Order and Disorder in Multiscale Substitution Tilings

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joint with Caltech and USC

Partially based on joint work with Yaar Solomon
Plan of Talk

• Introduction

• Multiscale substitution tilings

• Main results
Delone Sets

A uniformly discrete and relatively dense set $\Lambda \subseteq \mathbb{R}^d$ is called Delone.
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Examples Lattices, sets induced by tilings and cut-and-project sets

A basic problem is to classify and measure how ordered or disordered a given Delone set is, compared to a lattice.
Lattice-like Properties

For \( x \in \Lambda \), \( r > 0 \) the \( r \)-patch of \( \Lambda \) at \( x \) is \( P_{\Lambda, r}(x) = (\Lambda - x) \cap B(0, r) \)

- Finite local complexity (FLC):
  \[ \forall \, r > 0 \, \#\{P_{\Lambda, r}(x) \mid x \in \Lambda\} < \infty \]

From Baake and Grimm's *Aperiodic Order Vol. I*
Lattice-like Properties

For $x \in \Lambda$, $r > 0$ the $r$-patch of $\Lambda$ at $x$ is $P_{\Lambda,r}(x) = (\Lambda - x) \cap B(0,r)$.

- **Finite local complexity (FLC):**
  \[ \forall r > 0 \quad \#\{ P_{\Lambda,r}(x) \mid x \in \Lambda \} < \infty \]

- **Repetitivity:** $\forall r > 0 \exists R = R(r)$ so that every $R$-ball contains a copy of every $r$-patch. Linear repetitivity: $R(r)$ is linear. Uniform patch frequency: patches appear in well-defined frequencies.
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- Self-similarity: $\exists \alpha > 1$ so that $\alpha \Lambda \subset \Lambda$
Spaces and Dynamical Systems of Delone Sets

Set $X_{\Lambda} = \{ \Lambda + t | t \in \mathbb{R}^d \}$, where the closure is with respect to a natural topology on Delone sets (induced by the Hausdorff metric restricted to centered balls).

- $\Lambda$ is (almost) repetitive $\Rightarrow$ The dynamical system $(X_{\Lambda}, \mathbb{R}^d)$ is minimal (every orbit is dense).

- (almost) linear repetitivity $\Rightarrow$ unique ergodicity (unique invariant measure)

(Radin '92, Solomyak '97, Damanik '01, Lagarias '03, Frettloeh '14)
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Substitution Tilings

A tiling is a collection of tiles with disjoint interiors that covers $\mathbb{R}^d$.

A substitution rule on a set of prototiles is a tessellation of each prototile by rescaled prototiles, with a fixed scale $c \in (0,1)$.

Repeated applications of the substitution rule followed by a rescaling define larger and larger patches.
Incommensurable Multiscale Substitution Tilings

A multiscale substitution scheme $\sigma$ in $\mathbb{R}^d$ consists of a substitution rule on unit volume prototiles $T_1, \ldots, T_n$, where various different scales appear and satisfy a simple incommensurability condition.

A time-dependent substitution semiflow $F_t$ defines a family of patches: At time $t=0$ $F_t(T) = T$, and as $t$ increases the patch is inflated by $e^t$ and tiles of volume $>1$ are substituted.
Some Predecessors

- Rauzy’s fractal ’81
  - multiple (but commensurable) scales
- Conway and Radin’s pinwheel tiling ’94
  - $\theta = \arctan \frac{1}{2} \implies$ same triangle incommensurable directions
- Sadun’s generalized pinwheel tilings ’98
- $\alpha$-Kakutani sequences in $[0,1]$ ’76
  - always split longest interval
- S ’20: multiscale substitution Kakutani sequences of partitions
The Associated Graph $\mathcal{G}_0$

A directed weighted graph is defined according to $\mathcal{G}_0$

Vertices model the prototiles

Edges model the tiles appearing in the substitution rule with

Lengths $= \log(1/\text{scale})$

$\mathcal{G}_0$ is incommensurable if $\mathcal{G}_0$ contains two closed paths of lengths $\frac{a}{6} \notin \mathbb{Q}$. Incommensurable multiscale substitution schemes generate a new distinct class of tilings of $\mathbb{R}^d$. 
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Counting in Multiscale Substitution Tilings

Substitution \# tiles in patches = entries of powers of the substitution matrix \( S \)

\[ \Rightarrow S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \]

Multiscale \{ Tiles in \( F_t(T_i) \) \} \leftrightarrow \{ Directed walks of length \( t \) in \( G_6 \) originating at vertex \( i \) \}

Example the \( \frac{1}{3} \)-Kakutani scheme in \( \mathbb{R} \):

the patches \( F_0(I), F_{\log_{\frac{3}{2}}}(I), F_{2\log_{\frac{3}{2}}}(I) \) and their respective walks
Counting in Multiscale Substitution Tilings

**Theorem** (SS121, $S \geq 21$, relying on Kiro, Smilansky x2 '20)

\[
\# \{ \text{tiles in } F_t(T) \} = \frac{v^T(S_\sigma - V_\sigma) 1}{v^T H_\sigma 1} \cdot \frac{e^{dt}}{\text{vol}(F_t(T))} + \text{ERROR}\text{TERM}, \quad t \to \infty
\]

- **combinatorics** matrix

\[
(S_\sigma)_{ij} = \sum_{T \text{ of type } j} 1 \quad \# \text{reds in white}
\]

\[
S_\sigma = \begin{pmatrix} 8 & 5 \\ 1 & 3 \end{pmatrix}
\]

- **volume** matrix

\[
(V_\sigma)_{ij} = \sum_{T \text{ of type } j} \text{vol}(T) \quad \text{total red area in white}
\]

\[
V_\sigma = \begin{pmatrix} \frac{15}{25} & \frac{8}{25} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}
\]

