Math 354: H1 Summer 2009 **Solutions to Midterm #1**

1.(12 points)
a) Model the following problem as a LPP in standard form. Be sure to define all variables you use.

The administrator of a $300,000 trust fund set up by Mr. Smith's will must adhere to certain guidelines. The total amount of $300,000 need not be fully invested at any one time, but at least $50,000 must be invested. The money may be invested in three different types of securities: a utilities stock paying a 7% dividend, an electronics stock paying a 5% dividend, and a bond paying 3% interest. Suppose that the amount invested in the stocks cannot be more than a third of the total amount invested; the amount invested in the electronics stock cannot exceed $60,000; and the amount invested in the bond must be at least $80,000. What investment policy should be pursued to maximize the return?

Let x_1 = the dollar amount invested in the utilities stock
x_2 = the dollar amount invested in the electronics stock
x_3 = the dollar amount invested in the bond

We wish to maximize the return, $z = .07x_1 + .05x_2 + .03x_3$
Putting the constraints directly into mathematical form, not yet standard form, we have

$$\begin{aligned}
x_1 + x_2 + x_3 &\le 300{,}000 \\
x_1 + x_2 + x_3 &\ge 50{,}000 \\
x_1 + x_2 &\le (1/3)(x_1 + x_2 + x_3) \\
x_2 &\le 60{,}000 \\
x_3 &\ge 80{,}000 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}$$

Reworking into standard form produces

Maximize $z = .07x_1 + .05x_2 + .03x_3$
subject to

$$\begin{aligned}
x_1 + x_2 + x_3 &\le 300{,}000 \\
-x_1 - x_2 - x_3 &\le -50{,}000 \\
2x_1 + 2x_2 - x_3 &\le 0 \\
x_2 &\le 60{,}000 \\
-x_3 &\le -80{,}000 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}$$

b) Why is it necessary to use the two-phase method to solve this LPP? Why can't we just directly apply the simplex method?

After creating slack variables, making all slack variables basic and all nonslack variables nonbasic, i.e. attempting to consider $x_1 = x_2 = x_3 = 0$ as an initial solution, in fact produces an infeasible solution. This is the result of negative values on the right sides of the constraints as written in standard form. In this situation we must find a different initial solution, which requires the two phase method.

2. (12 points) Suppose the canonical form of a LPP is given by the constraint matrix A and the resource vector **b**, where

$$A=\begin{bmatrix} 3 & 1 & 1 & 0 & 2 \\ 1 & 3 & -1 & 2 & 4 \\ 2 & 4 & 3 & -1 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix}$$

Consider the solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 8 \\ -2 \end{bmatrix}$$

a) Is **x** a feasible solution? How can you tell?

x is not a feasible solution, since $x_5 = -2 < 0$. To be feasible, all entries of x must be nonnegative; this is implicit in the assumption that our LPP is in canonical form.

b) Is **x** a basic solution? How can you tell? Be specific.

A has 5 columns and 3 rows, therefore a basic solution has at least 2 entries equal to 0. Since the vector **x** is a solution (it is the case, shown by direct computation, that A**x**=**b**) and does have two entries equal to 0, it is a basic solution.

c) Does **x** represent an extreme point of the set of feasible solutions? How can you tell?

x is not a feasible solution, so the answer is no. If **x** were both basic and feasible, then by the correspondence between basic feasible solutions and extreme points of the set of feasible solutions, the answer would be yes.

3. (15 points) Solve the following linear programming problem by using the Extreme Point Theorem:

Maximize $z = 3x_1 + x_2$

subject to

$-3x_1 + x_2 \geq 6$

$3x_1 + 5x_2 \leq 15$

$x_1, x_2 \geq 0$

That is,

a) graph the set of feasible solutions
The feasible set is empty.

b) find the extreme points
The set of extreme points of the feasible set is also empty.

c) find the optimal solution(s).
There is no optimal solution, since there are no feasible solutions.

4. (20 points) Solve the following LPP using the simplex method (credit will not be given for any other method). Be sure to indicate in your answer both the basic feasible solution which is optimal, and the optimal value of the objective function.

Maximize $z = 3x_1 + 2x_2 + 4x_3$
subject to

$$x_1 + x_2 + 2x_3 \leq 4$$
$$2x_1 + 3x_3 \leq 5$$
$$x_1 + x_2 + 3x_3 \leq 7$$
$$x_1, x_2, x_3 \geq 0$$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	1	2	1	0	0	4
x_5	2	0	3	0	1	0	5
x_6	1	1	3	0	0	1	7
	-3	-2	-4	0	0	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	-1/3	1	0	1	-2/3	0	2/3
x_3	2/3	0	1	0	1/3	0	5/3
x_6	-1	1	0	0	-1	1	2
	-1/3	-2	0	0	4/3	0	20/3

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	-1/3	1	0	1	-2/3	0	2/3
x_3	2/3	0	1	0	1/3	0	5/3
x_6	-2/3	0	0	-1	-1/3	1	4/3
	-1	0	0	2	0	0	8

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	0	1	1/2	1	-1/2	0	3/2
x_1	1	0	3/2	0	1/2	0	5/2
x_6	0	0	1	-1	0	1	3
	0	0	3/2	2	1/2	0	21/2

The optimal solution is $x_1 = 5/2$, $x_2 = 3/2$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, $x_6 = 3$, with optimal objective function value 21/2.

