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Pic is a contracted functor

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Summary. We show that there is a natural decomposition

$$\mathrm{Pic}(A[t, t^{-1}]) \cong \mathrm{Pic}(A) \oplus N\,\mathrm{Pic}(A) \oplus N\,\mathrm{Pic}(A) \oplus H^1(A)$$

for any commutative ring A, where $\mathrm{Pic}(A)$ is the Picard group of invertible A-modules, and $H^1(A)$ is the étale cohomology group $H^1(\mathrm{Spec}(A), \mathbb{Z})$. A similar decomposition of $\mathrm{Pic}(X[t, t^{-1}])$ holds for any scheme X. This makes Pic a "contracted functor" in the sense of Bass. $H^1(A)$ is always a torsionfree group, and is zero if A is normal. For pseudo-geometric rings, $H^1(A)$ is an effectively computable, finitely generated free abelian group. We also show that $H^1(A[t, t^{-1}]) \cong H^1(A)$, i.e., $NH^1 = LH^1 = 0$. This yields the formula for group rings:

$$\mathrm{Pic}(A[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}) \cong \mathrm{Pic}(A) \oplus \coprod_{i=1}^{m} H^1(A) \oplus \coprod_{k=1}^{m} \coprod_{i=1}^{2^k\binom{m}{k}} N^k\,\mathrm{Pic}(A).$$

Introduction

In this paper we solve one of the problems left over from the "classical" period of algebraic K-theory: analyzing the Picard group of the Laurent polynomial ring over a commutative ring. (See [Bass, p. 670].) The Picard group $\mathrm{Pic}(A)$ is the group of isomorphism classes of rank one projective A-modules (i.e., invertible ideals of A). We prove that Pic is a "contracted functor" (on schemes) in the sense of Bass, so there is a natural decomposition

$$\mathrm{Pic}(A[t, t^{-1}]) \cong \mathrm{Pic}(A) \oplus N\,\mathrm{Pic}(A) \oplus N\,\mathrm{Pic}(A) \oplus L\,\mathrm{Pic}(A),$$

and we identify the mystery term $L\,\mathrm{Pic}(A)$. The surprise is that the solution requires étale cohomology, because for any scheme X:

$$L\,\mathrm{Pic}(X) = H^1_{\text{et}}(X, \mathbb{Z}).$$

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The other term in the decomposition, $N\,\mathrm{Pic}(A)$, is the quotient $\mathrm{Pic}(A[t])/\mathrm{Pic}(A)$. Its structure is fairly well understood; for example, $N\,\mathrm{Pic}(A)=0$ iff A_{red} is *seminormal* [Swan], and $N\,\mathrm{Pic}(A)$ is a module over the ring $W(A)$ of Witt vectors [DW].

A secondary goal of this paper is to give an elementary description of the mystery group $L\mathrm{Pic}$ and when it vanishes. For example, $L\mathrm{Pic}(X)=0$ if X is normal. Our proofs here are more elementary for rings than for schemes; we cite the result (7.6.1) that $L\mathrm{Pic}(X)=L\mathrm{Pic}(X_{\mathrm{red}})$, a result that is trivial for rings but subtle for schemes because $\mathrm{Pic}(X)\neq\mathrm{Pic}(X_{\mathrm{red}})$ in general. For this reason, and in order to make the paper more accessible, we restrict ourselves to commutative rings in the first half of the paper, only introducing schemes and étale cohomology when forced to. We have therefore organized our results as follows.

In section one, we recall Bass' notion of a contracted functor, and define $L\mathrm{Pic}(A)$. We include a result of independent interest: if

$$A=R[t_1,t_1^{-1},\ldots,t_m,t_m^{-1},x_1,\ldots,x_n]$$

for a 0-dimensional ring R, then every projective A-module of constant rank is free.

In section two, we construct the $H^0-L\mathrm{Pic}$ *sequence*, an exact sequence associated to a finite integral extension $A\subset B$. When A is reduced noetherian and has a finite integral closure B, this sequence reduces to

$$0\to\mathbb{Z}^d\to L\mathrm{Pic}(A)\to L\mathrm{Pic}(A/I)\to L\mathrm{Pic}(B/I),$$

$$d=h^0(B/I)-h^0(A/I)+h^0(A)-h^0(B),$$

where I is the conductor ideal from B to A, and $h^0(R)$ denotes the number of connected components of a ring R. This yields an effective algorithm for computing $L\mathrm{Pic}(A)$ for pseudo-geometric rings, and shows that in this case $L\mathrm{Pic}(A)\cong\mathbb{Z}^r$ for some r. We also use the $H^0-L\mathrm{Pic}$ sequence to show that $NL\mathrm{Pic}=L^2\mathrm{Pic}=0$, which yields a formula for the Picard group of a free abelian group:

$$\mathrm{Pic}(A[t_1,t_1^{-1},\ldots,t_m,t_m^{-1}])\cong\mathrm{Pic}(A)\oplus\coprod_{i=1}^{m}L\mathrm{Pic}(A)\oplus\coprod_{k=1}^{m}\coprod_{i=1}^{2^k\binom{m}{k}}N^k\mathrm{Pic}(A).$$

Sections 3 and 6 are scenic excursions, showing how the introduction of $L\mathrm{Pic}$ simplifies many of the results in the literature, particularly those involving Asanuma's notion of "anodal" domain. Let A be a domain. Then $L\mathrm{Pic}(A)=0$ implies that A is anodal; Asanuma's theorem becomes: if $\dim(A)=1$ then $L\mathrm{Pic}(A)=0$ iff A is anodal; the Onoda-Yoshida principal becomes: $L\mathrm{Pic}(A_P)=0$ for every prime ideal P iff A_P is anodal for every prime ideal P. We give an example of a 2-dimensional anodal domain which has $L\mathrm{Pic}\neq 0$, showing that being anodal is not a sufficient condition for $L\mathrm{Pic}$ to vanish.

In Sect. 4 we show that $N\,\mathrm{Pic}$ satisfies descent for the étale topology. In contrast, descent fails miserably for $L\mathrm{Pic}$, because $L\mathrm{Pic}$ vanishes for hensel local rings. Our proof is an adaptation of the methods of Vorst and van der Kallen, who showed that the groups NK_i satisfy descent for the étale topology.

Our main results are proven in Sect. 5 (for rings), and extended to schemes in Sect. 7. We have described these results above.

Finally, we conclude with some remarks in Sect. 8 about the connection to negative algebraic K-theory. For example, the γ-filtration and the action of the Adams operations ψ^k are related to vanishing conjectures for negative K-theory [W]. We also consider the Brown-Gersten spectral sequence for $K_*(X)$ and show that the E_2^{10} term not only lives to infinity but is naturally isomorphic to $L\mathrm{Pic}(X)$.

Acknowledgements. Many background results on contracted functors, and (1.7) in particular, were developed jointly with Sue Geller. I have benefitted greatly from conversations with Bob Thomason and Wolmer Vasconcelos, and also with Dan Grayson, Rick Jardine and Les Reid. I would also like to thank N. Onoda and K. Yoshida for writing the paper [OY], which rekindled my interest in the subject of $\mathrm{Pic}(A[t, t^{-1}])$.

§1. Contracted functors of rings

We begin by recalling some notation from [Bass, ch. XII]. Let $\mathscr{R}$ be the category of commutative rings. Given a functor F from $\mathscr{R}$ to some abelian category $\mathscr{A}$, we define functors NF and LF by the formulas:

$$NF(R) = N_t F(R) = \ker[F(t=1)\colon F(R[t]) \to F(R)]$$
$$\cong \operatorname{coker}[F(i_+)\colon F(R) \to F(R[t])];$$
$$LF(R) = L_t F(R) = \operatorname{coker}[F(R[t]) \oplus F(R[t^{-1}]) \xrightarrow{\text{add}} F(R[t, t^{-1}])].$$

These functors can be iterated; it is easy to see that there are natural isomorphisms $NLF(R) \cong LNF(R)$ and $L^2F(R) = L_s L_t F(R) \cong L_t L_s F(R)$. Clearly $F(R[t]) \cong F(R) \oplus NF(R)$. To decompose $F(R[t, t^{-1}])$, let $\mathrm{Seq}(F, R)$ denote the chain complex

$$0 \to F(R) \xrightarrow{(+,-)} F(R[t]) \oplus F(R[t^{-1}]) \xrightarrow{\text{add}} F(R[t, t^{-1}]) \to LF(R) \to 0.$$

Here $(+, -)$ denotes the map $(F(i_+), -F(i_-))$, which is a split injection. Hence $\mathrm{Seq}(F, R)$ is quasi-isomorphic to the subcomplex

$$0 \to F(R) \oplus NF_t(R) \oplus N_{t^{-1}} F(R) \xrightarrow{\text{add}} F(R[t, t^{-1}]) \to LF(R) \to 0.$$

Following Bass, we say F is a *contracted functor* (on $\mathscr{R}$) if the sequence $\mathrm{Seq}(F, R)$ is naturally split exact, i.e., exact and split by a map

$$h = h_t(R)\colon\ LF(R) \to F(R[t, t^{-1}])$$

which is natural in R and t. Such a splitting induces a natural decomposition

$$F(R[t, t^{-1}]) \cong F(R) \oplus NF(R) \oplus NF(R) \oplus LF(R).$$

Example 1.1 [Bass, pp. 671–673]. The group $U(R)$ of units of a commutative ring R is a contracted functor. The group $NU(R)$ is $(1+(t-1)J[t])^{\times}$, where J is the nilradical of R, and $LU(R)$ is the group

$$H^0(R)=H^0(\mathrm{Spec}(R);\mathbb{Z})$$

of continuous maps from the topological space $\mathrm{Spec}(R)$ to $\mathbb{Z}$. The splitting $h\colon H^0(R)\to U(R[t,t^{-1}])$ takes a continuous map f to t^f, the unit of $R[t,t^{-1}]$ which is t^n on the component of $\mathrm{Spec}(R[t,t^{-1}])$ on which f takes the value n.

Remark 1.1.1. By formal nonsense, using Example 1.2 below, we see that $L^iN^jU(R)=0$ if $i\geqq 2$ or if $i=1, j\neq 0$. Note that if R is reduced then $N^jU(R)=0$ for all $j\neq 0$. For a reduced ring R, this quickly yields the Bass-Murthy formula [BM, 5.12]:

$$U(R[t_1,t_1^{-1},\ldots,t_m,t_m^{-1},x_1,\ldots,x_n])\cong U(R)\times\coprod_{i=1}^{m}H^0(R).$$

(If R is not reduced, one has to add lots of $N^jU(R)$ terms.)

Bass proves on p. 661 of [Bass] that, whenever F is a contracted functor, both NF and LF are also contracted functors. In fact, whenever $E\to F$ is a morphism of contracted functors (a natural transformation commuting with h), the kernel, cokernel and image are also contracted functors.

