OPERATIONS IN ÉTALE AND MOTIVIC COHOMOLOGY

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Abstract. We classify all étale cohomology operations on $H^n_{\mathrm{et}}(-,\mu_\ell^{\otimes i})$, showing that they were all constructed by Epstein. We also construct operations P^a on the mod-ℓ motivic cohomology groups $H^{p,q}$, differing from Voevodsky's operations; we use them to classify all motivic cohomology operations on $H^{p,1}$ and $H^{1,q}$ and suggest a general classification.

In the last decade, several papers have given constructions of cohomology operations on motivic and étale cohomology, following the earlier work of Jardine [J], Kriz-May [KM] and Voevodsky [V2, V1]: see [BJ, BJ1, Jo, M1, V3, V4]. The goal of this paper is to provide, for each n and i, a classification of all such operations on the étale groups $H^n_{\mathrm{et}}(-,\mu_\ell^{\otimes i})$ and the motivic groups $H^{n,i}(-,\mathbb{F}_\ell)$, similar to Cartan's classification of operations on singular cohomology $H^n_{\mathrm{top}}(-,\mathbb{F}_\ell)$ in [C]. We succeed for étale operations and partially succeed for motivic operations.

We work over a fixed field k and fix a prime ℓ with $1/\ell \in k$. By definition, an (unstable) étale cohomology operation on $H^n_{\mathrm{et}}(-,\mu_\ell^{\otimes i})$ over k is a natural transformation $H^n_{\mathrm{et}}(-,\mu_\ell^{\otimes i}) \to H^p_{\mathrm{et}}(-,\mu_\ell^{\otimes q})$ of set-valued functors from the category of (smooth) simplicial schemes over k (for some p and q). Similarly, an (unstable) motivic cohomology operation on $H^{n,i}$ over k is a natural transformation $H^{n,i} \to H^{p,q}$ of functors defined on this category, where $H^{p,q}(X)$ denotes the Nisnevich cohomology $H^p_{\mathrm{nis}}(X,\mathbb{F}_\ell(q))$, and the cochain complex $\mathbb{F}_\ell(q)$ is defined in [V2] or [MVW]. Fixing k, n and i, the set of all unstable cohomology operations forms a ring; the product of θ_1 and θ_2 is the operation $x \mapsto \theta_1(x)\cdot\theta_2(x)$.

Our classification theorems describe the ring of all operations in terms of certain specific operations. Thus we begin with their construction, in Sections 1, 4 and 6.

In étale cohomology with constant coefficients, operations P^a were constructed by Epstein [E] and used by Raynaud in [R]. A second construction is given by Peter May in [M]. As both are based on Steenrod's original construction in [SE], they agree. The upshot is that Cartan's ring $H^*_{\mathrm{top}}(K_n)$ of operations on $H^n_{\mathrm{top}}(-,\mathbb{F}_\ell)$ embeds into the ring of all étale operations on $H^n_{\mathrm{et}}(-,\mu_\ell^{\otimes i})$; we refer the reader to Definition 0.1 below for a precise description of Cartan's ring.

For étale cohomology with twisted coefficients $\mu_\ell^{\otimes i}$, Epstein's approach [E] and May's approach [M] give apparently different constructions of étale operations P^a. We will show in Corollary 4.7 that the two constructions give the same operations.

Epstein's construction is more easily accessible to algebraic geometers, because it uses equivariant sheaf cohomology, and is an application of the method described in his 1966 paper [E]. After stating Epstein's result in Theorem 1.3, we indicate the key points in his construction that we will need to compare with May's construction.

The classification of étale cohomology operations is given in Sections 2 and 3. Theorem 3.5 gives the general result: the ring of all (unstable) étale operations

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on $H^n_{\rm et}(-,\mu_\ell^{\otimes i})$ over k is the tensor product $H^*_{\rm et}(k(\zeta),\mathbb{F}_\ell)\otimes H^*_{\rm top}(K_n)$, where ζ is a primitive ℓ^{th} root of unity. Thus all (unstable) étale operations on $H^n_{\rm et}(-,\mu_\ell^{\otimes i})$ over k are $H^*_{\rm et}(k(\zeta),\mathbb{F}_\ell)$-linear combinations of monomials in the operations P^I. Our proof starts with the special case in which $\zeta\in k$. In this case, it is a result of Breen and Jardine that the graded ring of all étale operations on $H^n_{\rm et}(-,\mu_\ell^{\otimes i})$ is $H^*_{\rm et}(k,\mu_\ell^{\otimes *})\otimes H^*_{\rm top}(K_n)$.

In Section 4, we present May's construction, using the notion of a suitable pair $(\mathcal{K},\theta)$ (see 4.2), and show in Corollary 4.7 that the étale operations P^a coincide. Our construction of motivic cohomology operations will use May's construction. The brief Section 5 relates the discussion of Section 4 to the operad-based approach of Hinich and Schechtman [HS].

In Section 6, we use the Norm Residue Theorem to construct motivic operations P^a (see 6.5). We show they are compatible with the étale operations and stable under simplicial suspension, and we verify the usual properties in Section 7. The operation P^0 is the Frobenius $H^{n,i}\to H^{n,i\ell}$ on motivic cohomology, induced by the ℓ^{th} power map $\mathbb{F}_\ell(i)\to\mathbb{F}_\ell(i\ell)$; see Proposition 8.4. One new result concerning Voevodsky's operations is that for $n>i$ and $x\in H^{2n,i}$ we have $P^n_V(x)=[\zeta^{\otimes(\ell-1)}]^{(n-i)}\cup x^\ell$, where $[\zeta^{\otimes(\ell-1)}]$ is the canonical element of $H^{0,\ell-1}(k)$ (see Corollary 8.10). This extends Lemma 9.8 of [V1], which states that $P^n(x)=x^\ell$ for $x\in H^{2n,n}(X)$.

The classification of motivic cohomology operations is complicated by the presence of more operations than those constructed by Voevodsky or via Steenrod-Epstein methods. One example is that an ℓ-torsion element t in the Brauer group of k gives an operation $H^{1,2}\to H^{3,3}$ by

$$H^{1,2}(X)\cong H^1_{\rm et}(X,\mu_\ell^{\otimes 2})\xrightarrow{\cup t}H^3_{\rm et}(X,\mu_\ell^{\otimes 3})\cong H^{3,3}(X).$$

Also unexpectedly, we may also use t and the Bockstein β to get an operation $H^{1,2}(X)\to H^{4,3}(X)$ (see Example 11.5 below). When k contains a primitive ℓ^{th} root of unity ζ, we also have an interesting operation $H^{1,2}(X)\to H^{2,1}(X)=\mathrm{Pic}(X)/\ell$: divide by the Bott element $[\zeta]\in H^{0,1}(k)$ and then apply the Bockstein; see Proposition 11.2.

In Section 10, we determine the ring of all motivic cohomology operations on $H^{n,1}$. If $\ell\neq 2$, it is the algebra $H^{*,*}(k)\otimes H^*_{\rm top}(K_n)$, where $H^{*,*}(k)$ is the motivic cohomology of k and $H^*_{\rm top}(K_n)$ is Cartan's ring, described in Definition 0.1 below. Many of these operations fail to be stable operations because they do not fit into a sequence of operations compatible with the motivic t-suspension $X\mapsto S^1_tX$; for this reason, we call them *unstable*.

In Section 11, we determine the ring of (unstable) cohomology operations on $H^{1,i}$. When k contains the ℓ^{th} roots of unity, this is the graded polynomial ring over $H^{*,*}(k)$ on operations $\gamma: H^{1,i}(X)\cong H^{1,1}(X)$ and its Bockstein, where γ is given by the Norm Residue Theorem 6.2. For general fields, it is the Galois-invariant subring. The operations on $H^{1,2}$ referred to above arise in this way.

Finally Section 12 contains a conjecture about what the general classification might be for $H^{n,i}$ when $n,i>1$.

Since it is the topological prototype of our classification theorem, we conclude this introduction with a description of the ring of all singular cohomology operations on $H^n_{\rm top}(-,\mathbb{F}_\ell)$. Serre observed in [S50, 28.1] (cf. [EM, p. 513]) that the ring of operations from $H^n_{\rm top}(-,\mathbb{F}_\ell)$ to $H^*_{\rm top}(-,\mathbb{F}_\ell)$ is isomorphic to the cohomology $H^*_{\rm top}(K_n)$

of the Eilenberg-Mac Lane space $K_n = K(\mathbb{F}_\ell, n)$; the structure of this ring was determined by Serre and Cartan in [S50] [C] [C1]. The following description is taken from [McC, 6.19].

Definition 0.1. For $\ell > 2$, let Λ_n denote the free graded-commutative $\mathbb{F}_\ell$-algebra generated by the elements $P^I(\iota_n)$, where $I = (\epsilon_0, s_1, \epsilon_1, ..., s_k, \epsilon_k)$ is an admissible sequence satisfying either $e(I) < n$, or else $e(I) = n$ and $\epsilon_0 = 1$.

Here the *excess* of I is defined to be $e(I) = 2\sum(s_i - \ell s_{i+1} - \epsilon_i) + \sum_{i=0}^k \epsilon_i$, where $s_i = 0$ for $i > k$, and I is *admissible* if $s_i \geq \ell s_{i+1} + \epsilon_i$ for all $i < k$.

When $\ell = 2$, Λ_n denotes the free graded-commutative $\mathbb{F}_2$-algebra generated by the elements $Sq^I(\iota_n)$, with $I = (s_1, ..., s_k)$ admissible ($s_i \geq 2s_{i+1}$) and $e(I) < n$, where the excess is $e(I) = \sum(s_i - 2s_{i+1}) = s_1 - \sum_{i>1} s_i$.

We will write $H^*_{\mathrm{top}}(K_n)$ for Λ_n because of the following result.

Theorem 0.2 (Cartan–Serre)**.** *The ring $H^*_{top}(K_n, \mathbb{F}_\ell)$ of cohomology operations from $H^n_{top}(-, \mathbb{F}_\ell)$ to $H^*_{top}(-, \mathbb{F}_\ell)$ is isomorphic to Λ_n.*

For example, every operation on $H^2_{\mathrm{top}}(-, \mathbb{F}_\ell)$ is a polynomial in id, β, the $P^I\beta$ and (if $\ell \neq 2$) the $\beta P^I \beta$ (where $P^I = P^{\ell^k} \cdots P^\ell P^1$). This is because the only admissible sequences with excess < 2 are 0, (1) and $(0, \ell^k, 0, \dots, \ell, 0, 1, 1)$.

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1. Epstein's étale construction

Cohomology operations in étale cohomology were constructed by David Epstein long ago in the 1966 paper [E], as a special case of operations constructed in an axiomatic framework; see Sections 10 and 11.1 of *loc. cit.* Epstein's construction was made explicit by Michèle Raynaud [R, 4.4] for étale cohomology with constant coefficients. Alternative constructions were later given by L. Breen [Br, III.4] and J.F. Jardine [J, 1.4], [J1, §2] and [M1].

In Epstein's approach, one starts with an $\mathbb{F}_\ell$-linear tensor abelian category $\mathcal{S}h$ (such as sheaves of $\mathbb{F}_\ell$-modules on a site), a left exact functor $H^0(X, -)$ (global sections over X) and a commutative associative ring object $\mathcal{A}$ of $\mathcal{S}h$.

In this section, we consider the situation in which $\mathcal{S}h$ is the category of étale sheaves of $\mathbb{F}_\ell$-modules on the big étale site of simplicial schemes $X_\bullet$ over a base S, which we may assume is Spec(k). The ring object $\mathcal{A}$ will be the graded étale sheaf $\oplus_{i=0}^\infty \mu_\ell^{\otimes i}$.

Definition 1.1. If $X_\bullet$ is a simplicial scheme over S, the étale site $Et(X_\bullet)$ is the category whose objects are pairs $(n, U \to X_n)$ with $U \to X_n$ étale. A morphism to (m, U') is an ordinal map $[m] \to [n]$, together with a map $U \to U'$ forming a commutative square with $X_n \to X_m$. A *covering* of (n, U) is a family of maps $U_i \to U$ over X_n so that the $U_i \to U$ are an étale cover of U.

The étale site of a simplicial scheme $X_\bullet$ first arose in [D, 5.1.8]; our description is based upon the definition of the étale site $Et(X_\bullet)$ in [F, 1.4].

Recall that if $X = X_\bullet$ is a simplicial scheme, and $\mathcal{F}$ is a sheaf of $\mathbb{F}_\ell$-modules, then the functor $\Gamma(\mathcal{F}) = \mathcal{F}(X)$ is defined as the equalizer of $\mathcal{F}(X_0) \rightrightarrows \mathcal{F}(X_1)$, and the cohomology functors $H^i_{\rm et}(X,\mathcal{F})$ are defined as its derived functors; see [D, 5.2.2] or [F, 2.3]. If $X_\bullet$ is a constant simplicial scheme, $H^i_{\rm et}(X,\mathcal{F})$ is the usual étale cohomology of X.

The derived functors of $H^0(X,-) : \mathcal{S}h \to \mathbb{F}_\ell$-mod are just the usual étale cohomology groups, because the usual Godement resolution of a sheaf $\mathcal{F}$ [Milne, p. 90] is a flasque resolution by sheaves of $\mathbb{F}_\ell$-modules which are injective objects of $\mathcal{S}h$.

The étale Bockstein $\beta : H^n_{\rm et}(X,\mu_\ell^{\otimes i}) \to H^{n+1}_{\rm et}(X,\mu_\ell^{\otimes i})$ is defined as the connecting map in the cohomology sequence for $0 \to \mu_\ell^{\otimes i} \to \mu_{\ell^2}^{\otimes i} \to \mu_\ell^{\otimes i} \to 0$. (This sequence is the tensor product of the sequence $0 \to \mathbb{Z}/\ell \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 0$ with $\mu_{\ell^2}^{\otimes i}$.) By definition, β is natural in X, but can depend on the choice of i; see Remark 3.3.1 below. If $\mathcal{B} = \oplus_{i=0}^\infty \mu_{\ell^2}^{\otimes i}$, we may also regard β as the connecting map in the cohomology sequence for $0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \to \mathcal{A} \to 0$.

Lemma 1.2. *The étale Bockstein is a derivation. That is, if $u \in H^n_{et}(X,\mu_\ell^{\otimes i})$ and $v \in H^m_{et}(X,\mu_\ell^{\otimes j})$ then $\beta(u \cup v) = \beta(u) \cup v + (-1)^n u \cup \beta(v)$.*

Proof. (Folklore) Choose a flasque Godement-style resolution $\mathcal{B} \to \mathcal{I}$ whose stalks are free (=injective) $\mathbb{Z}/\ell^2$-modules, and write $\bar{\mathcal{I}}$ for $\mathcal{I}/\ell\mathcal{I}$, so that $\mathcal{A} \to \bar{\mathcal{I}}$ is also a flasque resolution. Lifting cycles $\bar{u}$ and $\bar{v}$ representing u and v to chains $u' \in \mathcal{I}^n(X)$ and $v' \in \mathcal{I}^m(X)$, $\beta(u)$ and $\beta(v)$ are represented by u'' and v'', defined by $\delta(u') = i(u'')$ and $\delta(v') = i(v'')$.

The cup product $u \cup v$ is represented by the image of $\bar{u} \otimes \bar{v}$ under the map $m : \bar{\mathcal{I}} \otimes \bar{\mathcal{I}} \to \bar{\mathcal{I}}$ resolving $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$; see [D-4.5, 1.2.2]. Since the coboundary on $\mathcal{I} \otimes \mathcal{I}$ satisfies

$$\delta(u' \otimes v') = i(u'') \otimes v' + (-1)^n u' \otimes i(v'') = i(u'' \otimes \bar{v}) + (-1)^n i(\bar{u} \otimes v'')$$

it follows that $\beta(u \cup v)$ is represented by $m(u'' \otimes \bar{v}) + (-1)^n m(\bar{u} \otimes v'')$, i.e., by $\beta(u) \cup v + (-1)^n u \cup \beta(v)$. $\square$

Remark 1.2.1. The same proof works in the motivic setting to show that the motivic Bockstein is also a derivation, a fact stated in [V1, (8.1)].

Epstein defines an operation $P^0 : H^n_{\rm et}(X,\mu_\ell^{\otimes i}) \to H^n_{\rm et}(X,\mu_\ell^{\otimes i\ell})$ in [E, 7.1], and shows in [E, 7.3] that P^0 is the canonical Frobenius isomorphism induced by $\mu_\ell^{\otimes i} \cong \mu_\ell^{\otimes i\ell}$. (Our pairing $H^0_{\rm et}(X,\mu_\ell^{\otimes i}) \otimes H^0_{\rm et}(X,\mu_\ell^{\otimes j}) \to H^0_{\rm et}(X,\mu_\ell^{\otimes i+j})$ is the pairing $SA \otimes TB \to U(A \otimes B)$ in [E, (3.2.1)].) Epstein also defines an operation

$$Q^0 : H^n_{\rm et}(X,\mu_\ell^{\otimes i}) \to H^{n+1}_{\rm et}(X,\mu_\ell^{\otimes i\ell});$$

we'll see in Proposition 3.3 below that $Q^0 = \beta P^0$ and $Q^a = \beta P^a$, where β is the Bockstein on $H^*_{\rm et}(X,\mu_\ell^{\otimes i\ell})$. With this dictionary, Epstein's theorem specializes to yield:

Theorem 1.3. *For each odd prime ℓ, there are additive cohomology operations*

$$P^a : H^n_{et}(X,\mu_\ell^{\otimes i}) \to H^{n+2a(\ell-1)}_{et}(X,\mu_\ell^{\otimes i\ell}), a \geq 0,$$

natural in X, satisfying the usual relations: $P^a x = x^\ell$ if $n = 2a$, $P^a x = 0$ if $n < 2a$, the Cartan relation $P^a(xy) = \sum P^i(x)P^j(y)$ and Adem relations for both P^aP^b $(a < b\ell)$ and $P^a\beta P^b$ $(a \leq b\ell)$.

When $\ell = 2$, there are Steenrod operations $Sq^a : H^n_{et}(X, \mu_2^{\otimes i}) \to H^{n+a}_{et}(X, \mu_2^{\otimes 2i})$, or $H^n_{et}(X, \mathbb{Z}/2) \to H^{n+a}_{et}(X, \mathbb{Z}/2)$, satisfying the usual relations.

Proof. The existence and basic properties is given in Chapter 7 of [E]; additivity is 6.7. The Adem relations are established in [E, 9.7–8], using the dictionary that $Q^b = \beta P^b$ and $P^a \beta P^b = P^a Q^b$. Naturality follows from [E, 11.1(8)]. $\square$

Remark 1.3.1. (i) There are canonical isomorphisms $\mu_\ell \cong \mu_\ell^{\otimes \ell}$ and $\mu_\ell^{\otimes i} \cong \mu_\ell^{\otimes i\ell}$. We have emphasized the difference in weight because of our application to motivic operations. (ii) Epstein also constructs operations Q^a; they equal βP^a if we use the sheaf $\mathcal{A} = \oplus_{i=0}^\infty \mu_\ell^{\otimes i}$, but can differ if we use the smaller sheaf $A = \oplus_{i=0}^d \mu_\ell^{\otimes i}$, where $d = [k(\zeta_\ell) : k]$; see Propositions 2.4 and 3.3. (iii) If Z is a closed simplicial subscheme of X (see [D] or [F]), we get cohomology operations P^a on the relative groups $H^n_{\mathrm{et}}(X, Z; \mu_\ell^{\otimes i})$, natural in the pair (X, Z), by replacing $H^0(X, -)$ by the left exact functor $H^0_{\mathrm{et}}(X, Z; -)$.

In order to compare to May's construction, and to classify operations, we will need a rephrasing of one of the key results from [E], using the language of equivariant sheaf cohomology.

Definition 1.4. If G is a finite group, we write $\mathcal{S}h_G$ for the category of G-equivariant objects of $\mathcal{S}h$, i.e., objects B equipped with a homomorphism $G \to \mathrm{End}(B)$. If B is in $\mathcal{S}h_G$ then $H^0(X, B)$ is a G-module, and we define the left exact functor $H^0_G(X, -)$ on the category $\mathcal{S}h_G$ by the formula $H^0_G(X, B) = H^0(X, B)^G$. We write $H^*_G(X, -)$ for the derived functors of $H^0_G(X, -)$.

