Math 354:H1 Second Midterm Exam — Solutions

(1a) (6 points) If the dual problem has feasible solutions but the objective function is unbounded, what follows about the solutions of the primal problem?

Solution: The primal problem is infeasible, i.e. it has no feasible solutions at all. This makes sense, in light of the Weak Duality Theorem, which states the the objective function of the primal problem takes values (at feasible solutions) which are less than or equal to *all* values taken by the dual objective function (at feasible solutions). If the objective function of the dual problem takes values which decrease without bound, then there is "no room" for the primal objective function to take any values at all.

(1b) (6 points) Suppose that $\mathbf{x_0}$ and $\mathbf{w_0}$ are feasible, not necessarily optimal solutions of the primal and dual LPPs, and that the value of the dual objective function at $\mathbf{w_0}$ equals -2. What follows about the value of the primal objective function at $\mathbf{x_0}$?

Implicitly, the question presumes that the primal problem has been written as a maximization problem and the dual has been written as a minimization problem.
By the Weak Duality Theorem, the value of the primal objective function is less than or equal to -2 at any feasible solution of the primal LPP. Equality holds if and only if $\mathbf{x_0}$ and $\mathbf{w_0}$ are optimal solutions of the primal and dual problems, respectively.

2. (16 points) Consider the LPP

Maximize $z = 4x_1 + 2x_2 + 3x_3$ subject to

$2x_1 + 3x_2 + x_3 \leq 12$
$x_1 + 4x_2 + 2x_3 \leq 10$
$3x_1 + x_2 + x_3 \leq 10$
$x_1, x_2, x_3 \geq 0$

which has optimal solution $x_1 = 2, x_2 = 0, x_3 = 4$. Use the principle of complementary slackness and the duality theorem to find an optimal solution to the dual problem.
Be sure to find the values of all slack and surplus variables in the optimal solutions to the primal and dual problems.
What value will the objective function of the dual problem have at this optimal solution?
Solution: In canonical form, the primal problem is
Maximize $z_P = 4x_1 + 2x_2 + 3x_3$ subject to

$2x_1 + 3x_2 + x_3 + x_4 = 12$ (w_1 constraint)
$x_1 + 4x_2 + 2x_3 + x_5 = 10$ (w_2 constraint)
$3x_1 + x_2 + x_3 + x_6 = 10$ (w_3 constraint)
$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

and the dual problem is

Minimize $z_D = 12w_1 + 10w_2 + 10w_3$ subject to

$2w_1 + w_2 + 3w_3 - w_4 = 4$ (x_1 constraint)
$3w_1 + 4w_2 + w_3 - w_5 = 2$ (x_2 constraint)
$w_1 + 2w_2 + w_3 - w_6 = 3$ (x_3 constraint)
$w_1, w_2, w_3, w_4, w_5, w_6 \geq 0$

Based on the above, the correspondence between primal and dual variables is as follows:

x_1	w_4
x_2	w_5
x_3	w_6
x_4	w_1
x_5	w_2
x_6	w_3

Given the optimal solution $x_1 = 2, x_2 = 0, x_3 = 4$, we compute the values of the primal slack variables by substitution into the equations in the canonical form of the primal LPP:

$x_1 = 2$	w_4
$x_2 = 0$	w_5
$x_3 = 4$	w_6
$x_4 = 2$	w_1
$x_5 = 0$	w_2
$x_6 = 0$	w_3

By complementary slackness, since x_1, x_3, and x_4 take nonzero values at the optimal solution of the primal LPP, the corresponding dual variables $w_4, w_6,$, and w_1 are equal to 0 at the optimal solution of the dual LPP.