Example 1.2. $H^0(R)$ is a "trivially" contracted functor, i.e., $NH^0=LH^0=0$, because it is easy to see that

$$H^0(R)\cong H^0(R[t])\cong H^0(R[t,t^{-1}]).$$

Construction 1.2.1. *If R is noetherian, so that* $\mathrm{Spec}(R)$ *has only finitely many (say h) connected components, it is obvious that $H^0(R)\cong\mathbb{Z}^h$. More generally, $H^0(R)$ is always a free abelian group* [Fuchs, 97.7].

This fact does not seem to be well-known; it is listed as an open problem in [Pierce, p. 108], and was established the next year by Nöbeling [No]. The following construction of a basis of $H^0(R)$ is due to Bergman, and involves a clever ordering of the set $\mathscr{E}$ of idempotents of R. A well-ordering of $\mathscr{E}$ is called a *Specker ordering* if the first element is 1 and $\alpha,\beta<\gamma$ implies $e_\alpha e_\beta<e_\gamma$. Specker orderings exist; to construct one, first choose any well-ordering of $\mathscr{E}$ and then do a bubble sort. Given a Specker ordering of $\mathscr{E}$, let $\mathscr{B}$ be the subset of all idempotents e_β not in the subgroup of R generated by the e_α with $\alpha<\beta$. Interpreting idempotents as characteristic functions on $\mathrm{Spec}(R)$, hence as elements of $H^0(R)$, $\mathscr{B}$ becomes a basis of $H^0(R)$. With this remark, we can improve upon [BM, A.5]:

Proposition 1.3. *If $A\to B$ is any commutative ring map, the image of the map $H^0(A)\to H^0(B)$ is a direct summand, and the cokernel is a free abelian group. If A is a subring of B, then $H^0(A)$ injects into $H^0(B)$, and $H^0(A)\cong H^0(B)$ iff every idempotent of B lies in A.*

Proof. When A is a subring of B, first choose a Specker ordering for the idempotents of A, and extract a basis $\mathscr{A}$ of $H^0(A)$ using Construction 1.2.1. Then complete the Specker ordering to the remaining idempotents of B, and extract a basis $\mathscr{B}$ of $H^0(B)$. It is clear from (1.2.1) that $\mathscr{A}$ is a subset of $\mathscr{B}$, so $H^0(A)$ is a summand of $H^0(B)$. In the general case, let S be the image in B of the set of idempotents of A, and let $\langle S\rangle$ denote the subgroup of $H^0(B)$ spanned by S. It is clear that $\langle S\rangle$ is the image of $H^0(A)$. Specker-order S, then complete to a Specker ordering of all the idempotents of B. Construction 1.2.1 shows that $\langle S\rangle$ is a direct summand of $H^0(B)$. □

Example 1.4. The Fundamental Theorem of algebraic K-theory states that the functors $K_n(R)$ are contracted functors (even on the category of all rings) with $LK_n = K_{n-1}$. For example, $K_0(R)$ is the Grothendieck group of finitely generated projective R-modules, but we lack a familiar interpretation of the negative groups $K_{-n}(R) = L^n K_0(R)$. (See [Bass, p. 663] and [Q].)

Example 1.4.1. If R is a commutative ring, the rank of a projective module gives a morphism rank: $K_0(R) \to H^0(R)$ of contracted functors. This map is a split surjection, since $H^0(R)$ is generated by the characteristic functions of idempotents, and the idempotent e corresponds to the projective module eR. Let $\tilde{K}_0(R)$ denote the kernel of the rank map; it follows from the above that $K_0 \cong H^0 \oplus \tilde{K}_0$, that $\tilde{K}_0$ is a contracted functor, and that

$$L\tilde{K}_0(R) \cong LK_0(R) = K_{-1}(R).$$

In fact, $\tilde{K}_0(R)$ itself is $LSK_1(R)$, where $SK_1(R)$ is $SL(R)/[SL(R), SL(R)]$, and $SL(R)$ is the special linear group of matrices of determinant one. See [Bass, pp. 671–673].

Example 1.5. If R is a commutative ring, $\mathrm{Pic}(R)$ is the group of isomorphism classes of rank one projective R-modules, the group operation being tensor product. In this paper we shall prove that Pic is a contracted functor, that det: $\tilde{K}_0 \to \mathrm{Pic}$ is a morphism of contracted functors, and that $NL\mathrm{Pic}(R) = L^2\,\mathrm{Pic}(R) = 0$. As a first step, we mention the following result of Bass and Murthy.

Lemma 1.5.1 [Bass, p. 670]. *The sequence* Seq(Pic, R):

$$0 \to \mathrm{Pic}(R) \to \mathrm{Pic}(R[t]) \times \mathrm{Pic}(R[t^{-1}]) \to \mathrm{Pic}(R[t, t^{-1}]) \to L\mathrm{Pic}(R) \to 0$$

is exact for every commutative ring R.

Proof. Only exactness at the second spot is nontrivial. Suppose given $L_+ \in \mathrm{Pic}(R[t])$ and $L_- \in \mathrm{Pic}(R[t^{-1}])$ which become isomorphic in $\mathrm{Pic}(R[t, t^{-1}])$. By Quillen's Affine Horrocks Theorem [Lam][Q76], L_+ and L_- are extended from the same $L \in \mathrm{Pic}(R)$, proving exactness. □

Remark 1.5.2. The following basic facts about $N\,\mathrm{Pic}$ and $L\mathrm{Pic}$ will be used extensively in this paper. $N\,\mathrm{Pic}$ and $L\mathrm{Pic}$ commute with finite products and filtered colimits of rings, because Pic does. Since $\mathrm{Pic}(R) = \mathrm{Pic}(R_{\mathrm{red}})$, both $N\,\mathrm{Pic}(R) = N\,\mathrm{Pic}(R_{\mathrm{red}})$ and $L\mathrm{Pic}(R) = L\mathrm{Pic}(R_{\mathrm{red}})$ hold. If R is an integrally closed domain, $N\,\mathrm{Pic}(R) = L\mathrm{Pic}(R) = 0$, because $\mathrm{Pic}(R) \cong \mathrm{Pic}(R[t]) \cong \mathrm{Pic}(R[t, t^{-1}])$ holds by [BM, 5.10] (and a colimit argument if R isn't noetherian).

In order to quickly expand our repertoire of rings with $N\,\mathrm{Pic} = L\mathrm{Pic} = 0$, we include the following result here, which may be of independent interest:

Theorem 1.6. *Let R be a zero-dimensional commutative ring, and set*

$$S=R[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}, x_1, \ldots, x_n] \qquad (m \geqq 0, n \geqq 0).$$

(a) *Every finitely generated projective S-module of constant rank is free.*

(b) *Every finitely generated projective S-module is extended from R. In fact, it is a direct sum of modules eS, where e is an idempotent in R.*

(c) $\mathrm{Pic}(S)=\tilde{K}_0(S)=0$, *and* $K_0(S) \cong H^0(R)$.

Proof. We may assume that R is reduced, i.e., von Neumann regular. In particular, every localization R_p of R is a field. Every projective R-module P is a sum of modules eR by [Kap, Thm. 4], so if rank(P) is constant then P is free. Hence (b) will imply (a) and (c). If $m=0$, i.e., $S=R[x_1, \ldots, x_n]$, the result (b) is now an easy consequence of Quillen's Patching Theorem, as in [Lam, p. 136]. We shall prove the result for $m>0$ by induction on m, using the following observation of Swan [Sw, 1.3]: if P is a fin. gen. $A[t, t^{-1}]$-module, and if $P \otimes A(t)$ is extended from A, then P is extended from $A[t^{-1}]$. Here $A(t)$ is the localization of $A[t]$ at the multiplicative set of all monic polynomials, and "extended" means that $P \cong A[t, t^{-1}] \otimes Q$ for some fin. gen. projective $A[t^{-1}]$-module Q. Now write $S=A[t, t^{-1}]$, where $t=t_m$, and let P be a fin. gen. projective S-module. It is easy to see that $R(t)$ is also a von Neumann regular ring, because its localizations are all fields of the form $R_p(t)$. By induction on m, the projective $A \otimes_R R(t)$-module $P \otimes_R R(t)$ is extended from $R(t)$, hence from R, as every idempotent e in $R(t)$ lies in R. Since $A(t)$ is a localization of $A \otimes_R R(t)$, $P \otimes_S A(t)$ is extended from R. Hence P is extended from $A[t^{-1}]$. By induction on m, P is extended from R. □

Corollary 1.6.1. *For every zero-dimensional ring R,* $K_{-m}(R)=0$ *for all* $m \geqq 1$, *and*

$$N\,\mathrm{Pic}(R)=L\mathrm{Pic}(R)=0.$$

Example 1.7. Let $SK_0(R)$ denote the kernel of the determinant map "det" from $\tilde{K}_0(R)$ to Pic(R). By chasing the diagram connecting Seq(F, R) for $F=SK_0$, $\tilde{K}_0$ and Pic, and using (1.5.1), we see that Seq(SK_0, R) is exact, and that there is a short exact sequence

$$0 \to LSK_0(R) \to K_{-1}(R) \xrightarrow{L\det} L\mathrm{Pic}(R) \to 0. \tag{1.7.1}$$

I do not know whether "det" is a morphism of contracted functors, nor whether SK_0 is a contracted functor. However, from the calculation in (2.4) below that $NL\mathrm{Pic}=L^2\,\mathrm{Pic}=0$, we see that $NLSK_0(R) \cong NK_{-1}(R)$ and that $L^2SK_0(R) \cong K_{-2}(R)$. In addition, Seq$(LSK_0, R)$ is exact. To see this, use (1.4), (1.5.1) and the exact sequence of chain complexes

$$0 \to \mathrm{Seq}(LSK_0, R) \to \mathrm{Seq}(K_{-1}, R) \to \mathrm{Seq}(L\mathrm{Pic}, R) \to 0.$$

§ 2. The $H^0 - L\,$Pic sequence

To compute LPic(A) for any given ring A, we will use a modified version of the "Units-Pic" sequence. Recall from [Bass, p. 482] that for any commutative ring map $A \to B$ and any ideal I of A with $I \cong IB$, *the Units-Pic sequence* of $(A \to B, I)$ is the exact sequence

$$1 \to U(A) \to U(B) \times U(A/I) \to U(B/I) \xrightarrow{\partial} \mathrm{Pic}(A) \to \mathrm{Pic}(B) \times \mathrm{Pic}(A/I) \to \mathrm{Pic}(B/I),$$

where ∂ is constructed in *loc. cit.* Applying L, we obtain the chain complex:

$$(2.0) \qquad 0 \to H^0(A) \to H^0(B) \times H^0(A/I) \to H^0(B/I) \xrightarrow{\partial} L\mathrm{Pic}(A) \to L\mathrm{Pic}(B) \times L\mathrm{Pic}(A/I) \to L\mathrm{Pic}(B/I).$$

We shall call (2.0) *the* $H^0 - L\mathrm{Pic}$ *sequence* of $(A \to B, I)$. Exactness at the left two spots is due to Bass and Murthy. Once we know that Pic is a contracted functor, it will follow that the $H^0 - L\mathrm{Pic}$ sequence is exact. (We'll also see in (5.6) how to continue the sequence to the right.) For now, we record a partial result.