We will use the following result in Section 6.

Theorem 1.5. *Let A be a bounded below cochain complex of objects of $\mathcal{S}h$, on which a finite group G acts trivially. Then there is a natural isomorphism*

$$H^*(G, \mathbb{F}_\ell) \otimes H^*(X, A) \xrightarrow{\simeq} H^*_G(X, A).$$

If A is a sheaf of dg commutative algebras, this is an algebra isomorphism.

Proof. (See [E, 4.4.4].) Fix an injective resolution $A \xrightarrow{\sim} I^*$ in $\mathcal{S}h$. Choosing a resolution $F_* \to \mathbb{F}_\ell$ by finitely generated free $\mathbb{F}_\ell[G]$-modules, set $F^* = \mathrm{Hom}(F_*, \mathbb{F}_\ell)$, so $H^*(G, \mathbb{F}_\ell)$ is the cohomology of $(F^*)^G$. Since G acts trivially on A, we have quasi-isomorphisms of complexes in $\mathcal{S}h_G$: $A \xrightarrow{\sim} I^* = \mathbb{F}_\ell \otimes I^* \xrightarrow{\sim} \mathrm{Tot}(F^* \otimes I^*)$. Since each F^n is a free $\mathbb{F}_\ell[G]$-module of finite rank, $F^n \otimes I^q \cong \mathrm{Hom}(F_n, I^q)$ is injective in $\mathcal{S}h_G$, and $A \to \mathrm{Tot}(F^* \otimes I^*)$ is an injective resolution in $\mathcal{S}h_G$. Hence $H^*_G(X, A)$ is the cohomology of the total complex of

$$H^0_G(X, F^* \otimes I^*) = (F^*)^G \otimes I^*(X).$$

The Künneth formula tells us that $H^*((F^*)^G \otimes I^*(X))$ is the tensor product of the cohomology of $(F^*)^G$ and $I^*(X)$, *i.e.,* of $H^*(G, \mathbb{F}_\ell)$ and $H^*(X, A)$.

We omit the standard proof that a commutative associative product on A induces an algebra structure on $H^*(X, A)$ and $H^*_G(X, A)$, and that the isomorphism of Theorem 1.5 commutes with products. $\square$

Recall that for any sheaf (or complex) A, the symmetric group S_n acts on $A^{\otimes n}$ by permuting factors (with the usual sign change for tensor products of complexes). If $A \xrightarrow{\sim} B$ is a quasi-isomorphism, then so is $A^{\otimes n} \xrightarrow{\sim} B^{\otimes n}$, by the following useful lemma.

Lemma 1.6. *If C, C' and D are bounded below cochain complexes of sheaves of $\mathbb{F}_\ell$-modules, and $f : C \xrightarrow{\sim} C'$ is a quasi-isomorphism, then so is $f \otimes 1 : C \otimes D \longrightarrow C' \otimes D$.*

Proof. Let K denote the cone of f; it is acyclic. Since $\otimes$ is an exact functor, every sheaf of $\mathbb{F}_\ell$-modules is flat, and $K \otimes D$ is acyclic by the Künneth Formula [WH, 3.6.1]. Since $C \otimes D \longrightarrow C' \otimes D \longrightarrow K \otimes D$ is a distinguished triangle, the result follows. $\square$

Remark 1.6.1. If A is a sheaf of commutative $\mathbb{F}_\ell$-algebras, and $A \xrightarrow{\sim} I$ is an injective resolution, the choice of a lift $I^{\otimes 2} \to I$ of $A^{\otimes 2} \to A$ makes I a sheaf of homotopy commutative, homotopy unital, and homotopy associative dg algebras, as $A^{\otimes 2} \xrightarrow{\sim} I^{\otimes 2}$. Such a lift exists, and is unique up to chain homotopy, by the comparison theorem [WH, 2.3.7, 10.4.7].

Similar remarks hold when A is a sheaf of bounded below, homotopy associative and commutative dg algebras, using the total complex I of a Cartan-Eilenberg resolution; see [WH, 5.7.9 and Ex. 5.7.2].

Let π be a Sylow ℓ-subgroup of S_ℓ. Choosing an injective resolution $A^{\otimes \ell} \to J^*$ in $\mathcal{S}h_\pi$, the comparison theorem lifts the equivariant quasi-isomorphism $A^{\otimes \ell} \to I^{\otimes \ell}$ to an equivariant map $I^{\otimes \ell} \to J^*$, unique up to chain homotopy.

Since $H^*(X, A)$ is the cohomology of $I(X)$, we can represent any element of $H^n(X, A)$ by an n-cocycle $u \in I^n(X)$. The $n\ell$-cocycle $u \otimes \cdots \otimes u$ of $I(X)^{\otimes \ell}$ is π-invariant, because the generator of π acts as multiplication by $(-1)^{n(\ell-1)}$, which is the identity on any $\mathbb{F}_\ell$-module. Its image Pu in $J^{n\ell}(X)$ is also π-invariant. Epstein shows in [E, 5.1.3] that $P(u + dv) = Pu + dw$ for $v \in I^{n-1}(X)$ and $w \in J^{n\ell-1}(X)$, so the cohomology class of Pu is independent of the choice of cocycle u.

Definition 1.7. The *reduced power map* is defined to be the map on cohomology associated to $u \mapsto u \otimes \cdots \otimes u$:

$$P : H^n(X, A) \to H^{n\ell}_\pi(X, A^{\otimes \ell}).$$

Now let π denote the cyclic group of order ℓ. We will write $W_* \to \mathbb{F}_\ell$ for the standard periodic $\mathbb{F}_\ell[\pi]$-resolution [WH, 6.2.1], with generator e_k of $W_k \cong \mathbb{F}_\ell[\pi]$, and set $W^* = \mathrm{Hom}(W_*, \mathbb{F}_\ell)$; thus $H^*(\pi, \mathbb{F}_\ell)$ is the cohomology of $(W^*)^\pi$.

Now suppose that there is a π-equivariant map $A^{\otimes \ell} \xrightarrow{m} B$, and that π acts trivially on B. (When A is a commutative ring, multiplication $A^{\otimes \ell} \to A$ is a π-equivariant map.) We write m_* for the induced map $H^*_\pi(X, A^{\otimes \ell}) \to H^*_\pi(X, B)$. By Theorem 1.5, $m_* P(u) \in H^*_\pi(X, B)$ has an expansion $\sum w_k \otimes D_k(u)$, where $w_k \in H^k(\pi, \mathbb{F}_\ell)$ are the (dual) basis elements of [SE, V.5.2]: if $\ell > 2$ then $w_0 = 1$, $w_2 = \beta w_1$, $w_{2i} = w_2^i$ and $w_{2i+1} = w_1 w_2^i$. If $\ell > 2$ and $n \geq 2a$, Epstein defines

$$P^a : H^n(X, A) \to H^{n+2a(\ell-1)}(X, B), \quad P^a u = (-1)^a \nu_n D_{(n-2a)(\ell-1)}(u), \tag{1.8}$$

where

$$\nu_n = (-1)^r \left(\frac{\ell - 1}{2}\right)!^{-n} \text{ and } r = \frac{(\ell - 1)(n^2 + n)}{4}.$$

(See [E, 7.1], [SE, VII.6.1] and [SE-err].) If $n < 2a$ then Epstein defines $P^a = 0$.

When $\ell = 2$, Epstein defines operations Sq^i by: $Sq^i(u) = D_{n-i}(u)$ for $n \geq i$, and $Sq^i(u) = 0$ for $n < i$.

Remark 1.8.1. Epstein also defines operations $Q^a = (-1)^{a+1}\nu_n D_{(n-2a)(\ell-1)-1}(u)$ in this setting, with $Q^a = 0$ when $n \le 2a$, and establishes Adem relations for them as well.

Of course, Epstein's construction mimicks Steenrod's construction of D_k, P^a and Q^a (see [SE], VII.3.2 and VII.6.1). In Steenrod's setting one can lift to integral cochains; with this assumption, Steenrod proves that $\beta D_{2k} = -D_{2k+1}$ and hence that $\beta P^a = Q^a$; see [SE, VII.4.6] and [SE-err]. As we mentioned in Remark 1.3.1, the formula $Q^a = \beta P^a$ may not hold when the sheaf A fails to distinguish between $\mu_\ell^{\otimes i}$ and $\mu_\ell^{\otimes i\ell}$. (See Propositions 2.4 and 3.3, and Theorem 8.11.)

Recall that the simplicial suspension SX of a simplicial scheme X is again a simplicial scheme. There is a canonical isomorphism $\widetilde{H}^n_{\rm et}(X,\mu_\ell^{\otimes i}) \xrightarrow{\cong} \widetilde{H}^{n+1}_{\rm et}(SX,\mu_\ell^{\otimes i})$.

Proposition 1.9. *The operations P^a are simplicially stable in the sense that they commute with simplicial suspension: there are commutative diagrams for all X, and all n and i, with $N = n + 2a(\ell-1)$:*

$$\begin{array}{ccc}
\widetilde{H}^n_{et}(X,\mu_\ell^{\otimes i}) & \xrightarrow{P^a} & \widetilde{H}^N_{et}(X,\mu_\ell^{\otimes i\ell}) \\
\cong\downarrow & & \cong\downarrow \\
\widetilde{H}^{n+1}_{et}(SX,\mu_\ell^{\otimes i}) & \xrightarrow{P^a} & \widetilde{H}^{N+1}_{et}(SX,\mu_\ell^{\otimes i\ell}).
\end{array}$$

Proof. The proofs of Lemmas 1.2 and 2.1 of [SE] go through, using simplicial homotopy invariance ($H^*_{\rm et}(X) \cong H^*_{\rm et}(X\times\Delta^1)$) of étale cohomology and excision. □

If $f : X \to Y$ is a finite map, then $f_* : \mathcal{S}h(X) \to \mathcal{S}h(Y)$ is an exact functor. It follows that a π-equivariant map $A^{\otimes\ell} \xrightarrow{m} B$ of étale sheaves on X induces a π-equivariant map $f_*m : (f_*A)^{\otimes\ell} \cong f_*(A^{\otimes\ell}) \to f_*B$, and hence operations P^a, Q^a from $H^*(Y, f_*A)$ to $H^*(Y, f_*B)$.

Proposition 1.10. *Suppose that $f : X \to Y$ is a finite map, and $A^{\otimes\ell} \xrightarrow{m} B$ is a π-equivariant map of sheaves on X. Then Epstein's P^a and Q^a commute with the isomorphism $f_* : H^*(Y, f_*A) \xrightarrow{\simeq} H^n(X, A)$.*

Proof. Let I and J be the injective resolutions of A and $A^{\otimes\ell}$ in the construction 1.7 of the power map P for A. Then f_*I and f_*J are injective resolutions of f_*A and $f_*A^{\otimes\ell}$, and if u is a cocycle in $I^n(X) = (f_*I^n)(Y)$ then $u\otimes\cdots\otimes u$ is a cocycle in $I(X)^{\otimes\ell} \cong (f_*I)(Y)^{\otimes\ell}$, and $I^*(X) \to J^*(X)$ is identified with $f_*I^*(Y) \to f_*J^*(Y)$. It follows that the left square commutes in the diagram

$$\begin{array}{ccccc}
H^n(Y, f_*A) & \xrightarrow{P} & H^{\otimes n\ell}_\pi(Y, f_*A^{\otimes\ell}) & \xrightarrow{m_*} & H^{\otimes n\ell}(Y, f_*B) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
H^n(X, A) & \xrightarrow{P} & H^{\otimes n\ell}_\pi(X, A^{\otimes\ell}) & \xrightarrow{m_*} & H^{\otimes n\ell}(X, B).
\end{array}$$

The right square commutes by the functoriality of f_*. Expanding $m_*P(f_*u)$ as in Theorem 1.5, the result follows from (1.8) and 1.8.1. □

2. The étale Steenrod algebra when $\zeta_\ell \in k$

In this section and the next we determine the algebra of all étale cohomology operations $H^n_{\mathrm{et}}(-,\mu_\ell^{\otimes i}) \to H^*_{\mathrm{et}}(-,\mu_\ell^{\otimes *})$ for each n and i, over a field k containing $1/\ell$. We work in the big étale site of smooth simplicial schemes over k (see Definition 1.1.

Recall from SGA 4 (V.2.1.2 in [Ver]) that if M is a (simplicial) étale sheaf of $\mathbb{F}_\ell$-modules then the sheaf cohomology groups $H^*_{\mathrm{et}}(X,M)$ are isomorphic to the (hyper) Ext-groups $\mathrm{Ext}^*(\mathbb{F}_\ell[X],M)$ in the category of étale sheaves of $\mathbb{F}_\ell$-modules. (Here we regard M as a cochain complex using Dold-Kan.) If K is a second simplicial étale sheaf of $\mathbb{F}_\ell$-modules, one writes $H^*_{\mathrm{et}}(K,M)$ for $\mathrm{Ext}^*(K,M)$.

It is well known that cohomology operations $H^n_{\mathrm{et}}(-,L) \to H^*_{\mathrm{et}}(-,M)$ are in 1–1 correspondence with elements of $H^*_{\mathrm{et}}(K,M)$, where K denotes the standard simplicial Eilenberg-Mac Lane sheaf $K(L,n)$ associated to L. If M is a ring, these operations form a ring; the product of θ_1 and θ_2 is $x \mapsto \theta_1(x)\cdot\theta_2(x)$.

Now fix i and consider cohomology operations $H^n_{\mathrm{et}}(-,\mu_\ell^{\otimes i}) \to H^*_{\mathrm{et}}(-,\mu_\ell^{\otimes j})$. As observed above, they are in 1–1 correspondence with elements of $H^*_{\mathrm{et}}(K,\mu_\ell^{\otimes j})$, where K denotes the simplicial Eilenberg-Mac Lane scheme $K(\mu_\ell^{\otimes i},n)$. For example, the identity operation on $H^n_{\mathrm{et}}(-,\mu_\ell^{\otimes i})$ corresponds to $\iota_n \in H^n_{\mathrm{et}}(K,\mu_\ell^{\otimes i})$, and the étale Bockstein $\beta : H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i}) \to H^{n+1}_{\mathrm{et}}(X,\mu_\ell^{\otimes i})$ corresponds to $\beta(\iota_n) \in H^{n+1}_{\mathrm{et}}(K,\mu_\ell^{\otimes i})$.

We first discuss the case of constant coefficients ($M=\mathbb{F}_\ell$), which is known and due to Breen [Br, 4.3–4] and Jardine [J]. The graded ring of all unstable étale cohomology operations from $H^n_{\mathrm{et}}(-,\mathbb{F}_\ell)$ to $H^*_{\mathrm{et}}(-,\mathbb{F}_\ell)$ is isomorphic to the cohomology ring $H^*_{\mathrm{et}}(K_n,\mathbb{F}_\ell)$, where $K_n = K(\mathbb{F}_\ell,n)$ is the constant simplicial sheaf classifying elements of $H^n_{\mathrm{et}}(-,\mathbb{F}_\ell)$. By Theorem 1.3, there is a ring homomorphism from the classical unstable Steenrod algebra $H^*_{\mathrm{top}}(K_n)$ of Definition 0.1 to $H^*_{\mathrm{et}}(K_n,\mathbb{F}_\ell)$.

There is also a ring homomorphism from $H^*_{\mathrm{et}}(k,\mathbb{F}_\ell)$ to $H^*_{\mathrm{et}}(K_n,\mathbb{F}_\ell)$, sending a to the constant operation $\theta(x)=a$. It is injective, and is induced by $K_n \to \operatorname{Spec} k$. These induce a graded algebra homomorphism from $H^*_{\mathrm{et}}(k,\mathbb{F}_\ell)\otimes_{\mathbb{F}_\ell} H^*_{\mathrm{top}}(K_n)$ to $H^*_{et}(K_n,\mathbb{F}_\ell)$. Note that $H^*_{\mathrm{et}}(k,\mathbb{F}_\ell)\otimes_{\mathbb{F}_\ell} H^*_{\mathrm{top}}(K_n)$ is free as a left $H^*_{\mathrm{et}}(k,\mathbb{F}_\ell)$-module; a basis is given by those monomials P^I in the Steenrod operations P^a and βP^a for which I has excess $< n$, exactly as in the topological case.

We summarize the above discussion:

Theorem 2.1. *(Breen-Jardine) The ring of all étale cohomology operations on $H^n_{et}(-,\mathbb{F}_\ell)$ is the graded tensor product $H^*_{et}(k,\mathbb{F}_\ell)\otimes H^*_{top}(K_n)$: every operation is a polynomial in the operations P^I with coefficients in $H^*_{et}(k,\mathbb{F}_\ell)$.*

Examples 2.2. When $k=\mathbb{C}$, $H^*_{\mathrm{et}}(\mathbb{C},\mathbb{F}_\ell)\cong\mathbb{F}_\ell$ and étale operations are classifed by $H^*_{\mathrm{top}}(K_n)$. The action of the P^I is compatible with the canonical comparison isomorphism $H^*_{\mathrm{et}}(X,\mathbb{F}_\ell)\cong H^*_{\mathrm{top}}(X(\mathbb{C}),\mathbb{F}_\ell)$. This is clear from the constructions in [E] and [J].

When $k=\mathbb{R}$ and $\ell=2$, the ring of étale cohomology operations over $\mathbb{R}$ is the graded polynomial ring $H^*_{\mathrm{top}}(K_n)[\sigma]$, generated over $\mathbb{F}_2$ by σ in degree 1 and the $Sq^I(\iota_n)$ with I admissible and $e(I)<n$. This is because $H^*_{\mathrm{et}}(\mathbb{R},\mathbb{F}_2)=\mathbb{F}_2[\sigma]$.

Remark 2.2.1. Let $H^*_{\mathrm{top}}(K_n)^{ind}$ denote the indecomposable subspace of $H^*_{\mathrm{top}}(K_n)$; by 0.1 it has the $P^I(\iota_n)$ as a basis, and $H^p_{\mathrm{top}}(K_n)^{ind}$ injects into the degree $p-n$ part of the topological Steenrod algebra. It follows that the vector space $H^*_{\mathrm{et}}(k,\mathbb{F}_\ell)\otimes_{\mathbb{F}_\ell}$

$H^*_{\rm top}(K_n)^{ind}$ embeds into the algebra of *stable* étale operations with a degree shift; cf. [J]. Note that the multiplication $\circ$ in the stable algebra is different: $P^a \circ \lambda = \sum P^i(\lambda)P^j$, $\lambda \in H^*_{\rm et}(k,\mathbb{F}_\ell)$ and $a>0$, arising from the Cartan formula.

Example 2.3. When $k^\times$ contains μ_ℓ but not μ_{ℓ^2}, the étale sheaves $\mu_{\ell^2}^{\otimes i}$ are distinct nontrivial extensions of $\mathbb{F}_\ell$ by $\mathbb{F}_\ell$ ($i=0,...,\ell-1$), and the associated boundary maps ∂_i in the exact sequence

$$(2.3.1) \qquad \cdots H^n(X,\mathbb{F}_\ell) \to H^n(X,\mu_{\ell^2}^{\otimes i}) \to H^n(X,\mathbb{F}_\ell) \xrightarrow{\partial_i} H^{n+1}(X,\mathbb{F}_\ell)\cdots$$

are cohomology operations on $H^n_{\rm et}(-,\mathbb{F}_\ell)$; ∂_0 is the Bockstein β. By Theorem 2.1, the ∂_i are linear combinations of the generators β of $H^1_{\rm top}(K_n)$ and elements of $H^1_{\rm et}(k,\mathbb{F}_\ell)$, such as $z=\zeta^{-1}\beta(\zeta)$, $\zeta\in\mu_\ell$. Note that z is nontrivial as $\mu_{\ell^2}\not\subset k^\times$. It is an exercise to check that z is independent of the choice of generator ζ of μ_ℓ.