$x_1 = 2$	$w_4 = 0$

$x_2 = 0$ $\quad$ w_5
$x_3 = 4$ $\quad$ $w_6 = 0$
$x_4 = 2$ $\quad$ $w_1 = 0$
$x_5 = 0$ $\quad$ w_2
$x_6 = 0$ $\quad$ w_3

Going back to the constraint equations for the dual LPP and substituting 0 for each of w_1, w_4, and w_6, this leaves us

$w_2 + 3w_3 = 4$
$4w_2 + w_3 - w_5 = 2$
$2w_2 + w_3 = 3$

Solving the first and third of these equations simultaneously yields $w_2 = 1, w_3 = 1$, and substitution into the second equation reveals that $w_5 = 3$. Our final result is

$x_1 = 2$ $\quad$ $w_4 = 0$
$x_2 = 0$ $\quad$ $w_5 = 3$
$x_3 = 4$ $\quad$ $w_6 = 0$
$x_4 = 2$ $\quad$ $w_1 = 0$
$x_5 = 0$ $\quad$ $w_2 = 1$
$x_6 = 0$ $\quad$ $w_3 = 1$

with optimal objective function values $z_D = z_P = 20$.

3. Consider an LPP whose final simplex tableau is

		4	6	3	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
3	x_3	$\frac{1}{20}$	0	1	$\frac{1}{2}$	$\frac{3}{10}$	0	$-\frac{1}{5}$	125
0	x_6	$\frac{13}{4}$	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	-1	425
6	x_2	$\frac{13}{20}$	1	0	$\frac{1}{2}$	$-\frac{1}{10}$	0	$\frac{2}{5}$	25
		$\frac{1}{20}$	0	0	$\frac{7}{2}$	$\frac{3}{10}$	0	$\frac{9}{5}$	525

(3a) (3 points) The optimal solution to the dual LPP is ...
Solution: $w_1 = \frac{3}{10}, w_2 = 0, w_3 = \frac{9}{5}$. These values can be read off directly from the objective row of the final simplex tableau of the primal problem; the values of the dual LPP at its optimal solution appear in the columns of the slack variables of the primal, in the objective row.

(3b) (7 points) Within what range can we change the coefficient c_2 without changing the optimal solution?

Solution: The first crucial point to note is that x_2 is a basic variable. To find a *lower bound* for Δc_2 consider all *positive* entries in the row labeled by x_2 (which happens to be row 3) which occur in columns of nonbasic variables; there are three of these. Compute the ratios of corresponding objective row entries with these positive entries of row 3; the ratios are $\frac{1}{13}, \frac{7}{1} = 7$, and $\frac{9}{2}$. Of these, $\frac{1}{13}$ is the closest to zero, so we conclude that $-\frac{1}{13} \leq \Delta c_2$.

To find an *upper bound* for Δc_2 consider all *negative* entries in the row labeled by x_2 which occur in columns of nonbasic variables; there is only one of these. Compute the ratio of the corresponding objective row entries with this negative entry of row 3; the ratio is $-\frac{3}{1} = -3$. We conclude that $\Delta c_2 \leq 3$.

Putting these together we have $-\frac{1}{13} \leq \Delta c_2 \leq 3$. Any change in c_2 which exceeds these bounds will result in a change in the optimal solution.

(3c) (8 points) Within what range can we change the value of b_1 so that x_3, x_6, and x_2 are still basic variables in the optimal solution?

Solution:

Problem 3, continued Here again is the final simplex tableau of the original LPP:

		4	6	3	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
3	x_3	$\frac{1}{20}$	0	1	$\frac{1}{2}$	$\frac{3}{10}$	0	$-\frac{1}{5}$	125
0	x_6	$\frac{13}{4}$	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	-1	425
6	x_2	$\frac{13}{20}$	1	0	$\frac{1}{2}$	$-\frac{1}{10}$	0	$\frac{2}{5}$	25
		$\frac{1}{20}$	0	0	$\frac{7}{2}$	$\frac{3}{10}$	0	$\frac{9}{5}$	525

(3c) (10 points) The original resource vector **b** was $[550 \quad 900 \quad 200]^T$. If change the value of b_1 to 1000, What will be the basic variables in the optimal solution to this new problem?
Be sure to show all your work, and explain briefly how you obtain your answer. You do not have to find the new optimal solution.

Solution:

4. (12 points) At a certain company there are three jobs to be assigned and three workers available. We wish to match each of these three workers to exactly one of the three jobs. A point scale has been set up rating the value of assigning a particular worker to a particular job, which is presented in the following table:

Worker	Job 1	Job 2	Job 3
Worker 1	6	9	5
Worker 2	5	4	4
Worker 3	7	8	3

Formulate, but do not solve, an integer linear programming problem whose solution would indicate how to assign each worker to a job in a way which maximizes the total number of value points. Be very clear what your decision variables are, and what each one represents.