Lemma 2.1. *The* $H^0 - L\mathrm{Pic}$ *sequence is exact, except possibly at the* $L\mathrm{Pic}(B) \times L\mathrm{Pic}(A/I)$ *term. It is exact there too if* $N\,\mathrm{Pic}(B/I) = 0$.

Proof (Cf. [Bass, pp. 675–676], [Or, p. 792], [Isch, 3.2], [OY, 1.6]). Combine the horizontally written Units-Pic exact sequences for $(A \to B, I)$, $(A[t] \to B[t], I[t])$, ... together with the $H^0 - L\mathrm{Pic}$ sequence to form a commutative diagram whose column sequences are either $\mathrm{Seq}(U, R)$ or $\mathrm{Seq}(\mathrm{Pic}, R)$ for the rings $R = A$, B, A/I and B/I. The columns are exact by (1.1) and (1.5.1). The lemma now follows by diagram chasing. □

Corollary 2.1.1. *The image of* $H^0(B/I) \xrightarrow{\partial} L\mathrm{Pic}(A)$ *is a torsionfree abelian group.*

Proof. Given the lemma, this is [Bass, p. 487]. □

Exercise 2.1.2. *Let* $b \in B$ *be such that* $(b^2 - b) \in I$. *Its image* $\bar{b}$ *in* B/I *is idempotent. Interpreting* $\bar{b}$ *as an element of* $H^0(B/I)$, *show that* $\partial(\bar{b})$ *is the invertible* $A[t, t^{-1}]$*-module*

$$\{(x, y) \in B[t, t^{-1}] \times A/I[t, t^{-1}] : (1 - b + bt)\,x \equiv y \text{ in } B/I[t, t^{-1}]\},$$

considered as an element of $\mathrm{Pic}(A[t, t^{-1}])$. *If* $A \subset B$ *and* B *is a domain, show that this is isomorphic to the invertible ideal*

$$A[t, t^{-1}] \cap (1 - b + bt)\,B[t, t^{-1}].$$

Example 2.2 (The node). Let k be either a field or $\mathbb{Z}$, and set

$$A = k[a, c]/(c^2 = ac + a^3).$$

Mapping a to $(b^2 - b)$ and c to $ba = (b^3 - b^2)$ embeds A as the subring $k[b^2 - b, b^3 - b^2]$ of $B = k[b]$ with conductor ideal $I = (a, c)A$. Alternatively, A is $\{f \in B : f(0) = f(1)\}$. The $H^0 - L\mathrm{Pic}$ sequence quickly yields $L\mathrm{Pic}(A) \cong \mathbb{Z}$. By (2.1.2),

$$L = A[t, t^{-1}] \cap (1 - b + bt)\,B[t, t^{-1}]$$

is an invertible ideal which as an element of $\mathrm{Pic}(A[t, t^{-1}])$ generates $L\mathrm{Pic}(A)$. Similarly, the local domain A_I has $L\mathrm{Pic}(A_I) \cong \mathbb{Z}$, generated by L_I.

Example 2.2.1. Let R be the direct product of an infinite number of copies (say κ many) of a field k, and set $A_R = A \otimes_k R = \{f \in R[b] : f(0) = f(1)\}$. Since R is von Neumann regular, we see by (1.6) that the $H^0-L\mathrm{Pic}$ sequence for $A_R \to R[b]$ degenerates to yield $L\mathrm{Pic}(A) \cong H^0(R)$. This is a free abelian group of cardinality 2^κ.

Example 2.2.2 (The axes in the plane). The ring $A = k[x, y]/(xy)$ has $L\mathrm{Pic}(A) = 0$. This is immediate from the $H^0-L\mathrm{Pic}$ sequence for $i\colon A \hookrightarrow k[x] \times k[y]$ and $I = (x, y)A$, where $i(x) = x$ and $i(y) = y$. This example shows that the analytic type of the singularity is not relevant for $L\mathrm{Pic}$, a point we shall return to in (2.5).

Recall that a noetherian ring A is called *pseudo-geometric* if every reduced finite A-algebra B has finite normalization. (For more details, see [N]. In [EGA, IV] these rings are called noetherian universally japanese rings.) For example, any finitely generated algebra over a field, or over $\mathbb{Z}$, is pseudo-geometric.

Proposition 2.3. *If A is pseudo-geometric, and* $\dim(A)$ *is finite, then*

$$L\mathrm{Pic}(A) \cong \mathbb{Z}^r \text{ for some } r.$$

Proof. We proceed by induction on $\dim(A)$. As $L\mathrm{Pic}(A) = L\mathrm{Pic}(A_{\mathrm{red}})$, we may assume that A is reduced. If $\dim(A) = 0$, we cite (1.6.1). Otherwise, let B be the normalization of A. The conductor ideal I contains a nonzerodivisor of both A and B, so $\dim(A/I) < \dim(A)$. By induction, $L\mathrm{Pic}(A/I) \cong \mathbb{Z}^s$ for some s. By (2.1.1) we have the exact

$$0 \to \mathbb{Z}^d \to L\mathrm{Pic}(A) \to \mathbb{Z}^s,$$
$$d = [\dim H^0(B/I) - \dim H^0(A/I)] - [\dim H^0(B) - \dim H^0(A)].$$

Since every subgroup of $\mathbb{Z}^s$ is free abelian, $L\mathrm{Pic}(A) \cong \mathbb{Z}^r$ for some r, $d \leqq r \leqq d + s$. □

Corollary 2.3.1. $L\mathrm{Pic}(A)$ *is a torsionfree abelian group for every ring A.*

Proof. $L\mathrm{Pic}(A)$ is the direct limit of the $L\mathrm{Pic}$ groups of the fin. gen. subrings of A.

Exercise 2.3.2. *Fix a field k and a positive integer p. Let A_n be the coordinate ring of p^n affine lines linked together into a circle:*

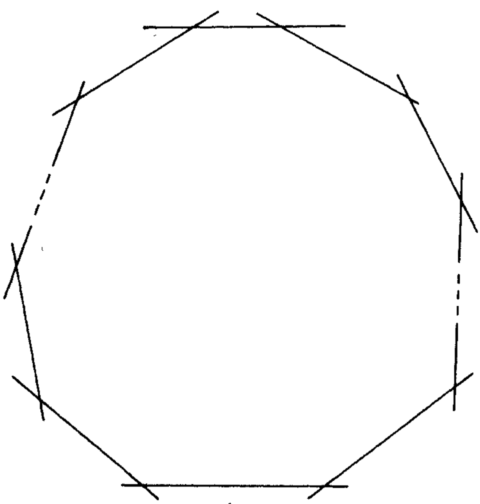

Fig. 1

and map A_n to A_{n+1} by the obvious degree p mapping of p^{n+1} lines to p^n lines. Show that $A=\varinjlim A_n$ is a 1-dimensional reduced k-algebra with $L\mathrm{Pic}(A)=\mathbb{Z}\left[\frac{1}{p}\right]$. (Hint: Show that the maps $L\mathrm{Pic}(A_n)\to L\mathrm{Pic}(A_{n+1})$ are $\mathbb{Z}\xrightarrow{p}\mathbb{Z}$.)

Remark 2.3.3. The proof of (2.3) yields an effective algorithm for computing $L\mathrm{Pic}(A)$. This is because by the $H^0-L\mathrm{Pic}$ sequence (5.3) the image of $L\mathrm{Pic}(A)$ in $\mathbb{Z}^s$ is the kernel of the map from $\mathbb{Z}^s\cong L\mathrm{Pic}(A/I)$ to $\mathbb{Z}^t\cong L\mathrm{Pic}(B/I)$.

Theorem 2.4. *For every commutative ring A, $NL\mathrm{Pic}(A)=L^2\mathrm{Pic}(A)=0$. That is, we have:*

$$L\mathrm{Pic}(A)\cong L\mathrm{Pic}(A[t])\cong L\mathrm{Pic}(A[t,t^{-1}]).$$

Proof. Since Pic commutes with filtered colimits of rings, we may assume A is finitely generated over $\mathbb{Z}$, i.e., pseudo-geometric. We now proceed by induction on $\dim(A)$. Since $\mathrm{Pic}(A)=\mathrm{Pic}(A_{\mathrm{red}})$, there is also no harm in assuming that A is reduced. If $\dim(A)=0$, i.e., A is a finite product of fields, the result is classical: we cite (1.6). If $\dim(A)>0$, the normalization B of A is a finite A-module, so the conductor ideal I contains a nonzerodivisor. Hence A/I and B/I have smaller Krull dimension than A. We now compare the $H^0-L\mathrm{Pic}$ sequences:

$$\begin{array}{ccccc}
H^0(B/I) & \longrightarrow & L\mathrm{Pic}(A) & \longrightarrow & L\mathrm{Pic}(A/I)\times L\mathrm{Pic}(B)\\
\downarrow & & \downarrow & & \downarrow\\
H^0(B/I[t]) & \longrightarrow & L\mathrm{Pic}(A[t]) & \longrightarrow & L\mathrm{Pic}(A/I[t])\times L\mathrm{Pic}(B[t])\\
\downarrow & & \downarrow & & \downarrow\\
H^0(B/I[t,t^{-1}]) & \longrightarrow & L\mathrm{Pic}(A[t,t^{-1}]) & \longrightarrow & L\mathrm{Pic}(A/I[t,t^{-1}])\\
 & & & & \times L\mathrm{Pic}(B[t,t^{-1}]).
\end{array}$$

The left verticals are isomorphisms by (1.2). The $L\mathrm{Pic}$ groups of the normal rings B, $B[t]$ and $B[t,t^{-1}]$ vanish by (1.5.2), so the right verticals are isomorphisms by the inductive hypothesis. Since the top row is a natural summand of the other rows, it follows that the complementary sumands of $L\mathrm{Pic}(A[t])$ and $L\mathrm{Pic}(A[t,t^{-1}])$ are zero. By definition, therefore, we have $N(L\mathrm{Pic})(A)=L(L\mathrm{Pic})(A)=0$. This completes the induction, and hence the proof. □

We have observed that normal domains and 0-dimensional rings are two classes of commutative rings for which $L\mathrm{Pic}$ vanishes. Hensel local rings form another class of such rings, as we now show. Recall that a local ring A is *hensel* if every finite A-algebra B is a direct product of local rings.

Theorem 2.5. $L\mathrm{Pic}(A)=0$ *for every hensel local ring A.*

Proof. We shall prove by induction on $\dim(A)$ that $L\mathrm{Pic}(A)=0$ for every pseudo-geometric hensel local ring A. This will be sufficient to prove the result in general, since every hensel local ring is the union of its finitely generated subrings A_α, hence of their henselizations A_α^h (with respect to $A_\alpha\to A$), and each A_α^h is pseudo-geometric by [N, 44.2] or [EGA, IV.18.7.3]. Since $L\mathrm{Pic}(A)=L\mathrm{Pic}(A_{\mathrm{red}})$, we may also assume that A is reduced. If $\dim(A)=0$, i.e., A is a field, then $L\mathrm{Pic}(A)=0$ by (1.6.1). If $\dim(A)>0$, let B be the normalization of A. As B is a finite A-algebra, the conductor ideal I contains a nonzerodivisor. Hence A/I is pseudo-geometric with $\dim(A/I)<\dim(A)$, so $L\mathrm{Pic}(A/I)=0$ by induction.