- **entropy** matrix

\[
(H_\sigma)_{ij} = \sum_{T \text{ of type } j} \text{vol}(T) \cdot \log \text{vol}(T) \quad \text{contribution of reds to entropy of white}
\]

\[
H_\sigma = \begin{pmatrix} -\frac{12}{5} \log \frac{1}{2} & \frac{6}{5} \log \frac{1}{2} \\ -\frac{1}{4} \log \frac{1}{4} & -\frac{3}{4} \log \frac{1}{4} \end{pmatrix}
\]

and $v^T = \text{left Perron-Frobenius eigenvector of } V_\sigma$

**Theorem** (SS121) $\exists k \in \mathbb{N} \quad \forall t_0 > 0 \exists t \geq t_0 : \frac{\text{ERRORTERM}}{c} \geq \frac{e^{dt}}{t^k}$
Bounded Displacement Equivalence

- Delone sets $\Lambda, \Gamma \subset \mathbb{R}^d$ are bounded displacement (BD) equivalent if $\exists$ bijection $\varphi : \Lambda \to \Gamma$ that moves every point a bounded distance.

- $\Lambda$ is uniformly spread if it is BD to $\alpha \mathbb{Z}^d$ for some $\alpha > 0$

- Not all Delone sets are uniformly spread
**Bounded Displacement Equivalence**

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- Sets associated with tilings with a single tile are uniformly spread $\Rightarrow$ lattices & periodic sets (Duneau, Oguey '90, Hall's marriage theorem)
Bounded Displacement Equivalence

Laczkovich '92 For a Delone set $A \subseteq \mathbb{R}^d$ the following are equivalent:

- $A$ is uniformly spread
- There exist $\alpha, C > 0$ so that $\forall A \in \mathbb{Q}_d = \{\text{finite unions of lattice cubes}\}$

$\text{discrepancy} \sim |\#(A \cap \Lambda) - \alpha \cdot \text{vol}(A)| \leq C \cdot \text{vol}_{d-1}(\partial A)$

$\Rightarrow$ Incommensurable multiscale substitution tilings are never uniformly spread.
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Theorem (SS2 '21) Let $X$ be a minimal space of Delone sets.
- Either $\exists \Lambda \subseteq X$ uniformly spread, and then every $\Lambda \subseteq X$ is such.
- Or $X$ contains continuously many distinct BD class representatives.

$\Rightarrow$ Incommensurable tiling spaces contain continuously many BD classes.
Dynamics in Multiscale Substitution Tilings

**Theorem** (SS121) Let $T$ be an incommensurable tiling in $\mathbb{R}^d$ and $(X_T, \mathbb{R}^d)$ with $\mathbb{R}^d$ acting by translations.

- $F_t(T-x) = F_t(T) - e^t x$ for $t \geq 0, x \in \mathbb{R}^d$ (horospheric and geodesic)
- $(X_T, \mathbb{R}^d)$ is minimal $\Rightarrow T$ is almost repetitive
- almost repetitivity is not linear (SS321)
- $(X_T, \mathbb{R}^d)$ is uniquely ergodic
- $T$ has uniform patch frequencies
- Lee-Solomyak's 19 "pixelization"
Dynamics in Multiscale Substitution Tilings

**Theorem (SS$^3$22)** Let $T$ be an incommensurable tiling in $\mathbb{R}^d$ and consider the semiflow $F_t$ on $X_T$ (scenery flow)

- there exist dense orbits.
- periodic orbits of $F_t \Rightarrow$ self similar tilings
- Prime orbit theorem $\pi_\sigma(t) \sim \frac{e^{dt}}{dt}$, $t \to \infty$

where $\pi_\sigma(t) = \{\text{orbits } \tau \text{ with minimal period } \lambda(\tau) \leq t\}$ \textit{(à la Parry, Pollicott)}

- tiling zeta function $\zeta_\sigma(s) = \prod_\tau (1 - e^{-\lambda(\tau)s})^{-1} \cdot \frac{1}{\det (I - M_\sigma(s))}$

where $(M_\sigma(s))_{ij} = \sum_{T \text{ of type } j \text{ in } T_i} \forall \ell(T)^s$
Counting in Multiscale Substitution Tilings

Theorem (SS121, $S \geq 21$, relying on Kiro, Smilansky $\times 2$ ’20)

Tiles and patches appear in a dense set of scales $\Rightarrow$ not FLC

Moreover, we give explicit formulas for asymptotic densities of:

- $\# \{\text{tiles of type } r \text{ and } \text{vol} \in [a,b] \text{ in } F_t(T)\}$

- $\text{volume}(U \{\text{tiles of type } r \text{ and } \text{vol} \in [a,b] \text{ in } F_t(T)\})$

- Expected values for random partitions
Theorem \((S \geq 21,\) relying on Kiro, Smilansky \(x 2 \ '20\))

- **Gap distribution** \(\Lambda\) - Delone set of tile boundaries in a 1-dim tiling

\[
\frac{\# \{\text{Neighbors in } \Lambda \cap [N, N] \text{ of distance } e [a, b]\}}{\# \{\Lambda \cap [N, N]\}} \to \int_a^b \frac{\nu^T C_\sigma(x) 1}{\nu^T H_\sigma 1} \, dx
\]

where \((C_\sigma(x))_{ij} = \sum_{T \text{ of type } j} \left\{ \begin{array}{ll} \frac{\text{vol } T}{x^2} & \text{for } \text{vol } T < x \leq 1 \\ 0 & \text{otherwise} \end{array} \right. \)

- **Numerics** for pair correlations are consistent with Poisson process
Thank You!