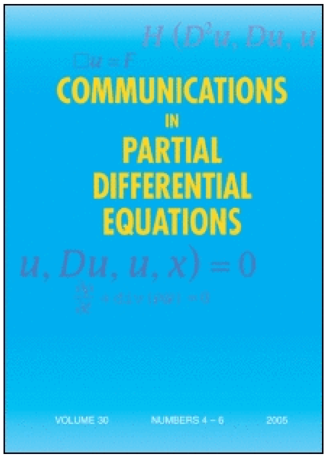

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DEGREE THEORY FOR SECOND ORDER NONLINEAR ELLIPTIC OPERATORS AND ITS APPLICATIONS

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Abstract

We introduce, along the lines of [4], an integer valued degree for second order fully nonlinear elliptic operators which is invariant under homotopy within elliptic operators. We also give some applications to the bifurcation problem for nonlinear elliptic equations. Applications to the existence of solutions of certain fully nonlinear elliptic equations on compact manifolds can be found in [7].

Introduction

Leray Schauder degree theory has been very useful in the study of quasilinear elliptic equations. It will certainly be useful if a degree theory can be introduced for fully nonlinear elliptic operators. Fitzpatrick and Pejsachowicz have introduced in [4] an integer valued degree for quasilinear Fredholm operator which is invariant up to sign under homotopy. They have also pointed out that it is impossible to have an integer valued degree which is invariant under homotopy within quasilinear Fredholm operators. But if we only restrict ourselves to second order elliptic operators, it is possible. In fact, following [4], we have used an elementary and constructive method to introduce an integer valued degree for second order fully nonlinear elliptic operator which is invariant under homotopy within elliptic operators. We also give some applications of this degree theory to the existence of solutions of certain fully nonlinear elliptic equations on compact manifolds (see [7]) and to the study of bifurcation problems of fully nonlinear elliptic equations. The author believes that such a degree will be useful in dealing with more delicate problems involving questions of multiplicities.

1 Degree of second order fully nonlinear elliptic operator with Dirichlet boundary data

We first give a brief review of the degree theory introduced in [4] for quasilinear Fredholm operators.

Let X, X_1 and Y be Banach spaces, with X compactly embedded in X_1. A map $f : X \rightarrow Y$ is called quasilinear Fredholm if f has a representation of the form:

$$f(x) = L_x(x) + C(x) \tag{1}$$

where

(i): For $x \in X_1, L_x \in L(X,Y)$ is Fredholm of index 0 and the map $x \longmapsto L_x$ is continuous from X_1 to $L(X,Y)$, the set of linear bounded operators from X to Y, and

(ii): $C : X \rightarrow Y$ is compact.

It is proved in [4] that each quasilinear map $f : X \rightarrow Y$ admits a representation of the form $f(x) = M_x(x) + C(x)$, where $x \longmapsto M_x$ is continuous from X_1 to $GL(X,Y)$, the subset of $L(X,Y)$ consisting of isomorphisms, and $C : X \rightarrow Y$ is compact. Moreover, the map $\bar{C} : X \rightarrow X$ defined by $\bar{C}(x) = M_x^{-1}(C(x))$ is compact.

The degree of f on O at 0, a bounded open set of X, is defined as

$$\deg(f,O,0) = \epsilon(M_0)\deg_{L.S.}(Id + \bar{C}, O, 0) \qquad (2)$$

where $\epsilon : GL(X,Y) \rightarrow \{-1,+1\}$ is an orientation of $GL(X,Y)$ (see [4]), $\deg_{L.S.}$ denotes the Leray Schauder degree.

The degree defined in (2) may change sign under homotopy within quasilinear Fredholm operators.

In this section we define a degree for second order fully nonlinear elliptic operators which is invariant under homotopy within elliptic operators.

Let $\Omega \subset R^n$ be a bounded domain with smooth boundary, $f \in C^{3,\alpha}(\bar{\Omega} \times R \times R^n \times R^{n^2})$, $0 < \alpha < 1$. Let F be a differential operator from $C_0^{4,\alpha}(\bar{\Omega})$ to $C^{2,\alpha}(\bar{\Omega})$ defined by

$$F : u \longmapsto f(\cdot, u, Du, D^2u).$$

Here and later repeated indices will denote sum over the indices.

Definition 1.1: $F : C_0^{4,\alpha}(\bar{\Omega}) \to C^{2,\alpha}(\bar{\Omega})$ is said to be elliptic on O , which is some bounded open set of $C_0^{4,\alpha}(\bar{\Omega})$, if there exists $\beta = \beta(O) > 0$, such that, for any $u \in O, x \in \bar{\Omega}, \xi \in R^n$

$$-f_{u_{ij}}(x, u, Du, D^2u)\xi_i\xi_j \geq \beta|\xi|^2$$

Let $O \subset C_0^{4,\alpha}(\bar{\Omega})$ be a bounded open set with $\partial O \cap F^{-1}(0) = \phi$ and suppose that F is elliptic on O. We want to define an integer valued degree for F on O at 0.

Consider

$$S \ : \ C^{2,\alpha}(\bar{\Omega}) \to C^{\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

$$u \longmapsto (-\Delta u, (D_j u\gamma_j + u)|_{\partial\Omega})$$

where $\gamma(x)$ denotes the outer unit normal of $\partial\Omega$ at x.

It is well known that S is an isomorphism.

Let $\widetilde{F}$ be the composite map of S and F, namely,

$$\widetilde{F} = (\widetilde{F}_{(1)}, \widetilde{F}_{(2)}) = S \circ F$$

$$\widetilde{F} : C_0^{4,\alpha}(\bar{\Omega}) \to C^{\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

Since S is an isomorphism, $F = 0$ is the same as $\widetilde{F} = 0$. We are going to define a degree for F by defining a degree for $\widetilde{F}$.