Lemma 2.3.2. *If $\mu_\ell \subset k^\times$, the cohomology operation ∂_i of* (2.3.1) *is $\beta - i\,z$.*

Proof. The sheaf ring $\oplus\mu_{\ell^2}^{\otimes j}$ acts on the extensions $0\to\mathbb{F}_\ell\to\mu_{\ell^2}^{\otimes i}\to\mathbb{F}_\ell\to 0$, so there is a pairing between $H^*_{\rm et}(k,R)$ and the direct sum of the cohomology sequences (2.3.1). Set $H^{0j}=H^0_{\rm et}(k,\mu_{\ell^2}^{\otimes j})$; when $j\not\equiv 0 \pmod{\ell}$, this group is $\mathbb{Z}/\ell$. This yields a commutative diagram for each i and j:

$$\begin{tikzcd}
H^{0j}\otimes H^n_{\rm et}(X,\mu_{\ell^2}^{\otimes i}) \arrow[r] \arrow[d, "\cup"'] & H^{0j}\otimes H^n_{\rm et}(X,\mathbb{F}_\ell) \arrow[r, "1\otimes\partial_i"] \arrow[d, "\cup"', "\cong"] & H^{0j}\otimes H^{n+1}_{\rm et}(X,\mathbb{F}_\ell) \arrow[d, "\cup"', "\cong"] \\
H^n_{\rm et}(X,\mu_{\ell^2}^{\otimes i+j}) \arrow[r] & H^n_{\rm et}(X,\mathbb{F}_\ell) \arrow[r, "1\otimes\partial_{i+j}"] & H^{n+1}_{\rm et}(X,\mathbb{F}_\ell).
\end{tikzcd}$$

Now set $j=\ell-i$, so $\mu_{\ell^2}^{\otimes i+j}=\mu_{\ell^2}^0=\mathbb{Z}/\ell^2$. Then the bottom right map ∂_{i+j} is the Bockstein β. For $u\in H^n_{\rm et}(X,\mathbb{F}_\ell)$ and $[\zeta^{\otimes j}]\in H^{0j}$ we have

$$\beta([\zeta^{\otimes j}]\cup u) = [\zeta^{\otimes j}]\cup\beta(u) + j\,z\cup[\zeta^{\otimes j}]\cup u = [\zeta^{\otimes j}]\cup(\beta(u)+j\,z).$$

As this equals $[\zeta^{\otimes j}]\cup\partial_i(u)$, we are done. $\square$

Proposition 2.4. *If $\mu_\ell\subset k$, Epstein's cohomology operation Q^0 agrees with the Bockstein β on $H^n_{\rm et}(X,\mathbb{F}_\ell)$, and $Q^a=\beta P^a$.*

Proof. Jardine's argument in [J, pp. 108–114] that Epstein's Sq^1 is the Bockstein when $\ell=2$ applies when $\ell>2$ as well, and proves that Epstein's Q^0 is the Bockstein operation. For Q^a, we invoke the Adem relation $Q^a=Q^aP^0=Q^0P^a$ [E, 9.8(4)]. $\square$

3. The étale Steenrod algebra when $\zeta_\ell\notin k$

We now discuss the twisted coeficient case ($M=\mu_\ell^{\otimes i}$). This reduces to Theorem 2.1 when k contains a primitive ℓ^{th} root of unity: since the sheaves $\mu_\ell^{\otimes i}$ are all isomorphic to $\mathbb{F}_\ell$, the ring of étale operations on $H^n_{\rm et}(-,\mu_\ell^{\otimes i})$ is just $H^*_{\rm et}(k,\mathbb{F}_\ell)\otimes H^*_{\rm top}(K_n)$. Since this is always the case when $\ell=2$, we shall restrict to the case of an odd prime ℓ.

Fix a field k with $1/\ell\in k$, and let G be the Galois group of the extension $k(\zeta)/k$, where ζ denotes a primitive ℓ^{th} root of unity. Then G is cyclic of order $d=[k(\zeta):k]$, $d\mid\ell-1$, and $1\mapsto\zeta^{\otimes d}$ defines an isomorphism $\mathbb{F}_\ell\xrightarrow{\cong}\mu_\ell^{\otimes d}$. Since

$i \equiv j \pmod{d}$ implies that $\mu_\ell^{\otimes i} \cong \mu_\ell^{\otimes j}$, we are led to consider the $\mathbb{Z}/d$-graded étale sheaf of Kummer algebras

$$A = \bigoplus_{i=0}^{d-1} \mu_\ell^{\otimes i}.$$

Thus our problem is to determine the ring $H^*_{\rm et}(K_n, A) = \oplus_{j=0}^{d-1} H^*_{\rm et}(K_n, \mu_\ell^{\otimes i})$.

Epstein defines cohomology operations P^a_A and Q^a_A on each $H^n_{\rm et}(-, A)$; setting $N = n + 2a(\ell-1)$, their components are operations

$$P^a_A : H^n_{\rm et}(X, \mu_\ell^{\otimes i}) \to H^N_{\rm et}(X, \mu_\ell^{\otimes i}) \quad\text{and}\quad Q^a_A : H^n_{\rm et}(X, \mu_\ell^{\otimes i}) \to H^{N+1}_{\rm et}(X, \mu_\ell^{\otimes i}).$$

By naturality with respect to the homomorphism $\mathcal{A} = \oplus_{i=0}^\infty \mu_\ell^{\otimes i} \to A$ [E, 6.2], the operations P^a and Q^a defined using $\mathcal{A}$ are compatible with the operations P^a_A and Q^a_A. Since the cup product with $\zeta^{\otimes i(\ell-1)} \in H^0_{\rm et}(k, \mu_\ell^{\otimes i(\ell-1)})$ induces an isomorphism $H^N_{\rm et}(X, \mu_\ell^{\otimes i}) \cong H^N_{\rm et}(X, \mu_\ell^{\otimes i\ell})$, this means that

$$P^a(u) = \zeta^{\otimes i(\ell-1)} \cup P^a_A(u) \quad\text{and}\quad Q^a(u) = \zeta^{\otimes i(\ell-1)} \cup Q^a_A(u). \tag{3.1}$$

Since ℓ does not divide $|G|$, Maschke's Theorem gives an identification of the étale sheaf $\mathbb{F}_\ell[G]$ with the direct sum of the sheaves of irreducible $\mathbb{F}_\ell[G]$-modules $\mu_\ell^{\otimes i}$, *i.e.*, with A. For any X, Shapiro's Lemma provides an isomorphism

$$H^*_{\rm et}(X, A) \xrightarrow{\cong} H^*_{\rm et}(X(\zeta), \mathbb{F}_\ell), \tag{3.2}$$

where $X(\zeta)$ denotes $X \times_k \mathrm{Spec}(k(\zeta))$. In fact, $A = \pi_*\mathbb{F}_\ell \simeq R\pi_*\mathbb{F}_\ell$, where $\pi : \mathrm{Spec}(k(\zeta))_{\rm et} \to \mathrm{Spec}(k)_{\rm et}$. Taking $X = \mathrm{Spec}(k)$ yields $H^*_{\rm et}(k, A) \cong H^*_{\rm et}(k(\zeta), \mathbb{F}_\ell)$.

Any $\mathbb{F}_\ell[G]$-module M is the sum of its isotypical summands, the isotypical summand for $\mu_\ell^{\otimes i}$ being $\mathrm{Hom}_G(\mu_\ell^{\otimes i}, M) \cong \mathrm{Hom}_G(\mathbb{F}_\ell, \mu_\ell^{\otimes -i} \otimes M)$. In particular, the action of G on $X(\zeta)$ decomposes $H^*_{\rm et}(X(\zeta), \mathbb{F}_\ell)$ into its isotypical pieces. Because the $\mu_\ell^{\otimes i}$ are the isotypical summands of $A = \pi_*\mathbb{F}_\ell$, the summand $H^*_{\rm et}(X, \mu_\ell^{\otimes i})$ in (3.2) is the isotypical summand of $H^*_{\rm et}(X(\zeta), \mathbb{F}_\ell)$ for $\mu_\ell^{\otimes i}$. Thus the injection

$$H^n_{\rm et}(X, \mu_\ell^{\otimes i}) \to H^n_{\rm et}(X(\zeta), \mu_\ell^{\otimes i}) \cong H^n_{\rm et}(X(\zeta), \mathbb{F}_\ell)$$

sends u to $\zeta^{-i} \otimes \pi^*(u)$, where $\pi^*(u)$ is the image of u in $H^n_{\rm et}(X(\zeta), \mu_\ell^{\otimes i})$.

Example 3.2.1. The product of $\zeta^{-1} \in H^0_{\rm et}(k(\zeta), \mu_\ell^{\otimes -1})$ and $\beta(\zeta) \in H^1_{\rm et}(k(\zeta), \mu_\ell)$ is an element $\zeta^{-1} \cup \beta(\zeta)$ of $H^1_{\rm et}(k(\zeta), \mathbb{F}_\ell)$. As it is fixed by G, it descends to an element of $H^1_{\rm et}(k, \mathbb{F}_\ell)$ which we will call z. Thus $\pi^*(z) = \zeta^{-1} \cup \beta(\zeta)$. As observed after (2.3.1), $z = 0$ iff $k(\zeta)$ contains primitive ℓ^2-roots of unity.

Because the Frobenius is the identity on A, P^0_A is the identity operation by [E, 8.3.4], and $P^0(u) = \zeta^{i(\ell-1)} \cup u$. We can now relate Epstein's operation Q^0 to the étale Bockstein β, and relate his Q^a to βP^a.

Proposition 3.3. *Suppose that $\zeta_\ell \notin k$, and let $z \in H^1_{et}(k, \mathbb{F}_\ell)$ be the element of Example 3.2.1. Then, as cohomology operations on $H^n_{et}(X, \mu_\ell^{\otimes i})$ for $(0 \le i < d)$:*

$$Q^0_A = \beta - i\,z \quad\text{and}\quad Q^0 = \beta P^0 = \zeta^{\otimes i(\ell-1)} \cup (\beta - i\,z);$$
$$Q^a_A = \beta P^a_A - i\,z \cup P^a_A \quad\text{and}\quad Q^a = \beta P^a \quad\text{for } a > 0.$$

The case when $\zeta_\ell \in k$ $(d = 1)$ is handled by Proposition 2.4. Note that by our convention, the β in $Q^a = \beta P^a$ is the Bockstein on $H^*(X, \mu_\ell^{\otimes i\ell})$.

Proof. Given the isomorphism (3.2), Proposition 1.10 for Q^0_A implies that the following diagram commutes:

$$\begin{array}{ccc}
\oplus_{i=0}^{d-1} H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i}) & \xrightarrow{Q^0_A} & \oplus_{0=0}^{d-1} H^{n+1}_{\mathrm{et}}(X,\mu_\ell^{\otimes i\ell}) \\
{\scriptstyle \zeta^{-i}\cup\pi^*}\downarrow\cong & & {\scriptstyle \zeta^{-i\ell}\cup\pi^*}\downarrow\cong \\
H^n_{\mathrm{et}}(X(\zeta),\mathbb{F}_\ell) & \xrightarrow{Q^0=\beta} & H^{n+1}_{\mathrm{et}}(X(\zeta),\mathbb{F}_\ell).
\end{array}$$

For all $u \in H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i})$, $\beta(\pi^* u) = \pi^*\beta(u)$ by naturality. Since $\pi^*(z) = \zeta^{-1}\beta(\zeta)$, the diagram implies that $(\zeta^{-i\ell}\cup\pi^*)(Q^0_A u)$ equals

$$\begin{aligned}
\beta(\zeta^{-i}\cup\pi^* u) =& \zeta^{-i}\cup\beta(\pi^* u) + \beta(\zeta^{-i})\cup\pi^*(u) \\
=& \zeta^{-i}\cup\pi^*\beta(u) - i\,\zeta^{-i-1}\beta(\zeta)\cup\pi^*(u) \\
=& (\zeta^{-i}\cup\pi^*)(\beta(u) - i\,z\cup u).
\end{aligned}$$

Since $\zeta^{-i}\cup\pi^*$ is the isomorphism (3.2), the identity $Q^0_A(u) = \beta(u) - i\,z\cup u$ follows. By (3.1), we have $Q^0 u = \zeta^{i(\ell-1)}\cup Q^0_A u$, which agrees with $\beta P^0 u = \beta(\zeta^{i(\ell-1)}\cup u)$.

Finally, the identity for Q^a_A and Q^a follows by invoking the Adem relations $Q^a_A P^0_A = Q^0_A P^a_A$, $Q^a P^0 = Q^0 P^a$ and $P^0 P^a = P^a P^0$ [E, 9.8(1,4)]:

$$\begin{aligned}
Q^a_A = Q^a_A P^0_A = Q^0_A P^a_A =& (\beta - i\,z)P^a_A \quad \text{and} \\
Q^a P^0 = Q^0 P^a =& \beta P^0 P^a = \beta P^a P^0.
\end{aligned}$$

Now use the fact that P^0 is invertible. □

Remark 3.3.1. If $i \equiv j \pmod d$, the étale sheaves $\mu_\ell^{\otimes i}$ and $\mu_\ell^{\otimes j}$ are isomorphic. This does not affect the operation Q^0_A, but it changes the Bockstein because the isomorphism $H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i}) \cong H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes j})$ sends u to $v = \zeta^{j-i}\cup u$. Setting $j = dq+i$, Lemma 1.2 yields

$$\beta(v) = \zeta^{\otimes dq}\cup\beta(u) + dq\zeta^{\otimes dq-1}\beta(\zeta)\cup u = \zeta^{\otimes dq}\cup\{\beta(u) + (dq)\,z\cup u\}\,.$$

Using the Bockstein and Epstein's operations P^a, we have operations P^I defined on $H^n_{\mathrm{et}}(-,\mu_\ell^{\otimes i})$ for every admissible sequence I in the sense of Definition 0.1.

In order to classify all operations on H^n_{et}, we first consider the case $n = 1$. In topology, the ring of operations on $H^1(-,\mathbb{F}_\ell)$ is $H^*_{\mathrm{top}}(K_1) \cong \mathbb{F}_\ell[u,v]/(u^2)$, where $u = P^0$ is in degree 1, corresponding to the identity operation, and v is in degree 2, corresponding to the Bockstein operation. By Theorem 1.3, there is a canonical map from $\mathbb{F}_\ell[u,v]/(u^2)$ to étale cohomology operations from $H^1_{\mathrm{et}}(-,\mu_\ell^{\otimes i})$ to $H^*_{\mathrm{et}}(-,\mu_\ell^{\otimes *})$, sending u to the identity, v to the Bockstein $\beta : H^1_{\mathrm{et}}(-,\mu_\ell^{\otimes i}) \to H^2_{\mathrm{et}}(-,\mu_\ell^{\otimes i})$ and v^m to $x \mapsto \beta(x)^m$.

For any i, the basechange $\mu_\ell^{\otimes i}(\zeta)$ of the algebraic group $\mu_\ell^{\otimes i}$ is isomorphic to $\mathbb{F}_\ell(\zeta)$, the constant sheaf $\mathbb{F}_\ell$ on the big étale site of $k(\zeta)$. The induced isomorphism $(B\mu_\ell^{\otimes i})(\zeta) \cong (B\mathbb{F}_\ell)(\zeta)$ induces an isomorphism of cohomology groups, which immediately yields the following calculation.

Recall that $A = \oplus_{i=0}^{d-1}\mu_\ell^{\otimes i}$.

Proposition 3.4. *The graded algebra of cohomology operations from $H^1_{et}(-,\mu_\ell^{\otimes i})$ to $\oplus_{j=0}^{d-1} H^*_{et}(-,\mu_\ell^{\otimes j})$ is isomorphic to the $H^*_{et}(k(\zeta),\mathbb{F}_\ell)$-module*

$$H^*(B\mu_\ell^{\otimes i}, A) \cong H^*(B\mu_\ell^{\otimes i}(\zeta),\mathbb{F}_\ell) \cong H^*(k(\zeta),\mathbb{F}_\ell)\otimes\mathbb{F}_\ell[u,v]/(u^2),\ \beta(u) = v.$$

Every operation on $H^1_{et}(-,\mu_\ell^{\otimes i})$ is uniquely a sum of operations $\phi(x)=cx^\varepsilon\beta(x)^m$, where $\varepsilon\in\{0,1\}$, $m\ge 0$ and $c\in H^*_{et}(k,\mu_\ell^{\otimes j})$ for some j.

The operations u and v on $H^1_{\rm et}$ are of course $u(x)=x$ and $v(x)=\beta(x)$. Proposition 3.4 is the case $n=1$ of the following result.

Theorem 3.5. *For each i and $n\ge 1$, the ring of all étale cohomology operations from $H^n_{et}(-,\mu_\ell^{\otimes i})$ to $H^*_{et}(-,A)$ is the free left $H^*_{et}(k(\zeta),\mathbb{F}_\ell)$-module $H^*_{et}(k(\zeta),\mathbb{F}_\ell)\otimes H^*_{top}(K_n)$.*

If $c\in H^*_{et}(k,\mu_\ell^{\otimes j})$, the operation coresponding to the monomial $c\,P^{I_1}\cdots P^{I_r}$ sends $H^n_{et}(-,\mu_\ell^{\otimes i})$ to $H^*_{et}(-,\mu_\ell^{\otimes ri+j})$.

Proof. We first show that the basechange $K(\mu_\ell^{\otimes i},n)\times_k \operatorname{Spec}(k(\zeta))$ is the space $K(\mu_\ell^{\otimes i},n)$ over $k(\zeta)$. This is clear for $n=0$, and follows inductively from the construction of $K(A,n+1)$ via the bar construction on $K(A,n)$, together with the observation that $(X\times_k Y)\times_k \operatorname{Spec}(k(\zeta))$ is $X(\zeta)\times_{k(\zeta)} Y(\zeta)$.

By (3.2), the cohomology of $K(\mu_\ell^{\otimes i},n)$ with coefficients in A is the same as the cohomology of $K(\mu_\ell^{\otimes i},n)\times_k \operatorname{Spec}(k(\zeta))$ with coefficients in $\mathbb{F}_\ell$. The Breen-Jardine result, Theorem 2.1, shows that this is $H^*_{\rm et}(k(\zeta),\mathbb{F}_\ell)\otimes H^*_{\rm top}(K_n)$. □

4. May's adjoint construction

A somewhat different approach to constructing cohomology operations was given by Peter May in [M]. Because we will need May's version of Kudo's Theorem (in 9.5 below), we need to know how the two constructions compare.

First, we need a chain level version of the Steenrod-Epstein function

$$m_*P: H^n(X,A)\to H^{n\ell}_\pi(X,A)$$

used in (1.8) to define P^a when A is a sheaf of commutative algebras. Fix an injective resolution $A\xrightarrow{\sim} I^*$ (in $\mathcal{S}h$), an injective resolution $A^{\otimes\ell}\xrightarrow{\sim} I^{\otimes\ell}\xrightarrow{\sim} J^*$ in the category $\mathcal{S}h_\pi$ of π-equivariant sheaves, and an injective resolution $I^{\otimes\ell}\xrightarrow{\sim} J^*_S$ in the category $\mathcal{S}h_{S_\ell}$ of S_ℓ-equivariant sheaves, as in Section 1.

The multiplication map $m: A^{\otimes\ell}\to A$ is equivariant for both the action of S_ℓ and its subgroup π on $A^{\otimes\ell}$. As we observed in the proof of Theorem 1.5, $A\to\operatorname{Tot}(W^*\otimes I^*)$ is an injective resolution in $\mathcal{S}h_\pi$. The comparison theorem lifts the resolution $A^{\otimes\ell}\to J^*$ to an equivariant map $J^*\to\operatorname{Tot}(W^*\otimes I^*)$ over $A^{\otimes\ell}\to A$; taking sections over X yields a map $J^*(X)\to\operatorname{Tot}(W^*\otimes I^*(X))$, natural in X. This induces an equivariant map of complexes of π-sheaves

$$m_\pi: I^{\otimes\ell}\xrightarrow{\sim} J^*\to\operatorname{Tot}(W^*\otimes I^*).$$

Consider the isomorphism $\eta: W^*\otimes I^*\to\operatorname{Hom}(W_*,I^*)$, defined on sections by

$$\eta_U(f\otimes x)(w)=(-1)^{|x|\,|w|}f(w)x,\qquad f\in W^*, w\in W_*, x\in I^*(U).$$

If $\{w_k\in W^k\}$ is the dual basis for $\{e_k\in W_k\}$ we have $\eta(w_j\otimes x)(e_k)=(-1)^{k|x|}\delta_{jk}\,x$. The composition $\eta\, m_\pi$ sends $I^{\otimes\ell}$ to $\operatorname{Hom}(W_*,I^*)$. It is the (signed) adjoint

$$\theta: W_*\otimes I^{\otimes\ell}\to I^* \tag{4.1}$$

of the map $\eta\, m_\pi(X)$ which forms the basis for May's approach; see [M, 2.1]. In this approach, we fix a projective resolution of $\mathbb{F}_\ell$ as an S_ℓ-module, $V_*\to\mathbb{F}_\ell$, and a π-equivariant map $j: W_*\to V_*$ over $\mathbb{F}_\ell$.

