THE DISTRIBUTION OF PATH LENGTHS ON DIRECTED WEIGHTED GRAPHS

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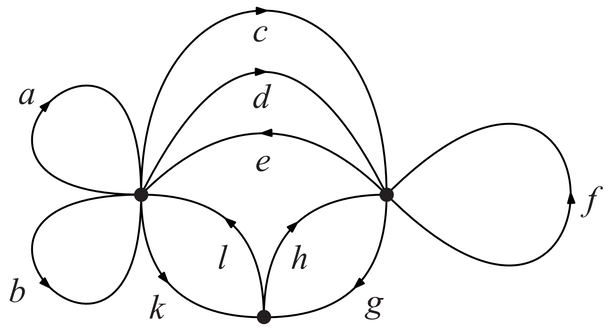
Consider a strongly connected **DIRECTED WEIGHTED MULTIGRAPH** G on a finite set of vertices



Let $N_{ij}(x)$ count the number of paths from vertex v_i to vertex v_j of length at most x.

QUESTION What is the asymptotic behavior of $N_{ij}(x)$?

Define the corresponding **GRAPH MATRIX** function

$$M_G(s) = \begin{pmatrix} e^{-as}+e^{-bs} & e^{-cs}+e^{-ds} & e^{-ks} \\ e^{-es} & e^{-fs} & e^{-gs} \\ e^{-ls} & e^{-hs} & 0 \end{pmatrix}$$

LEMMA Denote by λ_G the maximal real value for which the spectral radius of $M_G(s)$ is equal to 1. Then λ_G is a well defined positive real constant depending only on G.

MAIN RESULT

A graph G is called **INCOMMENSURABLE** if there exist two closed orbits of lengths A and B such that $A \notin \mathbb{Q}B$.

THEOREM Let G be an incommensurable graph, and let $M_G(s)$ and λ_G be as above, then

$$N_{ij}(x) = \frac{1}{\lambda_G} \cdot \frac{\left(\operatorname{adj}\left(I_n - M_G(\lambda_G)\right)\right)_{ij}}{-\operatorname{tr}\left(\operatorname{adj}\left(I_n - M_G(\lambda_G)\right) \cdot M_G'(\lambda_G)\right)} e^{\lambda_G x} + o\left(e^{\lambda_G x}\right)$$

as $x \to \infty$, where M_G' is the entry-wise derivative of M_G.

TOOLS AND METHODS

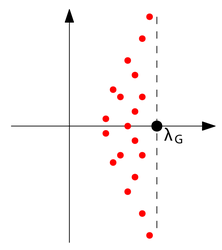
I The **PERRON-FROBEINUS** theory for primitive matrices.

II The **LAPLACE TRANSFORM** of the counting function

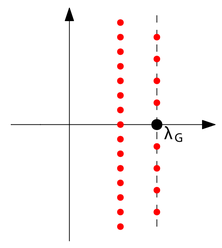
$$f_{ij}(s) = \int_0^\infty N_{ij}(x)\, e^{-xs} dx = \frac{1}{s} \cdot \frac{\left(\operatorname{adj}\left(I_n - M_G(s)\right)\right)_{ij}}{\det\left(I_n - M_G(s)\right)}$$

III The **WEINER-IKEHARA** Tauberian theorem which is used to deduce the main result from the pole structure of the Laplace transform.

Pole structure in the incommensurable case



Pole structure in the rational case



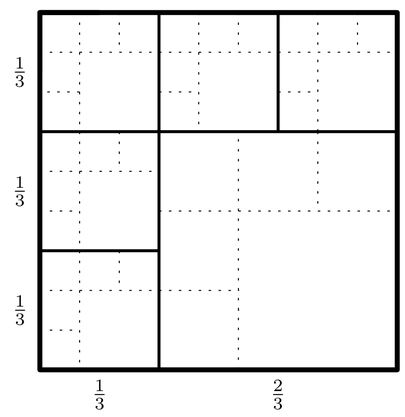
APPLICATIONS

I MATHEMATICAL MODELS OF QUASICRYSTALS

The **MULTISCALE SUBSTITUTION** construction is a generalization of the classical substitution tiling construction:

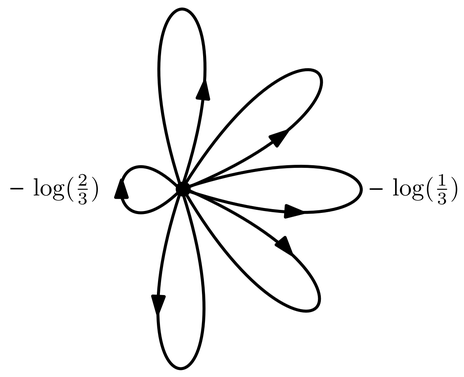
Let $\mathscr{B} = T_1, .., T_n$ be a finite collection of basic tiles in $\mathbb{R}^d$, such that each basic tile can be tiled by isometric copies of elements of $\mathscr{B}$ contracted by possibly different factors.

