and the $C^{1,\alpha}$ norm of the dividing surfaces) on the flat torus $\mathbb{R}^n/\mathbb{Z}^n$. For $A \in \widetilde{\mathcal{A}}$, consider for $0 < \varepsilon < 1$,

$$L_\varepsilon = -\partial_\alpha\left(A_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\partial_\beta\right).$$

In the following discussions, $A \in \widetilde{\mathcal{A}}$.

Let $\chi = (\chi_{ij}^\alpha)$ denote the corrector matrix, defined as the solution of

$$-\partial_\alpha\big(A_{ij}^{\alpha\beta}(x)\partial_\beta\chi_{jk}^\gamma\big) = \partial_\alpha(A_{ik}^{\alpha\gamma}) \quad \text{in } \mathbb{R}^n ,$$

$$\chi \text{ is 1-periodic in } x^1, \dots, x^n, \qquad \int\limits_{[0,1]^n} \chi = 0 .$$

For any $B \in \mathbb{R}^{nN}$, let $(x + \varepsilon\chi(x/\varepsilon))B$ denote the vector-valued function

$$\left[\left(x + \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\right)B\right]^j = x^\gamma B_\gamma^j + \chi_{jk}^\gamma B_\gamma^k .$$

It is easy to see that

$$L_\varepsilon\left(\left(x+\varepsilon\chi\left(\frac{x}{\varepsilon}\right)\right)B\right)=0\,, \tag{6.1}$$

i.e.,

$$\partial_\alpha\left(A_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\partial_\beta\left[\left(x+\varepsilon\chi\left(\frac{x}{\varepsilon}\right)\right)B\right]^j\right)=0\,.$$

By Theorem 1.1, χ satisfies

$$\|\nabla\chi\|_{L^\infty(\mathbb{R}^n)}\le C\,.$$

Let $\{u_\varepsilon\}$ satisfy

$$L_\varepsilon u_\varepsilon=0 \quad \text{in an open bounded set } D \text{ in } \mathbb{R}^n,$$

and, along a subsequence $\varepsilon\to 0$,

$$u_\varepsilon \text{ converging weakly to } u_0 \text{ in } H^1(D)\,.$$

It is known, following an argument in [3, chap. 1, sec. 3], that u_0 satisfies a homogenized system

$$L_0u_0=0 \quad \text{in } D\,,$$

where

$$L_0=-\partial_\alpha\left(A_{0ij}^{\alpha\beta}\partial_\beta\right)$$

is the homogenized operator with $\{A_{0ij}^{\alpha\beta}\}$ constants satisfying

$$|A_0|\le\Lambda \tag{6.2}$$

and

$$\int_D A_{0ij}^{\alpha\beta}\partial_\alpha\varphi^j\partial_\beta\varphi^i\ge\lambda\int_D|\nabla\varphi|^2 \quad \forall\varphi\in H_0^1(D,\mathbb{R}^N)\,.$$

It follows that

$$A_{0ij}^{\alpha\beta}\xi_\alpha\xi_\beta\eta^i\eta^j\ge\lambda|\xi|^2|\eta|^2 \quad \forall\xi,\eta\,. \tag{6.3}$$

We first establish the following:

THEOREM 6.1 *Given* $0<\nu<1$*, suppose that* u_ε *satisfies*

$$L_\varepsilon u_\varepsilon=0 \text{ in } B_1 \quad \text{and} \quad \|u_\varepsilon\|_{L^2(B_1)}<\infty\,.$$

Then

$$\|u_\varepsilon\|_{C^\nu(B_{1/2})}\le C\|u_\varepsilon\|_{L^2(B_1)}\,,$$

where C *depends only on* n, N, ν, λ, *and* Λ, *the number of the dividing surfaces* $\{\partial D_m\}$ *and their* $C^{1,\alpha}$ *norms, and the Hölder-continuity of* A *in each* $\overline{D}_m$.

We will use the notation $(\bar u_\varepsilon)_{x,r}=⨍_{B(x,r)}\bar u_\varepsilon$.