Similarly, suppose that $\mathcal{K}$ is a sheaf of bounded below, homotopy associative dg algebras, and $\mathcal{K} \to I$ is an injective replacement (so that a lift $I \otimes I \to I$ gives I the structure of a homotopy associative dg algebra). Given a π-equivariant map $m : \mathcal{K}^{\otimes \ell} \to \mathrm{Hom}(W_*, \mathcal{K})$, the comparison theorem lifts

$$\mathcal{K}^{\otimes \ell} \xrightarrow{m} \mathrm{Hom}(W_*, \mathcal{K}) \xrightarrow{\simeq} \mathrm{Hom}(W_*, I)$$

to an equivariant map $m_\pi : I^{\otimes \ell} \to \mathrm{Hom}(W_*, I)$, whose adjoint is again a map θ of the form given in (4.1).

Definition 4.2. Suppose that $\mathcal{K}$ is a sheaf of homotopy associative dg $\mathbb{F}_\ell$-algebras on some site, and $\theta : W_* \otimes \mathcal{K}^{\otimes \ell} \to \mathcal{K}$ is a morphism of complexes in $\mathcal{S}h_\pi$. We say that $(\mathcal{K}, \theta)$ is *suitable* if (i) the restriction of θ to $\mathcal{K}^{\otimes \ell} = \mathbb{F}_\ell\{e_0\} \otimes \mathcal{K}^{\otimes \ell}$ is chain homotopic to the iterated product $\mathcal{K}^{\otimes \ell} \to \mathcal{K}$ (in some order), and (ii) θ is chain homotopic to a composite

$$W_* \otimes \mathcal{K}^{\otimes \ell} \xrightarrow{j} V_* \otimes \mathcal{K}^{\otimes \ell} \xrightarrow{\phi} \mathcal{K},$$

where ϕ is the restriction to $\mathcal{S}h_\pi$ of a morphism of complexes in $\mathcal{S}h_{S_\ell}$.

A *morphism* of suitable pairs $(\mathcal{K}, \theta) \to (\mathcal{K}', \theta')$ is a morphism $f : \mathcal{K} \to \mathcal{K}'$ for which $f\theta$ is chain homotopic (over $\mathbb{F}_\ell[\pi]$) to $\theta'(1 \otimes f^{\otimes \ell})$. It is a *perfect* morphism if $f\theta = \theta'(1 \otimes f^{\otimes \ell})$. May writes $\mathcal{C}(\pi, \infty, \mathbb{F}_\ell)$ for the category of suitable pairs.

Taking sections over X, the pair $(\mathcal{K}(X), \theta_X)$ satisfies May's axioms in [M, 2.1], where θ_X is the induced map $W_* \otimes \mathcal{K}(X)^{\otimes \ell} \to W_* \otimes \mathcal{K}^{\otimes \ell}(X) \to \mathcal{K}(X)$. If f is a (perfect) morphism then for any X, $f_X : \mathcal{K}(X) \to \mathcal{K}'(X)$ is a (perfect) morphism in the sense of [M, 2.1].

Remark 4.2.1. If Λ is any commutative ring, Definition 4.2 makes sense for any sheaf of homotopy associative dg Λ-algebras; we say that (K, θ) is *suitable for* Λ. Following [M, p. 161], we say that a suitable (K, θ) is *reduced* if it is obtained by reduction mod ℓ from a pair $(\widetilde{K}, \tilde{\theta})$ which is suitable for $\mathbb{Z}/\ell^2$, such that $\widetilde{K}$ is a flat $\mathbb{Z}/\ell^2$-module. Since $0 \to K \to \widetilde{K} \to K \to 0$ is an exact sequence of chain complexes, this data suffices to yield a Bockstein $\beta : H^N(K) \to H^{n+1}(K)$.

Example 4.3. Let $\mathcal{C}$ be an acyclic operad of dg vector spaces over $\mathbb{F}_\ell$. Then we may take $V_* = \mathcal{C}(\ell)$, since it is a resolution of $\mathbb{F}_\ell$. If $\mathcal{C}$ acts on $\mathcal{K}$, $\mathcal{K}$ is homotopy associative, and θ is the composition of $j \otimes 1 : W_* \otimes \mathcal{K}^{\otimes \ell} \to \mathcal{C}(\ell) \otimes \mathcal{K}^{\otimes \ell}$ with the structure map $\mathcal{C}(\ell) \otimes \mathcal{K}^{\otimes \ell} \to \mathcal{K}$, then $(\mathcal{K}, \theta)$ is a suitable pair.

Definition 4.4 (May)**.** Suppose that $(\mathcal{K}, \theta)$ is suitable, and set $K = \mathcal{K}(X)$. Define the function $D_k^M : K^n \to K^{n\ell - k}$ by the formula

$$D_k^M(u) = \theta(e_k \otimes u^{\otimes \ell}),$$

and define the Steenrod operations $P_M^a : H^n(K) \to H^{n+2a(\ell-1)}(K)$ (and Q_M^a) by

$$P_M^a(u) = (-1)^a \nu_n D^M_{(n-2a)(\ell-1)}(u) \text{ and } Q_M^a(u) = (-1)^a \nu_n D^M_{(n-2a)(\ell-1)-1}(u)$$

(see [M, pp. 162, 182]; May's $\nu(-n)$ is our ν_n). As with Epstein's construction, $P_M^a = 0$ when $n < 2a$ and $Q_M^a = 0$ when $n \le 2a$. May notes in [M, 2.3–2.5] that the D_k^M and hence the P_M^a, Q_M^a are additive and functorial for morphisms of suitable pairs.

These operations are natural in X, because for every morphism $f : Y \to X$ in the site, the restriction $f^* : \mathcal{K}(X) \to \mathcal{K}(Y)$ gives a perfect morphism $(\mathcal{K}(X), \theta_X) \to (\mathcal{K}(Y), \theta_Y)$ of objects in the sense of [M, 2.1].

A morphism of suitable pairs $(\mathcal{K},\theta) \to (\mathcal{K}',\theta')$ induces a map $H^n\mathcal{K}(X) \to H^n\mathcal{K}'(X)$ compatible with the operations P^a_M on $H^n\mathcal{K}(X)$ and the corresponding operations P^a_M on $H^n\mathcal{K}'(X)$ in the sense that $P^a_M f^* = f^* P^a_M$. This follows from [M, 3.1(iii)].

Lemma 4.4.1. *If (K,θ) is reduced then $Q^a_M = \beta\, P^a_M$, where β is the Bockstein.*

Proof. The proof of [M, 2.3(v)] applies; it suffices to show that $\beta D_{2i} = D_{2i-1}$. Given u in K^n with $d(u)=0$, lift u to $\tilde u \in \widetilde{K}^n$ and let b be such that $d(\tilde u) = \ell\, b$, so $\beta(u) = b$. Since $d(\tilde u^{\otimes \ell}) = N(\ell\, b \otimes \tilde u^{\otimes \ell-1})$, where $N = \sum\{\sigma \in \pi\}$, and $\theta\beta = \beta\theta$, May's calculation in $W_* \otimes \widetilde{K}^{\otimes\ell}$ [M, p. 163] goes through to show that:

$$\beta D_{2i}(u) = \theta\beta(e_{2i} \otimes u^{\otimes\ell}) \equiv \theta\left(e_{2i-1} \otimes u^{\otimes\ell}\right) = D_{2i-1}(u) \text{ modulo boundaries.} \quad \square$$

Lemma 4.5. *If A is a sheaf of commutative dg $\mathbb{F}_\ell$-algebras, $A \xrightarrow{\sim} I$ an injective resolution and θ as in* (4.1)*, the pair (I,θ) is suitable. Hence this data yields cohomology operations $P^a_M : H^n(X,A) \to H^{n+2a(\ell-1)}(X,A)$, natural in X.*

Proof. By Remark 1.6.1, I is a sheaf of homotopy associative dg algebras. By construction, the restriction of θ to $I^{\otimes\ell} = W_0 \otimes I^{\otimes\ell}$ is chain homotopic to the given map $I^{\otimes\ell} \to I$. To see that axiom 4.2(ii) is satisfied, set $V^* = \mathrm{Hom}(V_*,\mathbb{F}_\ell)$ and note that, by the proof of Theorem 1.5, $A \to \mathrm{Tot}(V^* \otimes I)$ is an injective resolution in $\mathcal{S}h_{S_\ell}$. The construction before Definition 1.7 yields a map $m_S : I^{\otimes\ell} \xrightarrow{\sim} J^*_S \to V^* \otimes I$ whose adjoint $V_* \otimes I^{\otimes\ell} \to I$ is ϕ. The comparison theorem for $\mathcal{S}h_\pi$ provides a lift $J_S \to J$ over $I^{\otimes\ell}$, such that the map

$$I^{\otimes\ell} \xrightarrow{m_S} \mathrm{Hom}(V_*,I) \xrightarrow{j^*} \mathrm{Hom}(W_*,I)$$

is chain homotopic over $\mathbb{F}_\ell[\pi]$ to the map m_π, as required. The final assertion follows because $H^*(X,A) = H^*I(X)$. $\square$

In fact, the operations P^a_M are independent of the choice of resolution $A \xrightarrow{\simeq} I$; this follows from the following lemma, whose proof is the same as the proof of Lemma 4.5, with A replaced by $\mathcal{K}$. (See the discussion before Definition 4.2.)

Lemma 4.5.1. *Let $(\mathcal{K},\theta_0)$ be a suitable pair, where $\mathcal{K}$ is a sheaf of bounded below, homotopy associative dg $\mathbb{F}_\ell$-algebras. If $\eta : \mathcal{K} \xrightarrow{\sim} I$ is an injective replacement and θ is the map of* (4.1)*, then the pair (I,θ) is suitable, and η is a perfect morphism $(\mathcal{K},\theta_0) \to (I,\theta)$ of suitable pairs.*

The sign differences in the formulas for P^a and P^a_M (and for Q^a and Q^a_M) are explained by the following calculation.

Proposition 4.6. *For u in $H^n(X,A)$, $D^M_k = (-1)^k D_k$.*

Proof. Consider the isomorphism $\phi : W^* \otimes I^*(X) \to \mathrm{Hom}(W_*, I^*(X))$, defined above. The adjoint θ of $\phi\, m_\pi$ is the composite

$$W_* \otimes I(X)^{\otimes\ell} \cong I(X)^{\otimes\ell} \otimes W_* \xrightarrow{\phi\, m_\pi \otimes 1} \mathrm{Hom}(W_*, I(X)) \otimes W_* \xrightarrow{\eta} I(X),$$

where the first map is the signed symmetry isomorphism and η is evaluation. We now compute that

$$\begin{aligned} D_k^M(u) = \theta(e_k \otimes u^{\otimes \ell}) &= (-1)^{kn\ell}\eta\left(\phi[m_\pi(u^{\otimes \ell})] \otimes e_k\right) \\ &= (-1)^{kn\ell}\eta\left(\phi\left[\sum w_j \otimes D_j(u)\right] \otimes e_k\right) \\ &= (-1)^{kn\ell}\sum_j \phi\left[w_j \otimes D_j(u)\right](e_k) \\ &= (-1)^{kn\ell}(-1)^{k(n\ell-k)}D_k(u) = (-1)^k D_k(u). \qquad \square \end{aligned}$$

Recall that Epstein's operations P^a and Q^a are defined in (1.8) and 1.8.1.

Corollary 4.7. *May's operations P_M^a and Q_M^a coincide with Epstein's P^a and Q^a.*

Lemma 4.8. *Set $m = (\ell - 1)/2$. Then for each $u \in I^n(X)$:*
(i) $dP^a(u) = P^a(du)$ and $dQ^a(u) = -Q^a(du)$, and
(ii) if u is a cocycle representing $x \in H^n(X, A)$ then $P^a(u)$ and $Q^a(u)$ are cocycles representing $P^a(x)$ and $Q^a(x)$, respectively.

Proof. In Theorem 3.1 of [M], May shows that (i) $dP_M^a(u) = P_M^a(du)$ and $dQ_M^a(u) = -Q_M^a(du)$, and (ii) if u is a cocycle representing $x \in H^n(X, A)$ then $P_M^a(u)$ and $Q_M^a(u)$ are cocycles representing $P_M^a(x)$ and $Q_M^a(x)$. The result is immediate from Corollary 4.7. $\square$

Let $G \cong \pi \wr \pi$ denote a Sylow ℓ-subgroup of S_{ℓ^2}. Then there is a free $\mathbb{F}_\ell[G]$-module resolution $W_* \otimes W_*^{\otimes \ell} \to \mathbb{F}_\ell$. If $'V_* \to \mathbb{F}_\ell$ is a projective resolution of $\mathbb{F}_\ell$ as an S_{ℓ^2}-module, there is a G-module morphism $\sigma : W_* \otimes W_*^{\otimes \ell} \to {}'V_*$ over $\mathbb{F}_\ell$.

Definition 4.9. Following [M, 4.1], we say that a suitable $(\mathcal{K}, \theta)$ is an *Adem object* if there is a S_{ℓ^2}-equivariant morphism $'V_* \otimes \mathcal{K}^{\otimes \ell^2} \to \mathcal{K}$ whose composition with $\sigma \otimes 1$ is G-homotopic to

$$(W_* \otimes W_*^{\otimes \ell}) \otimes \mathcal{K}^{\otimes \ell^2} \cong W_* \otimes (W_* \otimes \mathcal{K}^{\otimes \ell})^{\otimes \ell} \xrightarrow{\theta^{\otimes \ell}} W_* \otimes \mathcal{K}^{\otimes \ell} \xrightarrow{\theta} \mathcal{K}.$$

May proves in [M, 4.7] that the P^a satisfy the Adem relations whenever $(\mathcal{K}, \theta)$ is an Adem object.

Example 4.9.1. Given $A \xrightarrow{\sim} I$, $A^{\otimes \ell} \to \mathrm{Tot}('V^* \otimes I)$ is an injective resolution in $\mathcal{S}h_{S_{\ell^2}}$, and we get a S_{ℓ^2}-equivariant map $I^{\otimes \ell^2} \to \mathrm{Hom}('V_*, I)$ whose adjoint $'V_* \otimes I^{\otimes \ell^2} \to I$ makes (I, θ) into an Adem object. We omit the routine details.

Similarly, if $(\mathcal{K}, \theta_0)$ is an Adem object, $\mathcal{K} \to I$ an injective replacement and (I, θ) as in Lemma 4.5.1, then the map $'V_* \otimes \mathcal{K}^{\otimes \ell^2} \to \mathcal{K} \to I$ lifts to an S_{ℓ^2}-equivariant map $'V_* \otimes I^{\otimes \ell^2} \to I$; this map makes (I, θ) into an Adem object. Again, we omit the routine details.

Definition 4.10. If $(\mathcal{K}_1, \theta_1)$ and $(\mathcal{K}_2, \theta_2)$ are suitable pairs, the tensor product $(\mathcal{K}_1 \otimes \mathcal{K}_2, \theta_1 \otimes \theta_2)$ is also a suitable pair, where $\theta_1 \otimes \theta_2$ is described in [M, 2.1].

A suitable pair $(\mathcal{K}, \theta)$ is a *Cartan object* if the product $(\mathcal{K} \otimes \mathcal{K}, \theta \otimes \theta) \to (\mathcal{K}, \theta)$ is a morphism of suitable pairs.

Example 4.10.1. If A is a sheaf of commutative algebras then so is $A \otimes A$, and the product $A \otimes A \to A$ makes A into a Cartan object. If $A \xrightarrow{\sim} I$ is an injective resolution, then the pair (I, θ) of Lemma 4.5 is also a Cartan object, because the two morphisms $W_* \otimes (I \otimes I)^{\otimes \ell} \to I$ are isomorphic to the two morphisms $W_* \otimes (A \otimes A)^{\otimes \ell} \to A$ in the derived category, and (because I is a bounded below complex of injectives) therefore chain homotopic.

For exactly the same reasons, if $(\mathcal{K}, \theta_0)$ is a Cartan object and $\mathcal{K} \to I$ is an injective replacement, then so is (I, θ).

5. Operads and operations

In [M1], May gave a different approach to power operations in sheaf cohomology. In this section, we give a short discussion of this approach.

Let $\mathcal{Z}$ denote the *Eilenberg-Zilber operad* in the category of $\mathbb{F}_\ell$-modules, defined by Hinich and Schechtman in [HS]. By an action of $\mathcal{Z}$ on a cochain complex of sheaves C we mean a collection of sheaf morphisms

$$\mathcal{Z}(n) \otimes C^{\otimes n} \longrightarrow C$$

satisfying appropriate equivariance, associativity and unit axioms. The choice of an element $m \in \mathcal{Z}(2)$ determines a product $C^{\otimes 2} \to C$, and makes C into a homotopy associative dg algebra. Because each $\mathcal{Z}(n)$ is an acyclic complex of $\mathbb{F}_\ell[S_n]$-modules, we have quasi-isomorphisms $W_* \xrightarrow{\simeq} \mathcal{Z}(\ell)$ and $'V_* \xrightarrow{\simeq} \mathcal{Z}(\ell^2)$. We thus have a natural map

$$\theta : W_* \otimes C^{\otimes \ell} \xrightarrow{\simeq} \mathcal{Z}(\ell) \otimes C^{\otimes n} \longrightarrow C.$$

Lemma 5.1. *If $\mathcal{Z}$ acts on a cochain complex C, then (C, θ) is a suitable pair in the sense of Definition 4.2. It is also an Adem and a Cartan object (Definitions 4.9 and 4.10).*

Proof. This is an exercise in the axioms of operads, left to the reader. The axiom about $\mathcal{Z}(\ell) \otimes \mathcal{Z}(\ell)^{\otimes \ell} \to \mathcal{Z}(\ell^2)$ is used to show (C, θ) is an Adem object, and the axiom about $\mathcal{Z}(2) \otimes \mathcal{Z}(\ell)^{\otimes 2} \to \mathcal{Z}(2\ell)$ is used to show it is a Cartan object. $\square$

Example 5.2. Let $F^\bullet \mathcal{F}$ denote the Godement resolution of a sheaf $\mathcal{F}$. Since every skyscraper sheaf of $\mathbb{F}_\ell$-modules is an injective sheaf, this is an injective resolution of $\mathcal{F}$. If A is any sheaf of commutative $\mathbb{F}_\ell$-algebras, the results of Section 4 apply. Alternatively, Hinich and Schechtman showed in [HS] that $\mathcal{Z}$ acts on $F^\bullet A$, so $(F^\bullet A, \theta)$ is not only suitable but is both an Adem and a Cartan object by the lemma above.

By Lemma 4.5, this data provides cohomology operations P^a on $H^*(X, A)$, natural in X. Of course, this construction is little more than a reinterpretation of May's procedure (in Theorem 4.8 of [M1]), where he shows that the Eilenberg-Zilber operad acts on the sections $F^\bullet A(X)$, giving cohomology operations.

Remark 5.3. If A is a sheaf of étale algebras, the direct image $\alpha_*(F^\bullet A)$ is also a complex of injective Nisnevich sheaves. However, although the operad $\mathcal{Z}$ acts on the good truncations $\tau^{\leq i} \alpha_*(F^\bullet A)$, it only acts up to homotopy on an injective replacement, such as the complex I_{nis} of Corollary 6.4 below. Thus one needs care when using this approach to construct motivic cohomology operations in the next section.

6. Motivic Steenrod operations

In this section we construct operations P^a on the motivic cohomology groups $H^{n,i}(X) = H^{n,i}(X, \mathbb{F}_\ell)$, $n \geq 2a$, compatible with the operations P^a in étale cohomology (defined in Theorem 1.3) in the sense that there are commutative diagrams

$$\begin{array}{ccc} H^{n,i}(X, \mathbb{F}_\ell) & \xrightarrow{P^a} & H^{n+2a(\ell-1), i\ell}(X, \mathbb{F}_\ell) \\ \downarrow & & \downarrow \\ H^n_{\text{et}}(X, \mu_\ell^{\otimes i}) & \xrightarrow{P^a} & H^{n+2a(\ell-1)}_{\text{et}}(X, \mu_\ell^{\otimes i\ell}). \end{array} \tag{6.1}$$

Let α_* denote the direct image functor from the étale site to the Nisnevich site. If A is any étale sheaf then we may regard $R\alpha_* A$ as a complex of Nisnevich sheaves such that $H^*_{\text{nis}}(X, R\alpha_* A) \cong H^*_{\text{et}}(X, A)$. If $A \to I$ is an injective resolution then $\alpha_* I$ is a complex of injective Nisnevich sheaves representing $R\alpha_* A$.