Because A is a hensel local ring, B and B/I must be finite products of local rings. Every continuous function from $\mathrm{Spec}(B/I)$ to $\mathbb{Z}$ must therefore lift to a continuous function from $\mathrm{Spec}(B)$ to $\mathbb{Z}$, i.e., $H^0(B)$ maps onto $H^0(B/I)$. The exact sequence of Lemma 2.1 now implies that $L\mathrm{Pic}(A)=0$, completing the inductive step. □

§ 3. Anodal extensions

It would be nice to have an elementary characterization of those rings A for which $L\mathrm{Pic}(A)=0$. This subject has been addressed sporadically over the last two decades in [BM], [P], [Or], [Isch], [G], [Rush], [OY], [OSY] and [And] under the guise of determining when $\mathrm{Pic}(A)=\mathrm{Pic}(A[t,t^{-1}])$, i.e., when $N\mathrm{Pic}(A)=L\mathrm{Pic}(A)=0$. (Starting with [G], such rings have been called *quasinormal*; cf. [BM, p. 33].) The introduction of $L\mathrm{Pic}$ simplifies many of the results found in *op. cit.*, because it eliminates the extraneous hypothesis that A be seminormal. (We will see in (5.4) that $L\mathrm{Pic}(A)=L\mathrm{Pic}(^+A)$ anyhow.) To illustrate this point, consider the following notion:

Definition 3.1 (T. Asanuma). An inclusion $A\subset B$ of rings is called *anodal*, or an *anodal extension*, if every $b\in B$ such that $(b^2-b)\in A$ and $(b^3-b^2)\in A$ belongs to A. That is, every solid diagram

$$\begin{array}{ccc} \mathbb{Z}[b^2-b,\, b^3-b^2] & \hookrightarrow & \mathbb{Z}[b] \\ \downarrow & \swarrow & \downarrow \\ A & \hookrightarrow & B \end{array}$$

can be completed by a dashed arrow, as shown. (Asanuma and [OY] use the phrase "A is u-closed in B" in the case that A is a domain; D. Grayson has pursuaded me that the term "anodal" is better.) Note that if b is an idempotent element of B not in A, then $A\subset B$ cannot be anodal. Using (1.3), we rephrase this as:

Lemma 3.1.1. *If $A\subset B$ is an anodal extension, then $H^0(A)\cong H^0(B)$.*

Lemma 3.2 (Onoda-Yoshida). *Suppose that $A\subset B$ is an extension satisfying $H^0(A)\cong H^0(B)$. Then the following are equivalent:*

(i) *$A\subset B$ is anodal;*

(ii) *For every finite A-algebra C contained in B, and every ideal I of A with $I\cong IC$,*

$$H^0(A/I)\cong H^0(C/I);$$

(iii) *For every finite A-algebra C contained in B, and every ideal I of A with $I\cong IC$,*

$$L\mathrm{Pic}(A)\to L\mathrm{Pic}(C)\times L\mathrm{Pic}(A/I) \quad \textit{is injective.}$$

Proof (Cf. [OY, 1.10]). The equivalence of (ii) and (iii) follows easily from the $H^0-L\mathrm{Pic}$ sequence. The implication (i) ⇒ (ii) is immediate from (1.3). Finally, suppose $b \in B$ is such that $b^2 - b$ and $b^3 - b^2$ lie in A, and set $C = A[b]$, $I = (b^2 - b)C$. If (ii) holds, the idempotent $\bar{b}$ of B/I belongs to A/I, forcing $b \in A$, so (i) holds. □

We shall say that a domain A is *anodal* if the extension $A \subset \bar{A}$ (or equivalently, $A \subset F$) is anodal. Here F denotes the field of fractions of A and $\bar{A}$ is the integral closure of A in F. Note that $H^0(A) \cong H^0(\bar{A}) \cong H^0(F) \cong \mathbb{Z}$.

Corollary 3.3. *If A is a domain, then $L\mathrm{Pic}(A) = 0$ implies that A is anodal.*

Example 3.3.1. If A is not a domain, $L\mathrm{Pic}(A) = 0$ does not imply that $A \subset \bar{A}$ is anodal, nor that $H^0(A) = H^0(\bar{A})$. This is illustrated by (2.2.2), the axes in the plane.

Theorem 3.4 (Asanuma). *Let A be a 1-dimensional domain. Then:*

$$L\mathrm{Pic}(A) = 0 \Leftrightarrow A \text{ is anodal.}$$

Proof (Cf. [OY, 1.14]). The $\Rightarrow$ direction is (3.3). For the $\Leftarrow$ direction, let L be a nontrivial element of $L\mathrm{Pic}(A)$. Then L vanishes in $L\mathrm{Pic}(B)$ for some finite A-subalgebra B. As the conductor I from B to A is nonzero, $\dim(A/I) = 0$. Hence $L\mathrm{Pic}(A/I) = 0$ by (1.6.1). The $H^0 - L\mathrm{Pic}$ sequence shows that L is in the image of $H^0(B/I)$, so $H^0(A/I) \neq H^0(B/I)$. By (3.2), $A \subset \bar{A}$ fails to be anodal. □

Exercise 3.4.1. *Show that $L\mathrm{Pic}(A)$ is a free abelian group for every 1-dimensional domain (Hint: Construct a family of extensions $A_{\lambda+1} = A_\lambda[b_\lambda]$ inside $\bar{A}$ with $\bigcup A_\lambda$ anodal. Use (1.3) and (1.6) to show that $L\mathrm{Pic}(A_\lambda) \to L\mathrm{Pic}(A_{\lambda+1})$ is onto, and that its kernel F_λ is free. Then show that $L\mathrm{Pic}(A) \cong \bigoplus F_\lambda$.)*

We now construct an example of a 2-dimensional seminormal domain which is anodal yet has $L\mathrm{Pic} \neq 0$. This shows that (3.4) does not extend very far into higher dimensions. The ring A will be obtained from $k[x, y]$ by glueing the coordinate axes in the plane together into a node, using the technique of [P].

Example 3.5. Let k be a field and set $R = k[b]$. Let $C = k[b^2 - b, b^3 - b^2]$ be the node, considered as a subring of R. Let $D = k[x, y]/(xy)$ be the axes in the plane, considered both as a quotient of $B = k[x, y]$ and as a subring of $\bar{D} = k[x] \times k[y]$ as in (2.2.2). The map $R \to \bar{D}$ sending b to $(x, 1-y)$ induces a map $i\colon C \to D$. Finally, define $A = B \times_D C$, so that

(3.5.1)
$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \overset{i}{\hookrightarrow} & D \end{array}$$

is a cartesian square. A is a 2-dimensional noetherian domain whose integral closure is B (the proof is a standard exercise we leave to the reader). Using Unit-Pic sequences, it is easy to see that A, B, C and D are all seminormal.

Proposition 3.5.2. *The seminormal domain A is anodal, yet $L\mathrm{Pic}(A) = \mathbb{Z}$.*

Proof. By (5.3) below, or by (2.1) and the seminormality of D, we see that the full $H^0-L\text{Pic}$ sequence for (3.5.1) is exact. Using (2.2) and (2.2.2), this yields

$$L\text{Pic}(A)\cong L\text{Pic}(C)\cong \mathbb{Z}.$$

To see that $A\to B$ is anodal, it suffices by Lemma (3.6) below to show that $C\to D$ is anodal. If $t\in D$ had t^2-t, $t^3-t^2\in C$, then t must lie in the subring R of $\bar{D}$; this follows from Lemma (3.7) below, because $\bar{D}\cong R\times R$ and (0, 1) and (1, 0) are not in D. We can write $t=u+\lambda b$ for $u\in C$, $\lambda\in k$, because every element of R has this form. The image of t in $\bar{D}$ is $i(u)+\lambda(x, 1-y)=i(u)+\lambda x-\lambda y+\lambda(0, 1)$, which lies in D iff $\lambda=0$. But $\lambda=0$ implies that $t\in C$, which proves that $C\to D$ is anodal. □

Lemma 3.6. *If $A\subset B$ is any ring extension, and I is an ideal satisfying $I\cong IB$, then:*

$$A\subset B \text{ is anodal} \Leftrightarrow (A/I)\subset(B/I) \text{ is anodal.}$$

Proof. Fix $b\in B$ and set $a=b^2-b$, $c=ab$. Observe that $b\in A$ iff $\bar{b}\in A/I$, $a\in A$ iff $\bar{a}\in A/I$ and $c\in A$ iff $\bar{c}\in A/I$. The lemma is now a syllogism. □

Lemma 3.7. *If R is any domain and $R\subset R\times R$ is the diagonal inclusion, then the only $t=(x, y)$ in $R\times R$ which satisfy $t^2-t\in R$ and $t^3-t^2\in R$ are: $t\in R$, $t=(0, 1)$ and $t=(1, 0)$.*

Proof. Set $r=x^2-x=y^2-y$. If $r\neq 0$, we can divide $x^3-x^2=y^3-y^2$ by r to get $x=y$, i.e., $t\in R$. If $r=0$ then x and y must both be either 0 or 1, which gives the 4 solutions: $t=0$, $t=1$, $t=(0, 1)$ and $t=(1, 0)$. □

§ 4. Descent for NPic

In this section we establish some local-global results about $N\text{Pic}(A)$ for the Zariski and étale topology. In order to postpone the scheme-theoretic invasion as long as possible, we first show that the methods of Vorst [V] and van der Kallen [vdK] allow us to prove the following two results:

Theorem 4.1 (Descent for N Pic). *Let i: $A\to B$ be an étale, faithfully flat extension of commutative rings. Then the standard descent complex*

$$0\to N\text{Pic}(A)\xrightarrow{i} N\text{Pic}(B)\xrightarrow{i\otimes 1-1\otimes i} N\text{Pic}(B\otimes_A B)\to N\text{Pic}(B\otimes_A B\otimes_A B)\to\cdots$$

is exact.

Theorem 4.2 (Vorst's Theorem). *Given $f\in A$, write $[f]$ for the endomorphism of $N\text{Pic}(A)$ and $\text{Pic}(A[t])$ induced by the substitution $t\mapsto ft$ on $A[t]$. Then*

$$N\text{Pic}(A)_{[f]}\cong N\text{Pic}(A_f).$$

Proof of 4.2. Since A is the union of all its finitely generated subrings which contain f, and $N\operatorname{Pic}(A)=N\operatorname{Pic}(A_{\mathrm{red}})$, we are reduced to the case in which A is reduced noetherian. In the square

$$\begin{array}{ccc} NK_0(A)_{[f]} & \longrightarrow & NK_0(A_f) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ N\operatorname{Pic}(A)_{[f]} & \longrightarrow & N\operatorname{Pic}(A_f) \end{array}$$

the top arrow is an isomorphism by [V, 1.4 and 1.6], and the "det" maps are onto, so the bottom arrow is also a surjection.