Clearly $\widetilde{F}$ can be expressed, even though not uniquely, as the following:

$$\left\{\begin{array}{ll} & \widetilde{F}_{(1)}(u) \\ = & a_{st}(x, u, Du, D^2u) D_{iist} u + C_0(x, u, Du, D^2u, D^3u) \\ & +C_1(x, u, Du, D^2u, D^3u)u + C_{1i}(x, u, Du, D^2u, D^3u)u_i \\ & +C_{2ij}(x, u, Du, D^2u, D^3u)u_{ij} + C_{3ijk}(x, u, Du, D^2, D^3u)u_{ijk} \\ & \widetilde{F}_{(2)}(u) \\ = & -a_{st}(x, u, Du, D^2u) D_{sti} u \gamma_i + E_0(x, u, Du, D^2u, \gamma) \end{array}\right. \tag{3}$$

Where

$$a_{st}(x, u, Du, D^2u) = -f_{u_{st}}(x, u, Du, D^2u)$$

By freezing coefficients in (3), the operator $\tilde{F}$ is of the form (1). Adding some lower order term in a canonical way we can actually invert $a_{st}D_{iist}+$ lower order term. Therefore $\tilde{F}$ will be reduced to an operator of the form $Id+$ Compact, hence we can define the degree by using Leray-Schauder degree. Before carrying out this, we should first study the following linear problem:

Theorem 1.1: Let $C_1, C_{1i}, C_{2ij}, C_{3ijk} \in C^{\alpha}(\bar{\Omega})$, $a_{st} \in C^{1,\alpha}(\bar{\Omega})$, where $1 \leq i,j,k,s,t \leq n$. There exists $\beta > 0$, such that, $a_{st}(x)\xi_i\xi_j \geq \beta|\xi|^2 \quad \forall \xi \in R^n, \; x \in \bar{\Omega}$. Let

$$\begin{array}{cccc} M^N: & C_0^{4,\alpha}(\bar{\Omega}) \rightarrow & C^{\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega) \\ & w \longmapsto & (M_{(1)}^N w, M_{(2)}^N w) \end{array}$$

where

$$\left\{ \begin{array}{ll} & M_{(1)}^N w \\ = & a_{st}D_{iist}w + C_{3ijk}D_{ijk}w + C_{2ij}D_{ij}w \\ & +C_{1i}D_i w + C_1 w - N a_{st}D_{st}w \\ & M_{(2)}^N w \\ = & -a_{st}D_{sti}w\gamma_i - \mu(a_{st}D_{st}w + D_i a_{st}D_{st}u\gamma_i) \end{array} \right. \qquad (4)$$

where N is a real number, $0 \leq \mu \leq 1$.

Then there exists some constant N_0, depending only on

$\|a_{st}\|_{C^{1,\alpha}(\bar{\Omega})}, \|C_1\|_{C^\alpha(\bar{\Omega})}, \|C_{1i}\|_{C^\alpha(\bar{\Omega})}, \|C_{2ij}\|_{C^\alpha(\bar{\Omega})}, \|C_{3ijk}\|_{C^\alpha(\bar{\Omega})}$,

n, β, such that, for any $N > N_0$, M^N is an isomorphism from $C_0^{4,\alpha}(\bar{\Omega})$ onto $C^\alpha(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega)$ .Furthermore M^N depends continuously on

$$C_1, C_{1i}, C_{2ij}, C_{3ijk} \in C^\alpha(\bar{\Omega}) \text{ , } a_{st} \in C^{1,\alpha}(\bar{\Omega})$$

with respect to the corresponding topologies.

Proof: According to [1] and [2], for any $N \in R$, M^N has finite dimensional kernel and hence is Fredholm. From the stability of the Fredholm index (see [6]), we know that the Fredholm index of M^N is the same as that of the following map:

$$C_0^{4,\alpha}(\bar{\Omega}) \to C^\alpha(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

$$w \longmapsto (\Delta^2 w, (-\frac{\partial(\Delta w)}{\partial\gamma} - \Delta w)|_{\partial\Omega})$$

which is of Fredholm index zero. To prove that M^N is an isomorphism we only need to prove that M^N has a trivial kernel.

Let $w \in C_0^{4,\alpha}(\bar{\Omega})$, $M^N w = 0$. It is not difficult to see that

$$\int_\Omega (M_{(1)}^N w)(-a_{lm} D_{lm} w) \geq \beta \int_\Omega |\nabla (a_{st} D_{st} w)|^2 + N \int |a_{st} D_{st} w|^2$$
$$-C\|w\|^2_{H^2(\partial\Omega)} - C\|w\|_{H^3(\Omega)}\|w\|_{H^2(\Omega)}$$

where C depends only on $\|C_1\|_{C^0}$, $\|C_{1i}\|_{C^1}$, $\|C_{2ij}\|_{C^0}$, $\|C_{3ijk}\|_{C^0}$, $\|a_{ij}\|_{C^{1,\alpha}}$.

Therefore there exists $N_0 > 0$, such that,

$$\left\{ \begin{array}{ll} & \int_\Omega (M_{(1)}^N w)(-a_{lm} D_{lm} w) \\ \geq & \frac{\beta}{2} \int_\Omega |\nabla (a_{st} D_{st} w)|^2 + (N - N_0) \int_\Omega |a_{st} D_{st} w|^2 \end{array} \right. \qquad (5)$$

We see from the calculation that N_0 depends only on $\|a_{st}\|_{C^{1,\alpha}(\bar{\Omega})}$, $\|C_1\|_{C^\alpha(\bar{\Omega})}$, $\|C_{1i}\|_{C^\alpha(\bar{\Omega})}$, $\|C_{1ij}\|_{C^\alpha(\bar{\Omega})}$, $\|C_{1ijk}\|_{C^\alpha(\bar{\Omega})}$,^{7}

It follows from (5) that for $N > N_0, M^N$ is injective, hence an isomorphism from $C_0^{4,\alpha}(\bar{\Omega})$ onto $C^\alpha(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega)$.

Having proved Theorem 1.1 we are in a position to define an integer valued degree for $F : O \rightarrow C^{2,\alpha}(\bar{\Omega})$.

$\tilde{F}$ can be represented as $\tilde{F} = L_u(u)$:

$$L_u : C_0^{4,\alpha}(\bar{\Omega}) \rightarrow C^\alpha(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

$$L_u = (L_u^{(1)}, L_u^{(2)})$$

$$\begin{cases} L_u^{(1)}w \\ = \quad a_{st}(x,u,Du,D^2u)D_{iist}w + C_0(x,u,Du,D^2u,D^3u) \\ +C_1(x,u,Du,D^2u,D^3u)w + C_{1i}(x,u,Du,D^2u,D^3u)w_i \\ +C_{2ij}(x,u,Du,D^2u,D^3u)w_{ij} + C_{3ijk}(x,u,Du,D^2u,D^3u) \\ L_u^{(2)}w \\ = \quad -a_{st}(x,u,Du,D^2u)D_{sti}w\gamma_i + E_0(x,u,Du,D^2u) \end{cases} \tag{6}$$

where $w \in C_0^{4,\alpha}(\bar{\Omega})$.