Solution: Let $x_{ij} = \begin{cases} 1 \text{ if worker i is assigned job j} \\ 0 \text{ otherwise} \end{cases}$
for $i = 1, 2, 3$ and $j = 1, 2, 3$.
These x_{ij} are the decision variables, since the decision the company must make is which worker to assign to which job.
Let c_{ij} = the number of value points associated with assigning worker i to job j.

The IPP is to maximize the profit,
$z = \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij}x_{ij} = 6x_{11} + 9x_{12} + 5x_{13} + 5x_{21} + 4x_{22} + 4x_{23} + 7x_{31} + 8x_{32} + 3x_{33}$

subject to

$x_{11} + x_{12} + x_{13} = 1$ (worker 1 must be assigned exactly one job)
$x_{21} + x_{22} + x_{23} = 1$ (worker 2 must be assigned exactly one job)
$x_{31} + x_{32} + x_{33} = 1$ (worker 3 must be assigned exactly one job)
$x_{11} + x_{21} + x_{31} = 1$ (job 1 must be assigned to exactly one worker)
$x_{12} + x_{22} + x_{32} = 1$ (job 2 must be assigned to exactly one worker)
$x_{13} + x_{23} + x_{33} = 1$ (job 3 must be assigned to exactly one worker)

Although the definition of x_{ij} implies this, it is wise to indicate here that $x_{ij} \geq 0$, integers for $i = 1, 2, 3$ and $j = 1, 2, 3$.

5. (20 points) Apply Gomory's cutting plane method to the following integer programming problem. Perform at most two steps, i.e. create at most two cutting planes.

The final simplex tableau of the problem regarded as an LPP is

		1	2	0	0	
		x_1	x_2	x_3	x_4	
1	x_1	1	0	$\frac{3}{2}$	$-\frac{1}{4}$	$\frac{9}{4}$
2	x_2	0	1	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{2}$
		0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{29}{4}$

Solution: Since the current value of x_2 has the greater fractional part than the current value of x_1, we base the first cutting plane on the second row (though this is not strictly necessary). The first cutting plane constraint is obtained by writing the second constraint
$x_2 - \frac{1}{2}x_3 + \frac{1}{4}x_4 = \frac{5}{2}$,
rounding each coefficient down to its floor,
$x_2 - x_3 + 0x_4 \leq 2$,
and introducing a new slack variable to restore equality,
$x_2 - x_3 + x_5 = 2$.

...

After two cutting planes, have you obtained an integer solution, and if so, what is it?
If not, what would you do next?

Solution: The dual simplex method indicates that no feasible solution exists.

6. Suppose that the branch and bound method is applied to a certain *pure* IPP. After several steps of branching, the method reaches the following point, at which there are four nodes to consider:

At node 4 we have $z = 28\frac{2}{3}, x_1 = 4, x_2 = 5, x_3 = 3\frac{1}{3}$.

At node 5 we have $z = 26, x_1 = 2\frac{2}{3}, x_2 = 6, x_3 = 2$.

At node 6 we have $z = 27\frac{1}{3}, x_1 = 3\frac{1}{3}, x_2 = 5\frac{2}{3}, x_3 = 3$.

At node 7 we have an infeasible solution.

(6a) (8 points) What is the next step in the branch and bound method? Explain briefly (if the algorithm terminates, briefly explain why and say what the optimal solution is, and if the algorithm continues, briefly explain what is done at the next step).

Solution: Branch from node 4, since among all dangling nodes it represents the solution with greatest z value. This means that we will create two new tableaux, one by adding the constraint $x_3 \leq 3$, the other by adding the constraint $x_3 \geq 4$

(6b) (4 points) What if this problem is instead a *mixed* IPP, in which only x_2 and x_3 are required to be integers? In this case what is the next step in the branch and bound method?

Solution: Even though node 5 represents a solution in which x_2 and x_3 have integer values, we branch from node 4. It is possible that doing so will result in a solution in which $z > 26$ and x_2 and x_3 have integer values. If node 4 does not lead to such an outcome, we must still branch from node 6, for the same reason.