Consider a strongly connected Directed Weighted Multigraph \( G \) on a finite set of vertices.

\[ \begin{array}{ccc}
\text{a} & \text{b} & \text{c} \\
\text{d} & \text{e} & \text{f} \\
\text{g} & \text{h} & \text{i} \\
\end{array} \]

Let \( N_{ij}(x) \) count the number of paths from vertex \( v_i \) to vertex \( v_j \) of length at most \( x \). 

**Question** What is the asymptotic behavior of \( N_{ij}(x) \)?

**Main Result**

A graph \( G \) is called **Incommensurable** if there exist two closed orbits of lengths \( A \) and \( B \) such that \( A \not\equiv QB \).

**Theorem** Let \( G \) be an incommensurable graph, and let \( M_G(s) \) and \( \lambda_G \) be as above, then

\[
N_{ij}(x) = \frac{1}{\lambda_G} \left( \frac{\text{det}(I_n - M_G(s))_{ij}}{\text{det}(I_n - M_G(s)) \cdot \lambda_G} \right) e^{\lambda_G x} + o(e^{\lambda_G x})
\]

as \( x \to \infty \), where \( M_G' \) is the entry-wise derivative of \( M_G \).

**Applications**

**I Mathematical Models of Quasicrystals**

The **Multiscale Substitution** construction is a generalization of the classical substitution tiling construction:

Let \( \mathcal{A} = T_1, \ldots, T_n \) be a finite collection of basic tiles in \( \mathbb{R}^d \), such that each basic tile can be tiled by isometric copies of elements of \( \mathcal{A} \) contracted by possibly different factors.

For example, in \( \mathbb{R}^2 \) let \( T_1 \) be the unit square with the following substitution rule:

\[
\begin{array}{c}
\text{a} & \text{b} \\
\text{c} & \text{d} \\
\end{array}
\]

When done carefully, repeated substitutions and rescalings produce an infinite tiling of the space as a limit object in the topology of all closed subsets. Such a tiling is typically of infinite local complexity.

Questions regarding the asymptotic density of tiles and the frequency of local patterns can be approached using the **Accompanying Graph**:

\[
\begin{array}{c}
\text{a} \quad \text{b} \\
\text{c} \quad \text{d} \\
\end{array}
\]

For graphs accompanying a multiscale substitution tiling in \( \mathbb{R}^d \) we have \( \lambda_G = d \).

**Tools and Methods**

**I** The Perron-Frobenius theory for primitive matrices.

**II** The Laplace Transform of the counting function

\[
f_{ij}(s) = \int_0^\infty N_{ij}(x) e^{-sx} dx = \frac{1}{\text{det}(I_n - M_G(s))} \frac{\text{det}(I_n - M_G(s))_{ij}}{\text{det}(I_n - M_G(s))} e^{\lambda_G x} + o(e^{\lambda_G x})
\]

**III** The Weiner-Ikehara Tauberian theorem which is used to deduce the main result from the pole structure of the Laplace transform.

**Applications**

**I Distribution of Transit Times Through Quantum Graphs**

See separate poster for details.

**Related Topics**

**I Closed Orbits of a Suspension of a Shift of Finite Type**

Let \( X \) be the unit interval, \( \mathcal{T} \) be the circle-doubling map \( \mathcal{T}(x) = 2x \mod 1 \) and \( f \) a piecewise constant function assuming the values \( a \) and \( b \).

The \( f \)- suspension \( X^f \) is the space \( \{(x,t) \mid x \in X, 0 \leq t \leq f(x)\} \) with the points \( (x,f(x)) \) and \( (\mathcal{T}(x),0) \) identified, and the suspension flow \( \mathcal{T}^f \) is as described in the figure. The flow is weakly mixing if \( a \) and \( b \) are incommensurable.

**II Summation in Pascal’s Triangle**

This well known triangular array of binomial coefficients contains many patterns of numbers and properties of combinatorial interest.

Counting the number of closed orbits of \( \mathcal{T}^f \) and summing the binomial coefficients in the triangle \( OBA \) of sides \( OA = \frac{a}{2} \) and \( OB = \frac{b}{2} \) with angle \( \angle COAB = \arctan \frac{b}{a} \) are both equivalent to counting paths in

\[
\begin{array}{c}
a \quad b \\
\end{array}
\]