Let

$$C_u^N : C_0^{4,\alpha}(\bar{\Omega}) \to C^\alpha(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

be defined by

$$\begin{aligned} & C_u^N(w) \\ = \quad & (\ Na_{st}(u,Du,D^2u)D_{st}w + C_0(x,u,Du,D^2u,D^3u)\ , \\ & E_0(x,u,Du,D^2u)|_{\partial\Omega}\ ) \end{aligned} \tag{7}$$

Let

$$M_u^N : C_0^{4,\alpha}(\bar{\Omega}) \to C^\alpha(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

$$M_u^N := L_u - C_u^N \tag{8}$$

It is easy to see that for any $u \in O, L_u, M_u^N$ are linear bounded operators and C_u^N is a compact operator. According to Theorem 1.1, there exists some positive number $N_0 = N_0(O,\beta)$, such that, M_u^N is an isomorphism for any $u \in O,\ N > N_0$. Furthermore L_u, M_u^N, C_u^N continuously depend on $u \in O$ with respect to its $C^{3,\alpha}$ norm.

$L_u(u) = 0$ is the same as $u + (M_u^N)^{-1}C_u^N(u) = 0$ and $u \longmapsto (M_u^N)^{-1}C_u^N(u)$ is a compact operator from O to $C_0^{4,\alpha}(\bar{\Omega})$. Therefore we can define the degree of F as the Leray Schauder degree of the map $u \longmapsto u + (M_u^N)^{-1}\, C_u^N(u)$. More precisely we have the following definition.

Definition 1.2: Let $F : O \rightarrow C^{2,\alpha}(\bar{\Omega})$ be an elliptic operator, where $O \subset C_0^{4,\alpha}(\bar{\Omega})$ is a bounded open set with $\partial O \cap F^{-1}(0) = \phi$, we define a degree of F on O at 0 by

$$\deg(F, O, 0) := \deg_{L.S.}(u \longmapsto u + (M_u^N)^{-1}C_u^N(u), O, 0)$$

where $N > N_0(O,\beta), u \longmapsto u + (M_u^N)^{-1}C_u^N(u)$ is a map from O to $C_0^{4,\alpha}(\bar{\Omega})$.

We justify the above definition by making the following remarks.

Remark 1.1: The map $u \longmapsto u + (M_u^N)^{-1} C_u^N(u)$ is of the form Id+Compact : $\mathcal{O} \to C_0^{4,\alpha}(\bar{\Omega})$.

Remark 1.2: The degree is independent of $N > N_0(\mathcal{O}, \beta)$ according to the homotopy invariance of the Leray Schauder degree.

Remark 1.3: The degree is independent of the representation we choose in (6).

Remark 1.1 and Remark 1.2 are quite clear, we will only explain Remark 1.3.

If we have another representation in (6), then we have $\widetilde{C}_u^N$ and $\widetilde{M}_u^N$ in (7) and (8). We need to prove that for N large enough

$$\left\{ \begin{array}{rl} & \deg_{L.S.}(u \longmapsto u + (M_u^N)^{-1} C_u^N(u), \mathcal{O}, 0) \\ = & \deg_{L.S.}(u \longmapsto u + (\widetilde{M}_u^N)^{-1} \widetilde{C}_u^N(u), \mathcal{O}, 0) \end{array} \right. \tag{9}$$

Consider a homotopy

$$u \longmapsto u + (tM_u^N + (1-t)\widetilde{M}_u^N)^{-1}(tC_u^N(u) + (1-t)\widetilde{C}_u^N(u))$$

where $0 \le t \le 1$.

Use Theorem 1.1, it is not difficult to check that for N

sufficiently large the above homotopy is an admisable homotopy for Leray Schauder degree . Therefore (9) follows from the homotopy invariance of the Leray Schauder degree.

The degree we have defined above has the following properties:

Proposition 1.1: Let F be an elliptic operator from O to $C^{2,\alpha}(\bar{\Omega})$, where $O \subset C_0^{4,\alpha}(\bar{\Omega})$ is a bounded open set, $\partial O \cap F^{-1}(0) = \phi$. Suppose that $\bar{U} \subset O$ and $\bar{U} \cap F^{-1}(0) = \phi$, then

$$\deg(F, O, 0) = \deg(F, O \setminus \bar{U}, 0)$$

Proposition 1.2: Let $O \subset C_0^{4,\alpha}(\bar{\Omega})$ be a bounded open set and suppose that $H \in C([0,1] \times \bar{\Omega} \times R^n \times R^{n^2})$ and $t \longmapsto H(t, \cdot)$ is continuous from $[0,1]$ to $C^{3,\alpha}(\bar{\Omega} \times R^n \times R^{n^2})$, $0 < \alpha < 1$, there exists $\beta = \beta(O) > 0$, such that, for any $u \in O, x \in \bar{\Omega}$, $t \in [0,1], \xi \in R^n$. we have

$$-H_{u_{ij}}(t, x, u(x), Du(x), D^2u(x))\xi_i\xi_j \geq \beta|\xi|^2$$

and for any $u \in \partial O, 0 \leq t \leq 1$,

$$H(t, \cdot, u, Du, D^2u) \neq 0 \in C^{2,\alpha}(\bar{\Omega})$$

Then

$$\begin{aligned}&\deg(u\longmapsto H(0,\cdot,u,Du,D^2u)\ ,\ O,\ 0)\\ =\ &\deg(u\longmapsto H(1,\cdot,u,Du,D^2u)\ ,\ O,\ 0)\end{aligned}$$

Proposition 1.1 and Proposition 1.2 follow from the definition of our degree and the corresponding properties of the Leray Schauder degree.

Proposition 1.3: Let $F : C_0^{4,\alpha}(\bar{\Omega}) \to C^{2,\alpha}(\bar{\Omega})$ be elliptic on some neighborhood of 0, satisfying $F(0) = 0, F$ is Frechèt differentiable at 0 and $F'(0)$ is invertible. Then $u = 0$ is an isolated point of $F^{-1}(0)$. Furthermore $\deg(F, O, 0)$ $=\deg\,(F'(0), O, 0)$ where O is any open neighborhood of 0 in $C_0^{4,\alpha}(\bar{\Omega})$ which does not contain any other points of $F^{-1}(0)$ and on which F is elliptic.