Let $\tau^{\leq i} A$ denote the good truncation of A in cohomological degrees at most i; $H^n(\tau^{\leq i} A)$ is $H^n(A)$ for $i \leq n$, and zero for $n > i$. (cf. [WH, 1.2.7]). If $B^n = 0$ for $n < 0$, there is a natural transformation $(\tau^{\leq i} A) \otimes B \to \tau^{\leq i}(A \otimes B)$.

The following theorem, due to Voevodsky and Rost, is sometimes known as the Beilinson Conjecture; it is equivalent to the Norm Residue Theorem; see [SV], [W], [V4, 6.17], or [HW, Thm. C].

Norm Residue Theorem 6.2. *For any field of characteristic $\neq \ell$, the canonical map $\mathbb{F}_\ell(i) \to \tau^{\leq i} R\alpha_* \mu_\ell^{\otimes i}$ is a quasi-isomorphism of complexes of Nisnevich sheaves on the category of smooth simplicial schemes. Hence for any X we have*

$$H^{n,i}(X, \mathbb{F}_\ell) \cong H^n_{nis}(X, \tau^{\leq i} R\alpha_* \mu_\ell^{\otimes i}).$$

In particular, if $n \leq i$ then $H^{n,i}(X, \mathbb{F}_\ell) \cong H^n_{et}(X, \mu_\ell^{\otimes i})$.

We apply the above constructions to the étale sheaf of dg algebras $\mathcal{A} = \oplus_{i=0}^\infty \mu_\ell^{\otimes i}$. Choose injective resolutions $\mu_\ell^{\otimes i} \to I(i)$ and write I for the complex of étale sheaves $\oplus_{i=0}^\infty I(i)$. Then $\mathcal{A} \xrightarrow{\sim} I$ is an injective resolution; by Remark 1.6.1, I is an étale sheaf of homotopy-associative graded dg algebras, and (I, θ) is suitable by Lemma 4.5.

Now consider the Nisnevich sheaf of dg algebras $\alpha_* I$. Using the natural transformation $(\alpha_* I)^{\otimes \ell} \to \alpha_*(I^{\otimes \ell})$, we obtain a map $\alpha_* \theta : W_* \otimes (\alpha_* I)^{\otimes \ell} \to \alpha_* I$ such that $(\alpha_* I, \alpha_* \theta)$ is suitable in the sense of Definition 4.2. It is also clear that $(\alpha_* I(X), \alpha_* \theta_X)$ is both an Adem object and a Cartan object, natural in X (see Examples 4.9.1 and 4.10.1).

Of course, nothing new has happened; for each X we have $\alpha_* I(X) = I(X)$ and $(\alpha_* \theta)_X = \theta_X$ by the definition of direct image, so the resulting cohomology operations on $H^* \alpha_* I(X) = H^*_{\text{et}}(X, \mu_\ell^{\otimes *})$ are the same as those constructed in Definition 4.4 and hence (by 4.7) are the same as those in Theorem 1.3.

We now consider the effect of truncation. If $\mathcal{K} = \oplus_{i=0}^\infty \mathcal{K}_i$ is a sheaf of (homotopy associative) graded dg algebras, set $\tau\mathcal{K} = \oplus \tau^{\leq i} \mathcal{K}_i$. The products

$$\tau^{\leq i_1} \mathcal{K}_{i_1} \otimes \tau^{\leq i_2} \mathcal{K}_{i_2} \to \tau^{\leq i_1 + i_2}(\mathcal{K}_{i_1} \otimes \mathcal{K}_{i_2}) \to \tau^{\leq i_1 + i_2} \mathcal{K}_{i_1 + i_2}$$

make $\tau\mathcal{K}$ a sheaf of (homotopy associative) graded dg algebras. If $(\mathcal{K}, \theta)$ is suitable (Definition 4.2), and $\sum i_r = n$, the component maps

$$W_* \otimes \bigotimes_{r=1}^{\ell} (\tau^{\leq i_r} \mathcal{K}_{i_r}) \to \tau^{\leq n}(W_* \otimes \bigotimes \mathcal{K}_{i_r}) \xrightarrow{\theta} \tau^{\leq n} \mathcal{K}_n$$

assemble to define a map $W_* \otimes (\tau\mathcal{K})^{\otimes \ell} \to \tau\mathcal{K}$ we shall call $\tau\theta$.

Lemma 6.3. *If $(\mathcal{K},\theta)$ is suitable, then so is $(\tau\mathcal{K},\tau\theta)$, and $\tau\mathcal{K}\to\mathcal{K}$ defines a perfect morphism of suitable pairs.*

If $(\mathcal{K},\theta)$ is an Adem object (resp., a Cartan object), so is $(\tau\mathcal{K},\tau\theta)$.

Proof. Axiom (i) is trivial, and axiom (ii) follows from the commutative diagram

$$\begin{array}{ccccc} W_*\otimes\bigotimes_{r=1}^{\ell}(\tau^{\le i_r}\mathcal{K}_{i_r}) & \to & \tau^{\le j}(W_*\otimes\bigotimes\mathcal{K}_{i_r}) & & \\ \downarrow j & & \downarrow j & \searrow^{\tau\theta} & \\ V_*\otimes\bigotimes_{r=1}^{\ell}(\tau^{\le i_r}\mathcal{K}_{i_r}) & \to & \tau^{\le j}(V_*\otimes\bigotimes\mathcal{K}_{i_r}) & \xrightarrow{\phi} & \tau^{\le j}\mathcal{K}_j. \end{array}$$

The final two assertions are easily verified using similar diagrams. $\square$

Example 6.3.1. When $\mathcal{K}=\alpha_*I=\oplus\alpha_*I(i)$, we see that $(\tau\,\alpha_*I,\tau\theta)$ is suitable. where $\tau\,\alpha_*I=\oplus\tau^{\le i}I(i)$. Since (α_*I,θ) is both an Adem object and a Cartan object, so is $(\tau\,\alpha_*I,\tau\theta)$.

Because each $\alpha_*I(i)^n$ is an injective Nisnevich sheaf and the sheaf $Z^i(\alpha_*I(i))$ of i-cycles has an injective resolution starting with $\alpha_*I(i)^i$, $\tau^{\le i}\alpha_*I(i)$ is quasi-isomorphic to a chain complex $I(i)^*_{\rm nis}$ of injective Nisnevich sheaves on X with $I(i)^n_{\rm nis}=\alpha_*I(i)^n$ for $n\le i$, and $I(i)_{\rm nis}$ represents $\tau^{\le i}R\alpha_*\mu_\ell^{\otimes i}$. By Theorem 6.2, $H^{n,i}(X,\mathbb{F}_\ell)$ is the n^{th} cohomology of the cochain complex $I(i)^*_{\rm nis}(X)$.

By Remark 1.6.1, $I_{\rm nis}=\oplus I(i)_{\rm nis}$ is a Nisnevich sheaf of homotopy-associative dg algebras, and the products $I(i)_{\rm nis}\otimes I(j)_{\rm nis}\to I(i+j)_{\rm nis}$, representing the pairings

$$\tau^{\le i}R\alpha_*\mu_\ell^{\otimes i}\otimes\tau^{\le j}R\alpha_*\mu_\ell^{\otimes j}\to\tau^{\le i+j}R\alpha_*\mu_\ell^{\otimes i+j},$$

induce the product in motivic cohomology, by [SV, 7.1].

Corollary 6.4. *The pair (I_{nis},θ') is suitable, and is both an Adem and a Cartan object. In addition, $\tau\alpha_*I\to I_{nis}$ defines a perfect morphism $(\tau\alpha_*I,\tau\theta)\to(I_{nis},\theta')$.*

Proof. By Example 6.3.1, $(\tau\alpha_*I,\tau\theta)$ is suitable and both an Adem and a Cartan object. By Lemma 4.5.1, $(I_{\rm nis},\theta')$ is suitable, and η is a perfect morphism. As observed in Examples 4.9.1 and 4.10.1, $(I_{\rm nis},\theta')$ is both an Adem and a Cartan object. $\square$

Remark 6.4.1. By the comparison theorem, $\tau\alpha_*I\to\alpha_*I$ lifts to a morphism $f:I_{\rm nis}\to\alpha_*I$. We will see in Lemma 6.6 that f defines a morphism of suitable pairs, $(I_{\rm nis},\theta')\to(\alpha_*I,\alpha_*\theta)$.

Recall that $H^{n,i}(X,\mathbb{F}_\ell)=H^n_{\rm nis}(X,\mathbb{F}_\ell(i))$. By the Norm Residue Theorem 6.2,

$$\oplus_iH^{n,i}(X,\mathbb{F}_\ell)=\oplus_iH^n_{\rm nis}(X,I(i)_{\rm nis})=H^nI_{\rm nis}(X).$$

Definition 4.4, applied to $(I_{\rm nis},\theta')$, yields operations P^a and Q^a on the cohomology of $I_{\rm nis}(X)$, sending $\oplus_iH^{n,i}(X)$ to $\oplus_iH^{N,i\ell}(X)$ and $\oplus_iH^{N+1,i\ell}(X)$. By construction, these operations multiply the weight i by ℓ. We shall now formalize this observation.

Definition 6.5 (P^a)**.** The motivic cohomology operations are the operations

$$P^a:H^{n,i}(X)\to H^{N,i\ell}(X),\ \text{ and }\ Q^a:H^{n,i}(X)\to H^{N+1,i\ell}(X),$$

$N=n+2a(\ell-1)$, defined by the suitable pair $(I_{\rm nis},\theta')$ of Corollary 6.4, using the identification of $H^{n,i}(X)$ with $H^nI_{\rm nis}(X)$. If $n<2a$, we define $P^a=0$.

If $\ell=2$, we write Sq^{2a} for P^a and Sq^{2a+1} for βP^a, so that Sq^a takes $H^{n,i}(X)$ to $H^{n+a,2i}(X)$. If $n<a$ then Sq^a is zero on $H^{n,i}(X)$.

By Lemma 4.5, these operations are natural in X.

Remark 6.5.1. These motivic cohomology operations are almost surely the operations defined by Kriz and May in [KM, I.7.2], and by Joshua in [Jo, §8]; compare with [BJ].

Lemma 6.6. *The motivic cohomology operations P^a and Q^a are compatible with the étale cohomology operations P^a and Q^a in the sense that the diagram* (6.1) *commutes.*

Proof. We need to show that $f\theta'$ is chain homotopic to $(\alpha_*\theta)(1\otimes f^{\otimes\ell})$. Consider the following diagram of complexes in $\mathcal{S}h_\pi$.

$$\begin{tikzcd} W_*\otimes(\tau\alpha_*I)^{\otimes\ell} \arrow[r, "1\otimes\epsilon^{\otimes\ell}"] \arrow[d, "\tau\alpha_*\theta"] & W_*\otimes I_{\mathrm{nis}}^{\otimes\ell} \arrow[r, "1\otimes f^{\otimes\ell}"] \arrow[d, "\theta'"] & W_*\otimes(\alpha_*I)^{\otimes\ell} \arrow[d, "\alpha_*\theta"] \\ \tau\alpha_*I \arrow[r, "\epsilon"] & I_{\mathrm{nis}} \arrow[r, "f"] & \alpha_*I \end{tikzcd}$$

By Lemma 6.3, the bottom composite $\tau\alpha_*I\to\alpha_*I$ defines a perfect morphism of suitable pairs, meaning that the outer square commutes (see 4.2). By Corollary 6.4, $\epsilon:\tau\alpha_*I\xrightarrow{\sim}I_{\mathrm{nis}}$ defines a perfect morphism of suitable pairs, meaning that the left square commutes. Because ϵ is a quasi-isomorphism, so is $1\otimes\epsilon^{\otimes\ell}$, by Lemma 1.6. It follows that the right square commutes in the derived category. Because α_*I is a bounded below complex of injectives, this implies that the right square commutes up to chain homotopy equivalence; see [WH, 10.4.7]. □

Remark 6.6.1. It is an easy exercise to show that the motivic and étale Bockstein operations are compatible, using $\tau^{\le i}\mu_{\ell^2}^{\otimes i}$. We omit the details.

7. Motivic formulas

We now show that the motivic cohomology operations P^a of Definition 6.5 enjoy familiar properties.

Proposition 7.1. *If $u\in H^{2n,i}(X)$ then $P^n(u)=u^\ell$.*

Proof. This is [M, 2.4]. □

We now turn to the Adem relations. Recall that by convention $\binom{n}{k}$ is zero if $k<0$. Thus the sums below run over $t\le a/\ell$.

Theorem 7.2 (Adem Relations). *If $\ell>2$ and $a<b\ell$ then*

$$P^aP^b=\sum_{s+t=b}(-1)^{a+t}\binom{(\ell-1)s-1}{a-t\ell}P^{a+s}P^t;$$

$$P^a\beta P^b=\sum_{s+t=b}(-1)^{a+t}\binom{(\ell-1)s}{a-t\ell}\beta P^{a+s}P^t+(-1)^{a+t}\binom{(\ell-1)s-1}{a-t\ell-1}\beta P^{a+s}\beta P^t.$$

If $\ell=2$ and $a<2b$ then

$$Sq^aSq^b=\sum_{s+t=b}\binom{s-1}{a-2t}Sq^{a+s}Sq^t.$$

Proof. By Corollary 6.4, $(I_{\mathrm{nis}},\theta')$ is an Adem object. As noted in [M, 4.7] (see Definition 4.9), this implies that the P^a and Q^a satisfy the Adem relations. We have replaced Q^a by βP^a (using Theorem 8.11) for the sake of familiarity. □

Bistable Operations 7.3. In [V1], Voevodsky defines (bistable) cohomology operations P_V^a on $H^{n,i}(-,\mathbb{F}_\ell)$ of bidegree $(2a(\ell-1), a(\ell-1))$. These satisfy: $P_V^0 x = x$ for all x; $P_V^a x = x^\ell$ for $x \in H^{2a,a}(X,\mathbb{F}_\ell)$; $P_V^a = 0$ on $H^{n,i}$ if $i \le a$ and $n < i+a$; the usual Adem relations hold when $\ell > 2$. The cohomological degrees of P^a and P_V^a are the same, namely $2a(\ell-1)$, but the weights differ if $a \neq i$: if $a < i$ then P_V^a has lower weight, but if $a > i$ then P^a has lower weight.

When $\ell = 2$, Voevodsky's operations Sq_V^{2i} have bidegree$(2i,i)$ and $Sq_V^{2i+1} = \beta Sq^{2i}$. They satisfy a modified Cartan formula [V1, 9.7] which differs from our Cartan formula (Theorem 7.4) by the presence of a factor of $[\zeta]$ in some terms.

Remark 7.3.1. Brosnan and Joshua have observed in [BJ, 2.1] and [BJ1, 1.1(iii)] that the motivic-to-étale map sends P_V^a to P^a and Sq_V^a to Sq^a. The key is to observe that Voevodsky's total power operation [V1, 5.3] is compatible with Epstein's reduced power map (Definition 1.7 above).

Theorem 7.4 (Cartan Formula)**.** *Let $u \in H^{n,i}(X)$ and $v \in H^{m,j}(Y)$. Then in $H^{*,(i+j)\ell}(X \times Y)$ we have:*

$$P^a(u \cup v) = \sum_{s+t=a} P^s(u) \cup P^t(v), \quad \ell > 2,$$

and $Sq^a(u \cup v) = \sum_{s+t=a} Sq^s(u) \cup Sq^t(v)$ when $\ell = 2$. There is a similar formula for $Q^a(u \cup v)$.

Proof. By Corollary 6.4, $(I_{\text{nis}}, \theta')$ is a Cartan object (Definition 4.10). The formula now follows from [M, 2.7] (for both P^a and Q^a). □

Cohomology operations on $H^{n,0}$ are easy to describe because of the following characterization. Recall that $\pi_0 X$ denotes the set of connected components of a scheme X; if $X_\bullet$ is a simplicial scheme, $\pi_0 X_\bullet$ denotes the simplicial set $n \mapsto \pi_0(X_n)$.

Lemma 7.5. *Let A be any abelian group. If $X_\bullet$ is a smooth simplicial scheme, the motivic cohomology ring $H^{*,0}(X_\bullet, A)$ is isomorphic to the topological cohomology $H^*_{top}(\pi_0 X_\bullet, A)$ of the simplicial set $\pi_0 X_\bullet$.*

Proof. For smooth connected X we have $H^{n,0}(X,A) = H^n_{\text{nis}}(X,A) = 0$ for $n > 0$ and $H^{0,0}(X,A) = A$, almost by definition; see [MVW, 3.4]. Hence the spectral sequence $E_1^{p,q} = H^q(X_p, A) \Rightarrow H^{p+q,0}(X)$ degenerates to the cohomology of the chain complex $\text{Hom}(\pi_0 X_\bullet, A)$, which is $H^*_{\text{top}}(\pi_0 X_\bullet, A)$. For a simplicial set K such as $\pi_0 X_\bullet$, the construction of the product in motivic cohomology [MVW, 3.11] shows that $H^*_{\text{top}}(K) \cong H^{*,0}(K)$ is an isomorphism of rings. □

Any simplicial set $X = X_\bullet$ may be regarded as a discrete simplicial scheme; in degree n it is the disjoint union of copies of $\text{Spec}(k)$, indexed by X_n.

Theorem 7.6. *For any simplicial set X, there is a natural isomorphism*

$$H^{*,*}(k,\mathbb{F}_\ell) \otimes H^*_{top}(X,\mathbb{F}_\ell) \cong H^{*,*}(k,\mathbb{F}_\ell) \otimes H^{*,0}(X,\mathbb{F}_\ell) \xrightarrow{\cup} H^{*,*}(X,\mathbb{F}_\ell).$$

Proof. By Lemma 7.5, $H^*_{\text{top}}(X,\mathbb{F}_\ell) \cong H^{*,0}(X,\mathbb{F}_\ell)$. Thus the map exists and is natural in X by the above remarks. It is an isomorphism for spheres by [V1, (2.7)]. If $X^{(i)}$ denotes the i-skeleton of X, the cone C_i of $X^{(i-1)} \to X^{(i)}$ is a bouquet of i-spheres, and the map is compatible with the exact sequence

$$\cdots \tilde{H}^n(C_i,\mathbb{F}_\ell) \to H^n(X^{(i)},\mathbb{F}_\ell) \to H^n(X^{(i-1)},\mathbb{F}_\ell) \cdots$$

of [V2, Lemma 8.2]. The result now follows by induction on i. □

Applying this to the classifying space K_n for simplicial cohomology, we obtain:

Corollary 7.7. *The ring of motivic cohomology operations on $H^{n,0}(-,\mathbb{F}_\ell)$ is isomorphic to $H^{*,*}(k,\mathbb{F}_\ell)\otimes H^*_{top}(K_n)\cong H^{*,*}(K_n,\mathbb{F}_\ell)$ as a free left $H^{*,*}(k,\mathbb{F}_\ell)$-module.*

If K is a simplicial set, the isomorphism $H^{*,0}(K,\mathbb{F}_\ell)\cong H^*_{\mathrm{top}}(K,\mathbb{F}_\ell)$ is compatible with the action of the P^I. This is clear from Lemma 6.6 and Remark 2.2.

Example 7.7.1. Let Δ^1 denote the simplicial 1-simplex and $s\in H^{1,0}(\Delta^1,\partial\Delta^1)$ the generator. By the above comparison with topology, $P^0(s)=s$. By definition, $P^a(s)=0$ for $a>0$.

Recall that the simplicial suspension SX of a pointed simplicial scheme X is again a simplicial scheme. Multiplication by the element s of Example 7.7.1 induces a canonical isomorphism $H^{n,i}(X,\mathbb{F}_\ell)\xrightarrow{\cong}H^{n+1,i}(SX,\mathbb{F}_\ell)$. (Compare to Lemmas 1.2 and 2.1 of [SE].)