To see that $N\operatorname{Pic}(A)_{[f]}\to N\operatorname{Pic}(A_f)$ is an injection, hence an isomorphism, let L be an element of $N\operatorname{Pic}(A)$ which dies in $N\operatorname{Pic}(A_f)$. Then $L_f\cong A_f[t]$ as a rank one projective module, so the element $\lambda=[L]-[A[t]]$ of $NK_0(A)$ dies in $NK_0(A_f)$. Hence λ dies in the localization $NK_0(A)_{[f]}$, forcing its determinant $\det(\lambda)=L$ to die in $N\operatorname{Pic}(A)_{[f]}$. □

The key technical point in the proof of Theorem 4.1 is the projection formula for the action of $W(A)$, the ring of big Witt vectors, on $N\operatorname{Pic}(A)$. By [St], the group $NK_0(A)$ is a $W(A)$-module; by [DW, 4.4] the group $N\operatorname{Pic}(A)$ is also a $W(A)$-module and $\det_A\colon NK_0(A)\to N\operatorname{Pic}(A)$ is a $W(A)$-module map.

Lemma 4.3 (Projection Formula for $N\operatorname{Pic}$). *Let C be a finite A-algebra, free as an A-module. The map $i\colon A\to C$ induces transfer maps $i^*\colon N\operatorname{Pic}(C)\to N\operatorname{Pic}(A)$ and $i^*\colon W(C)\to W(A)$, as well as the usual base change map $i_*\colon N\operatorname{Pic}(A)\to N\operatorname{Pic}(C)$. For $\omega\in W(C)$ and $\alpha\in N\operatorname{Pic}(A)$, the projection formula*

$$(i^*\omega)\cdot\alpha=i^*[\omega\cdot i_*(\alpha)]$$

holds in $N\operatorname{Pic}(A)$.

Proof. It is proven in [DW, 3.2] that $i^*(L)=\det_A(L)$ defines a transfer map $i^*\colon \operatorname{Pic}(C[t])\to\operatorname{Pic}(A[t])$ and that the following diagram commutes:

$$\begin{array}{ccc} NK_0(C) & \xrightarrow{i^*} & NK_0(A) \\ \downarrow{\scriptstyle \det_C} & & \downarrow{\scriptstyle \det_A} \\ N\operatorname{Pic}(C) & \xrightarrow{i^*} & N\operatorname{Pic}(A). \end{array}$$

In [vdK, 1.18 and 1.19], van der Kallen constructs the transfer map on Witt vectors and proves the projection formula $(i^*\omega)\cdot\lambda=i^*[\omega\cdot i_*(\lambda)]$ for NK_0. Choose a lift $\lambda\in NK_0(A)$ of α and apply det to van der Kallen's formula. □

Theorem 4.1 is now an immediate consequence of the following more general result, which we state here in the generality for which van der Kallen's proof of [vdK, Thm. 1.2] remains valid.

Theorem 4.4 (van der Kallen). *Let F be any functor from commutative rings to abelian groups which commutes with filtered colimits. Suppose that $NF(A)$*

is a $W(A)$-module in such a way that for $f \in A$ the Witt vector $1-ft$ acts on $F(A[t])$ via the endomorphism $[f]$ induced by the substitution $t \mapsto ft$. Suppose in addition:

(i) *Vorst's Theorem holds: $NF(A)_{[f]} \cong NF(A_f)$ for all $f \in A$;*

(ii) *The Projection Formula holds: whenever C is a finite free A-algebra, there is a transfer map $i^*: NF(C) \to NF(A)$ so that for $\omega \in W(C)$ and $\alpha \in NF(A)$*

$$(i^* \omega) \cdot \alpha = i^* [\omega \cdot i_*(\alpha)].$$

Then the standard descent complex

$$0 \to NF(A) \to NF(B) \to NF(B \otimes_A B) \to NF(B \otimes_A B \otimes_A B) \to \dots$$

is exact for every étale faithfully flat A-algebra B.

Corollary 4.5. *If A is a local ring, essentially of finite type over a field or over an excellent discrete valuation domain, then $N\,\mathrm{Pic}(A)$ injects into both $N\,\mathrm{Pic}(A^h)$ and $N\,\mathrm{Pic}(\hat{A})$, where A^h is the henselization and $\hat{A}$ is the completion of A.*

Proof. This follows from Artin approximation, as in [vdK, 1.11]. □

This last result suggests a cohomological interpretation of $N\,\mathrm{Pic}(A)$ as the global sections of a sheaf on the étale site of Spec(A). In fact, this works on the Zariski site as well, and we shall give a self-contained proof, making no use of Vorst or van der Kallen's results. The cohomological invasion has now arrived.

Definition 4.6. Let X be a scheme. The étale sheafification of the presheaf $U \mapsto N\,\mathrm{Pic}(U)$ will be denoted $\mathcal{N}\,\mathcal{P}ic$. The stalk of $\mathcal{N}\,\mathcal{P}ic$ at a geometric point $\bar{x}$ is $N\,\mathrm{Pic}(\mathcal{O}_{X,\bar{x}})$, where $\mathcal{O}_{X,\bar{x}}$ denotes the strict henselization of $\mathcal{O}_{X,x}$ at $\bar{x}$. Note that, since $N\,\mathrm{Pic}(\mathcal{O}_{X,\bar{x}}) = N\,\mathrm{Pic}(\mathcal{O}_{X_{\mathrm{red}},\bar{x}})$, the sheaves $\mathcal{N}\,\mathcal{P}ic(X)$ and $\mathcal{N}\,\mathcal{P}ic(X_{\mathrm{red}})$ agree.

Theorem 4.7. *If X is any scheme, the restriction of $\mathcal{N}\,\mathcal{P}ic$ to the Zariski site is the Zariski sheafification of the presheaf $U \mapsto N\,\mathrm{Pic}(U)$. Moreover,*

$$H^0_{\mathrm{et}}(X, \mathcal{N}\,\mathcal{P}ic) = H^0_{\mathrm{zar}}(X, \mathcal{N}\,\mathcal{P}ic) = N\,\mathrm{Pic}(X_{\mathrm{red}}).$$

Finally, if $X = \mathrm{Spec}(A)$ is affine then $H^0(X, \mathcal{N}\,\mathcal{P}ic) = N\,\mathrm{Pic}(A)$.

Proof. First suppose that X is reduced, and write π for the étale structure map $X[t] \to X$. From (1.1) we see that $\pi_*(\mathcal{O}^*_{X[t]}) \cong \mathcal{O}^*_X$. The étale sheaf $\mathcal{N}\,\mathcal{P}ic$ is the derived sheaf $R^1\pi_*(\mathcal{O}^*_{X[t]})$. Since $H^i(X, \mathcal{O}^*_X)$ is a summand of $H^i(X[t], \mathcal{O}^*_{X[t]})$, the Leray spectral sequence degenerates enough to yield

$$N\,\mathrm{Pic}(X) = H^1_{\mathrm{et}}(X[t], \mathcal{O}^*_{X[t]})/H^1_{\mathrm{et}}(X, \mathcal{O}^*_X) \cong H^0_{\mathrm{et}}(X, \mathcal{N}\,\mathcal{P}ic).$$

This proves that for any scheme X we have $H^0_{\mathrm{et}}(X, \mathcal{N}\,\mathcal{P}ic) = N\,\mathrm{Pic}(X_{\mathrm{red}})$. When X is affine, $N\,\mathrm{Pic}(X) = N\,\mathrm{Pic}(X_{\mathrm{red}}) = H^0_{\mathrm{et}}(X, \mathcal{N}\,\mathcal{P}ic)$.

It remains to consider the site change $\tau: X_{\mathrm{et}} \to X_{\mathrm{zar}}$ and show that $\tau_*\mathcal{N}\,\mathcal{P}ic$, the restriction of $\mathcal{N}\,\mathcal{P}ic$ to the Zariski site, is the sheafification $\mathcal{N}\,\mathcal{P}ic^{\mathrm{zar}}$ of the presheaf $N\,\mathrm{Pic}$. At the Zariski point $x \in X$ the stalk of $\tau_*\mathcal{N}\,\mathcal{P}ic$ is $H^0_{\mathrm{et}}(\mathrm{Spec}(\mathcal{O}_{X,x}), \mathcal{N}\,\mathcal{P}ic) = N\,\mathrm{Pic}(\mathcal{O}_{X,x})$, which is also the stalk of $\mathcal{N}\,\mathcal{P}ic^{\mathrm{zar}}$. Since the map from $\mathcal{N}\,\mathcal{P}ic^{zar}$ to $\tau_*\mathcal{N}\,\mathcal{P}ic$ is an isomorphism on stalks, it is an isomorphism. Finally, we deduce that:

$$H^0_{zar}(X, \mathcal{N}\,\mathcal{P}ic^{zar}) = H^0_{zar}(X, \tau_*\mathcal{N}\,\mathcal{P}ic) = H^0_{et}(X, \mathcal{N}\,\mathcal{P}ic). \quad \square$$

Remark 4.7.1. If X is not reduced, the surjection $\pi_*(\mathcal{O}^*_{X[t]}) \to \mathcal{O}^*_X$ is no longer an isomorphism; its kernel is $\mathcal{N}\mathbf{G}_m$, the étale sheaf associated to the presheaf $U \mapsto NU(U)$. Consequently, there is an exact sequence (for étale cohomology)

$$0 \to H^1(X, \mathcal{N}\mathbf{G}_m) \to N\operatorname{Pic}(X) \to N\operatorname{Pic}(X_{red}) \to H^2(X, \mathcal{N}\mathbf{G}_m) \ldots$$

We would like to close this section by observing that several of the last few results also follow from Descent for NPic. Indeed, the proof of [vdK, 1.9] goes through in the current context to yield the following result:

Proposition 4.8. *The Zariski and étale cohomology groups of $\mathcal{N}\,\mathcal{P}ic$ agree:*

$$H^*_{et}(X, \mathcal{N}\,\mathcal{P}ic) = H^*_{zar}(X, \mathcal{N}\,\mathcal{P}ic).$$

Moreover, if $X = \operatorname{Spec}(A)$ is affine, then $H^0(X, \mathcal{N}\,\mathcal{P}ic) = N\operatorname{Pic}(A)$ and

$$H^i(X, \mathcal{N}\,\mathcal{P}ic) = 0 \quad \text{if } i \neq 0.$$

§ 5. The contraction of Pic

In order to prove that Pic is a contracted functor on commutative rings, we need to consider sheaves on Spec(A) for the étale topology. For any scheme X, let $\mathcal{P}ic[T]$ denote the étale sheaf on X associated to the presheaf $U \mapsto \operatorname{Pic}(U[t, t^{-1}])$. The stalk of this sheaf at a geometric point $\bar{x}$ of X is $\operatorname{Pic}(R[t, t^{-1}])$, where R is the strict henselization $\mathcal{O}^{sh}_X$ of the corresponding local ring $\mathcal{O}_{X,x}$. By (1.5.1) and (2.5), this is isomorphic to $N_t\operatorname{Pic}(R) \oplus N_{t^{-1}}\operatorname{Pic}(R)$. Since $N\operatorname{Pic}(R)$ is the stalk of the étale sheaf $\mathcal{N}\,\mathcal{P}ic$ of (4.6), this proves:

Proposition 5.1. *There is an isomorphism of étale sheaves on every scheme:*

$$\mathcal{N}_t\mathcal{P}ic \oplus \mathcal{N}_{t^{-1}}\mathcal{P}ic \xrightarrow{\simeq} \mathcal{P}ic[T].$$

Theorem 5.2. Pic *is a contracted functor on the category $\mathcal{R}$ of commutative rings, and the natural splitting of the exact sequence* Seq(Pic, A) *is provided by the global sections map*

$$\operatorname{Pic}(A[t, t^{-1}]) \to H^0_{et}(\operatorname{Spec}(A), \mathcal{P}ic[T]) \cong N_t\operatorname{Pic}(A) \oplus N_{t^{-1}}\operatorname{Pic}(A).$$

Proof. By (4.7), the map from $N\operatorname{Pic}(A)$ to $H^0(\operatorname{Spec}(A), \mathcal{N}\,\mathcal{P}ic)$ is an isomorphism. By (5.1), the injection $N_t\operatorname{Pic}(A) \oplus N_{t^{-1}}\operatorname{Pic}(A) \to \operatorname{Pic}(A[t, t^{-1}])$ is split by the global sections map. The theorem now follows from the generalities on contracted functors discussed in § 1. $\square$

Remark 5.2.1. Unlike the situation with $\mathcal{N}\,\mathcal{P}ic$, the Zariski sheafification $\mathcal{S}$ of $U \mapsto \operatorname{Pic}(U[t, t^{-1}])$ need not be a sheaf for the étale topology. For example, consider the node discussed in (2.2). Here $\mathcal{S}(U)$ is the skyscraper sheaf $i_*\mathbb{Z}$ at the singular point, while the étale sheaf $\mathcal{P}ic[T]$ is zero.

Recall from [Nis] [Nis2] [KS, 1.1] [TT, Appendix E] that the *henselian topology* on X (discovered by Nisnevich in 1974 and also called the *Nisnevich topology*) is intermediate between the Zariski and étale topologies. Its site is the category of étale schemes over X, but a family $\{V_\alpha \to U\}$ is a covering when every point x of U has some point $y_\alpha \in V_\alpha$ lying over it with the same residue field as x. Hence the "points" of X are the Zariski points on "all" the schemes étale over X, and the henselian stalk of $\mathcal{O}_X$ at a point $x \in U$, $U \to X$ étale, is the henselization $\mathcal{O}_U^h$ of the local ring $\mathcal{O}_{U,x}$. The main advantage for us of the henselian topology over the étale topology is that the cohomological dimension of a scheme is at most its Zariski dimension. In particular, $H^i_{\text{hen}}(F, -)=0$ for $i \neq 0$ when F is a field.

Remark 5.2.2. The argument given above in (5.1) for the étale site applies to the henselian site as well, and shows that $\mathcal{P}ic[T]$ is also the sheaf associated to the presheaf $U \mapsto \operatorname{Pic}(U[t,t^{-1}])$ for the henselian topology. In particular, the étale and henselian cohomology groups of $\mathcal{P}ic[T]$ agree.

Corollary 5.3. *The $H^0-L\mathrm{Pic}$ sequence* (2.0) *is exact for every $A \to B$ and every ideal I of A with $I \cong IB$.*

Proposition 5.4. *For every reduced commutative ring A,*

$$L\operatorname{Pic}(A) \cong L\operatorname{Pic}({}^+A),$$

where ${}^+A$ denotes the seminormalization of A.

Proof. Since $L\mathrm{Pic}$ commutes with filtered colimits, the construction of ${}^+A$ in [Swan, 4.1] shows that it suffices to prove that whenever $b, c \in A$ satisfy $b^3=c^2$,

$$L\operatorname{Pic}(A) \to L\operatorname{Pic}(A[x]/(b-x^2, c-x^3))$$

is an isomorphism. Set $B=A[x]/(b-x^2, c-x^3)$; the map $A \to B$ is injective by [Swan, 4.3], so the ideal $I=(b,c)A$ is isomorphic to $IB=x^2B$. Since $(A/I)_{\text{red}} \cong (B/I)_{\text{red}}$, $H^0(A/I) \cong H^0(B/I)$ and $L\operatorname{Pic}(A/I) \cong L\operatorname{Pic}(B/I)$. The isomorphism $L\operatorname{Pic}(A) \cong L\operatorname{Pic}(B)$ now follows from the $H^0-L\mathrm{Pic}$ sequence (5.3). □

Remark 5.4.1. Here is another proof of Theorem 5.4, using the cohomological characterization of $L\mathrm{Pic}$ below. If $A \to B$ is étale, then ${}^+A \otimes_A B$ is the seminormalization of B by [Rush, 1.10] or [G, 1.6]. Therefore, there is a bijection between the henselian points of $\operatorname{Spec}(A)$ and the henselian points of $\operatorname{Spec}({}^+A)$. This implies that $H^*_{\text{hen}}(\operatorname{Spec}(A), \mathbb{Z})$ is isomorphic to $H^*_{\text{hen}}(\operatorname{Spec}({}^+A), \mathbb{Z})$. Now we cite (5.5) below. □

In order to obtain a more explicit description of the map h from $L\operatorname{Pic}(A)$ to $\operatorname{Pic}(A[t,t^{-1}])$, we focus our attention on the Leray spectral sequence for the étale (or henselian) cohomology of $\mathbf{G}_{m,Y}$ on $Y=\operatorname{Spec}(A[t,t^{-1}])$ via the map $\pi\colon Y \to X$, $X=\operatorname{Spec}(A)$. For convenience we assume A is reduced, so that by (1.1.1) we have

$$\pi_* \mathbf{G}_{m,Y} \cong \mathbf{G}_m \times \mathbb{Z}.$$

By construction, $\mathcal{P}ic[T]$ is the sheaf $R^1\pi_* \mathbf{G}_{m,Y}$ (see [Milne]). The Leray spectral sequence therefore yields an exact sequence

$$0 \to H^1(X, \mathbf{G}_m) \times H^1(X, \mathbb{Z}) \to \mathrm{Pic}(Y) \to H^0(X, \mathcal{P}ic[T]).$$

We know that the righthand map is split surjective by (5.2), and that $H^1(X, \mathbf{G}_m) = \mathrm{Pic}(A)$. In summary, we have proven the following result.

Theorem 5.5. *For any commutative ring A,*

$$L\mathrm{Pic}(A) \cong H^1_{\text{ét}}(\mathrm{Spec}(A), \mathbb{Z}) \cong H^1_{\text{hen}}(\mathrm{Spec}(A), \mathbb{Z}).$$

The splitting map h: $L\mathrm{Pic}(A) \to \mathrm{Pic}(A[t, t^{-1}])$ is given by the edge map in the Leray spectral sequence:

$$L\mathrm{Pic}(A) = H^1(X, \mathbb{Z}) \to H^1(X, \pi_* \mathbf{G}_m) \to H^1(X[t, t^{-1}], \mathbf{G}_m) = \mathrm{Pic}(A[t, t^{-1}]).$$

Remark 5.5.1. Let A be a normal domain with quotient field F. From [SGA1, 1.10.1], the étale sheaf $\mathbb{Z}$ on $X = \mathrm{Spec}(A)$ is $g_* \mathbb{Z}$, where g: $\mathrm{Spec}(F) \to X$ is the generic point. From [Milne, p. 106] we calculate that

$$H^1_{\text{ét}}(X, \mathbb{Z}) \cong H^1_{\text{ét}}(F, \mathbb{Z}) \cong \mathrm{Hom}_{\text{top}}(G, \mathbb{Z}) = 0,$$

where G is the separable Galois group of F. This provides an alternative proof that $L\mathrm{Pic}(A) = 0$ for a normal domain A. Notice the advantage of the henselian topology here: F has cohomological dimension zero, so:

$$H^i_{\text{hen}}(X, \mathbb{Z}) \cong H^i_{\text{hen}}(F, \mathbb{Z}) = 0 \quad \text{for } i \neq 0.$$

In contrast, $H^2_{\text{ét}}(X, \mathbb{Z})$ is a subgroup of $H^2_{\text{ét}}(F, \mathbb{Z}) \cong \mathrm{Hom}_{\text{top}}(G, \mathbb{Q}/\mathbb{Z})$, which is large.

Remark 5.5.2. Let $X = \mathrm{Spec}(A)$ be the node of (2.2) and i: $x \to X$ the singular point. By [Artin, p. 102], there is an exact sequence of étale sheaves on X:

$$0 \to \mathbb{Z} \to g_* \mathbb{Z} \to i_* \mathbb{Z} \to 0.$$

Here g: $\mathrm{Spec}(F) \to X$ is the generic point of X. This provides another proof that $L\mathrm{Pic}(A) \cong H^1_{\text{ét}}(X, \mathbb{Z})$ is $\mathbb{Z}$. In contrast, $H^1_{\text{zar}}(X, \mathbb{Z}) = 0$, because A is a domain.

We can expand on these remarks in order to extend the $H^0 - L\mathrm{Pic}$ sequence. Let $A \to B$ be a ring map, and I an ideal with $I \cong IB$. Set $X = \mathrm{Spec}(A)$, $\tilde{X} = \mathrm{Spec}(B)$ and write π, i, j for the maps $\tilde{X} \to X$, $\mathrm{Spec}(A/I) \to X$ and $\mathrm{Spec}(B/I) \to \tilde{X}$.

