Name: ____________________

Workshop 8

Problem 1: Consider the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$.

(a) Use linear approximation to approximate the value of $f(3.01)$.

Solution. $f(3.01)$ can be linearly approximated by $f(3.01) \approx f(3) + f'(3) \cdot (3.01 - 3)$.

Since $f'(x) = \underline{2x}$, we have $f'(3) = \underline{6}$.

Moreover, $f(3) = \underline{9}$, we conclude that $f(3) + f'(3) \cdot (3.01 - 3) = 9 + 6 \times 0.01 = \underline{9.06}$. □

(b) Use a calculator to find the actual value of $f(3.01)$. Is this value larger or smaller than your approximated value?

Solution. $f(3.01) = 3.01^2 = 9.0601 \underline{>} 9.06$. □

(c) Use linear approximation to approximate the value of $g(4.01)$?

Solution. $g(4.01)$ can be linearly approximated by $g(4.01) \approx g(4) + g'(4) \cdot (4.01 - 4)$.

Since $g'(x) = \underline{\frac{1}{2}x^{-\frac{1}{2}}}$, we have $g'(4) = \underline{\frac{1}{4}}$.

Moreover, $g(4) = \underline{2}$, we conclude that $g(4) + g'(4) \cdot (4.01 - 4) = 2 + \frac{1}{4} \times (0.01) = \underline{2.0025}$ □

(d) Use a calculator to find the actual value of $g(4.01)$. Is this value larger or smaller than your approximated value?

Solution. $g(4.01) = \sqrt{4.01} \approx 2.00249843945 \underline{<} 2.0025$. □

(e) Now, we'll try to explain what's happening here. Compute the functions $f''(x)$ and $g''(x)$. What do you notice about these functions? Are they positive or negative?

Solution. $f''(x) = (f'(x))' = \underline{2}$;

$g''(x) = \underline{-\frac{1}{4}x^{-\frac{3}{2}}}$.

Thus, $f''(x)$ is always $\underline{>}$ 0 ($>$ or $<$?)

$g''(x)$ is always $\underline{<}$ 0 ($<$ or $>$?) □

(f) (*optional/will not be graded*) Using the fact that the second derivative is the rate of change of the first derivative, explain why your approximated answers were larger or smaller than the actual functions values for the functions f and g above.

Solution. The Mean Value Theorem states that:

"Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point $c \in (a, b)$ at which

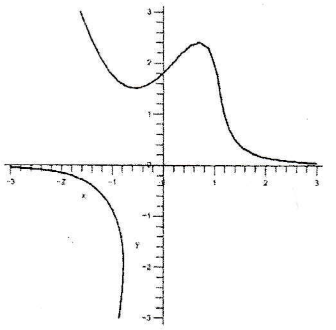
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

" Take $f(x) = x^2, a = 3, b = 3.01$ as an example, by the Mean Value Theorem, there exists a $c \in (3, 3.01)$ such that $f'(c) = \frac{f(3.01) - f(3)}{3.01 - 3}$. It follows that $f(3.01) = f(3) + f'(c) \cdot 0.01$. On the other hand, in part (a), we are using $f(3) + f'(3) \cdot 0.01$ to approximate $f(3.01)$. In part (e), we have found that $f''(x)$ is always positive, so $f'(c) > f'(3)$. This explains why $f(3.01) > f(3) + f'(3) \cdot 0.01$.

Similarly, one can argue that $g(4.01) < g(4) + g'(4) \cdot 0.01$ by using $g''(x) < 0$ instead.

□

Problem 2: Below is part of the graph of $5x^3y - 3xy^2 + y^3 = 6$. Since when $(x, y) = (1, 2)$, we have $5x^3y - 3xy^2 + y^3 = 5 \cdot (1^3) \cdot 2 - 3 \cdot 1 \cdot 2^2 + 2^3 = 6$, we conclude that **$(1, 2)$ is on the graph of $5x^3y - 3xy^2 + y^3 = 6$.**



(a) There's a nearby point on the curve whose coordinates are $(1.07, u)$. What is an approximate value for u?

Solution. Idea: To approximate u, we realize y locally as a function of x and use the derivative $y' = \frac{dy(x)}{dx}$ at $(1, 2)$ to approximate u.

Taking the derivative of $5x^3y - 3xy^2 + y^3 = 6$ with respect to x gives us $5(3x^2y + x^3y') - 3(y^2 + x(2y)y') + 3y^2y' = 0$. We are interested in the value of y' when $(x, y) = (1, 2)$. Plugging $(x, y) = (1, 2)$ into $5(3x^2y + x^3y') - 3(y^2 + x(2y)y') + 3y^2y' = 0$ gives us $y'|_{(1,2)} = -18/5$. Now we approximate u as $u \approx 2 + y'|_{(1,2)} \cdot (1.07 - 1) = 1.748$. □

(b) There's a nearby point on the curve whose coordinates are $(0.98, v)$. Mimic part (a) to give an approximate value for v?

Sol. $v \approx 2 + y'|_{(1,2)} \cdot (0.98 - 1) = 2.072.$

□

(c) There's a nearby point on the curve whose coordinates are $(w, 2.04)$. What is an approximate value for w?

Hint: Similar to part (a), regard x locally as a function of y and find the value of $x' = \frac{dx(y)}{dy}$ at $(x, y) = (1, 2)$ by taking the implicit differentiation. We can approximate w as $w \approx 1 + x'|_{(1,2)} \cdot (2.04 - 2)$.