Proposition 7.8. *The motivic operations P^a and Q^a are simplicially stable in the sense that they commute with simplicial suspension: there are commutative diagrams for all X, n and i, with $N=n+2a(\ell-1)$, the diagram for P^a being:*

$$\begin{array}{ccc} H^{n,i}(X) & \xrightarrow{P^a} & H^{N,i\ell}(X) \\ \cong\big\downarrow & & \cong\big\downarrow \\ H^{n+1,i}(SX) & \xrightarrow{P^a} & H^{N+1,i\ell}(SX). \end{array}$$

Proof. By the Cartan formula 7.4, $P^a(sx)=P^0(s)P^a(x)=s\cdot P^a(x)$. $\square$

Recall from [V1] that although each $H^{n,i}(-,\mathbb{F}_\ell)$ is defined as a contravariant functor on the category of smooth simplicial schemes, it is homotopy invariant and factors through the pointed motivic homotopy category $\mathrm{Ho}_\bullet$. It is an elementary observation that any natural transformation between homotopy invariant functors, defined on the category of smooth simplicial schemes, must factor through $\mathrm{Ho}_\bullet$. In particular, cohomology operations $H^{n,i}(-,\mathbb{F}_\ell)\to H^{p,q}(-,\mathbb{F}_\ell)$ may be regarded as natural transformations between functors defined on $\mathrm{Ho}_\bullet$.

Example 7.9. The classifying space $K=K(\mathbb{F}_\ell(i),n)$ for $H^{n,i}(-,\mathbb{F}_\ell)$ is an object of $\mathrm{Ho}_\bullet$; as observed in [V1, p. 3] we have $H^{n,i}(X,\mathbb{F}_\ell)\cong\mathrm{Hom}_{\mathrm{Ho}_\bullet}(X_+,K)$. By the Yoneda Lemma, cohomology operations $H^{n,i}(-,\mathbb{F}_\ell)\to H^{p,q}(-,\mathbb{F}_\ell)$ correspond to elements of $H^{p,q}(K,\mathbb{F}_\ell)$. For example, under the map from $H^{p,q}=H^{p,q}(k,\mathbb{F}_\ell)$ to $H^{p,q}(K,\mathbb{F}_\ell)$ induced by the structure map $K\to\mathrm{Spec}(k)$, $c\in H^{p,q}$ corresponds to the constant operation sending every element of $H^{n,i}(X,\mathbb{F}_\ell)$ to the image of c in $H^{p,q}(X,\mathbb{F}_\ell)$. Similarly, the canonical class $\iota\in H^{n,i}(K,\mathbb{F}_\ell)$ corresponds to the identity operation.

If $n\geq i$, we see from [V3, 3.27] that the summands of the motive $\mathbb{F}_{\ell\mathrm{tr}}(K)$ having smallest weight or degree are $\mathbb{F}_\ell(i)[n]$ and $\mathbb{F}_\ell(i)[n+1]$. It follows that: every cohomology operation $H^{n,i}\to H^{p,q}$ with $p\leq n+1$ is an $\mathbb{F}_\ell$-linear combination of the Bockstein, the identity, and constant operations. More precisely:

a) if $p<n$ then $H^{p,*}(K)=H^{p,*}$, corresponding to constant operations;

b) $H^{n,*}(K)\cong H^{0,*}\cdot\iota\oplus H^{n,*}$, with $(b\,\iota,c)$ corresponding to the operation $\phi(x)=bx+c$;

c) $H^{n+1,*}(K)\cong H^{0,*}\cdot\beta(\iota)\oplus H^{1,*}\cdot\iota\oplus H^{n+1,*}$, with (a,b,c) corresponding to the operation $\phi(x)=a\beta(x)+bx+c$.

If $n < i$, this is no longer the case. In Example 11.5 below, we show that there is a weight-reducing operation $H^{1,2}(-,\mathbb{F}_\ell) \to H^{2,1}(-,\mathbb{F}_\ell)$ for all k, and a weight-preserving operation $H^{1,2}(-,\mathbb{F}_\ell) \to H^{3,2}(-,\mathbb{F}_\ell)$ for most k. For another example, suppose that $\zeta \in k$ and $n \le i$. Then cupping with $[\zeta] \in H^{0,1}(k,\mathbb{F}_\ell)$ is an isomorphism by Theorem 6.2; its inverse (defined when $n < i$) is an operation $H^{n,i}(-,\mathbb{F}_\ell) \to H^{n,i-1}(-,\mathbb{F}_\ell)$.

8. Relation to periodicity

Sometimes we can deduce motivic operations from étale operations. For example, if $n \le i$ (and hence $n \le i\ell$) then the diagram (6.1) allows us to identify the motivic operation $P^0 : H^{n,i}(X) \to H^{n,i\ell}(X)$ with the étale operation $P^0 : H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i}) \cong H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i\ell})$, and thus conclude that P^0 is an isomorphism in this range. The same reasoning, using the Norm Residue Theorem 6.2, shows that if $n \le i$ and $n + 2a(\ell-1) \le i\ell$, the motivic and étale operations P^a agree on $H^{n,i}(X) \cong H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i})$, and also agree with $b^{(i-a)(\ell-1)/d}P^a_V$, where $b \in H^{0,d}(k)$ is defined as follows.

Fix a primitive ℓ^{th} root of unity, ζ, in an extension field of k; this choice determines a generator $[\zeta]$ of $H^0(k(\zeta),\mu_\ell)$. If $[k(\zeta):k] = d$ then $H^{0,d}(k) \cong H^0_{\mathrm{et}}(k,\mu_\ell^{\otimes d}) \cong H^0_{\mathrm{et}}(k,\mathbb{F}_\ell)$, and the element $[\zeta]^d = [\zeta^{\otimes d}]$ descends to a "periodicity" element b in $H^{0,d}(k)$. By abuse of notation, if $d|m$ we write $[\zeta]^m$ for the element $b^{m/d}$ of $H^{0,m}(k)$. (If $d = 1$ then $b = [\zeta]$.)

Note that multiplication by b is a map from $H^{n,i}(X)$ to $H^{n,i+d}(X)$; by the Norm Residue Theorem 6.2, it is an isomorphism when $i \ge n$. By construction, this is the map in cohomology induced by the change-of-truncation map

$$\mathbb{F}_\ell(i) \cong \tau^{\le i} R\alpha_*\mu_\ell^{\otimes i} \to \tau^{\le i+d} R\alpha_*\mu_\ell^{\otimes i} \cong \mathbb{F}_\ell(i+d). \tag{8.1}$$

associated to the isomorphism of étale sheaves $\mu_\ell^{\otimes i} \to \mu_\ell^{\otimes i+d}$ sending the generator $\zeta^{\otimes i}$ to the generator $\zeta^{\otimes i+d}$.

Write $H^{n,i}(X)[1/b]$ for the colimit of

$$H^{n,i}(X) \xrightarrow{b} H^{n,i+d}(X) \xrightarrow{b} \cdots \xrightarrow{b} H^{n,i+jd}(X) \xrightarrow{b} \cdots.$$

From the diagram

$$\begin{array}{ccccccc}
H^{n,i}(X) & \xrightarrow{b} & H^{n,i+d}(X) & \xrightarrow{b} & \cdots H^{n,i+jd}(X) & \xrightarrow{b} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i}) & \xrightarrow{\cong} & H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i+d}) & \xrightarrow{\cong} & \cdots H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i+jd}) & \xrightarrow{\cong} & \cdots
\end{array}$$

we obtain a natural transformation from $H^{n,i}(X)[1/b]$ to $H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i})$.

We can formulate this in the motivic derived category DM, using the étale-to-Nisnevich change of topology map α. Recall from [MVW, 10.2] that $H^n_{\mathrm{et}}(X,\mu_\ell^{\otimes i})$ is isomorphic to

$$\mathrm{Hom}_{DM_{\mathrm{et}}}(\alpha^*\mathbb{F}_{\ell,\mathrm{tr}}X, \mathbb{F}_\ell(i)[n]) \cong \mathrm{Hom}_{DM}(\mathbb{F}_{\ell,\mathrm{tr}}X, R\alpha_*\mu_\ell^{\otimes i}).$$

The map $\mathbb{F}_\ell(i) = \tau^{\le i}R\alpha_*\mu_\ell^{\otimes i} \to R\alpha_*\mu_\ell^{\otimes i}$ is compatible with the map $\mathbb{F}_\ell(i) \to \mathbb{F}_\ell(i+d) \to R\alpha_*\mu_\ell^{\otimes i+d}$, so it factors through a map $\mathbb{F}_\ell(i)[1/b] \to R\alpha_*\mu_\ell^{\otimes i}$, where $\mathbb{F}_\ell(i)[1/b]$ denotes the (homotopy) colimit in DM of

$$\mathbb{F}_\ell(i) \xrightarrow{b} \mathbb{F}_\ell(i+d) \xrightarrow{b} \mathbb{F}_\ell(i+2d) \xrightarrow{b} \cdots \xrightarrow{b} \mathbb{F}_\ell(i+jd) \xrightarrow{b} \cdots.$$

The following calculation is originally due to Levine [L].

Theorem 8.2. *For each i, $\mathbb{F}_\ell(i)[1/b] \to R\alpha_*\mu_\ell^{\otimes i}$ is an isomorphism in DM.*
For X smooth and all n, $H^{n,i}(X)[1/b] \longrightarrow H^n_{et}(X,\mu_\ell^{\otimes i})$ is an isomorphism.

Proof. Any complex C is the homotopy colimit of the change-of-truncation maps $\tau^{\le m}C \to \tau^{\le m+1}C$. For $C = R\alpha_*\mu_\ell^{\otimes i}$, this yields the first assertion. The second assertion is an immediate consequence of this and the fact that $\mathbb{F}_{\ell,\mathrm{tr}}X$ is a compact object in DM, so $\mathrm{Hom}_{DM}(\mathbb{F}_{\ell,\mathrm{tr}}X, -)$ commutes with homotopy colimits. □

Our next goal is to compare P^0 to the cohomology of the change-of-truncation map $\tau^{\le i}R\alpha_*\mu_\ell^{\otimes i} \to \tau^{\le i\ell}R\alpha_*\mu_\ell^{\otimes i}$ of (8.1).

Lemma 8.3. *The Frobenius map $\mathbb{F}_\ell(i) \xrightarrow{\Phi} \mathbb{F}_\ell(i\ell)$ in motivic cohomology is chain homotopic to the change-of-truncation map*

$$\mathbb{F}_\ell(i) \cong \tau^{\le i}R\alpha_*\mu_\ell^{\otimes i} \to \tau^{\le i\ell}R\alpha_*\mu_\ell^{\otimes i} \cong \mathbb{F}_\ell(i\ell).$$

The Frobenius $H^{n,i}(X) \xrightarrow{\Phi} H^{n,i\ell}(X)$ is multiplication by $b^{i(\ell-1)/d} = [\zeta^{\otimes i(\ell-1)}]$.

Proof. The Frobenius endomorphism is the identity on the étale sheaf of rings $A = \oplus_{i=0}^d \mu_\ell^{\otimes i}$, so if we fix i and an injective replacement $\mu_\ell^{\otimes i} \to I$, the Frobenius on $\mu_\ell^{\otimes i}$ lifts to a map $f_i : I \to I$ which is chain homotopic to the identity. Since the product in motivic cohomology is induced from the product on $R\alpha_*\mu_\ell^{\otimes i} = \alpha_* I$, the Frobenius in motivic cohomology is represented by the good truncation in degrees at most $i\ell$ of the composite $\tau^{\le i}\alpha_*I \subset \alpha_*I \xrightarrow{f_i} \alpha_*I$. Since good truncation preserves chain homotopy, it is chain homotopic to the canonical map $\tau^{\le i}\alpha_*I \subset \tau^{\le i\ell}\alpha_*I$.

The final assertion follows from (8.1). □

Proposition 8.4. *The map $P^0 \colon H^{n,i}(X) \to H^{n,i\ell}(X)$ is multiplication by $b^{i(\ell-1)/d}$. Equivalently, P^0 is the cohomology of the change-of-truncation map*

$$\tau^{\le i}R\alpha_*\mu_\ell^{\otimes i} \to \tau^{\le i\ell}R\alpha_*\mu_\ell^{\otimes i}.$$

Proof. Recall that the Godement resolution $\mathcal{F} \to S^\bullet(\mathcal{F})$ is a functorial simplicial resolution of any sheaf $\mathcal{F}$ by flasque sheaves. Letting $S_i^\bullet$ denote the total complex of the Godement resolution of $\tau^{\le i}R\alpha_*\mu_\ell^{\otimes i}$, it follows that the product on $\oplus\tau^{\le i}R\alpha_*\mu_\ell^{\otimes i}$ induces a product pairing $S_i^n \otimes S_j^n \to S_{i+j}^n$ for all n. In particular, the Frobenius on $R\alpha_*\mu_\ell^{\otimes i}$ induces a map $S_i^n \to S_{i\ell}^n$.

In [E, 11.1], Epstein shows that the Godement resolution satisfies the conditions of his section 8. By functoriality, the equivariant map $(R\alpha_*\mu_\ell^{\otimes i})^{\otimes\ell} \to R\alpha_*\mu_\ell^{\otimes i\ell}$ constructed after Theorem 6.2 lifts to an equivariant map $(S_i^\bullet)^{\otimes\ell} \to S_{i\ell}^\bullet$. This is the analogue of [E, 8.3.2], and is exactly what we need in order for the proof of [E, 8.3.4] to work. Thus if we represent $v \in H^{n,i}(X)$ by a cocycle u in the algebra $H^0(X, S^n)$, then $P^0 v$ is represented by the element u^ℓ of $H^0(X,S^n)$. Therefore P^0 is represented by the Frobenius. □

Example 8.4.1. Recall that $b \in H^{0,d}(k)$. Since $P^0(b) = b^\ell$ (by 8.4), the Cartan formula 7.4 yields $P^a(bx) = b^\ell P^a(x)$.

Recall from [V3, 2.60] that a *split proper Tate motive* is a direct sum of Tate motives $\mathbb{F}_\ell(i)[2i+j]$ with $j \ge 0$. If the weights i are at least n then we say the motive has weight $\ge n$. Note that the cohomology of $\mathbb{F}_\ell(i)[2i+j]$ is a free bigraded $H^{*,*}$-module of rank 1 with a generator in bidegree $(2i+j, i)$.

It follows that we have a Künneth formula (see [W, 4.1]): if $\mathbb{F}_{\ell,\mathrm{tr}}(Y)$ is a split proper Tate motive then $H^{*,*}(Y)$ is a free bigraded $H^{*,*}$-module, and

$$H^{*,*}(X\times Y)\cong H^{*,*}(X)\otimes_{H^{*,*}}H^{*,*}(Y). \tag{8.5}$$

Example 8.6. Let $K=K(\mathbb{F}_\ell(i),n)$ be the Eilenberg-Mac Lane space classifying $H^{n,i}(-,\mathbb{F}_\ell)$; if $n\ge 2i\ge 0$ then $M=\mathbb{F}_{\ell,\mathrm{tr}}(K)$ is a split proper Tate motive of weight $\ge i$, by [V3, 3.28]. It follows that $H^{*,*}(K^{\otimes p})$ is the p-fold tensor product of $H^{*,*}(K)$ with itself over $H^{*,*}$.

Recall from [MVW, 3.1] that $\mathbb{F}_\ell(i)[i]$ is represented by the abelian presheaf $\mathbb{F}_{\ell,\mathrm{tr}}(\mathbb{G}_m^{\wedge i})$ so $\mathbb{F}_\ell(i)[n]$ is represented by the simplicial abelian presheaf associated to $\mathbb{F}_{\ell,\mathrm{tr}}(\mathbb{G}_m^{\wedge i})[n-i]$ when $n\ge i$. From the adjunction

$$\mathrm{Hom}_{\mathrm{Ho}_\bullet}(X,u\mathbb{F}_\ell(i)[n])\cong\mathrm{Hom}_{DM}(\mathbb{F}_{\ell,\mathrm{tr}}(X),\mathbb{F}_\ell(i)[n])=H^{n,i}(X),$$

(see Example 7.9) we see that the classifying space $K(\mathbb{F}_\ell(i),n)$ of $H^{n,i}(-,\mathbb{F}_\ell)$ is the simplicial abelian presheaf $G=u\mathbb{F}_\ell(i)[n]$ underlying $\mathbb{F}_\ell(i)[n]$; see [V3, p. 5].

Lemma 8.7. *If $\mathbb{F}_{\ell,tr}(Y)$ is a split proper Tate motive then multiplication by b^e is an injection from $H^{p,q}(Y,\mathbb{F}_\ell)$ into $H^{p,q+de}(Y,\mathbb{F}_\ell)$, and hence into $H^{p,q}(Y,\mathbb{F}_\ell)[1/b]$.*

Proof. It suffices to consider $\mathbb{F}_{\ell,\mathrm{tr}}(Y)=\mathbb{F}_\ell(i)[2i+j]$. There is no harm in increasing e so that $(\ell-1)|de$. Set $p'=p-2i-j$ and $q'=q-i$. Since $H^{*,*}(Y)$ is a free $H^{*,*}(k)$-module of rank 1, the assertion for $H^{p,q}(Y)$ amounts to the assertion that either $H^{p',q'}(k)=0$ (and injectivity is obvious) or else $0\le p'\le q'$ and $H^{p',q'}(k)\cong H^{p'}_{\mathrm{et}}(k,\mu_\ell^{\otimes q'})$. In the latter case, we also have $H^{p',q'+de}(k)\cong H^{p'}_{\mathrm{et}}(k,\mu_\ell^{\otimes q'+de})$ and the isomorphism is induced from the isomorphism $\mu_\ell^{\otimes q'}\cong\mu_\ell^{\otimes q'+de}$. □

In the next proposition, we write K for $K(\mathbb{F}_\ell(i),n)$. For each p and q, there is a canonical map $H^{p,q}(K,\mathbb{F}_\ell)\to H^p_{\mathrm{et}}(K,\mu_\ell^{\otimes q})$. It sends the motivic operations P^a of Definition 6.5 to the étale operations P^a of Theorem 1.3.

Proposition 8.8. *If $n\ge 2i$, the canonical map is an injection, from the set $H^{p,q}(K,\mathbb{F}_\ell)$ of motivic cohomology operations $H^{n,i}\to H^{p,q}$ to the set $H^p_{et}(K,\mu_\ell^{\otimes q})$ of étale cohomology operations $H^n_{et}(-,\mu_\ell^{\otimes i})\to H^p_{et}(-,\mu_\ell^{\otimes q})$.*

Proof. By the usual transfer argument, we may assume that $\zeta\in k$. Let K denote the Eilenberg-Mac Lane space classifying $H^{n,i}(-,\mathbb{F}_\ell)$. By Example 8.6, $\mathbb{F}_{\ell,\mathrm{tr}}(K)$ is a split proper Tate motive. By Lemma 8.7 and Levine's Theorem 8.2, $H^{p,q}(K,\mathbb{F}_\ell)$ injects into $H^{p,q}(K,\mathbb{F}_\ell)[1/b]\cong H^p_{\mathrm{et}}(K,\mu_\ell^{\otimes q})$. Thus the group $H^{p,q}(K,\mathbb{F}_\ell)$ of motivic cohomology operations injects into the group $H^p_{\mathrm{et}}(K,\mu_\ell^{\otimes q})$ of étale cohomology operations. □

Corollary 8.9. *Suppose that $n\ge 2i$ and $n\ge 2a$. Then for $x\in H^{n,i}(X)$:*

(1) *If $a\le i$, $P^a(x)=[\zeta]^{(i-a)(\ell-1)}\cup P^a_V(x)$;*
(2) *If $a\ge i$, $P^a_V(x)=[\zeta]^{(a-i)(\ell-1)}\cup P^a(x)$.*

Proof. (Cf. [BJ, Thm. 1.1]) The two sides have the same bidegree, and agree with $P^a(x)$ in étale cohomology by Lemma 6.6 and Remark 7.3.1. □

Corollary 8.10. *If $n\ge i$ and $x\in H^{2n,i}$ then $P^n_V(x)=[\zeta]^{(n-i)(\ell-1)}\cup x^\ell$.*

Proof. This is the case $a=n$ of Corollary 8.9, as $P^n(x)=x^\ell$ (Proposition 7.1). □

We can now show that the motivic Q^a equals βP^a.

