Notes on the Revised Simplex Method

Motivation: Why do we want to use the Revised Simplex Method?

- Because when we are trying to find the optimal (basic) solution, it suffices to find out the values of the basis variables, since non-basis variables are evaluated to be zeros.

  However, if we work with the entire tableau using the simplex method, especially when there are ~~way more~~ too many variables, ~~compared~~ we end up spending most of our time computing coefficients of non-basis variables which are not necessary.

  So the idea is that we only maintain a mini-tableau, while using the more convenient matrices to keep track of other necessary numbers for the simplex method.

<u>Example</u> (same as the lecture notes)

$$\text{Max: } Z = x_1 + x_2 + x_3 + 2x_4$$

$$\begin{cases} 3x_1 - x_2 + 4x_3 - x_4 \le 4 \\ 2x_2 + x_4 \le 5 \end{cases}$$

$$\vec{x} \ge 0$$

The whole tableau is:

	x_1	x_2	x_3	x_4	x_5	x_6	
z	-1	-1	-1	-2	0	0	0
x_5	3	-1	4	-1	1	0	4
x_6	0	2	0	1	0	1	5

Choose x_5, x_6 as basis variables, and we <u>only</u> keep track of the basis variables and their coefficients as well as the entering variable in the mini-tableau:

B	A_B^{-1}		x_1	$\vec{p}$
x_5	1	0	(3)	4
x_6	0	1	0	5

Notes: B: basis; A_B^{-1}: the coefficient matrix of the <u>new</u> (basis & non-basis) variables at each step, or simply take it to be what we have in the following mini-tableaux after normalizing; $\vec{p}$: constants; x_1: entering, since $-1 < 0$ and we are doing in Bland's fashion.

	B	A_B^{-1}		x_1	$\vec{p}$
☆	x_1	$\frac{1}{3}$	0	(1)	$\frac{4}{3}$
	x_6	0	1	0	5

Note: normalizing ↙ to be 1; x_5 is replaced with x_1

Now we need to decide the next entering variable.

Compute the vector: $\vec{r} = \vec{c}_B^{\,T} A_B^{-1} A_N - \vec{c}_N^{\,T}$, where $\vec{c}_B^{\,T}$ denotes the coefficient vector of the basis variables at this step;
$\vec{c}_N$ denotes the coefficient vector of the non-basis variables at this step; ~~in our case, it is~~

A_B^{-1} as usual, the coefficient matrix of our mini-tableau at this step;

A_N denotes the coefficient matrix of the non-basis variables in the ~~of~~ <u>initial</u> tableau;

~~$\vec{c}_B$ is the coefficient vector of~~ .

In our case, the initial tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	
z	-1	-1	-1	-2	0	0	0
x_5	3	-1	4	-1	1	0	4
x_6	0	2	0	1	0	1	5

$-1<0$, so let x_1 enter

basis variables

The mini-tableau is:

B	A_B^{-1}		x_1 (entering)	$\vec{P}$
x_1	$\frac{1}{3}$	0	1	$\frac{4}{3}$
x_6	0	1	0	5

$\vec{c}_N^{\,T} = (1\ 1\ 2\ 0)$

So $\vec{c}_B^{\,T} = (1, 0)$, $A_B^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$, $A_N = \begin{pmatrix} -1 & 4 & -1 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix}$, and we have

$$\vec{r} = (1, 0)\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 4 & -1 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix} - (1\ 1\ 2\ 0) = \left(-\frac{4}{3}\ \ \frac{1}{3}\ \ -\frac{7}{3}\ \ \frac{1}{3}\right)$$

Since the first entry of $\vec{r}$ corresponds to x_2 and is non-positive, we can choose x_2 as the next entering variable.

And the column for x_2 is given by

$$A_B^{-1} A_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ 2 \end{pmatrix}$$

where A_2 is the column in A_N that corresponds to x_2, so it is the first column.

The mini-tableau becomes:

	B	A_B^{-1}		x_2 (entering ↓)	$\vec{P}$
	x_1	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{4}{3}$
leaving ←	x_6	0	1	2	5

Now we determine the leaving variable. Since in x_2 column above, 2 is the only one that > 0, so only x_6 can leave.

We have:

B	A_B^{-1}		x_2	$\vec{P}$
x_1	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{13}{5}$
x_2	0	$\frac{1}{2}$	1	$\frac{5}{2}$

Draft:

$0 + \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$\frac{4}{3} + \frac{1}{3} \times \frac{5}{2} = \frac{13}{6}$

Iterate the above process:

$$\vec{r} = \vec{c}_B^{\,T} A_B^{-1} A_N - \vec{c}_N^{\,T} = (1 \ \ 1)\begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} - (1 \ \ 2 \ \ 0 \ \ 0)$$

$$= \left(\tfrac{1}{3} \ \ -\tfrac{5}{3} \ \ \tfrac{1}{3} \ \ \tfrac{2}{3}\right)$$

Only $-\frac{5}{3} < 0$ and it corresponds to x_4. So x_4 will be the new entering variable. Compute the column of x_4: $A_B^{-1} A_4 = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{2} \end{pmatrix}$

	B	A_B^{-1}		x_4 (entering ↓)	$\vec{p}$
	x_1	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{13}{6}$
leaving ←	x_2	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{2}$

Since $\frac{1}{2}$ is the only entry in the x_4 column that is > 0, only x_2 can leave.

The updated mini-tableau:

B	A_B^{-1}		x_4	$\vec{p}$
x_1	$\frac{1}{3}$	$\frac{1}{3}$	0	3
x_4	0	1	1	5

Draft:

$\frac{1}{6} \times 5 + \frac{13}{6} = 3$

$\frac{1}{6} \times 1 + \frac{1}{6} = \frac{1}{3}$

Finally, compute $\vec{r} = \vec{c}_B^{\,T} A_B^{-1} A_N - \vec{c}_N^{\,T}$ in this step:

$$\vec{r} = \vec{c}_B^{\,T} A_B^{-1} A_N - \vec{c}_N^{\,T} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{10}{3} & \frac{1}{3} & \frac{1}{3} & \frac{7}{3} \end{pmatrix} > 0$$

Thus we are done since $\vec{r} > 0$.

And when the non-basis variables take zeros, i.e., $x_2 = x_3 = x_5 = x_6 = 0$, we look at $\vec{p}$ in the last mini-tableau. This $\vec{p}$ is exactly equal to $(x_1 \;\; x_4)^T$.

So the optimal solution is given by $\vec{x} = (3 \;\; 0 \;\; 0 \;\; 5 \;\; 0 \;\; 0)^T$. □