Math 311 Workshop

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Recall: The (axiomatic)definition of real numbers $\mathbb{R}$:

Let $\mathbb{R}$ be a field extension of $\mathbb{Q}$, such that:

(1) The set $\mathbb{R}$ is a field: (*roughly* meaning:) For any element r in $\mathbb{R}$, additive inverse exists($-r$), multiplicative inverse exists if it is nonzero!($1/r, r \neq 0$) addition and multiplication of real numebrs are commutative($x + y = y + x, xy = yx$), associative($x + (y + z) = (x + y) + z, (xy)z = x(yz)$), and the distibutive property($x(y + z) = xy + xz$) holds.

(2) The field $\mathbb{R}$ is ordered, meaning there exists a total order $\geq$ such that for all real numbers x, y and z:

  (a) if $x \geq y$, then $x + z \geq y + z$;

  (b) if $x \geq 0$, and $y \geq 0$, then $xy \geq 0$.

(3) **Axiom of completeness. Every nonempty set of real numbers that is bounded above has a least upper bound.**

Remark: A **partial order** on a set S is a subset R of the set $S \times S$, such that:

1. For any element $x \in S, (x, x) \in R$ (**Reflexive**, in other words $x \geq x$);

2. If for $x, y \in S$, $(x, y) \in R$ and $(y, x) \in R$, then we have $x = y$ (**Antisymmetric**, if $x \geq y$ and $y \geq x$, then $x = y$);

3. If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$ (**Transitive**, if $x \geq y$ and $y \geq z$, then $x \geq z.$).

Well, a **total order** is a partial order where every two elements in the set S are comparable, namely, it satisfies the additional condition:

4*. Either $(x, y) \in R$ or $(y, x) \in R$ (**Strongly connected**, either $x \geq y$ or $y \geq x$).

Example 1.3.7. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}.$$

Then $\sup(c + A) = c + \sup A$.

Solution: To show that $\sup(c+A) = c+\sup(A)$, we need to show that (1) $(c+\sup A)$ is an upper bound; and (2) for any upper bound u of $c + A$, we have $u \geq c + \sup A$.

(1): For any element $c + a \in c + A$, by the definition of $\sup A$, we have $\sup A \geq a$. By axiom (2)(a), for any $a \in A$, we have $c + \sup A \geq c + a$(add c on both sides of the inequality $\sup A \geq a$).

(2): Since u is an upper bound of $c + A$, then for any $a \in A$, $u \geq c + a$, which implies $(u + (-c) =) = u - c \geq a(= c + a + (-c))$. So, $u - c$ is an upper bound of A. But by the definition of $\sup A$, $u - c \geq \sup A$. Hence $u \geq c + \sup A$. Because for any $a \in A$ we have $\sup A \geq a$ which implies $u \geq c + \sup A \geq c + a$. We are done.

□

Remark: This is vacuously true if $A = \varnothing$, since the infimum of the empty does not exist.

Workshop 1

Problems 1 and 2 are due tomorrow(Tuesday 9/14) at 10 pm. You must submit your work through Canvas.

Problem 1. Let A be a bounded set of reals and z a real number such that the following holds:

(†) **For every** $\varepsilon > 0$**, there is an** $a \in A$ **such that** $a < z + \varepsilon$.

<u>Hint</u>: (f) and (†) are actually equivalent! But if you want to use this, you must prove it first! Otherwise, you have to give case-by-case reasons.