Theorem 8.11. *The motivic operations Q^a on $H^{n,i}$ satisfy $Q^a = \beta P^a$.*

Proof. Set $K = K(\mathbb{F}_\ell(i), n)$, so that motivic cohomology operations $H^{n,i} \to H^{N,i\ell}$ correspond to elements of $H^{N,i\ell}(K)$ (see 8.6). Now the identity on $H^{n,i}$ is represented by the canonical element ι of $H^{n,i}(K)$, and the motivic cohomology operations Q^a and βP^a are represented by the elements $Q^a(\iota)$ and $\beta P^a(\iota)$ of $H^{N,i\ell}(K)$.

We first consider Q^0. The map $H^{n+1,i\ell}(K) \to H^{n+1}_{\text{et}}(K, \mu_\ell^{\otimes i\ell})$ is an isomorphism if $n \le i\ell$ by Theorem 6.2, and is an injection if $n \ge 2i$ by Proposition 8.8. By Lemma 6.6 and Remark 6.6.1, we have a commutative diagram:

$$\begin{array}{ccc}
H^{n,i}(K) & \xrightarrow{Q^0-\beta P^0} & H^{n+1,i\ell}(K) \\
\downarrow & & \text{into}\downarrow \\
H^n_{\text{et}}(K,\mu_\ell^{\otimes i}) & \xrightarrow{Q^0-\beta P^0} & H^{n+1}_{\text{et}}(K,\mu_\ell^{\otimes i\ell}).
\end{array}$$

The bottom map is zero by Proposition 3.3. It follows that $Q^0(\iota) = \beta P^0(\iota)$ in $H^{n+1,i\ell}(K)$, and hence that $Q^0 = \beta P^0$ as cohomology operations.

Now suppose that $a > 0$, and set $N = n + a(\ell - 1) + 1$. If $n \ge 2i$, we consider the commutative diagram:

$$\begin{array}{ccc}
H^{n,i}(K) & \xrightarrow{Q^a-\beta P^a} & H^{N,i\ell}(K) \\
\downarrow & & \text{into}\downarrow \\
H^{n,i}_{\text{et}}(K,\mu_\ell^{\otimes i}) & \xrightarrow{Q^a-\beta P^a} & H^{N}_{\text{et}}(K,\mu_\ell^{\otimes i\ell}).
\end{array}$$

The lower left horizontal map is zero by Proposition 3.3. The right vertical map is an injection by Proposition 8.8, because $\mathbb{F}_{\ell,\text{tr}}(K)$ is a split proper Tate motive (by Example 8.6). It follows that $Q^a(\iota) = \beta P^a(\iota)$ in $H^{N,i\ell}(K)$, and hence that $Q^a = \beta P^a$ as cohomology operations on $H^{n,i}$.

If $n < 2i$, we consider suspension $S^d K$, where $d = 2i - n$. By Proposition 7.8, we have a commutative diagram with vertical isomorphisms:

$$\begin{array}{ccc}
H^{n,i}(K) & \xrightarrow{Q^a-\beta P^a} & H^{N,i\ell}(K) \\
\cong\downarrow & & \cong\downarrow \\
H^{2i,i}(S^dK) & \xrightarrow{Q^a-\beta P^a} & H^{N+d,i\ell}(SK).
\end{array}$$

By the above argument, $Q^a = \beta P^a$ on $H^{2i,i}(S^d K)$. It follows that $Q^a = \beta P^a$ on $H^{n,i}(K)$, and hence as cohomology operations on $H^{n,i}$. □

9. Borel's Theorem

In order to go from $H^{1,*}$ to $H^{n,*}$, we need a slight generalization of Borel's theorem [McC, 6.21], one which accounts for the coefficient ring $H^{*,*} = H^{*,*}(\operatorname{Spec} k)$.

Definition 9.1. Let H^* be a graded-commutative $\mathbb{F}_\ell$-algebra. If W^* is a graded H^*-algebra, an *ℓ-simple system of generators* of W^* over H^* is a totally ordered set of elements x_i, such that W^* is a free left H^*-module on the monomials $x_{i_1}^{m_1} \cdots x_{i_k}^{m_k}$, where the $i's$ are in order and $0 \le m_j < \ell$ (with $m_j \le 1$ if $\deg(x_j)$ is odd).

Theorem 9.2. *Let H^* be a graded-commutative $\mathbb{F}_\ell$-algebra with $H^0 = \mathbb{F}_\ell$, and suppose that $\{E_r^{*,*}, d_r\}$ is a 1^{st}-quadrant spectral sequence of graded-commutative H^*-algebras converging to H^*. Set $V^* = E_2^{*,0}$ and $W^* = E_2^{0,*}$, and suppose that (i) $E_2^{*,*} \cong W^* \otimes_{H^*} V^*$ as algebras, and that (ii) the H^*-algebra W^* has an ℓ-simple system of generators $\{x_i\}$, each of which is transgressive.*

Then V^* is the tensor product of H^* and a free graded-commutative $\mathbb{F}_\ell$-algebra on generators $y_i = \tau(x_i)$ and (when $\ell \neq 2$ and $\deg(x_j)$ is even) $z_j = \tau(y_j \otimes x_j^{\ell-1})$. (Here τ is the transgression.)

Proof. The proof of Borel's Theorem in [McC, 6.21] goes through. □

Let G be a simplicial sheaf of groups, such as $B\mu_\ell$. We use the bar construction to form the bisimplicial classifying spaces $B_\bullet G$ (with G^p in simplicial degree p) of a simplicial sheaf of groups G and $E_\bullet G$ (with G^{p+1} in simplicial degree p). We write the canonical projection as $E_\bullet G \xrightarrow{\pi} B_\bullet G$. The Leray spectral sequence [Milne, III.1.18] becomes

$$E_2^{p,q} = H^p(B_\bullet G, R^q\pi_* A) \Rightarrow H^{p+q}(E_\bullet G, A). \tag{9.3}$$

Proposition 9.4. *Suppose that G is a simplicial sheaf of groups on a site over k, and A is a sheaf of homotopy-associative dg $\mathbb{F}_\ell$-algebras satisfying $H^0(G, A) = H^0(k, A)$ as well as the Künneth condition that*

$$H^*(U, A) \otimes_{H^*(k,A)} H^*(G, A) \xrightarrow{\cong} H^*(U \times G, A)$$

is an isomorphism for all U in the site of $B_\bullet G$. Then the Leray spectral sequence (9.3) *satisfies condition (i) of Borel's Theorem 9.2 with*

$$E_2^{p,q} = H^p(B_\bullet G, A) \otimes_{H^*(k,A)} H^q(G, A).$$

Proof. For simplicity of notation, let us write $\otimes_H$ for $\otimes_{H^*(k,A)}$. We first claim that the higher direct images $R^q\pi_*(A)$ are $A \otimes_H H^*(G, A)$. To see this, recall that $R^q\pi_*(A)$ is the sheafification of the presheaf that to a map $U \to B_pG$ associates $H^q(\pi^{-1}U, A)$, where $\pi^{-1}U = E_\bullet G \times_{B_pG} U$ is $U \times G$. By hypothesis, $H^*(\pi^{-1}U, A)$ is $H^*(U, A) \otimes_H H^*(G, A)$. The claim follows, since sheafification commutes with $\otimes_H H^*(G, A)$ and the sheaf associated to $H^q(-, A)$ is A if $q = 0$ and zero for $q > 0$. Thus we have $E_2^{p,q} = H^p(B_\bullet G, A) \otimes_H H^q(G, A)$.

It remains to check the Borel condition. By hypothesis, $H^0(G, A) = H^0(k, A)$, so $E_2^{p,0} = H^p(B_\bullet G, A)$ and $H^0(B_\bullet G, A) = H^0(k, A)$. Thus we have

$$E_2^{0,q} = H^0(B_\bullet G, A) \otimes_H H^q(G, A) = H^q(G, A).$$

The fact that the spectral sequence is multiplicative follows from the fact that A is a sheaf of algebras, and the work of Massey [Mass]. □

Kudo's Theorem 9.5. *Suppose G and A satisfy the hypotheses of Proposition 9.4. If $x \in H^n(G, A)$ transgresses to $y \in H^{n+1}(B_\bullet G, A)$ then*

(1) *$\beta(x)$ transgresses to $-\beta(y)$;*
(2) *$P^a(x)$ transgresses to $P^a(y)$; and*
(3) *if $n = 2a$ then $x^{\ell-1} \otimes y$ transgresses to $-Q^a(y)$.*

Any simplicially stable operation commutes with the transgression; see [McC, 6.5]. Hence part (2) of Theorem 9.5 is immediate whenever we know that P^a is simplicially stable. This is so for the operations P^a in étale and motivic cohomology (by 1.9 and 7.8).

Proof. (Cf. [M, 3.4]) As in the proof of Theorem 1.5, we fix a quasi-isomorphism $A \xrightarrow{\sim} I^*$. Let $f = \pi^*$ and $g = i^*$ be the canonical maps $I(G) \xleftarrow{g} I(E_\bullet G) \xleftarrow{f} I(B_\bullet G)$ coming from $G \xrightarrow{i} E_\bullet G \xrightarrow{\pi} B_\bullet G$. The assertion that x transgresses to y means that there is a cocycle b in $I^{n+1}(B_\bullet G)$ representing y, and an element u in $I^n(E_\bullet G)$, such that $f(b) = du$ and $g(u)$ is a cocyle representing x,

Since the Bockstein satisfies $g(\beta u) = \beta g(u)$ and $f(\beta b) = \beta(du) = -d(\beta u)$, we see that $\beta(x)$, which is represented by $g(\beta u)$, transgresses to $-\beta(y)$.

Recall from Section 4 that b and u determine a cocyle $P^a(b)$ in $I^*(B_\bullet G)$ representing $P^a(y)$ and an element $P^a(u)$ in $I^*(E_\bullet G)$ so that $P^a(x)$ is represented by $P^a g(u) = gP^a(u)$. By Lemma 4.8, we have

$$fP^a(b) = P^a f(b) = P^a(du) = dP^a(u).$$

It follows that $P^a(x)$ trangresses to $P^a(y)$.

Since b is a cocycle, $Q^a(b)$ represents $Q^a(y)$, and by Lemma 4.8 we have

$$fQ^a(b) = Q^a f(b) = Q^a(du) = -\,d(Q^a u).$$

Thus the class of $Q^a(u)$ transgresses to $-\,Q^a(y)$, and it suffices to show that $Q^a(u)$ represents $x^{\ell-1} \otimes y$ under the isomorphism $E_2^{p,q} \cong H^p(B_\bullet G) \otimes H^q(G)$ of 9.4.

Recall from (1.8) that $\nu_n = (-1)^r m!^{-n}$, where $m = (\ell-1)/2$. We have $\nu_n = (-1)^a$, because $n = 2a$, $(m!)^2 = (-1)^{m+1}$ and $r \equiv am \pmod 2$. We now follow p. 167 of [M] up to (9). Starting from $u \in I^n(X)$, May produces elements t_i in $I^{\otimes \ell}(X)$ and a family of elements $\{c_a\}$, $\{c'_a\}$ in $C_* \otimes I^{\otimes \ell}(X)$, depending naturally on u, such that

$$Q^a_M(u) = (-1)^a \nu(1-n)\,\theta(c'_a) = m!\,\theta(c'_a).$$

The analysis of the terms in c'_a on top of p. 171 of [M] shows that there is a term c'' such that $c' - d(c'')$ is $(-1)^m m!\,z$ plus terms mapped by θ into lower parts of the filtration, where $z = e_0 \otimes u \otimes \cdots \otimes u \otimes du$, and that $\theta(z)$ represents $x^{\ell-1} \otimes y$. Therefore, up to terms in lower parts of the filtration we have $Q^a_M(u) = m!\,\theta(c'_a) = (-1)^m (m!)^2 \theta(z) = -\theta(z)$. Since we saw in Corollary 4.7 that $Q^a(u) = Q^a_M(u)$, the result follows. □

We illustrate the use of Proposition 9.4 with the étale topology. First, consider the étale sheaf $G = \mu_\ell$. If μ_ℓ is connected then it does not satisfy the Künneth condition of Proposition 9.4 for $U = \operatorname{Spec}(\bar k)$. Indeed, $H^0(\mu_\ell, \mathbb{F}_\ell) = \mathbb{F}_\ell$ yet $H^0(G \times \operatorname{Spec} \bar k, \mathbb{F}_\ell) = \prod_1^\ell \mathbb{F}_\ell$. However, things change if we consider the étale sheaf $\mathcal{A} = \bigoplus_{i=0}^{d-1} \mu_\ell^{\otimes i}$ of Section 2.

Lemma 9.6. $H^*_{et}(X \times \mu_\ell^{\otimes i}, \mathcal{A}) \cong H^*_{et}(X, \mathcal{A}) \otimes_{H^*(k,\mathcal{A})} H^*_{et}(\mu_\ell^{\otimes i}, \mathcal{A})$.

Proof. As an étale sheaf of $\mathbb{F}_\ell$-modules, constant over $k(\zeta)$, $\mathbb{F}_\ell[\mu_\ell^{\otimes i}]$ is a direct sum of the locally constant sheaves $\mu_\ell^{\otimes \alpha}$, each of which is an invertible object. Because $\mathbb{F}_\ell[X \times \mu_\ell^{\otimes i}] \cong \mathbb{F}_\ell[X] \otimes \mathbb{F}_\ell[\mu_\ell^{\otimes i}]$, $H^n_{\rm et}(X \times \mu_\ell^{\otimes i}, \mu_\ell^{\otimes q})$ equals

$$\begin{aligned}
\operatorname{Ext}^n(\mathbb{F}_\ell[X] \otimes \mathbb{F}_\ell[\mu_\ell^{\otimes i}], \mu_\ell^{\otimes q}) &\cong \operatorname{Ext}^n(\mathbb{F}_\ell[X], \mathcal{RH}om(\mathbb{F}_\ell[\mu_\ell^{\otimes i}], \mu_\ell^{\otimes q})) \\
&\cong \operatorname{Ext}^n(\mathbb{F}_\ell[X], \mathcal{RH}om(\mathbb{F}_\ell, \oplus \mu_\ell^{\otimes q-\alpha})) \\
&\cong \oplus_\alpha \operatorname{Ext}^n(\mathbb{F}_\ell[X], \mu_\ell^{\otimes q-\alpha}) \cong \oplus_\alpha H^n_{\rm et}(X, \mu_\ell^{\otimes q-\alpha}).
\end{aligned}$$

The pairing $H^*_{\rm et}(X,\mathcal{A})\otimes_{\mathbb{F}_\ell} H^*_{\rm et}(\mu_\ell^{\otimes i},\mathcal{A})\to H^*_{\rm et}(X\times\mu_\ell^{\otimes i},\mathcal{A})$ is the direct sum over α, s and t of the top row in the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^*(\mathbb{F}_\ell[X],\mu_\ell^{\otimes s})\otimes_{\mathbb{F}_\ell}\mathrm{Ext}^*(\mu_\ell^{\otimes\alpha},\mu_\ell^{\otimes t}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^*(\mathbb{F}_\ell[X]\otimes\mu_\ell^{\otimes\alpha},\mu_\ell^{\otimes s+t}) \arrow[d, "\cong"] \\
\mathrm{Ext}^*(\mathbb{F}_\ell[X],\mu_\ell^{\otimes s})\otimes_{\mathbb{F}_\ell}\mathrm{Ext}^*(\mathbb{F}_\ell,\mu_\ell^{\otimes t-\alpha}) \arrow[r] & \mathrm{Ext}^*(\mathbb{F}_\ell[X],\mu_\ell^{\otimes s+t-\alpha}).
\end{tikzcd}$$

Since $H^*_{\rm et}(k,\mathcal{A})=\mathrm{Ext}^*(\mathbb{F}_\ell,\mathcal{A})\cong\oplus_t\,\mathrm{Ext}(\mathbb{F}_\ell,\mu_\ell^{\otimes t-\alpha})$ for each α, setting $s=q-\alpha$ and summing over s and t yields the result. $\square$

Corollary 9.7. *If Y is a coproduct of schemes which are finite products of $\mu_\ell^{\otimes i}$, then*

$$H^*_{et}(X\times Y,\mathcal{A})\cong H^*_{et}(X,\mathcal{A})\otimes_{H^*(k,\mathcal{A})}H^*_{et}(Y,\mathcal{A}).$$

Example 9.8. The ring of all étale cohomology operations from $H^2_{\rm et}(-,\mu_\ell^{\otimes i})$ to $H^*_{\rm et}(-,\mu_\ell^{\otimes *})$ is the free left $H^*_{\rm et}(k,\mathcal{A})$-module on generators in $H^{*,i}(K_2)$: monomials in the identity ($\mathrm{id}\in H^2_{\rm et}(K_2,\mu_\ell^{\otimes i})$), β, the $P^I\beta$ and the $\beta P^I\beta$ ($P^I=P^{\ell^\nu}\cdots P^\ell P^1$). This result, proven in Theorem 3.5 above, can also be obtained from the Leray spectral sequence (9.3).

Each term in the simplicial sheaf $B_\bullet\mu_\ell^{\otimes i}$ is a coproduct of products of $\mu_\ell^{\otimes i}$, so Corollary 9.7 and Proposition 9.4 imply that the Leray spectral sequence satisfies condition (i) of Borel's Theorem 9.2. The explicit description of $H^*_{\rm et}(B\mu_\ell^{\otimes i},\mathcal{A})$ in Proposition 3.4 as $H^*_{\rm et}(k,\mathcal{A})\otimes\mathbb{F}_\ell[u,v]/(u^2)$, shows that it has an ℓ-simple system of generators: u, and the $x_\nu=v^{\ell^\nu}$ for $\nu\ge0$. The transgression τ sends u to ι, so $v=\beta u$ transgresses to $-\beta$, by Kudo's Theorem 9.5(1). Thus condition (ii) is also satisfied, and Borel's Theorem states that $H^*_{\rm et}(K_2,\mathcal{A})$ is the free graded-commutative $H^{*,*}$-algebra on generators $\iota\in H^{2,i}(K_2)$,

$$y_\nu=\tau(x_\nu)\in H^{2\ell^\nu+1}(K_2,\mu_\ell^{\otimes i\ell^\nu})\quad\text{and}\quad z_\nu=\tau(x_\nu^{\ell-1}\otimes y_\nu)\in H^{2\ell^{\nu+1}+2}(K_2,\mu_\ell^{\otimes i\ell^{\nu+1}}).$$

Note that $y_0=\beta(\iota)$. Since $x_{\nu+1}=x_\nu^{\ell^\nu}=P^{\ell^\nu}x_\nu$, Kudo's Theorem 9.5(2) and an inductive argument show that $y_{\nu+1}$ is $P^{\ell^\nu}y_\nu$ and also $P^{\ell^\nu}\cdots P^\ell P^1\beta$. This completes the proof for $\ell=2$.

For $\ell>2$, it remains to show that $-z_\nu$ is $\beta P^{\ell^\nu}(y_\nu)=\beta P^{\ell^\nu}\cdots P^\ell P^1\beta$. This follows from Kudo's Theorem 9.5(3), using Proposition 3.3 to write βP^a for Q^a.

10. Motivic operations on weight 1 cohomology

We now turn to natural operations defined on the motivic cohomology groups with weight 1, *i.e.*, $H^{n,1}(X)=H^{n,1}(X,\mathbb{F}_\ell)$. We begin with the case $n=1$.

Let μ_ℓ be the group scheme of ℓ^{th} roots of unity. On pp. 130–131 of [MV], Morel and Voevodsky define a simplicial Nisnevich sheaf $B_{et}\mu_\ell$ and observe that it classifies the étale cohomology group $H^1_{\rm et}(-,\mu_\ell)$, and hence the motivic group $H^{1,1}$ by Theorem 6.2, in the sense that $[X_+,B_{et}\mu_\ell]\cong H^{1,1}(X)$ for every smooth simplicial scheme X over k.

Following [V1, p. 17], we write $B\mu_\ell$ for the geometric classifying space of μ_ℓ, constructed in [MV, p. 133] (where the notation $B_{gm}\mu_\ell$ was used). By [MV, 4.2.7], $B\mu_\ell$ is $\mathbb{A}^1$-equivalent to $B_{et}\mu_\ell$, so it also classifies $H^{1,1}$.