Sol. Taking the derivative of $5x^3y - 3xy^2 + y^3 = 6$ with respect to y gives us

$5(3x^2x'y + x^3) - 3(x'y^2 + x(2y)) + 3y^2 = 0$. Plugging in $(x, y) = (1, 2)$ gives

$x'|_{(1,2)} = -\frac{5}{18}$. So, $w \approx 1 - \frac{5}{18} \cdot (0.04) = 1 - \frac{1}{90} \approx 1 - 0.0111\cdots = 0.9888\cdots$

□

Problem 3: Imagine that in the future you are on a long flight and your phone battery is at 1%, rendering it useless. You thought you packed your phone charger, but realize that it is in your checked luggage. For some reason, you need an approximation to $\sqrt[3]{9}$ and all you have to help you is the blank paper and pencil that the flight attendant has graciously given you.

(a) What is the value of 2^3? And what is the value of $\sqrt[3]{8}$?

Sol.

$$2^3 = 8\ ,\quad \sqrt[3]{8} = 2 \qquad \square$$

(b) Define $f(x) = \sqrt[3]{x}$. What is the derivative $f'(x)$ of $f(x)$ with respect to x? What is $f'(8)$?

Sol. $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$.

$$f'(8) = \frac{1}{12}. \qquad \square$$

(c) Define $L(x) = f(8) + f'(8) \cdot (x - 8)$. Compute $L(9)$ and we may take $L(9)$ as an approximation of $f(9)$.

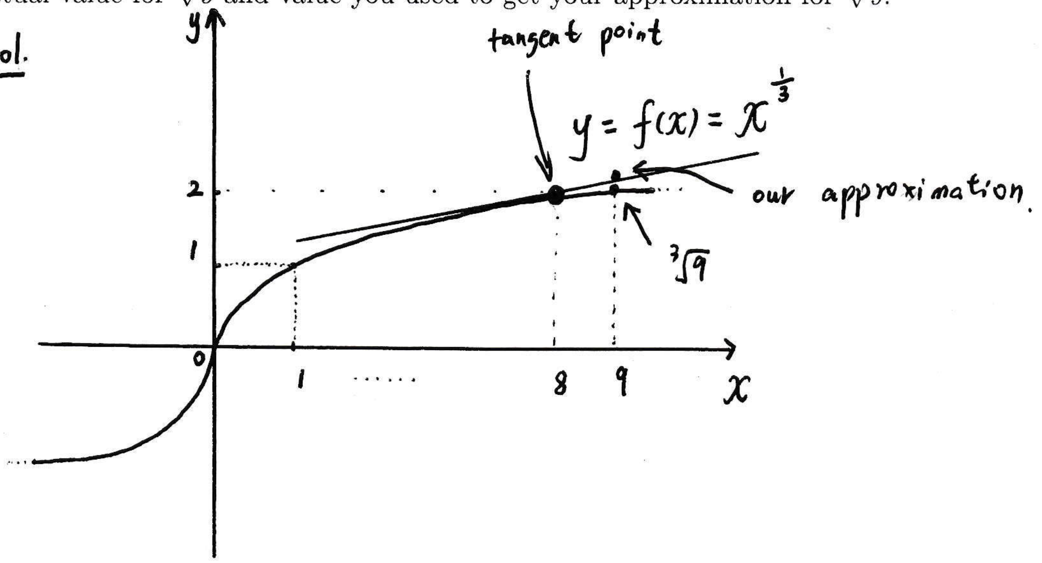
Sol.

$$L(9) = 2 + \frac{1}{12} \times (9-8) = 2 + \frac{1}{12} \approx 2.083.$$

$\square$

(d) Make a sketch of the graphs of $f(x)$ and $L(x)$ together on the same axes. Indicate in the sketch the actual value for $\sqrt[3]{9}$ and value you used to get your approximation for $\sqrt[3]{9}$.

Sol.



$\square$

Math 151 Quiz 8

Score:___0___1___2___3___4___5

(1 point) **First Name**(PRINT):________ **Last Name**(PRINT):________ **Session**:________ **NetID**:________

(2 points)**1.** Suppose $f(x)$ is a differentiable function defined for all real numbers x. The tangent line to the graph of $y = f(x)$ at $x = 1$ is $y = \frac{5}{2}x + \frac{1}{2}$. Find an appropriate linearization $L(x)$ at $x = 1$ and use it to estimate $f(1.1)$. (Make sure you box both of your answer for $L(x)$ and your approximation of $f(1.1)$.)

<u>Sol.</u> $L(x) = \frac{5}{2}x + \frac{1}{2}$

$$f(1.1) \approx L(1.1) = \frac{5}{2}\cdot(1.1) + \frac{1}{2} = \frac{5}{2} + \frac{5}{2}\cdot(0.1) + \frac{1}{2}$$

$$= 2.5 + 0.25 + 0.5$$

$$= 3.25 .$$

□

(2 points)**2.** Use differentials to approximate $(.9)^{3/4}$. (Your answer should be a rational number, preferably expressed as a single simplified fraction.)

Solution:

Let $f(x) = x^{3/4}$. $f'(x) = \frac{3}{4}x^{-1/4}$ and $f'(1) = \frac{3}{4}$, $f(1) = 1$.

Let $L(x) = f(1) + f'(1)\cdot(x-1)$

$$= 1 + \frac{3}{4}(x-1)$$

$$L(0.9) = 1 - \frac{3}{4}\cdot(0.1) = 1 - 0.075 = 0.925 .$$

□