LEMMA 6.2 *For every* $0 < \nu < 1$, *there exist* θ, $\varepsilon_0 \in (0,1)$, *depending only on* n, N, ν, λ, *and* Λ, *such that if* $u_\varepsilon \in H^1(B_1, \mathbb{R}^N)$ *satisfy*

$$L_\varepsilon u_\varepsilon = 0 \quad \text{in } B_1 \,,$$

then, for $0 < \varepsilon \le \varepsilon_0$,

$$\fint_{B_\theta} |u_\varepsilon - (\bar{u}_\varepsilon)_{0,\theta}|^2 \le \theta^{2\nu} \fint_{B_1} |u_\varepsilon|^2 \,. \tag{6.4}$$

PROOF: Fix a $\nu' \in (\nu, 1)$, and let $L_0 = -\partial_\alpha(A_{0ij}^{\alpha\beta}\partial_\beta)$ with A_0 constant and satisfying (6.2) and (6.3). By the interior gradient estimates of solutions of elliptic systems with constant coefficients, there exists sufficiently small $\theta > 0$, depending only on n, N, ν', λ, and Λ, such that if $u_0 \in H^1(B_1, \mathbb{R}^N)$ is a solution of

$$L_0 u_0 = 0 \quad \text{in } B_1 \,, \tag{6.5}$$

then

$$\fint_{B_\theta} |u_0 - (\bar{u}_0)_{0,\theta}|^2 \le C\theta^2 \fint_{B_1} |u_0|^2 \le \theta^{2\nu'} \fint_{B_1} |u_0|^2 \,. \tag{6.6}$$

To prove (6.4), we argue by contradiction. Suppose the contrary, that there is a sequence of $L_{\varepsilon_j}^j$ in our class and $u_{\varepsilon_j} \in H^1(B_1, \mathbb{R}^N)$ satisfying

$$L_{\varepsilon_j}^j u_{\varepsilon_j} = 0 \text{ in } B_1 \,, \quad \fint_{B_1} |u_{\varepsilon_j}|^2 = 1 \,, \quad \varepsilon_j \to 0 \,,$$

but for which

$$\fint_{B_1} |u_{\varepsilon_j} - (\bar{u}_{\varepsilon_j})_{0,\theta}|^2 > \theta^{2\nu} \,. \tag{6.7}$$

By ellipticity,

$$\|u_{\varepsilon_j}\|_{H^1(B_\theta)} \le C$$

for some C independent of j. After passing to a subsequence, for some $u_0 \in H^1_{\text{loc}}(B_1, \mathbb{R}^N)$, we have

$$u_{\varepsilon_j} \text{ converges weakly to } u_0 \text{ in } H^1(B_\theta, \mathbb{R}^N) \,.$$

As explained earlier, u_0 satisfies (6.5) with some L_0 as above. Passing to the limit in (6.7) and using (6.6), we have

$$\theta^{2\nu} \le \fint_{B_\theta} |u_0 - (\bar{u}_0)_{0,\theta}|^2 \le \theta^{2\nu'} \fint_{B_1} |u_0|^2 \le \theta^{2\nu'} \,,$$

a contradiction. Hence (6.4) holds for some $\varepsilon_0 > 0$. □

LEMMA 6.3 *Given* $0 < \nu < 1$*, let* θ *and* ε_0 *be as in Lemma* 6.2*. Then, for all* u_ε *satisfying*

$$L_\varepsilon u_\varepsilon = 0 \text{ in } B_1 , \quad \|u_\varepsilon\|_{L^2(B_1)} < \infty,$$

and for all $k \geq 1$ *such that* $\varepsilon/\theta^{k-1} \leq \varepsilon_0$*, we have*

$$\fint_{B_{\theta^k}} |u_\varepsilon - (\bar{u}_\varepsilon)_{0,\theta^k}|^2 \leq \theta^{2k\nu} \fint_{B_1} |u_\varepsilon|^2 . \tag{6.8}$$