For example, in $\mathbb{R}^2$ let T_1 be the unit square with the following substitution rule:



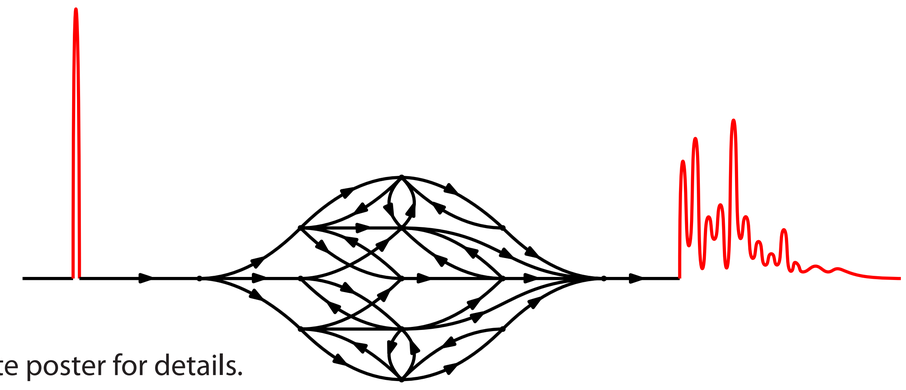
When done carefully, repeated substitutions and rescalings produce an infinite tiling of the space as a limit object in the topology of all closed subsets. Such a tiling is typically of infinite local complexity.

Questions regarding the asymptotic density of tiles and the frequency of local patterns can be approached using the **ACCOMPANYING GRAPH:**



For graphs accompanying a multiscale substitution tiling in $\mathbb{R}^d$ we have $\lambda_G = d$.

II DISTRIBUTION OF TRANSIT TIMES THROUGH QUANTUM GRAPHS



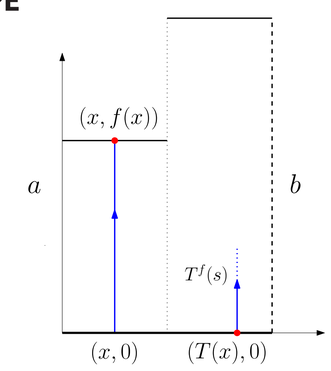
See separate poster for details.

RELATED TOPICS

I CLOSED ORBITS OF A SUSPENSION OF A SHIFT OF FINITE TYPE

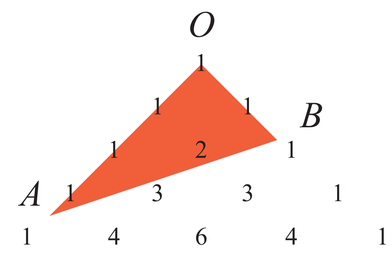
Let X be the unit interval, T be the circle-doubling map $T(x) = 2x \mod 1$ and f a piecewise constant function assuming the values a and b.

The f suspension X^f is the space $\{(x,t) \mid x \in X,\ 0 \le t \le f(x)\}$ with the points $(x, f(x))$ and $(T(x), 0)$ identified, and the suspension flow T^f is as described in the figure. The flow is weakly mixing if a and b are incommensurable.

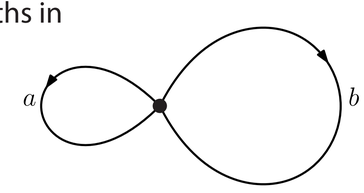


II SUMMATION IN PASCAL'S TRIANGLE

This well known triangular array of binomial coefficients contains many patterns of numbers and properties of combinatorial interest.



Counting the number of closed orbits of T^f and summing the binomial coefficients in the triangle OBA of sides $OA = \frac{x}{a}$ and $OB = \frac{x}{b}$ with angle $\angle OAB = \arctan\frac{a}{b}$ are both equivalent to counting paths in



Joint work with . For more details: *The distribution of path lengths on directed weighted graphs* http://arxiv.org/abs/1608.00150 Avner Kiro, Yotam Smilansky and Uzy Smilansky