Proof: Proposition 1.3 is straight forward once we apply Proposition 1.2 to the following homotopy:

$$H(t,u) = \begin{cases} \dfrac{1}{t}F(tu) & 0 < t \leq 1 \\ F'(0)u & t = 0 \end{cases}$$

Proposition 1.4: Suppose that an elliptic operator F happens to be a linear operator, namely, $F : u \longmapsto -a_{ij}(x)D_{ij}u+$

$b_i(x)D_iu+c(x)u$, where a_{ij}, b_i, c are sufficiently smooth functions, there exists some $\beta > 0$, such that, for any $\xi \in R^n$, $x \in \bar{\Omega}$, $a_{ij}(x)\xi_i\xi_j \geq \beta|\xi|^2$. If furthermore 0 is not in $F^{-1}(0)$ we have

$$\deg(F, O, 0) = \sum_{\lambda_i<0}(-1)^{\beta_i}$$

where λ_i is an eigenvalue of F with algebraic multiplicity β_i for $i = 1,2,3,\ldots$, $\lambda_1 < \lambda_2 < \ldots$, O is any open set of $C_0^{4,\alpha}(\bar{\Omega})$ containing 0.

Proof: Use the notation as before only notice that the operators L_u, M_u^N, C_u^N can be chosen as linear operators independent of u:

$$M^N = (M^N_{(1)}, M^N_{(2)})$$

$$\begin{aligned} M^N_{(1)}u &= \Delta(a_{ij}(x)D_{ij}u) - Na_{ij}D_{ij}u + C_{3ijk}(x)D_{ijk}u \\ &\quad + C_{2ij}(x)D_{ij}u + C_{1i}(x)D_iu + C_1(x)u \\ M^N_{(2)}u &= -a_{ij}(x)D_{ijk}u\gamma_k \end{aligned}$$

$$C^N = \tilde{F} - M^N$$

where $\tilde{F} = S \circ F$ and S is the operator we defined before.

According to our definition,

$$\deg(F, O, 0) = \deg_{L.S.}(\mathrm{Id} + (M^N)^{-1}C^N, O, 0)$$

where O is any open neighborhood of $0 \in C_0^{4,\alpha}(\bar{\Omega})$.

Let

$$\begin{array}{rccc} H: & C_0^{4,\alpha}(\bar{\Omega}) & \to & C^{2,\alpha}(\bar{\Omega}) \\ & u & \longmapsto & -a_{ij}(\cdot)D_{ij}u \end{array}$$

$$G := SH : C_0^{4,\alpha}(\bar{\Omega}) \to C^{\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

Then

$$\mathrm{Id} + (M^N)^{-1}C^N = (M^N)^{-1}GH^{-1}F$$

$$\begin{aligned} & \deg_{L.S.}(\mathrm{Id} + (M^N)^{-1}C^N, O, 0) \\ = \; & \deg_{L.S.}((M^N)^{-1}G, O, 0)\deg_{L.S.}(H^{-1}F, O, 0) \end{aligned}$$

It is well known that

$$\deg\,(H^{-1}F, O, 0) = \sum_{\lambda_i<0}(-1)^{\beta_i}$$

To conclude the proof we only need to prove that for large

N,

$\deg_{L.S.}((M^N)^{-1}G, O, 0) = 1.$

Notice that

$$G : u \longmapsto (\Delta(a_{ij}(x)D_{ij}u), \frac{\partial}{\partial\gamma}(-a_{ij}(x)D_{ij}u)|_{\partial\Omega} + (-a_{ij}(x)D_{ij}u)|_{\partial\Omega})$$

Define the following homotopy:

$$\begin{aligned} M_t^N u = \quad & (\Delta(a_{ij}(x)D_{ij}u) - N a_{ij}D_{ij}u + tC_{3ijk}(x)D_{ijk}u \\ & +tC_{2ij}(x)D_{ij}u + tC_{1i}(x)D_i u + tC_1(x)u, \\ & -a_{ij}(x)D_{ijk}u\gamma_k + \\ & (1-t)(-a_{ij}(x)D_{ij}u|_{\partial\Omega} - D_k a_{ij}(x)D_{ij}u\gamma_k) \end{aligned}$$

where $0 \leq t \leq 1$.

Applying Theorem 1.1 we know that for N sufficiently large,

$$M_t^N : C_0^{4,\alpha}(\bar{\Omega}) \to C^{\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega) \text{ are isomorphisms,}$$

where $0 \leq t \leq 1$. It is also clear that

$$M_t^N - G : C_0^{4,\alpha}(\bar{\Omega}) \to C^{\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

is compact, hence $(M_t^N)^{-1}G$ is of the form Id+compact. According to the homotopy invariance of the Leray Schauder degree we have, for N large enough that

$$\deg_{L.S.}((M_1^N)^{-1}G, O, 0) = \deg_{L.S.}((M_0^N)^{-1}G, O, 0)$$

Clearly $M_1^N = M^N$, therefore we only need to prove that

$$\deg_{L.S.}((M_0^N)^{-1}G, O, 0) = 1 \tag{10}$$

Since $(M_0^{tN})^{-1}G,\ 0 \leq t \leq 1$ is an admissable homotopy and $M_0^0 = G$, use the homotopy invariance of the Leray Schauder degree we have (10).

2 Degree of second order fully nonlinear elliptic operator on compact Riemanian manifold

We have defined a degree for second order fully nonlinear elliptic operator with Dirichlet boundary data in section 1. In this section we define such a degree for second order fully nonlinear elliptic operator on compact Riemanian manifold.

Let (M, g) be a n-dimensional compact Riemanian manifold and suppose $f^* \in C^{3,\alpha}(M \times R \times R^n \times R^{n^2}), 0 < \alpha < 1$. Let F^* be an operator from $C^{4,\alpha}(M)$ to $C^{2,\alpha}(M)$ defined by:

$$F^* : u \longmapsto f^*(\cdot, u, \nabla u, \nabla^2 u)$$

where $\nabla u, \nabla^2 u$ denote the covariant derivatives of u (See [3]).

Definition 2.1: Let $O^* \subset C^{4,\alpha}(M)$ be a bounded open set, F^* is said to be elliptic on O^* if there exists $\beta^* = \beta^*(O^*) > 0$, such that, for any $u \in O^*, x \in M, \xi \in R^n$

$$-\frac{\partial f^*}{\partial \nabla_{ij} u}(x, u, \nabla u, \nabla^2 u)\xi_i \xi_j \geq \beta^* |\xi|^2$$

Let $O^* \subset C^{4,\alpha}(M)$ be a bounded open set with $\partial O^* \cap F^{*-1}(0) = \phi$ and suppose that F^* is elliptic on O^*. We proceed as in section 1 to define an integer valued degree for F^* on O^* at 0.