PROOF: The proof is by induction on k. By Lemma 6.2, (6.8) holds for $k = 1$. Assume that (6.8) holds for k. For k satisfying $\varepsilon/\theta^k \leq \varepsilon_0$, set

$$w_\varepsilon(x) = u_\varepsilon(\theta^k x) - (\bar{u}_\varepsilon)_{0,\theta^k} , \quad x \in B_1 . \tag{6.9}$$

Then

$$L_{\varepsilon/\theta^k} w_\varepsilon = 0 \quad \text{in } B_1$$

and, by the induction hypothesis,

$$\fint_{B_1} |w_\varepsilon|^2 \leq \theta^{2k\nu} \fint_{B_1} |u_\varepsilon|^2 .$$

Since $\varepsilon/\theta^k \leq \varepsilon_0$, we may apply Lemma 6.2 to obtain

$$\fint_{B_\theta} |w_\varepsilon - (\bar{w}_\varepsilon)_{0,\theta}|^2 \leq \theta^{2\nu} \fint_{B_1} |w_\varepsilon|^2 \leq \theta^{2(k+1)\nu} \fint_{B_1} |u_\varepsilon|^2 . \tag{6.10}$$

Rewriting (6.10) and using (6.9), we have

$$\fint_{B_{\theta^{k+1}}} |u_\varepsilon - (\bar{u}_\varepsilon)_{0,\theta^{k+1}}|^2 \leq \theta^{2(k+1)\nu} \fint_{B_1} |u_\varepsilon|^2 ;$$

i.e., (6.8) holds for $k + 1$. Lemma 6.3 is established. □

PROOF OF THEOREM 6.1: We denote by C a generic constant depending on admissible parameters, i.e., the parameters specified in Theorem 6.1. We need only prove that

$$\fint_{B_r(x)} |u_\varepsilon - (\bar{u}_\varepsilon)_{x,r}|^2 \leq C r^{2\nu} \|u_\varepsilon\|^2_{L^2(B_1)} \quad \forall 0 < r \leq \frac{1}{4},\ |x| < \frac{1}{2} . \tag{6.11}$$

Without loss of generality (making a translation), we only need to establish (6.11) for $x = 0$. By Lemma 6.3, (6.11) with $x = 0$ holds for $r \geq \varepsilon/\varepsilon_0$. Set

$$w_\varepsilon(x) = u_\varepsilon(\varepsilon x) - (\bar{u}_\varepsilon)_{0,2\varepsilon/\varepsilon_0} .$$

Then

$$L_1 w_\varepsilon = 0 \quad \text{in } B_{2/\varepsilon_0}$$

and, by (6.11) with $\bar{r} = 2\varepsilon/\varepsilon_0$ and $x = 0$ in (6.11), we have

$$\|w_\varepsilon\|_{L^2(B_{2/\varepsilon_0})} \le C\bar{r}^\nu \|u_\varepsilon\|_{L^2(B_1)} .$$

We have interior gradient estimates for w_ε (Theorem 1.1), in particular C^ν estimates for w_ε, so

$$\fint_{B_s} |w_\varepsilon - (\bar{w}_\varepsilon)_{0,s}|^2 \le Cs^{2\nu} \|w_\varepsilon\|^2_{L^2(B_{2/\varepsilon_0})} \quad \forall s \le \frac{1}{\varepsilon_0} .$$

It follows, by setting $r = s\varepsilon$, that

$$\fint_{B_r} |u_\varepsilon - (\bar{u}_\varepsilon)_{0,r}|^2 \le Cr^{2\nu} \|u_\varepsilon\|^2_{L^2(B_1)} \quad \forall r \le \frac{\varepsilon}{\varepsilon_0} .$$

We have established (6.11) for $x = 0$. As pointed out earlier, (6.11) is established. □

6.1 Gradient Estimates for u_ε

In this section we establish Theorem 1.9, gradient estimates for u_ε.

