# Quantum $K$-theory of Incidence Varieties

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## Summary

We prove a conjecture of Buch and Mihalcea in the case of the incidence variety $X = \mathrm{Fl}(1, n-1; n)$ and determine the structure of its ($T$-equivariant) quantum $K$-theory ring. Our results are an interplay between geometry and combinatorics. The geometric side concerns Gromov-Witten varieties of 3-pointed genus 0 stable maps to $X$ with markings sent to Schubert varieties, while on the combinatorial side are formulas for the (equivariant) quantum $K$-theory ring of $X$. We prove that the Gromov-Witten variety is rationally connected when one of the defining Schubert varieties is a divisor and another is a point. This implies that the (equivariant) $K$-theoretic Gromov-Witten invariants defined by two Schubert classes and a Schubert divisor class can be computed in the ordinary (equivariant) $K$-theory ring of $X$. We derive a positive Chevalley formula for the equivariant quantum $K$-theory ring of $X$ and a positive Littlewood-Richardson rule for the non-equivariant quantum $K$-theory ring of $X$. The Littlewood-Richardson rule in turn implies that non-empty Gromov-Witten varieties given by Schubert varieties in general position have arithmetic genus 0.

## The Incidence Variety

$$\begin{aligned} X &= \mathrm{Fl}(1, n-1; n) = \mathrm{SL}(\mathbb{C}^n)/P \\ &= \{U \subset V \subset \mathbb{C}^n : \dim U = 1, \dim V = n-1\} \\ &= \{x_1y_1 + \cdots + x_ny_n = 0\} \subset \mathbb{P}(\mathbb{C}^n) \times \mathbb{P}(\mathbb{C}^{n*}) \end{aligned}$$

**Schubert varieties** in $X$ are indexed by

$$W^P := \{[i,j] : 1 \le i \ne j \le n\}.$$

$X_{[i,j]} = \{x_{i+1} = \cdots = x_n = y_1 = \cdots = y_{j-1} = 0\} \subseteq X$,
$X^{[i,j]} = \{x_1 = \cdots = x_{i-1} = y_{j+1} = \cdots = y_n = 0\} \subseteq X$,
$D^{[1]} := X^{[2,n]} = \{x_1 = 0\}$, $D^{[2]} := X^{[1,n-1]} = \{y_n = 0\}$.

## $\overline{M}_{0,3}(X,d)$ and Related Constructions

Fix $d \in H_2(X)^+ = \mathbb{Z}^2_{\ge 0}$.
$M_d := \overline{M}_{0,3}(X,d) := \overline{\{f : \mathbb{P}^1 \to X \mid f_*[\mathbb{P}^1] = d\}}$.
$\mathrm{ev}_1,\ \mathrm{ev}_2,\ \mathrm{ev}_3 : M_d \to X$ (evaluate at markings $0,\ 1,\ \infty \in \mathbb{P}^1$)

For $u,\ v,\ w \in W^P$,
$M_d(X_u, X^v) := \mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v) \subseteq M_d$,
$\Gamma_d(X_u, X^v) := \mathrm{ev}_3(M_d(X_u, X^v)) \subseteq X$,
$\Gamma_d(X_u) := \mathrm{ev}_2(\mathrm{ev}_1^{-1}(X_u))$ is again a Schubert variety,
$M_d(X_u, X^v, g.X^w) := \mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v) \cap \mathrm{ev}_3^{-1}(g.X^w)$.

Assume $g \in \mathrm{SL}(\mathbb{C}^n)$ general, then
the **cohomological Gromov-Witten Invariant**

$$I_d([X_u], [X^v], [X^w]) = \# M_d(X_u, X^v, g.X^w);$$

the **$K$-theoretic Gromov-Witten Invariant**

$$\begin{aligned} I_d([\mathcal{O}_{X_u}], [\mathcal{O}_{X^v}], [\mathcal{O}_{X^w}]) &= \chi(\mathcal{O}_{M_d(X_u, X^v, g.X^w)}) \\ &= \chi_{M_d}(\mathrm{ev}_1^*[\mathcal{O}_{X_u}] \cdot \mathrm{ev}_2^*[\mathcal{O}_{X^v}] \cdot \mathrm{ev}_3^*[\mathcal{O}_{X^w}]) \in K(\mathrm{pt}); \end{aligned}$$

the **$T$-equivariant $K$-theoretic Gromov-Witten Invariant**

$$\begin{aligned} I_d^T([\mathcal{O}_{X_u}], [\mathcal{O}_{X^v}], [\mathcal{O}_{X^w}]) \\ = \chi^T_{M_d}(\mathrm{ev}_1^*[\mathcal{O}_{X_u}] \cdot \mathrm{ev}_2^*[\mathcal{O}_{X^v}] \cdot \mathrm{ev}_3^*[\mathcal{O}_{X^w}]) \in K^T(\mathrm{pt}), \end{aligned}$$

where $T \subset \mathrm{SL}(\mathbb{C}^n)$ is the maximal torus of diagonal matrices.
$K^T(\mathrm{pt})$ can be identified with the representation ring of $T$.
$[\mathcal{O}_{X_u}]$ and $[\mathcal{O}_{X^v}]$ form $K^T(\mathrm{pt})$-bases for $K_T(X)$.

$X$ can be replaced by any flag variety $G/P$.