Consider

$$S^* \ : \ C^{2,\alpha}(M) \rightarrow C^{\alpha}(M)$$
$$u \longmapsto -\Delta u + u$$

It is well known that S^* is an isomorphism.

Let $\tilde{F}^*$ be the composite map of S^* and F^*, namely,

$$\tilde{F}^* = S^* \circ F^* : C^{4,\alpha}(M) \rightarrow C^{\alpha}(M)$$

Since S^* is an isomorphism, $F^* = 0$ is the same as $\tilde{F}^* = 0$. As in section 1 we define the degree of F^* by defining a degree for $\tilde{F}^*$.

Clearly $\widetilde{F}^*$ can be expressed as the following:

$$\widetilde{F}^*(u) = a^*_{st}(x, u, \nabla u, \nabla^2 u)\nabla_{iist}u + C^*_0(x, u, \nabla u, \nabla^2 u, \nabla^3 u) \tag{11}$$

Where

$$a^*_{st}(x, u, \nabla u, \nabla^2 u) = -\frac{\partial f^*}{\partial \nabla_{st} u}(x, u, \nabla u, \nabla^2 u)$$

As in section 1 we need to study the following linear problem first :

Theorem 2.1: Let $a^*_{st} \in C^{1,\alpha}(M)$, where $1 \le s, t \le n$. There exists $\beta^* > 0$, such that, $a^*_{st}(x)\xi_i\xi_j \ge \beta^*|\xi|^2 \quad \forall \xi \in R^n,\ x \in M$. Let

$$M^{*N}: \ C^{4,\alpha}(M) \rightarrow \ C^{\alpha}(M)$$

where

$$M^{*N} w = a^*_{st}\nabla_{iist} w - N \triangle w + N w \tag{12}$$

where N is some real number.

Then there exists some constant N^*_0, depending only on $||a^*_{st}||_{C^{1,\alpha}}$, n,β^*. such that, for any $N > N^*_0, M^{*N}$ is an isomorphism from $C^{4,\alpha}(M)$ onto $C^{\alpha}(M)$. Furthermore M^{*N}

depends continuously on $a_{st} \in C^{1,\alpha}(M)$ with respect to the corresponding topologies.

Proof: According to [1] and [2], for any $N \in R$, M^{*N} has finite dimensional kernel and hence is Fredholm. From the stability of the Fredholm index (see [6]), we know that the Fredholm index of M^{*N} is the same as that of the following map:

$$C^{4,\alpha}(M) \to C^{\alpha}(M)$$

$$w \longmapsto \Delta^2 w - \Delta w + w$$

which is of Fredholm index zero. To prove that M^{*N} is an isomorphism we only need to prove that M^{*N} has a trivial kernel.

Let $w \in C^{4,\alpha}(M), M^{*N}w = 0$. It is not difficult to see that

$$\begin{aligned} &\int_M (M^{*N} w)(- \Delta w + w) \\ \geq\ &\int_M a^*_{st} \nabla_s(\Delta w) \nabla_t(\Delta w) + N \int_M |\Delta w|^2 + \\ &2N \int_M |\nabla w|^2 + N \int_M w^2 - C \|w\|_{H^3(M)} \|w\|_{H^2(M)} \end{aligned}$$

where C depends only on $\|a^*_{ij}\|_{C^{1,\alpha}}$.

Therefore there exists $N_0^* > 0$, such that,

$$\left\{\begin{array}{ll} & \int_M (M^{*N} w)(-\triangle w + w) \\ \geq & \frac{\beta^*}{2}\int_M |\nabla(\triangle w)|^2 + (N - N_0^*)\int_M |\triangle w|^2 \\ & +(N - N_0^*)\int_M |\nabla w|^2 + (N - N_0^*)\int_M w^2 \end{array}\right. \tag{13}$$

We see from the calculation that N_0^* depends only on $\|a_{st}^*\|_{C^{1,\alpha}}, n, \beta^*$ It follows from (13) that for $N > N_0^*, M^{*N}$ is injective, hence an isomorphism from $C^{4,\alpha}(M)$ onto $C^\alpha(M)$.

Now we are ready to define an integer valued degree for $F^* : O^* \to C^{2,\alpha}(M)$.

$\widetilde{F}^*$ can be represented as $\widetilde{F}^* = L_u^*(u)$ where

$$L_u^* : C^{4,\alpha}(M) \to C^\alpha(M)$$

is defined as:

$$L_u^* w = a_{st}^*(x, u, \nabla u, \nabla^2 u) D_{iist} w + C_0^*(x, u, \nabla u, \nabla^2 u, \nabla^3 u) \tag{14}$$

where $w \in C^{4,\alpha}(M)$.

Let

$$C_u^{*N} : C^{4,\alpha}(M) \to C^\alpha(M)$$

be defined by

$$C^{*N}_u(w) = N \triangle w - Nw + C_0^*(x, u, \nabla u, \nabla^2 u, \nabla^3 u) \tag{15}$$

Let

$$M^{*N}_u : C^{4,\alpha}(M) \to C^{\alpha}(M)$$

$$M^N_u := L_u - C^{*N}_u \tag{16}$$

It is easy to see that for any $u \in O^*, L^*_u, M^{*N}_u$ are linear bounded operators and C^{*N}_u is a compact operator. According to Theorem 2.1, there exists some positive number $N_0^* = N_0^*(O^*, \beta^*)$, such that, M^{*N}_u is an isomorphism for any $u \in O^*$, $N > N_0^*$. Futhermore $L^*_u, M^{*N}_u, C^{*N}_u$ continuously depend on $u \in O^*$ with respect to its $C^{3,\alpha}$ topology.

Definition 2.2: Let $F^* : O^* \to C^{2,\alpha}(M)$ be an elliptic operator, where $O^* \subset C^{4,\alpha}(M)$ is a bounded open set with $\partial O^* \cap F^{*-1}(0) = \phi$, we define the degree of F^* on O^* at 0 by

$$\deg(F^*, O^*, 0) := \deg_{L.S.}(u \longmapsto u + (M^{*N}_u)^{-1} C^{*N}_u(u), O, 0)$$

where $N > N_0^*(O, \beta^*), u \longmapsto u + (M^{*N}_u)^{-1} C^{*N}_u(u)$ is a map from O^* to $C^{4,\alpha}(M)$.