LEMMA 6.4 *There exist* $0 < \theta < 1$ *and* $0 < \varepsilon_0 < 1$, *which depend on admissible parameters, such that if* $u_\varepsilon \in H^1(B_1, \mathbb{R}^N)$ *satisfies*

$$L_\varepsilon u_\varepsilon = 0 \quad \text{in } B_1 ,$$

then, for $0 < \varepsilon \le \varepsilon_0$,

$$\sup_{|x|<\theta} \left| u_\varepsilon(x) - u_\varepsilon(0) - \left(x + \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\right)(\overline{\nabla u_\varepsilon})_\theta \right| \le \theta^{5/4} \|u_\varepsilon\|_{L^\infty(B_1)} , \tag{6.12}$$

where χ *is defined at the beginning of this section.*

PROOF: Let L_0 be any operator that is obtained from a sequence of L_ε with $A_\varepsilon \in \widetilde{\mathcal{A}}$. Then L_0 is a constant-coefficient operator with ellipticity under control. Therefore there exists $0 < \theta < 1$, depending only on n, N, λ, and Λ, such that for any

$$L_0 u_0 = 0 \quad \text{in } B_1 ,$$

we have

$$\sup_{|x|<\theta} \left| u_0(x) - u_0(0) - x \cdot (\overline{\nabla u_0})_\theta \right| \le C\theta^2 \|u_0\|_{L^\infty(B_1)} \le \theta^{3/2} \|u_0\|_{L^\infty(B_1)} . \tag{6.13}$$

Fixing this value of θ, we prove (6.12) by a contradiction argument. Suppose on the contrary that there exist $A_j \in \widetilde{\mathcal{A}}$ and $\varepsilon_j \to 0$ such that

$$L^j_{\varepsilon_j} u_{\varepsilon_j} = 0 \ \text{ in } B_1 , \quad \|u_{\varepsilon_j}\|_{L^\infty(B_1)} = 1 ,$$

and

$$\sup_{|x|<\theta} \left| u_{\varepsilon_j}(x) - u_{\varepsilon_j}(0) - \left(x + \varepsilon_j\chi\left(\frac{x}{\varepsilon_j}\right)\right)(\overline{\nabla u_{\varepsilon_j}})_\theta \right| > \theta^{5/4} . \tag{6.14}$$

Passing to a subsequence,

$$u_{\varepsilon_j} \text{ converges weakly to some } u_0 \text{ in } H^1_{\mathrm{loc}}(B_1)\,,$$

and, by Theorem 6.1,

$$u_{\varepsilon_j} \text{ converges to } u_0 \text{ in } C^0_{\mathrm{loc}}(B_1)\,.$$

As explained at the beginning of this section, u_0 satisfies a homogenized equation

$$L_0 u_0 = 0 \quad \text{in } B_1\,,$$

where L_0 is as described earlier.

Clearly

$$\|u_0\|_{L^\infty(B_1)} \le 1\,.$$

By (6.13),

$$\sup_{|x|<\theta} \left| u_0(x) - u_0(0) - x \cdot (\overline{\nabla u_0})_\theta \right| \le \theta^{3/2}\,.$$

Since $|(\overline{\nabla u_{\varepsilon_j}})_\theta| \le C(\theta)$ by the H^1 bound of u_ε,

$$\sup_{|x|<\theta} \left| \varepsilon_j \chi\left(\frac{x}{\varepsilon_j}\right)(\overline{\nabla u_{\varepsilon_j}})_\theta \right| \le \varepsilon_j C(\theta) \to 0\,.$$

Sending j to infinity in (6.14), we have

$$\sup_{|x|<\theta} \left| u_0(x) - u_0(0) - x \cdot (\overline{\nabla u_0})_\theta \right| \ge \theta^{5/4}\,,$$

so we have

$$\theta^{5/4} \ge \theta^{3/2}\,,$$

which contradicts the fact that $\theta < 1$. Estimate (6.12) is established, and so is Lemma 6.4. □