## Results Part 1

**Theorem 1** (X '21). *The general fibre of*

$$\mathrm{ev}_3 : M_d(X_{[i,j]}, D^{[k]}) \to \Gamma_d(X_{[i,j]}, D^{[k]})$$

*is rationally connected.*

Using a result of Kollár, this implies that

$$\mathrm{ev}_{3*}[\mathcal{O}_{M_d(X_{[i,j]}, D^{[k]})}] = [\mathcal{O}_{\Gamma_d(X_{[i,j]}, D^{[k]})}]$$

in (equivariant) $K$-theory, proving a conjecture of Buch and Mihalcea for $X$.

In the following, $\mathcal{O}_{[i,j]} := [\mathcal{O}_{X_{[i,j]}}]$, $\mathcal{O}^{[k]} := [\mathcal{O}_{D^{[k]}}] \in K_T(X)$.

**Corollary: "quantum equals classical" formula** (X '21).

$$\begin{aligned} I_d^T(\mathcal{O}_{[i,j]}, \mathcal{O}^{[k]}, \sigma) &= \chi_X^T([\mathcal{O}_{\Gamma_d(X_{[i,j]}, D^{[k]})}] \cdot \sigma) \\ &= \begin{cases} \chi_X^T([\mathcal{O}_{\Gamma_d(X_{[i,j]})}] \cdot \sigma) & \text{if } d_k > 0 \\ \chi_X^T([\mathcal{O}_{\Gamma_d(X_{[i,j]})}] \cdot \mathcal{O}^{[k]} \cdot \sigma) & \text{if } d_k = 0 \end{cases}. \end{aligned}$$

The right hand side is easily computable using Lenart and Postnikov's Chevalley formula for $K_T(X)$.

## Quantum $K$-theory

Quantum $K$-theory was introduced by Givental and Lee as a $K$-theoretic analogue of quantum cohomology.
The (small) $T$-equivariant quantum $K$-theory ring of $X$ is an algebra $QK_T(X)$ over $K^T(\mathrm{pt})[\![q_1, q_2]\!]$ with a $K^T(\mathrm{pt})[\![q_1, q_2]\!]$-basis consisting of $\mathcal{O}^w$ for $w \in W^P$. Multiplication in $QK_T(X)$ is defined using $I_d^T(\sigma_1, \sigma_2, \sigma_3)$.
$\widetilde{W^P} := \{[i,j] \in \mathbb{Z} \times \mathbb{Z} : i \not\equiv j \bmod n\}$. For $w \in \widetilde{W^P}$,

$$\overline{w} := [\bar{i}, \bar{j}] \in W^P \text{ is defined by } \bar{i} \equiv i,\ \bar{j} \equiv j,$$

$$d(w) := \left(\frac{i - \bar{i}}{n}, \frac{\bar{j} - j}{n}\right),$$

$$\mathcal{O}^w := q^{d(w)}[\mathcal{O}_{X^{\overline{w}}}] \in QK_T(X)_q := QK_T(X) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q, q^{-1}].$$

We write $\mathbb{C}_{\varepsilon_i}$ for the 1-dimensional representation of $T$ given by the character $\varepsilon_i$ which records the $i$-th diagonal entry.
We let $\varepsilon_i := \varepsilon_{\bar{i}}$ for $i \in \mathbb{Z}$.

## Results Part 2

**Equivariant Chevalley Formula** (X '21).
*In $QK_T(X)_q$, for $[i,j] \in \widetilde{W^P}$, $\mathcal{O}^{[i,j]} \star \mathcal{O}^{[1]}$ equals*

$(1 - [\mathbb{C}_{\varepsilon_i - \varepsilon_1}])\mathcal{O}^{[i,j]} + [\mathbb{C}_{\varepsilon_i - \varepsilon_1}]\mathcal{O}^{[i+1,j]}$ *when* $i + 1 \not\equiv j$ *mod* $n$,

$(1 - [\mathbb{C}_{\varepsilon_i - \varepsilon_1}])\mathcal{O}^{[i,j]} + [\mathbb{C}_{\varepsilon_i - \varepsilon_1}](\mathcal{O}^{[i+1,j-1]} + \mathcal{O}^{[i+2,j]} - \mathcal{O}^{[i+2,j-1]})$

*when* $i + 1 \equiv j$ *mod* $n$.
*The formula for $\mathcal{O}^{[i,j]} \star \mathcal{O}^{[2]}$ is analogous.*

The non-equivariant case ($[\mathbb{C}_\varepsilon] = 1$) is in Rosset's thesis ('20).
A formula for two-step varieties is in Kouno-Lenart-Naito-Sagaki ('21).

**Non-equivariant Littlewood-Richardson Rule** (X '21).
*In $QK(X)$, for $[i,j],\ [k,l] \in W^P$, $\mathcal{O}^{[i,j]} \star \mathcal{O}^{[k,l]}$ equals*

$$\mathcal{O}^{[x,y]} \text{ when } x - y < n[\chi(i > j) + \chi(k > l)],$$

$$\mathcal{O}^{[x,y-1]} + \mathcal{O}^{[x+1,y]} - \mathcal{O}^{[x+1,y-1]} \text{ otherwise},$$

*where $x = i + k - 1$, $y = j + l - n$.*

**Corollary 2** (X '21). *For $d \in H_2(X)^+$, a general $g \in \mathrm{SL}(\mathbb{C}^n)$, and $u,\ v,\ w \in W^P$, $M_d(g.X^u, X^v, X_w)$ has arithmetic genus 0 whenever it is non-empty.*

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