We justify the above definition by making the following remarks.

Remark 2.1: The map $u \longmapsto u + (M^{*N}_u)^{-1} C^{*N}_u(u)$ is of the form Id+Compact : $O^* \to C^{4,\alpha}(M)$.

Remark 2.2: The degree is independent of $N > N_0^*(O, \beta^*)$ according to the homotopy invariance of the Leray Schauder degree.

Remark 2.1 and Remark 2.2 are valid due to the similar reasons as in section 1.

The degree we have defined above has the following properties:

Proposition 2.1: Let F^* be an elliptic operator from O^* to $C^{2,\alpha}$, where $O^* \subset C^{4,\alpha}(M)$ be a bounded open set, $\partial O^* \cap F^{*-1}(0) = \phi$. Suppose that $\bar{U}^* \subset O^*$ and $\bar{U}^* \cap F^{*-1}(0) = \phi$, then

$$\deg(F^*, O^*, 0) = \deg(F^*, O^* \setminus \bar{U}^*, 0)$$

Proposition 2.2: Let $O^* \subset C^{4,\alpha}(M)$ be a bounded open set and suppose that $H^* \in C([0,1] \times M \times R^n \times R^{n^2})$

and $t \longmapsto H^*(t,\cdot)$ is continuous from $[0,1]$ to $C^{3,\alpha}(M \times R^n \times R^{n^2})$, $0 < \alpha < 1$, there exists $\beta^* = \beta^*(O^*) > 0$, such that, for any $u \in O^*, x \in M, t \in [0,1], \xi \in R^n$. we have

$$-\frac{\partial H^*}{\partial \nabla_{ij} u}(t, x, u(x), \nabla u(x), \nabla^2 u(x))\xi_i\xi_j \geq \beta^* |\xi|^2$$

and for any $u \in \partial O^*, 0 \leq t \leq 1$,

$$H^*(t, \cdot, u, \nabla u, \nabla^2 u) \neq 0 \in C^{2,\alpha}(M)$$

Then

$$\begin{aligned} & \deg(u \longmapsto H^*(0, \cdot, u, \nabla u, \nabla^2 u)\ ,\ O,\ 0) \\ = \quad & \deg(u \longmapsto H^*(1, \cdot, u, \nabla u, \nabla^2 u)\ ,\ O,\ 0) \end{aligned}$$

Proposition 2.3: Let $F^* : C^{4,\alpha}(M) \to C^{2,\alpha}(M)$ be elliptic on some neighborhood of 0, satisfying $F^*(0) = 0, F^*$ is Frechèt differentiable at 0 and $F^{*'}(0)$ is invertible. Then $u = 0$ is an isolated point of $F^{*-1}(0)$. Furthermore $\deg(F^*, O, 0) = \deg\ (F^{*'}(0), O, 0)$ where O^* is any open neighborhood of 0 in $C^{4,\alpha}(M)$ which does not contain any other points of $F^{*-1}(0)$ and on which F^* is elliptic.

Proposition 2.4: Suppose that an elliptic operator F^* happens to be a linear operator, namely, $F^* : u \longmapsto -a^*_{ij}(x)\nabla_{ij} u +$

$b_i^*(x)\nabla_i u + c^*(x)u$, where a_{ij}^*, b_i^*, c^* are sufficiently smooth functions, there exists some $\beta^* > 0$, such that, for any $\xi \in R^n$, $x \in M$, $a_{ij}^*(x)\xi_i\xi_j \geq \beta^*|\xi|^2$. If furthermore 0 is not in $F^{*-1}(0)$ we have

$$\deg(F^*, O^*, 0) = \sum_{\lambda_i<0} (-1)^{\beta_i}$$

where λ_i is an eigenvalue of F^* with algebraic multiplicity β_i for $i = 1, 2, 3, \ldots$, $\lambda_1 < \lambda_2 < \ldots$, O^* is any open set of $C^{4,\alpha}(M)$ containing 0.

The proof of Proposition 2.1-2.4 is similar to that of Proposition 1.1-1.4.

3 Global bifurcation for second order fully nonlinear elliptic equations

Rabinowitz has proved in [9] a beautiful global bifurcation result for compact operators, which applies to quasilinear elliptic equations. Using the degree we have introduced for fully nonlinear second order nonlinear elliptic operators, we prove a similar result for nonlinear elliptic equations.

Suppose that

$$g \in C(R \times \bar{\Omega} \times R^n \times R^{n^n})$$

and

$$\lambda \longmapsto g(\lambda, \cdot)$$

be continuous from R to $C^{3,\alpha}(\bar{\Omega}, R \times R^n \times R^{n^2})$. There exists $\beta \in C(R \times \bar{\Omega} \times R \times R^n \times R^{n^2}, (0, +\infty))$, such that, $-g_{u_{ij}}(\lambda, x, u, Du, D^2u)\xi_i\xi_j \geq \beta(\lambda, x, u, Du, D^2u)|\xi|^2$ for all $\xi \in R^n$, where $\lambda \in R, u \in C_0^{4,\alpha}(\bar{\Omega})$, $x \in \bar{\Omega}$.

Let

$$\begin{aligned} G: \quad C_0^{4,\alpha}(\bar{\Omega}) \times R \quad &\rightarrow C^{2,\alpha}(\bar{\Omega}) \\ (u, \lambda) \quad &\longmapsto g(\lambda, \cdot, u, Du, D^2u). \end{aligned}$$

We also suppose that

$$G(0, \lambda) = 0 \quad \forall \ \lambda \in R \tag{17}$$

Definition 3.1: $\lambda_0 \in R$ is called a nonbifurcation point of G if there exists an open neighborhood of λ_0 in $C_0^{4,\alpha}(\bar{\Omega}) \times R$, say U, such that,

$$U \cap \{(u,\lambda) \in C_0^{4,\alpha}(\bar{\Omega}) \times R : G(u,\lambda) = 0\}$$

$$= U \cap \{(0,\lambda) \in C_0^{4,\alpha}(\bar{\Omega}) \times R\}$$

Otherwise λ_0 is called a bifurcation point.