Lemma 5.6. *The following sequence of étale (or henselian) sheaves on X is exact:*

$$0 \to \mathbb{Z} \to \pi_* \mathbb{Z} \oplus i_* \mathbb{Z} \to \pi_* j_* \mathbb{Z} \to 0. \tag{5.6.1}$$

If B is finite over A, the long exact sequence for its (étale or henselian) cohomology starts with the $H^0 - L\mathrm{Pic}$ sequence, and continues as:

$$\begin{aligned} L\mathrm{Pic}(B) \times L\mathrm{Pic}(A/I) \to L\mathrm{Pic}(B/I) \to H^2(X, \mathbb{Z}) \to H^2(\tilde{X}, \mathbb{Z}) \\ \times H^2(\mathrm{Spec}(A/I), \mathbb{Z}) \to \dots \end{aligned}$$

Proof. We shall give the proof for the henselian topology; the proof for the étale topology is the same. The sequence of stalks of (5.6.1) at a point x is

$$0 \to H^0(A^h) \to H^0(B^h) \times H^0(A^h/IA^h) \to H^0(B^h/IB^h) \to 0.$$

This is exact by (2.1) and (2.5), so (5.6.1) is exact. Here A^h and B^h are the henselizations of A and B at the point x. To see that the cohomology exact sequence coincides with the $H^0 - L\mathrm{Pic}$ sequence, we need only check that the two maps $H^0(B/I) \to L\mathrm{Pic}(A)$ coincide. This is a routine calculation, using (2.1.2) and the boundary map for Čech cohomology, which we leave to the reader. □

§ 6. The Zariski topology

We devote a quick section to a discussion of the local-to-global behavior of LPic for the Zariski topology. Remark (5.5.2) shows that $H^1_{\mathrm{zar}}(\mathrm{Spec}(A), \mathbb{Z})$ does not completely determine the structure of LPic. Our first result refines this:

Proposition 6.1. *In general, $H^1_{\mathrm{zar}}(\mathrm{Spec}(A), \mathbb{Z})$ is a subgroup of $L\mathrm{Pic}(A)$. However, if $L\mathrm{Pic}(A_P)=0$ for every prime ideal P of A, then*

$$L\mathrm{Pic}(A) \cong H^1_{\mathrm{zar}}(\mathrm{Spec}(A), \mathbb{Z}).$$

Proof. Set $X = \mathrm{Spec}(A)$, and consider the Leray spectral sequence for $\pi\colon X_{\text{ét}} \to X_{\mathrm{zar}}$ [Milne, III. 1.18]. Since $\pi_* \mathbb{Z} = \mathbb{Z}$ and $R^1 \pi_* \mathbb{Z}$ is the Zariski sheaf $\mathscr{L}\mathscr{P}ic$ associated to the presheaf $U \mapsto L\mathrm{Pic}(U)$, we extract the exact sequence:

$$0 \to H^1_{\mathrm{zar}}(X, \mathbb{Z}) \to L\mathrm{Pic}(A) \to H^0_{\mathrm{zar}}(X, \mathscr{L}\mathscr{P}ic).$$

Under the hypothesis that the stalks $L\mathrm{Pic}(A_P)$ of $\mathscr{L}\mathscr{P}ic$ vanish, $\mathscr{L}\mathscr{P}ic = 0$. □

Example 6.1.1 (Triangle). If $A = k[x, y]/(xy(x+y-1)=0)$ then $L\mathrm{Pic}(A_P)=0$ for all P, and $L\mathrm{Pic}(A) = H^1_{\mathrm{zar}}(X, \mathbb{Z}) = \mathbb{Z}$.

We now focus on domains, so that $H^1_{\mathrm{zar}}(X, \mathbb{Z}) = 0$ and $L\mathrm{Pic}(A) \cong H^0_{\mathrm{zar}}(X, \mathscr{L}\mathscr{P}ic)$.

Corollary 6.2 (Onoda-Yoshida [OY, 2.6]). *If A is a domain, then*

$$L\mathrm{Pic}(A_P)=0 \quad \textit{for all } P \in \mathrm{Spec}(A) \quad \Rightarrow \quad L\mathrm{Pic}(A)=0.$$

Example 6.2.1 (Pedrini [P, p. 98] [OSY, p. 250]). Let A be obtained from $k[x, y]$ by glueing the axes in the plane together into a line L. That is,

$$A = \{f(x, y) \in k[x, y] : f(t, 0) = f(0, t) \ \forall t \in k\} = k[x+y, xy, x^2 y].$$

Then A is a noetherian domain, $L\mathrm{Pic}(A_P) = \mathbb{Z}$ for all P on the line L except one, yet $L\mathrm{Pic}(A)=0$. Thus although $\mathscr{L}\mathscr{P}ic$ is not zero, it has no global sections.

Lemma 6.3 (Rush). *Let A be a domain and F its field of fractions. Suppose that $\{A_\lambda\}$ is a family of subrings of F such that $A = \bigcap A_\lambda$. Then*

$$L\mathrm{Pic}(A) \to \prod L\mathrm{Pic}(A_\lambda) \textit{ is injective.}$$

Proof. Since Pic is contracted, this follows from (1.3) and (1.8) of [Rush]. □

Proposition 6.4 (Onoda-Yoshida [OY, 2.13]). *If A is a domain, the following are equivalent:*

(i) $L\mathrm{Pic}(A_P)=0$ *for every* $P\in\mathrm{Spec}(A)$;

(ii) A_P *is anodal for every* $P\in\mathrm{Spec}(A)$.

Proof. By (3.3), (i) implies (ii). If (i) fails, and $\dim(A)$ is finite, choose a prime ideal P of A minimal with respect to the property that $L\mathrm{Pic}(A_P)\neq 0$, and set $C=\bar{A}_P\cap\bigcap\{A_Q: Q\subset P, Q\neq P\}$. Note that $L\mathrm{Pic}(C)=0$ by (6.3), so there is a finite overring B of A_P contained in C such that $L\mathrm{Pic}(A_P)$ does not inject into $L\mathrm{Pic}(B)$. By the construction of C, $A_Q=B_Q$ for every $Q\subset P$, $Q\neq P$. Since B is finite over A_P the conductor ideal I cannot lie in any such Q. Therefore, $\dim(A_P/I)=0$, so $L\mathrm{Pic}(A_P/I)=0$ by (1.6.1). By (3.2), A_P cannot be anodal, so (ii) fails.

If $\dim(A)=\infty$ and (i) fails, the argument is more delicate. Consider the nonempty poset of pairs (x,P), where P is a prime ideal and x is a nonzero element of $L\mathrm{Pic}(A_P)$; $(x,P)<(y,Q)$ if $Q\subset P$ and x maps to y under $L\mathrm{Pic}(A_P)\to L\mathrm{Pic}(A_Q)$. If $P=\bigcap P_i$ then $L\mathrm{Pic}(A_P)=\varinjlim L\mathrm{Pic}(A_{P_i})$, so Zorn's Lemma applies, and the poset has a minimal element (x,P). Construct C as above and note that the image of x in $L\mathrm{Pic}(C)$ is zero by (6.3). Hence x vanishes in some finite overring B contained in C, and we conclude as before that (ii) fails. □

Exercise 6.4.1. *Modify Example 3.5 to give an example of a 2-dimensional seminormal local domain A_P which is anodal, yet $L\mathrm{Pic}(A_P)\cong\mathbb{Z}$.*

§7. L Pic on schemes

We can extend many of the above results from rings to schemes, writing $X[t]$ and $X[t,t^{-1}]$ for $X\otimes_{\mathbb{Z}}\mathbb{Z}[t]$ and $X\otimes_{\mathbb{Z}}\mathbb{Z}[t,t^{-1}]$. In particular, it makes sense to ask if a contravariant functor on schemes is contracted.

Example 7.1. Let X be a scheme. The group $H^0(X,\mathbb{Z})$ of continuous maps from the topological space underlying X to $\mathbb{Z}$ is a contracted functor, isomorphic to

$$H^0_{\mathrm{zar}}(X,\mathbb{Z})=H^0_{\mathrm{\acute{e}t}}(X,\mathbb{Z})=\check{H}^0_{\mathrm{top}}(X,\mathbb{Z}).$$

In fact, $NH^0(X,\mathbb{Z})=LH^0(X,\mathbb{Z})=0$, because it is easy to see that

$$H^0(X,\mathbb{Z})\cong H^0(X[t],\mathbb{Z})\cong H^0(X[t,t^{-1}],\mathbb{Z}).$$

If X is quasicompact, then $H^0(X,\mathbb{Z})$ is always a free abelian group by [Fuchs, 97.7]; in fact, $H^0(X,\mathbb{Z})\cong H^0(A)$ for $A=H^0(X,\mathcal{O}_X)$. However, if X is the disjoint union of an infinite number of copies of $\mathrm{Spec}(F)$, F a field, then $H^0(X,\mathbb{Z})$ is $\prod\mathbb{Z}$, which is not free abelian.

Proposition 7.2. *The global units functor $U(X)=H^0_{\mathrm{zar}}(X,\mathcal{O}_X^*)$ is a contracted functor on the category of schemes, $LU(X)=H^0(X,\mathbb{Z})$, and the splitting map*

$$h\colon H^0(X,\mathbb{Z})\to H^0_{\mathrm{zar}}(X[t,t^{-1}],\mathcal{O}^*_{X[t,t^{-1}]})$$

is multiplication by the global section t. Therefore, (7.1) *implies that*

$$L^2U(X)=LNU(X)=0,\quad \textit{and}\quad L^iN^jU(X)=0\quad \textit{if } i\geqq 2 \textit{ or if } i=1 \textit{ and } j\neq 0.$$

Proof. Set $Y=X[t,t^{-1}]$, and let π: $Y\to X$ be the structure map. Similarly, let π^+ and π^- denote the structure maps from $Y^+=X[t]$ and $Y^-=X[t^{-1}]$ to X. By (1.1),

$$0\to\mathcal{O}_X^*\to\pi_*^+(\mathcal{O}_{X[t]}^*)\times\pi_*^-(\mathcal{O}_{X[t^{-1}]}^*)\to\pi_*(\mathcal{O}_Y^*)\to\mathbb{Z}\to 0$$

is a naturally split exact sequence of Zariski sheaves on X. (It is exact because the stalks at each $x\in X$ form the exact sequence $\mathrm{Seq}(U,\mathcal{O}_{X,x})$; the splitting is given by two maps: $\pi_*(\mathcal{O}_Y^*)\to\mathcal{O}_X^*$ is evaluation at $t=1$, and the map $\mathbb{Z}\to\pi_*(\mathcal{O}_Y^*)$ is multiplication by t.) Taking global sections therefore yields a naturally split exact sequence. This sequence is $\mathrm{Seq}(U,X)$ because, for example, $U(Y)$ is $H^0(X,\pi_*\mathcal{O}_Y^*)$. The rest is a formal consequence of the definitions, given (7.1). □

Proposition 7.3. *For any scheme* X, $U(\mathbb{P}_X^1)\cong U(X)$ *and* $\mathrm{Pic}(\mathbb{P}_X^1)\cong\mathrm{Pic}(X)\times H^0(X,\mathbb{Z})$. *Moreover, the sequence* $\mathrm{Seq}(\mathrm{Pic},X)$:

$$1\to\mathrm{Pic}(X)\to\mathrm{Pic}(X[t])\times\mathrm{Pic}(X[t^{-1}])\to\mathrm{Pic}(X[t,t^{-1}])\to L\mathrm{Pic}(X)\to 1$$

is exact, and $L\mathrm{Pic}(X)$ *is a subgroup of both* $H_{\mathrm{zar}}^2(\mathbb{P}_X^1,\mathcal{O}_X^*)$ *and* $H_{\mathrm{\acute{e}t}}^2(\mathbb{P}_X^1,\mathbf{G}_m)$.