LEMMA 6.5 *Let θ and ε_0 be as in Lemma* 6.4. *Suppose that $u_\varepsilon \in H^1(B_1, \mathbb{R}^N)$ satisfies*

$$L_\varepsilon u_\varepsilon = 0 \quad \text{in } B_1\,.$$

Then, for all k with $\varepsilon \le \varepsilon_0 \theta^{k-1}$, there exists $a_k^\varepsilon \in \mathbb{R}$ and $B_k^\varepsilon \in \mathbb{R}^n$ such that

$$|a_k^\varepsilon| \le C_1 \|u_\varepsilon\|_{L^\infty(B_1)}\,, \quad |B_k^\varepsilon| \le C_2\left(1 + \sum_{j=0}^{k-1} \theta^{j/4}\right)\|u_\varepsilon\|_{L^\infty(B_1)} \tag{6.15}$$

(C_1 *and* C_2 *are generic constants, depending only on* θ, ε_0, *and admissible parameters*) *and*

$$\sup_{|x|<\theta^k} \left| u_\varepsilon(x) - u_\varepsilon(0) - \varepsilon a_k^\varepsilon - \left(x + \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\right) B_k^\varepsilon \right| \le \theta^{5k/4}\|u_\varepsilon\|_{L^\infty(B_1)}\,. \tag{6.16}$$

PROOF: We argue by induction. In the following, C, C_1, and C_2 have the ordering $C \ll C_2 \ll C_1$. By Lemma 6.4, estimate (6.16) holds for $k = 1$ with

$$a_1^\varepsilon = 0 \quad \text{and} \quad B_1^\varepsilon = (\overline{\nabla u_\varepsilon})_\theta \,.$$

Suppose (6.16) holds for some k. For $\varepsilon \le \varepsilon_0 \theta^k$, define on B_1

$$w_\varepsilon(x) = \theta^{-5k/4} \|u_\varepsilon\|_{L^\infty(B_1)}^{-1} \left[u_\varepsilon(\theta^k x) - u_\varepsilon(0) - \varepsilon a_k^\varepsilon - \left(\theta^k x + \varepsilon \chi\left(\frac{\theta^k x}{\varepsilon}\right)\right) B_k^\varepsilon \right].$$

Then, by using (6.1) and the equation of u_ε,

$$L_{\frac{\varepsilon}{\theta^k}} w_\varepsilon = 0 \quad \text{in } B_1 \,.$$

By (6.16) (the induction hypothesis), $\|w_\varepsilon\|_{L^\infty(B_1)} \le 1$. Applying Lemma 6.4, we have

$$\sup_{|x|<\theta} \left| w_\varepsilon(x) - w_\varepsilon(0) - \left(x + \frac{\varepsilon}{\theta^k} \chi\left(\frac{\theta^k x}{\varepsilon}\right)\right) (\overline{\nabla w_\varepsilon})_\theta \right| \le \theta^{5/4} \,, \tag{6.17}$$

and, by ellipticity,

$$|(\overline{\nabla w_\varepsilon})_\theta| \le C \,.$$

Rewriting (6.17) in terms of u_ε, we have

$$\begin{aligned} \sup_{|x|<\theta} \Bigg| & u_\varepsilon(\theta^k x) - u_\varepsilon(0) + \varepsilon \chi(0) B_k^\varepsilon - \left(\theta^k x + \varepsilon \chi\left(\frac{\theta^k x}{\varepsilon}\right)\right) B_k^\varepsilon \\ & - \|u_\varepsilon\|_{L^\infty(B_1)} \theta^{5k/4} \left(x + \frac{\varepsilon}{\theta^k} \chi\left(\frac{\theta^k x}{\varepsilon}\right)\right) (\overline{\nabla w_\varepsilon})_\theta \Bigg| \\ \le{} & \|u_\varepsilon\|_{L^\infty(B_1)} \theta^{5(k+1)/4} \,. \end{aligned} \tag{6.18}$$

Define

$$a_{k+1}^\varepsilon = -\chi(0) B_k^\varepsilon \,, \quad B_{k+1}^\varepsilon = B_k^\varepsilon + \|u_\varepsilon\|_{L^\infty(B_1)} \theta^{k/4} (\overline{\nabla w_\varepsilon})_\theta \,. \tag{6.19}$$