Theorem 3.1:(Local bifurcation)

Let G be as above, α, β are not bifurcation points, $\alpha < \beta$. If

$$\deg(F(\cdot,\alpha), O, 0) \neq \deg\,(F(\cdot,\beta), O, 0)$$

where O is a small neighborhood of 0 in $C_0^{4,\alpha}(\bar{\Omega})$, such that,

$$O \cap G(\cdot,\alpha)^{-1}(0) = \{0\}, O \cap G(\cdot,\beta)^{-1}(0) = \{0\}$$

Then there exists at least one bifurcation point in (α,β).

Theorem3.2: (Global bifurcation)

Let G be as above, S be the closure of $\{(u,\lambda\} \in C_0^{4,\alpha}(\bar{\Omega}) \times R : G(u,\lambda) = 0\,,\ u \neq 0\}$ in $C_0^{4,\alpha}(\bar{\Omega})$. Suppose that α, β are not bifurcation points of $G, \alpha < \beta$. Let $\tilde{S} = S \cup \{0\} \times [\alpha,\beta], \tilde{C}$ be the connected component of $\tilde{S}$ to which $\{0\} \times [\alpha,\beta]$ belongs to. If deg $(G(\cdot,\alpha), O, 0) \neq$ deg $(G(\cdot,\beta), O, 0)$, where O is a small neighborhood of $C_0^{4,\alpha}(\bar{\Omega})$, such that,

$$O \cap G(\cdot, \alpha)^{-1}(0) = \{0\}$$

$$O \cap G(\cdot, \beta)^{-1}(0) = \{0\}$$

Then $\widetilde{C}$ is either unbounded or containing a point $(0, \lambda^*)$ with $\lambda^* \in R \setminus [\alpha, \beta]$.

Proof of Theorem 3.1:

We prove it by contradiction argument. If not, then there exists $\epsilon > 0$, such that, $G(u, \lambda) \neq 0$ for $\alpha \leq \lambda \leq \beta$ and $0 < \|u\| \leq \epsilon$.

Let

$$B_\epsilon(0) = \{u \in C_0^{4,\alpha}(\bar{\Omega}) \ : \|u\| < \epsilon\}$$

we have

$$\deg\ (G(\cdot, \alpha), B_\epsilon(0), 0) = \deg\ (G(\cdot, \beta), B_\epsilon(0), 0)$$

Since $G(\cdot, \lambda)$ is an admissable homotopy for $\alpha \leq \lambda \leq \beta$. This is a contradiction to our assumption.

The proof of Theorem 3.2 is similar to that of Theorem 1.4 in [9], we only need to apply our degree instead of Leray Schauder degree.

In the following we state a result more general than Theorem 3.2.

Suppose that G^* satisfies all the hypotheses of G except to replace (17) by the following:

There exists some constants $\alpha_1^* < \alpha^* < \beta^* < \beta_1^*$, such that

$$G^*(0, \lambda) = 0 \qquad \forall\, \alpha_1^* \leq \lambda \leq \beta_1^*$$

Then we have the following result.

Theorem 3.3:

Let G^* be as above and $S^* = \{(u, \lambda) \in C_0^{4,\alpha}(\bar{\Omega}) \times R : G^*(u, \lambda) = 0\}$. Suppose that α^*, β^* are not bifurcation points of G^*. Let $\tilde{S}^* = S^* \setminus \{(0, \alpha^*), (0, \beta^*)\}$ and $\tilde{C}^*$ be the connected component of $\tilde{S}^*$ to which $\{0\} \times (\alpha^*, \beta^*)$ belongs to. If $\deg(G^*(\cdot, \alpha^*), O^*, 0) \neq \deg(G^*(\cdot, \beta^*), O^*, 0)$, where O^* is a small neighborhood of $C_0^{4,\alpha}(\bar{\Omega})$, such that,

$$O^* \cap G^*(\cdot, \alpha^*)^{-1}(0) = \{0\}$$

$$O^* \cap G^*(\cdot, \beta^*)^{-1}(0) = \{0\}$$

Then $\tilde{C}^*$ is unbounded.

The proof of Theorem 3.3 is similar to that of Theorem 3.2.

4 Some more applications

Consider the following nonlinear eigenvalue problem:

$$\begin{cases} f(x,u,Du,D^2u) = \lambda(a(x)u + h(x,u,Du,\lambda)) \\ u|_{\partial\Omega} = 0 \end{cases} \tag{18}$$

where $\Omega \subset R^n$ is a smooth bounded domain, $u \in C_0^{4,\alpha}(\bar\Omega)$, $0 < \alpha < 1$, . f, h, a are sufficiently smooth functions.

Let

$$G : C_0^{4,\alpha}(\bar\Omega) \times R \rightarrow C^{2,\alpha}(\bar\Omega)$$

be the following map:

$$G : (u,\lambda) \longmapsto f(\cdot,u,Du,D^2u) - \lambda(a(\cdot)u + h(\cdot,u,Du,\lambda))$$

Suppose that for any $u \in C_0^{4,\alpha}(\bar\Omega)$, $x \in \bar\Omega$, $\xi \in R^n$,

$$-f_{u_{ij}}(x,u,Du,D^2u)\xi_i\xi_j \geq \beta(x,u,Du,D^2u)|\xi|^2 \tag{19}$$

where $\beta \in C(R \times \bar\Omega \times R \times R^n \times R^{n^2}, (0,+\infty))$.

Suppose also that

$$h(x,u,p,\lambda) = o(\sqrt{|u|^2+|p|^2}) \quad \text{as } (u,p)\to 0 \text{ in } R\times R^n \quad (20)$$

uniformly for λ in any finite interval.

$$f(x,0,0,0) = 0 \quad \forall x \in \bar{\Omega} \qquad (21)$$

$$a(x) \geq a_0 > 0 \quad \forall x \in \bar{\Omega} \qquad (22)$$

where a_0 is some positive constant.

Look at the linearized equation of (26) at $u = 0$:

$$\begin{cases} Lu = \lambda a(x)u \\ u|_{\partial\Omega} = 0 \end{cases} \qquad (23)$$

where

$$Lu := f_{u_{ij}}(x,0,0,0)D_{ij}u + f_{u_i}(x,0,0,0)D_iu + f_u(x,0,0,0)u$$

Let λ_1 be the first eigenvalue of (23). It is well known that λ_1 is a simple eigenvalue.