When $\ell=2$, the generator $[\zeta]$ of $H^{0,1}(k)=\mu_2(k)$ and its Bockstein, the element $[-1]\in H^{1,1}(k)=k^\times/k^{\times\ell}$, play an important role.

Proposition 10.1. *There are elements $u \in H^{1,1}(B\mu_\ell)$, $v \in H^{2,1}(B\mu_\ell)$ such that*

$$H^{*,*}(B\mu_\ell) \cong \begin{cases} H^{*,*}(k) \otimes \mathbb{F}_\ell[u,v]/(u^2), & \ell \neq 2 \\ H^{*,*}(k) \otimes \mathbb{F}_\ell[u,v]/(u^2 + [-1]u + [\zeta]v), & \ell = 2. \end{cases}$$

Thus every cohomology operation on $H^{1,1}(X)$ is uniquely a sum of the operations $x \mapsto cx^\varepsilon \beta(x)^m$, where $c \in H^{s,j}(k)$, $m \geq 0$ and $0 \leq \varepsilon \leq 1$.

Proof. This is the special case $F_\bullet = S^0$ in Proposition 6.10 of [V1]. Note that the operation $cx^\varepsilon(\beta x)^m$ has bidegree $(s + 2m + \varepsilon - 1, j + m + \varepsilon - 1)$. □

As in Example 9.8, we can use this as the starting point to describe all motivic operations on $H^{*,1}$. For example, $G = B\mu_\ell$ is a simplicial group-scheme whose class in DM is a split proper Tate motive, so the Künneth formula (8.5) holds. Since $H^{0,q}(B\mu_\ell) \cong H^{0,q}(k)$ by 10.1, Proposition 9.4 applies to show that the Leray spectral sequence has the form

$$E_2^{p,q} = H^{p,*}(B_\bullet B\mu_\ell) \otimes_{H^{*,*}(k)} H^{q,*}(B\mu_\ell) \Rightarrow H^{p+q,*}(E_\bullet B\mu_\ell) \cong H^{p+q,*}(k). \tag{10.2}$$

Corollary 10.3. *If $\ell \neq 2$, the ring of cohomology operations on $H^{2,1}$ is the tensor product of $H^{*,*}(k, \mathbb{F}_\ell)$ and the free graded-commutative algebra generated by: the identity of $H^{2,1}$, the Bockstein β, the $P^I\beta$ and the $\beta P^I \beta$ where $P^I = P^{\ell^\nu} \cdots P^\ell P^1$.*

For $\ell = 2$, the ring of cohomology operations on $H^{2,1}$ is the tensor product of $H^{*,*}(k, \mathbb{F}_2)$ and the free graded-commutative algebra generated by: the identity of $H^{2,1}$, Sq^1, ..., $Sq^I = Sq^I_V$, where $Sq^I = Sq^{2^\nu} \cdots Sq^2 Sq^1$.

Proof. By [MV, 4.1.16], $B\mu_\ell$ is $K(\mathbb{F}_\ell(1), 1)$. From the sequence

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ho}_\bullet}(X_+, E_\bullet B\mu_\ell) \to \mathrm{Hom}_{\mathrm{Ho}_\bullet}(X_+, B_\bullet B\mu_\ell) \to \mathrm{Hom}_{\mathrm{Ho}_\bullet}(X_+, B\mu_\ell[1]) \\ \to \mathrm{Hom}_{\mathrm{Ho}_\bullet}(X_+, E_\bullet B\mu_\ell[1]) \end{aligned}$$

it follows that the simplicial scheme $B_\bullet B\mu_\ell$ is $K(\mathbb{F}_\ell(1), 2)$. Thus we merely need to compute the motivic cohomology of $K(\mathbb{F}_\ell(1), 2)$ using (10.2).

By Proposition 10.1, $H^{*,*}(B\mu_\ell)$ has an ℓ-simple system of generators over $H^{*,*}$ consisting of u and the $x_\nu = v^{\ell^\nu}$ for $\nu \geq 0$. Since the Künneth formula (8.5) holds, Proposition 9.4 implies that the hypotheses of Borel's Theorem 9.2 and Kudo's Theorem 9.5 hold for (10.2). Therefore $H^{*,*}(B_\bullet B\mu_\ell)$ is the tensor product of $H^{*,*}$ and the free graded-commutative $\mathbb{F}_\ell$-algebra on generators ι, $y_\nu = \tau(x_\nu)$ — and z_ν if $\ell > 2$. Since u transgresses to ι, $x_0 = v = \beta(u)$ and $x_{\nu+1} = P^{\ell^\nu} x_\nu$, Kudo's Theorem 9.5 implies (by induction) that $y_0 = \beta(\iota)$, $y_{\nu+1} = P^{\ell^\nu} y_\nu = P^I \beta(\iota)$ and (using Theorem 8.11) that z_ν is $-\beta P^I \beta(\iota)$. □

To describe cohomology operations on $H^{n,1}$, we use the algebra $H^*_{\mathrm{top}}(K_n)$, defined in 0.1. We bigrade it by giving it the weight grading that P^I has weight $\ell^k - 1$, where $I = (\epsilon_0, s_1, \epsilon_1, ..., s_k, \epsilon_k)$.

Theorem 10.4. *For each $n \geq 1$, the ring of all motivic cohomology operations on $H^{n,1}$ is isomorphic to the free left $H^{*,*}$-module $H^{*,*}(k) \otimes H^*_{top}(K_n)$ in which the P^I are bigraded according to Definition 6.5.*

Thus every cohomology operation on $H^{n,1}(X)$ is a sum of the operations $x \mapsto c(P^{I_1}x)(P^{I_2}x) \cdots (P^{I_s}x)$, where $c \in H^{*,*}(k)$ and each I_j satisfies the excess condition of Definition 0.1.

Proof. We proceed by induction on n, the cases $n = 1, 2$ being given above. Set $K_n = K(\mathbb{F}_\ell(1), n)$, so $K_{n+1} = B_\bullet(K_n)$, and suppose inductively that the algebra $H^{*,*}(K_n)$ is given as described in the theorem, so that it has an ℓ-simple system of generators consisting of the $P^I(\iota_n)$ with I admissible and $e(I) < n$ (or $e(I) = n$ and $\epsilon_1 = 1$), and ℓ^ν powers of the $P^I(\iota_n)$ of even degree.

Since $\mathbb{F}_{\ell,\mathrm{tr}}(K_n)$ is a split proper Tate motive by [V3, 3.28], the Künneth condition (8.5) of Proposition 9.4 holds. Hence the hypotheses of Borel's Theorem 9.2 are satisfied and the Leray spectral sequence (9.3) has the form

$$E_2^{p,q} = H^{p,*}(K_{n+1}) \otimes_{H^{*,*}(k)} H^{q,*}(K_n) \Rightarrow H^{p+q,*}(E) \cong H^{p+q,*}(k).$$

Therefore $H^{*,*}(K_{n+1})$ is the tensor product of $H^{*,*}$ and a free graded-commutative $\mathbb{F}_\ell$-algebra on certain generators; it remains to establish that they are the ones describe in the theorem. But, except for weight considerations, this is exactly the same as in the topological case, as presented on p. 200 of [McC]. Of course, the weight of the $x_I = P^I(\iota_n)$ is the same as the weight of $y_I = P^I(\iota_{n+1})$. Inspection of the weights of the new generators $P^{\ell^t s} \cdots P^s y_I$ (when x_I has degree $2s$) shows that each additional $P^{\ell^t s}$ multiplies the weight by ℓ, as required. $\square$

11. Motivic operations on degree 1 cohomology

We now turn to operations defined on $H^{1,*}$. Here we encounter new cohomology operations arising from the Norm Residue Theorem 6.2, representing a negative twist. Here are a couple of examples.

Example 11.1. There are operations $H^{1,r(\ell-1)}(X) \xrightarrow{\sim} H^{1,1}(X)$, since both groups are naturally isomorphic to $H^1_{\mathrm{et}}(X, \mu_\ell)$. An element $\eta \in H^1_{\mathrm{et}}(k, \mu_\ell^{\otimes 2-i})$ determines a natural transformation $H^{1,i}(X) \to H^{2,2}(X)$.

The case $k = k(\zeta)$. If k contains a primitive ℓ^{th} root of unity ζ, the classification is immediate from Proposition 10.1. Let $[\zeta]$ be the class of ζ in $H^{0,1}(k) \cong \mu_\ell$.

Proposition 11.2. *Suppose that $\zeta \in k$ and $i > 1$. Then there is a natural isomorphism $\gamma : H^{1,i}(X) \xrightarrow{\sim} H^{1,1}(X)$, and $[\zeta]^{i-1} \cup \gamma(x) = x$.*

Every motivic cohomology operation on $H^{1,i}$ is uniquely a sum of the operations $x \mapsto c(\gamma x)^\varepsilon \beta(\gamma x)^m$, where $c \in H^{*,*}(k)$, $0 \le \varepsilon \le 1$ and $m \ge 0$.

Proof. By Theorem 6.2, $H^{1,i}(X) \cong H^1_{\mathrm{et}}(X, \mu_\ell^{\otimes i})$ for all $i > 0$. Since multiplication by $[\zeta^{i-1}]$ is an isomorphism between $H^n_{\mathrm{et}}(X, \mu_\ell)$ and $H^n_{\mathrm{et}}(X, \mu_\ell^{\otimes i})$, its inverse isomorphism γ is natural. Via γ, operations on $H^{1,i}$ correspond to operations on $H^{1,1}$, which are described in Proposition 10.1. $\square$

For example if $i \ge 2$ and $\eta \in H^1_{\mathrm{et}}(k, \mu_\ell^{\otimes 2-i})$ then the operation $H^{1,i} \to H^{2,2}$ of Example 11.1 is the operation $x \mapsto c(\gamma x)$ of Proposition 11.2, where $c = \eta \cup [\zeta]^{i-1}$.

Remark 11.2.1. If $c \in H^{s,j}(k)$ then $\phi(x) = c(\gamma x)^\varepsilon (\beta\gamma x)^m$ is a cohomology operation of bidegree $(s + \varepsilon + 2m - 1, j + \varepsilon + m - i)$. In particular γ is a cohomology operation of bidegree $(0, 1 - i)$.

Galois descent. We now consider the situation in which $\mu_\ell \not\subset k$. Clearly, not all cohomology operations defined over $k(\zeta)$ are defined over k. However, some of these operations do descend, such as those in Example 11.1.

It is convenient to consider the étale cohomology of k as being bigraded, by integers $n \geq 0$ and $i \in \mathbb{Z}$, with $H^n_{\mathrm{et}}(k, \mu_\ell^{\otimes i})$ in bidegree (n, i). Thus the motivic cohomology ring $H^{*,*}(k)$ is a bigraded subring of $H^*_{\mathrm{et}}(k, \mu_\ell^{\otimes *})$.

Definition 11.3. For each integer b, let $\zeta^{-b}H^{*,*}(k)$ denote the direct sum of all $H^s_{\mathrm{et}}(k, \mu_\ell^{\otimes t})$ with $0 \leq s \leq t+b$. This is a bigraded $H^{*,*}(k)$-submodule of $H^*_{\mathrm{et}}(k, \mu_\ell^{\otimes *})$. It is a cyclic module if and only if $\zeta^b \in k$, when it is the $H^{*,*}(k)$-submodule generated by $[\zeta^b] \in H^0_{\mathrm{et}}(k, \mu_\ell^{\otimes -b})$.

Theorem 11.4. *Fix an integer $i \geq 2$. Then the ring of cohomology operations on $H^{1,i}$ is the direct sum of copies of $\zeta^{-b}H^{*,*}(k)$, $b = (i-1)(\varepsilon+m)$, over integers $m \geq 0$ and $\varepsilon \in \{0,1\}$ If $0 \leq s \leq t+b$, the operation corresponding to $c \in H^s_{et}(k, \mu_\ell^{\otimes t})$, m and ε sends $H^{1,i}(X)$ to $H^{s+\varepsilon+2m, t+b+\varepsilon+m}(X)$:*

$$\phi(\zeta^{i-1} \cup y) = (\zeta^b \cup c) y^\varepsilon \beta(y)^m.$$

Proof. Let G denote the Galois group of $k(\zeta)/k$. Since $H^{*,*}(X)$ is the G-invariant summand of $H^{*,*}(X(\zeta))$, a motivic operation $H^{1,i}(X) \to H^{*,*}(X)$ is the same thing as a G-invariant operation $H^{1,i}(X) \to H^{*,*}(X(\zeta))$. Given $x \in H^{1,i}(X)$, there is a unique $y \in H^{1,1}(X(\zeta))$ so that $x = [\zeta]^{i-1} \cup y$, where $[\zeta] \in H^{0,1}(k(\zeta))$. By Proposition 11.2, we are reduced to determining when G acts trivially on $c' y^\varepsilon (\beta y)^m$. Since $y^\varepsilon (\beta y)^m$ is in the summand of $H^{*,*}(k(\zeta))$ which is isotypical for $\mu_\ell^{\otimes -b}$, this holds if and only if c' is in the summand of $H^{s,j}(k(\zeta))$ which is isotypical for $\mu_\ell^{\otimes b}$. By (3.2), there is a unique $c \in H^{s,j-b}(k)$ so that $c' = [\zeta]^b \cup c$. $\square$

Example 11.5 ($b = 1$)**.** An element c in $H^1_{\mathrm{et}}(k, \mathbb{F}_\ell) = \mathrm{Hom}(\mathrm{Gal}(\bar{k}/k), \mathbb{F}_\ell)$ determines operations $C : H^{1,2}(X) \to H^{2,1}(X)$ and $\phi : H^{1,2}(X) \to H^{3,2}(X)$. If $y \in H^{1,1}(X(\zeta))$ is such that $x = [\zeta] \cup y$ then, regarding ζc as an element of $H^{1,1}(k(\zeta))$, we have $C(x) = (\zeta c)y$ and $\phi(x) = (\zeta c)\beta(y)$. Of course, we can identify C with the map $H^1_{\mathrm{et}}(X, \mu_\ell) \xrightarrow{\cup c} H^2_{\mathrm{et}}(X, \mu_\ell)$.

An element t in $H^2_{\mathrm{et}}(k, \mu_\ell)$ (the ℓ-torsion subgroup of the Brauer group of k) determines operations $H^{1,2}(X) \to H^{3,3}(X)$ and $H^{1,2}(X) \to H^{4,3}(X)$. Writing $x = [\zeta] \cup y$ in $H^{1,2}(X(\zeta))$, the operations followed by the inclusion $H^{*,*}(X) \subset H^{*,*}(X(\zeta))$ send x to $([\zeta] \cup t)y$ and $([\zeta] \cup t)\beta(y)$, respectively. As mentioned in the introduction, we can identify the first operation with $H^1_{\mathrm{et}}(X, \mu_\ell^{\otimes 2}) \xrightarrow{\cup t} H^3_{\mathrm{et}}(X, \mu_\ell^{\otimes 3})$.

12. Conjectural matter

In the preceeding two sections we have classified motivic cohomology operations on $H^{n,i}$ when $n = 1$ or $i = 1$. We have also classified operations whose targets lie inside the "étale zone" where $n \leq i$. We know little about the intermediate zone where $i < n < 2i$. In this section we make some guesses about operations in the "topological zone" where $n \geq 2i$.

Example 12.1. There are many operations defined on $H^{n,2}$, $n \geq 2$. Let us compare Voevodsky's operation P^1_V (landing in $H^{n+2\ell-2,\ell+1}$) with our operation P^1 (landing in $H^{n+2\ell-2,2\ell}$). Thus P^1 has the same bidegree as $[\zeta]^{\ell-1}P^1_V$, where

$[\zeta] \in H^{0,1}(k)$. If $n \geq 4$, we have $P^1 = [\zeta]^{\ell-1}P^1_V$ by Corollary 8.9. If $n = 2$ we also have $P^1 = [\zeta]^{\ell-1}P^1_V$ because they induce the same étale operation (P^1) from $H^{2,2}(X) \cong H^2_{\text{et}}(X, \mu_\ell^{\otimes 2})$ to $H^{2\ell,2\ell}(X) \cong H^{2\ell}_{\text{et}}(X, \mu_\ell^{\otimes 2\ell})$. We do not know if P^1 and $[\zeta]^{\ell-1}P^1_V$ agree on $H^{3,2}$.

Suppose that ϕ is a motivic cohomology operation on $H^{n,i}$ where $n \geq 2i$. Passing to étale cohomology sends ϕ to an étale operation, which by Theorem 3.5 is a polynomial in the étale operations P^I. By Proposition 8.8, some multiple of the Bott element b sends ϕ to operations $b^N\phi$ which are in the subalgebra generated by the motivic operations P^I defined in 6.5. It remains to determine what those powers are.

The following result of Voevodsky [V1, 3.6–7] shows that all non-trivial operations in the topological zone increase n.

Lemma 12.2. *[Voevodsky] There are no motivic cohomology operations from $H^{2i,i}$ to $H^{n,j}$ when $j < i$, or when $i = j$ and $(n, j) \neq (2i, i)$. The module of motivic cohomology operations from $H^{2i,i}$ to $H^{*,i}$ is isomorphic to $\mathbb{F}_\ell$, on the identity.*

Conjecture 12.3. Assume that k contains all primitive ℓ^{th} roots of unity, and that $n \geq 2i$. Then the module of all motivic cohomology operations on $H^{n,i}(-, \mathbb{F}_\ell)$ is the tensor product of $H^{*,*}$ and a free graded polynomial algebra over $\mathbb{F}_\ell$ with generators all $P^I P^J_V$, where $I = (\epsilon_0, s_1, \epsilon_1, ..., s_k, \epsilon_k)$, $J = (s_{k+1}, \epsilon_{k+1}, ..., s_m, \epsilon_m)$ subject to the conditions that (a) the concatenation IJ is admissible with excess $e(IJ)$ either < 4 or else $\epsilon_0 = 1$ and $e(IJ) = 4$; and (b) for all $j > k$, $s_j < i + (\ell - 1)\sum_{j+1}^m s_i$.

For $(n, i) = (4, 2)$ this conjecture implies that among the polynomial generators for the motivic operations on $H^{4,2}$ we find $P^{\ell^2+\ell+1}\beta P^{\ell+1}_V \beta P^1_V \beta$. If $\ell = 2$, we may rewrite this operations as $Sq^{14}Sq^7_V Sq^3_V Sq^1_V$; compare with [V3, 3.57].

Lemma 12.4. *If Conjecture 12.3 holds for $H^{2i,i}$ then it holds for all $H^{n,i}$ with $n \geq 2i$.*

Proof. We consider the Leray spectral sequence (9.3) for $G = K(\mathbb{F}_\ell(i), n)$ and $K = B_\bullet G = K(\mathbb{F}_\ell(i), n + 1)$ when $n \geq 2i$. By induction, $H^{*,*}(G)$ is a polynomial algebra over $H^{*,*}$ with an ℓ-simple system $\{x_i\}$ of generators. By [V3, 3.28], $\mathbb{F}_{\ell,\text{tr}}(G)$ is a split proper Tate motive, so the Künneth condition of Proposition 9.4 holds, and Borel's Theorem 9.2 implies that $H^{*,*}(K)$ is the tensor product of $H^{*,*}$ and a free graded-commutative $\mathbb{F}_\ell$-algebra on generators $y_i = \tau(x_i)$ and, when $\deg(x_j)$ is even and $\ell > 2$, $z_j = \tau(x_j^{\ell-1} \otimes y_j)$.

We now use the fact that the transgression commutes with any (S^1-)stable cohomology operation, such as P^J_V; see [McC, 6.5]. Since the tautological element ι_n of $H^{n,i}(G)$ transgresses to the tautological element ι_{n+1} of $H^{n+1,i}(K)$, the generator $x_j = P^I P^J_V(\iota_n)$ transgresses to $y_j = P^I P^J_V(\iota_{n+1})$ by Kudo's Theorem 9.5. This finishes the proof for $\ell = 2$.

If ℓ is odd and $x_j = P^I P^J_V(\iota_n)$ has degree $2a$, the transgression z_j of $x_j^{\ell-1} \otimes y_j$ is $-\beta P^a P^I P^J_V(\iota_{n+1})$ by Kudo's Theorem 9.5(3). This finishes the proof for ℓ odd. □

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