It follows, by the induction hypotheses, that

$$\left|a_{k+1}^\varepsilon\right| \le C \left|B_k^\varepsilon\right| \le C C_2 \left(1 + \sum_{j=0}^{k-1} \theta^{j/4} \right) \|u_\varepsilon\|_{L^\infty(B_1)} \le C_1 \|u_\varepsilon\|_{L^\infty(B_1)}$$

and

$$\left|B_{k+1}^\varepsilon\right| \le \left|B_k^\varepsilon\right| + C\theta^{k/4} \|u_\varepsilon\|_{L^\infty(B_1)} \le C_2 \left(1 + \sum_{j=0}^{k} \theta^{j/4} \right) \|u_\varepsilon\|_{L^\infty(B_1)} \,.$$

So a_{k+1}^ε and B_{k+1}^ε also satisfy (6.15) with $k+1$ instead of k. Estimate (6.15) has been established for all $k \ge 1$.

Substituting (6.19) into (6.18) and making a change of variables $\theta^k x \to x$, we obtain (6.16) with $k+1$ instead of k. Lemma 6.5 is established. □

PROOF OF THEOREM 1.9: Let k be a positive integer with

$$\frac{\varepsilon}{\theta^k} \le \varepsilon_0 \le \frac{\varepsilon}{\theta^{k+1}} .$$

By Lemma 6.5,

$$\sup_{|x|<\varepsilon/\varepsilon_0} \left| u_\varepsilon(x) - u_\varepsilon(0) - \varepsilon a_k^\varepsilon - \left(x + \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\right) B_k^\varepsilon \right| \le \theta^{5k/4} \|u_\varepsilon\|_{L^\infty(B_1)} .$$

Rescaling the above, by (6.15),

$$\sup_{|x|<1/\varepsilon_0} \left| \frac{u_\varepsilon(\varepsilon x) - u_\varepsilon(0)}{\varepsilon} \right| \le C \|u_\varepsilon\|_{L^\infty(B_1)} .$$

Define

$$v_\varepsilon(x) = \frac{u_\varepsilon(\varepsilon x) - u_\varepsilon(0)}{\varepsilon} , \quad |x| < \frac{1}{\varepsilon_0} ; \tag{6.20}$$

then

$$L_1 v_\varepsilon = 0 \text{ in } B_{1/\varepsilon_0} \quad \text{and} \quad \|v_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} \le C \|u_\varepsilon\|_{L^\infty(B_1)} .$$

By Theorem 1.1,

$$\|\nabla v_\varepsilon\|_{L^\infty(B_{1/(2\varepsilon_0)})} \le C \|u_\varepsilon\|_{L^\infty(B_1)} ,$$

which, by (6.20), implies

$$\|\nabla u_\varepsilon\|_{L^\infty(B_{\varepsilon/(2\varepsilon_0)})} \le C \|u_\varepsilon\|_{L^\infty(B_1)} .$$

This estimate can be done in $B_{\varepsilon/(2\varepsilon_0)}(x)$ for any $x \in B_{1/2}$. Theorem 1.9 is established. □

Appendix: L^p-Integrability

For $0 < \lambda \le \Lambda < \infty$, let $A \in \mathcal{A}(\lambda, \Lambda)$; i.e., $\{A_{ij}^{\alpha\beta}(x)\}$ satisfies (6.12) and (6.13), with $D = \Omega := (-1, 1)^n$.

THEOREM A.1 *Let A be as above. There exists some $p_0 > 2$, depending only on n, N, λ, and Λ, such that for a solution $u \in H_0^1(\Omega, \mathbb{R}^N)$ of*

$$-\partial_\alpha\left(A_{ij}^{\alpha\beta}(x)\partial_\beta u\right) = \partial_\beta g_i^\beta , \quad 1 \le i \le N , \quad \text{in } \Omega ,$$

and for $2 < p < p_0$, we have $\nabla u \in L^p(\Omega)$ and

$$\int_\Omega |\nabla u|^p \le C \int_\Omega |g|^p .$$

PROOF: Let $B_{2R} = B_{2R}(x)$ be a ball of radius $2R$ contained in Ω, and let η be a smooth function with $\eta = 1$ in B_R and $\eta = 0$ outside B_{2R}. Multiplying the equation by $\eta^2 u$ and integrating by parts leads to

$$\int_{B_R} |\nabla u|^2 \le \frac{C}{R^2} \int_{B_{2R}} u^2 + \int_{B_{2R}} |g|^2 .$$

