Workshop 6

1. One of the Bessel functions used to describe the vibration of a circular plate is defined by this infinite series:

$$J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\,.$$

(1) Show that this series converges absolutely for all values of x.
(2) Here are individual terms of the series for two values of x and for some values of n.

$\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$x=1$	1	$-\frac{1}{4}$	$\frac{1}{64}$	$-\frac{1}{2{,}304}$	$\frac{1}{147{,}456}$	$-\frac{1}{14{,}745{,}600}$
$x=4$	1	-4	4	$-\frac{16}{9}$	$\frac{4}{9}$	$-\frac{16}{225}$

Use entries of this table and facts about the series to explain why $J(1)$ must be positive and $J(4)$ must be negative.

Hint: Select an N for each x and split the sum: $\sum_{n=0}^{\infty} = \sum_{n=0}^{N} + \sum_{n=N+1}^{\infty}$. Evaluate the finite sum explicitly and estimate the infinite tail $\sum_{n=N+1}^{\infty}$.

2. a) Use the formula $\frac{a}{1-r} = a + ar + ar^2 + ar^3 + \cdots$ valid for $|r| < 1$ to express each of the following functions as a power series $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$.

Give a formula for the coefficient a_n in each case.

$$f(x) = \frac{x}{1-x}; \quad g(x) = \frac{2}{3x^4 + 16}.$$

b) Determine the interval of x values in which each series in part a) converges (be sure to consider the endpoints).

c) Use your answer to a) to express $\int_0^1 \frac{2}{3x^4+16}\,dx$ as the sum of an infinite series.

3. Suppose that you have a power series $\sum_{n=1}^{\infty} a_n x^n$, whose interval of convergence is $(-1, 1]$.

(1) Using the same numbers a_n (adjusting them a little, if required), come up with a new power series whose interval of convergence is $(0, 2]$.

(2) Using the same numbers a_n (adjusting them a little, if required), come up with a new power series whose interval of convergence is $(-2, 2]$.

(3) Using the same numbers a_n (adjusting them a little, if required), come up with a new power series whose interval of convergence is $[-1, 1)$. (**Hint:** What "transformation" would turn the interval $(-1, 1]$ into $[-1, 1)$?)

(4) Putting together ideas from previous parts of this question, come up with a new power series whose interval of convergence is $[10, 20)$.

In each case, give reasons for your answers.