5. (24 points)
a) Set up the initial tableau for phase 1 of the 2-phase method applied to the following LPP:

Maximize $z = 2x_1 + 3x_2 - 5x_3$ subject to

$$\begin{aligned} x_1 + x_2 + 4x_3 &\geq 3 \\ x_1 - 2x_2 + 7x_3 &\leq 10 \\ -2x_1 + 4x_2 - x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

First write out the auxiliary problem:

Minimize $z' = y_1 + y_2$ subject to

$$\begin{aligned} x_1 + x_2 + 4x_3 - x_4 + y_1 &= 3 \\ x_1 - 2x_2 + 7x_3 + x_5 &= 10 \\ -2x_1 + 4x_2 - x_3 - x_6 + y_2 &= 1 \\ x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2 &\geq 0 \end{aligned}$$

Then write the initial tableau for the auxiliary problem (i.e. for phase 1):

First arranging all coefficients into a tableau produces

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
y_1	1	1	4	-1	0	0	1	0	3
x_5	1	-2	7	0	1	0	0	0	10
y_2	-2	4	-1	0	0	-1	0	1	1
	0	0	0	0	0	0	1	1	0

This is not yet the initial tableau for the auxiliary problem, since y_1 and y_2 are basic variables, so the 1s appearing below them in the objective row must be eliminated through row operations:

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	
y_1	1	1	4	-1	0	0	1	0	3
x_5	1	-2	7	0	1	0	0	0	10
y_2	-2	4	-1	0	0	-1	0	1	1
	1	-5	-3	1	0	1	0	0	-4

Now phase 1 can begin.

b) At some point during the first phase of the 2-phase method applied to a certain LPP (not related to any other on this exam), the following tableau was obtained:

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
y_1	-2	1	2	0	3	1	-1	2
x_4	1	-3	0	1	-3	0	3	1
	1	2	4	0	5	0	1	-2

What can you infer about this LPP, and how can you tell from looking at the current tableau?

We can infer that the original LPP has no feasible solutions. The reasoning is as follows. Since there are no negative entries in the objective row, phase 1 has ended, but the artificial variable y_1 does not equal 0. It follows that there are no feasible solutions of the original LPP.

c) The following tableau was obtained at the end of the first phase of the 2-phase method applied to a certain LPP (not related to any other on this exam).

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
x_3	-2	0	1	3	3	0	2	4
x_2	1	1	0	-1	-3	2	-1	1
	0	0	0	0	0	1	1	0

The original problem was to maximize $z = 2x_1 - x_2 + 3x_3$ subject to certain constraints.

Now write the initial tableau for phase 2 (you might want to do this in two steps):

First discard the columns labeled by artificial variables, and replace the objective row with that of the original LPP:

	x_1	x_2	x_3	x_4	x_5	
x_3	-2	0	1	3	3	4
x_2	1	1	0	-1	-3	1
	-2	1	-3	0	0	0

Next, since x2 and x3 are basic variables, we must perform row operations to eliminate the nonzero entries below them in the objective row. The resulting tableau, with which we begin phase 2, is

	x_1	x_2	x_3	x_4	x_5	
x_3	-2	0	1	3	3	4
x_2	1	1	0	-1	-3	1
	-9	0	0	10	12	11

What basic feasible solution does the initial tableau for phase 2 represent?

$x_1 = 0$, $x_2 = 1$, $x_3 = 4$, $x_4 = 0$, $x_5 = 0$, with objective function value $z = 11$.
Notice that because we begin phase 2 with a solution in which not all decision variables (nonslack variables) are 0, the value of z is already greater than 0.**6.** (12 points) Consider the following LPP:

6. Maximize $z_P = 5x_1 - 3x_2$
subject to

$2x_1 + x_2 \leq 15$
$-x_1 + 3x_2 \geq 3$
$x_1 + 3x_2 = 10$
$2x_1 - x_2 \geq 0$
x_1 unrestricted, $x_2 \geq 0$

Write the dual LPP:

First rewrite the given LPP as
Maximize $z_P = 5x_1 - 3x_2$
subject to

$2x_1 + x_2 \leq 15$
$x_1 - 3x_2 \leq -3$
$x_1 + 3x_2 = 10$
$-2x_1 + x_2 \leq 0$
x_1 unrestricted, $x_2 \geq 0$

The dual LPP is

Minimize $z_D = 15w_1 - 3w_2 + 10w_3$
subject to

$2w_1 + w_2 + w_3 - 2w_4 = 5$
$w_1 - 3w_2 + 3w_3 + w_4 \geq -3$
$w_1, w_2, w_4 \geq 0$, w_3 unrestricted

7. (5 points) If you reached the following tableau while applying the simplex method to a LPP, and you made the mistake of making x_3 the departing variable, what would go wrong at the next step?

	x_1	x_2	x_3	x_4	x_5	
x_3	0	1	1	3	3	6
x_1	1	2	0	-1	-3	4
	0	-4	0	2	1	10

Pivoting would produce an infeasible solution in the next step.