Substituting u by $u-\bar{u}$, where $\bar{u}$ is the average of u on B_{2R}, we may assume without loss of generality that the average of u on B_{2R} is zero. Thus, by the Poincaré inequality, we have

$$\int_{B_R} |\nabla u|^2 \le \frac{C}{R^2}\left(\int_{B_{2R}} |\nabla u|^{\frac{2n}{n+2}} \right) + \int_{B_{2R}} |g|^2 ,$$

i.e.,

$$\frac{1}{R^n}\int_{B_R} |\nabla u|^2 \le C\left(\frac{1}{R^n}\int_{B_{2R}} |\nabla u|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} + \frac{1}{R^n}\int_{B_{2R}} |g|^2 .$$

By the reverse Hölder inequality,

$$\frac{1}{R^n}\int_{B_R} |\nabla u|^p \le C\left(\frac{1}{R^n}\int_{B_{2R}} |\nabla u|^2 \right)^{p/2} + \frac{C}{R^n}\int_{B_{2R}} |g|^p , \tag{A.1}$$

where $2 \le p < p_0$, $p_0 > 2$, and C has the dependence stated in the theorem.

For any ball $B_R(x)$, we would like to show that for some $p_0 > 2$ (possibly smaller than the one above but having the same dependence) and any $2 < p < p_0$,

$$\frac{1}{R^n}\int_{B_R(x)} |\nabla u|^p \le C\left(\frac{1}{R^n}\int_{B_{2R}(x)} |\nabla u|^2 \right)^{p/2} + C\frac{1}{R^n}\int_{B_{2R}(x)} |g|^p .$$

Here u has been extended as zero outside Ω.

There are three cases: Case 1, where $B_{\frac{3}{2}R}(x) \cap \Omega = \varnothing$, is the interior case, and has been settled in (A.1). Case 2, where $B_{\frac{3}{2}R}(x) \subset \Omega$, is trivial. We only consider case 3, where $B_{\frac{3}{2}R}(x) \cap \partial\Omega \neq \varnothing$.

Let η be the same cutoff function. Multiplying the equation by $\eta^2 u$ and integrating by parts, we still have

$$\int_{B_R(x)} |\nabla u|^2 \le \frac{C}{R^2}\int_{B_{2R}(x)} u^2 + \int_{B_{2R}(x)} |g|^2 .$$

Since $B_{2R}(x) \cap \partial\Omega$ has a big enough portion and $u = 0$ on $\partial\Omega$, we have, by the Sobolev inequality,

$$\int_{B_{2R}(x)} u^2 \le C\left(\int_{B_{2R}(x)} |\nabla u|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} .$$

Thus we still have

$$\frac{1}{R^n}\int_{B_R(x)} |\nabla u|^2 \le C\left(\frac{1}{R^n}\int_{B_{2R}(x)} |\nabla u|^{\frac{2n}{n+2}} \right) + \frac{1}{R^n}\int_{B_{2R}(x)} f^2 .$$

The desired inequality still follows from the reverse Hölder inequality.

It follows that for some $p > 2$, the L^p norm of $|\nabla u|$ is controlled by the L^2 norm of $|\nabla u|$ and the L^p norm of g. On the other hand, we know that the L^2 norm of $|\nabla u|$ is controlled by the L^2 norm of g. Therefore we have shown that, for some $p > 2$,

$$\int_\Omega |\nabla u|^p \le C \int_\Omega |g|^p .$$

□

Acknowledgments. YanYan Li was partially supported by National Science Foundation Grant DMS-9706887 and a Rutgers University Research Council grant.

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YANYAN LI
Rutgers University
Department of Mathematics
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
E-mail: yyli@math.rutgers.edu

LOUIS NIRENBERG
Courant Institute
251 Mercer Street
New York, NY 10012-1185
E-mail: nirenl@cims.nyu.edu

Received November 2001.