According to the classical theory of linear elliptic equations, there exists $\delta > 0$, such that,

$$G(\cdot,\lambda)'(0) \text{ is invertible for } \lambda \in (\lambda_1-\delta,\lambda_1+\delta)\setminus\{\lambda_1\} \quad (24)$$

where $G(\cdot,\lambda)'(0)$ denotes the Frenchè derivative of $G(\cdot,\lambda)$: $C_0^{4,\alpha}(\bar{\Omega}) \to C^{2,\alpha}(\bar{\Omega})$ at 0, namely,

$$G(\cdot,\lambda)'(0) = L - \lambda a(x)$$

We deduce from (24) the existence of a positive continuous function $\rho(\lambda), \lambda \in (\lambda_1 - \delta, \lambda_1 + \delta) \setminus \{\lambda_1\}$, such that,

$$F(\cdot,\lambda)^{-1}(0) \cap B_{\rho(\lambda)} = 0 \qquad (25)$$

where

$$\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta) \setminus \{\lambda_1\}$$

$$B_{\rho(\lambda)} = \{u \in C_0^{4,\alpha}(\bar{\Omega}) \ : \|u\|_{C_0^{4,\alpha}(\bar{\Omega})} \leq \rho(\lambda)\}$$

Apply Proposition 1.3 and Proposition 1.4 and the fact that λ_1 is a simple eigenvalue of (23) we have:

$$\deg\, (G(\cdot,\alpha), B_{\rho(\alpha)} \cap B_{\rho(\beta)}, 0) = 1$$

$$\deg\, (G(\cdot,\beta), B_{\rho(\alpha)} \cap B_{\rho(\beta)}, 0) = -1$$

where $\alpha = \lambda_1 - \dfrac{\delta}{2}$, $\beta = \lambda_1 + \dfrac{\delta}{2}$.

Let S be the closure of $\{(u,\lambda) \in C_0^{4,\alpha}(\bar{\Omega}) \times R : u \neq 0, G(u,\lambda) = 0\}$ in $C_0^{4,\alpha}(\bar{\Omega})$, $\tilde{S} = S \cup \{0\} \times [\alpha,\beta]$, $\tilde{C}$ be the connected component of $\tilde{S}$ to which $\{0\} \times [\alpha,\beta]$ belongs to. Apply Theorem 3.2, $\tilde{C}$ is either unbounded or containing a point $(0,\lambda^*)$ with $\lambda^* \in R \setminus [\alpha,\beta]$.

Theorem 4.1: Under the hypotheses (18) through (22), $\tilde{C}$ is actually unbounded.

To prove Theorem 4.1 we only need to rule out the possibility that $\tilde{C}$ contains a point $(0,\lambda^*)$ with $\lambda^* \in R \setminus [\alpha,\beta]$.

Let

$$P^+ = \{u \in C_0^{4,\alpha}(\bar{\Omega}) \ : u(x) > 0, \forall x \in \Omega, \frac{\partial u}{\partial \gamma}|_{\partial\Omega} < 0\}$$

$$P^- = \{u \in C_0^{4,\alpha}(\bar{\Omega}) \ : -u \in P^+\}$$

$$P = P^+ \cup P^-$$

where $\frac{\partial u}{\partial \gamma}$ denotes the outer normal derivative of u on $\partial\Omega$.

Lemma 4.1: There exists an open neighborhood of $(0,\lambda_1) \in C_0^{4,\alpha}(\bar{\Omega}) \times R$, say U, such that,

$$u \in P \qquad \forall (u,\lambda) \in U \cap \tilde{C}$$

Proof: We prove it by contradiction argument. If not, then there exists a sequence $\{(u^k,\lambda^k)\} \subset \tilde{C}$, such that,

$$u^k \text{ does not belong to } P, \qquad k = 1,2,3,\ldots$$

$$\lim_{k \to +\infty} (u^k,\lambda^k) = (0,\lambda_1) \quad \text{in } C_0^{4,\alpha}(\bar{\Omega}) \times R$$

$$G(u^k,\lambda^k) = 0 \quad k = 1,2,3,\ldots$$

Divide the equation by $\|u^k\|_{C_0^{4,\alpha}(\bar\Omega)}$ and let k go to $+\infty$ we obtain

$$L\frac{u^k}{\|u^k\|_{C_0^{4,\alpha}(\bar\Omega)}} - \lambda_1 a(x)\frac{u^k}{\|u^k\|_{C_0^{4,\alpha}(\bar\Omega)}} \longrightarrow 0 \text{ in } C^{2,\alpha}(\bar\Omega) \quad (26)$$

Due to the fact that the embedding from $C_0^{4,\alpha}(\bar\Omega)$ to $C^{3,\alpha}(\bar\Omega)$ is compact, there exists $v \in C_0^{3,\alpha}(\bar\Omega)$, such that,

$$\frac{u^k}{\|u^k\|_{C_0^{4,\alpha}(\bar\Omega)}} \longrightarrow v \quad \text{strongly in } C_0^{3,\alpha}(\bar\Omega) \text{ along a subsequence}$$

Pass to the limit in (26) we have

$$Lv = \lambda_1 a(x) v, \quad v \in C_0^{3,\alpha}(\bar\Omega) \setminus \{0\}$$

It is well known that v belongs to P. Therefore

$$\frac{u^k}{\|u^k\|_{C_0^{4,\alpha}(\bar\Omega)}} \in P \quad \text{for } k \text{ large enough}$$

This is a contradiction.

Lemma 4.2: Suppose that u is not identically zero and there exists λ, such that $(u,\lambda) \in \tilde{C}$, then $u \in P$.

Proof: We prove it by contradiction argument. If not, according to Lemma 4.1, there exists $(u,\lambda^*) \in \tilde{C}, u \in \partial P$. By the strong maximum principle and Hopf lemma (See [GT]), $u = 0 \in C_0^{4,\alpha}(\bar\Omega)$. This is a contradiction.

If $\tilde{C}$ is not unbounded, then according to Theorem 3.2, there exists $\lambda^* \in R \setminus [\alpha, \beta]$, $(0, \lambda^*) \in \tilde{C}$. Therefore, there exists a sequence $\{(u_n, \lambda_n)\} \in \tilde{C}$, u_n is not identically zero, $u_n \in P$, $(u_n, \lambda_n) \rightarrow (0, \lambda^*)$. Use a similar argument as that in the proof of Lemma 4.1, we can prove that λ^* is an eigenvalue of of (23) with an eigenfunction in $\bar{P}$, which is clearly a contradiction.

Theorem 4.1 follows from Lemma 4.1 and Lemma 4.2.

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