Proof. The Mayer-Vietoris sequence for the (Zariski or ètale) cohomology of $\mathcal{O}_Y^*$, $Y=\mathbb{P}_X^1$, relative to the covering of $Y=\mathbb{P}_X^1$ by $X[t]$ and $X[t^{-1}]$ is

$$\begin{aligned}1&\to U(\mathbb{P}_X^1)\to U(X[t])\times U(X[t^{-1}])\to U(X[t,t^{-1}])\xrightarrow{\partial}\mathrm{Pic}(\mathbb{P}_X^1)\\&\to\mathrm{Pic}(X[t])\times\mathrm{Pic}(X[t^{-1}])\to\mathrm{Pic}(X[t,t^{-1}])\to H^2(\mathbb{P}_X^1,\mathcal{O}_Y^*)\ldots\end{aligned}$$

It is easy to calculate directly that $\partial(t)$ is $\mathcal{O}_Y(1)$. *I* claim that $\pi_*(\mathcal{O}_Y^*)=\mathcal{O}_X^*$ and $R^1\pi_*(\mathcal{O}_Y^*)=\mathbb{Z}$ as Zariski sheaves on X. Checking this stalkwise amounts to considering the special case $X=\mathrm{Spec}(A)$, where A is a local ring. Comparing the Mayer-Vietoris with the exact $\mathrm{Seq}(U,A)$ and $\mathrm{Seq}(\mathrm{Pic},A)$, we see that $U(\mathbb{P}_A^1)\cong U(A)$ and that $\mathrm{Pic}(\mathbb{P}_A^1)\cong\mathbb{Z}$, on $\mathcal{O}_Y(1)$. This proves the claim.

The Leray spectral sequence for the (Zariski or ètale) cohomology of $\mathcal{O}_Y^*$ yields $U(Y)\cong H^0(X,\pi_*\mathcal{O}_Y^*)\cong U(X)$ and the exact sequence

$$0\to\mathrm{Pic}(X)\to\mathrm{Pic}(Y)\to H^0(X,\mathbb{Z}).$$

The righthand map is the left inverse of the map $LU(X)=H^0(X,\mathbb{Z})\xrightarrow{\partial}\mathrm{Pic}(Y)$. This yields the calculation of $\mathrm{Pic}(Y)$, and shows that $\mathrm{Seq}(U,X)$ splits off from the beginning of the Mayer-Vietoris sequence, leaving $\mathrm{Seq}(\mathrm{Pic},X)$. □

Remark 7.3.1. According to [EGA, II.4.2.7], this result will appear in EGA V.

Remark 7.3.2. The group $H_{\mathrm{\acute{e}t}}^2(Y,\mathbf{G}_m)$ is called the *cohomological Brauer group*, and its torsion subgroup is related to the usual Brauer group, $\mathrm{Br}(Y)$ [Milne, IV.2]. Since $L\mathrm{Pic}(X)$ is often torsionfree (see (7.9.1) below), it would appear that there is little connection between $L\mathrm{Pic}(X)$ and $\mathrm{Br}(\mathbb{P}_X^1)/\mathrm{Br}(X)$.

Proposition 7.4. *If* X *is a normal scheme, then* $L\mathrm{Pic}(X)=N\mathrm{Pic}(X)=0$.

Proof. We compute $\mathrm{Pic}(Y)$, $Y=X[t,t^{-1}]$, using the Leray spectral sequence for π: $Y\to X$ and invoking (7.3). Let A be the local ring of X at some point; A is an integrally closed domain [EGA, 0_I(4.1.4)]. Thus $U(A[t,t^{-1}])=U(A)$, and $\mathrm{Pic}(A[t,t^{-1}])=\mathrm{Pic}(A)=0$ by (1.5.2). Hence as Zariski sheaves $\pi_*(\mathcal{O}_Y^*)=\mathcal{O}_X^*$

and $R^1\pi_*(\mathcal{O}_Y^*)=0$. The Leray spectral sequence therefore degenerates enough to yield

$$\mathrm{Pic}(X[t,t^{-1}])=H^1_{\mathrm{zar}}(X[t,t^{-1}],\mathcal{O}_Y^*)=H^1_{\mathrm{zar}}(X,\pi_*(\mathcal{O}_Y^*))=\mathrm{Pic}(X). \quad \square$$

Theorem 7.5. Pic *is a contracted functor on the category of reduced schemes, and*

$$L\mathrm{Pic}(X)\cong H^1_{\text{ét}}(X,\mathbb{Z})\cong H^1_{\mathrm{hen}}(X,\mathbb{Z}).$$

The splitting of Seq(Pic, X) *is provided by both the global sections map:*

$$\mathrm{Pic}(X[t,t^{-1}])\to H^0_{\text{ét}}(X,\mathcal{P}ic[T])\cong N_t\,\mathrm{Pic}(X)\oplus N_{t^{-1}}\,\mathrm{Pic}(X)$$

and the edge map in the Leray spectral sequence:

$$L\mathrm{Pic}(X)=H^1(X,\mathbb{Z})\to H^1(X,\pi_*\mathbf{G}_m)\to H^1(X[t,t^{-1}],\mathbf{G}_m)=\mathrm{Pic}(X[t,t^{-1}]).$$

Proof. Given (4.7) and (5.1), the proofs of (5.2) and (5.5) are valid here. $\square$

We cannot be so naïve when dealing with schemes which are not reduced, as Remark (4.7.1) indicates. Adapting the notation used in (4.7.1) and in the proof of (7.2) for the étale topology, we see from (1.1) that now we have:

$$\pi_*^+(\mathbf{G}_{m,Y^+})\cong\mathbf{G}_m\times\mathcal{N}_t\mathbf{G}_m,$$

and

$$\pi_*\mathbf{G}_{m,Y}\cong\mathbf{G}_m\times\mathcal{N}_t\mathbf{G}_m\times\mathcal{N}_t\mathbf{G}_m\times\mathbb{Z}.$$

Theorem 7.6. Pic *is a contracted functor on the category of all schemes, and*

$$L\mathrm{Pic}(X)\cong H^1_{\text{ét}}(X,\mathbb{Z})\cong H^1_{\mathrm{hen}}(X,\mathbb{Z}).$$

The splitting map is the edge map in the Leray spectral sequence:

$$L\mathrm{Pic}(X)=H^1(X,\mathbb{Z})\to H^1(X,\pi_*\mathbf{G}_m)\to H^1(X[t,t^{-1}],\mathbf{G}_m)=\mathrm{Pic}(X[t,t^{-1}]).$$

Proof. We compare the Leray spectral sequences for π^+, π^- and π, using (5.1) and the observation that $R^1\pi_*^+(\mathbf{G}_{m,Y^+})$ is $\mathcal{N}_t\mathcal{P}ic$ and $R^1\pi_*\mathbf{G}_{m,Y}$ is $\mathcal{P}ic[T]$, to construct the following exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0\to & & \mathrm{Pic}(X)\oplus H^1(X,\mathcal{N}_t\mathbf{G}_m)\oplus H^1(X,\mathcal{N}_{t^{-1}}\mathbf{G}_m) & \to & H^1(X,\pi_*\mathbf{G}_{m,Y}) & \to H^1(X,\mathbb{Z}) & \to 0\\
 & & \downarrow & & \downarrow & \downarrow & \\
0\to & & \mathrm{Pic}(X)\oplus N_t\,\mathrm{Pic}(X)\oplus N_{t^{-1}}\,\mathrm{Pic}(X) & \to & \mathrm{Pic}(X[t,t^{-1}]) & \to L\mathrm{Pic}(X) & \to 0\\
 & & \downarrow & & \downarrow & & \\
 & & N_t\,\mathrm{Pic}(X_{\mathrm{red}})\oplus N_{t^{-1}}\,\mathrm{Pic}(X_{\mathrm{red}}) & = & N_t\,\mathrm{Pic}(X_{\mathrm{red}})\oplus N_{t^{-1}}\mathrm{Pic}(X_{\mathrm{red}}) & & \\
 & & \downarrow & & \downarrow & & \\
0\to & & H^2(X,\mathbf{G}_m\times\mathcal{N}_t\mathbf{G}_m\times\mathcal{N}_{t^{-1}}\mathbf{G}_m) & \to & H^2(X,\mathbf{G}_m\times\mathcal{N}_t\mathbf{G}_m\times\mathcal{N}_{t^{-1}}\mathbf{G}_m\times\mathbb{Z}). & &
\end{array}$$

A diagram chase now shows that $H^1(X,\mathbb{Z})\to L\mathrm{Pic}(X)$ is an isomorphism. $\square$

Corollary 7.6.1. *For every scheme X, let ${}^{+}X$ denote the seminormalization of the reduced scheme X_{red}. Then*

$$L\mathrm{Pic}(X) \cong L\mathrm{Pic}(X_{\mathrm{red}}) \cong L\mathrm{Pic}({}^{+}X).$$

Proof. The étale cohomology of X and X_{red} agree [Milne], giving the first isomorphism. For the second, copy the proof (5.4.1), citing (7.6). □

Proposition 7.7. *For every scheme X, $NL\mathrm{Pic}(X) = L^2\,\mathrm{Pic}(X) = 0$.*

Proof. Consider the Leray spectral sequences converging to $H^i_{\mathrm{et}}(Y, \mathbb{Z})$ for $Y = X[t]$ and $X[t, t^{-1}]$ via $\pi\colon Y \to X$. The stalks of $R^p\pi_*\mathbb{Z}$ are just $H^p(A[t], \mathbb{Z})$ or $H^p(A[t, t^{-1}], \mathbb{Z})$ for some hensel local ring A. By (1.2), (2.4) and (2.5) we see that $\pi_*\mathbb{Z} = \mathbb{Z}$ and $R^1\pi_*\mathbb{Z} = 0$. Thus $L\mathrm{Pic}(Y) \cong H^1(Y, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \cong L\mathrm{Pic}(X)$. □

Our next topic is the Units-Pic sequence for an affine map $\pi\colon \tilde{X} \to X$ of schemes. If there is a closed subscheme $i\colon Y \hookrightarrow X$ with ideal sheaf $\mathcal{I}$, so that $\mathcal{I} \cong \mathcal{I} \cdot \pi_*(\mathcal{O}_{\tilde{X}})$, we shall call Y a *conducting subscheme* of X. If so, let us set $\tilde{Y} = Y \times_X \tilde{X}$ with structure map $j\colon \tilde{Y} \hookrightarrow \tilde{X}$. The reader should bear in mind the paradigm (5.6), in which $X = \mathrm{Spec}(A)$, $\tilde{X} = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A/I)$ and $\tilde{Y} = \mathrm{Spec}(B/I)$. The sequence of (Zariski, henselian or étale) sheaves on X

$$1 \to \mathcal{O}_X^* \to \pi_*(\mathcal{O}_{\tilde{X}}^*) \times i_*(\mathcal{O}_Y^*) \to \pi_* j_*(\mathcal{O}_{\tilde{Y}}^*) \to 1$$

is exact, because the stalks at $x \in X$ form the start of the Units-Pic sequence for $\mathcal{O}_{X,x} \to \mathcal{O}_{\tilde{X},x}$, and Pic of the local ring $\mathcal{O}_{X,x}$ is zero. Similarly, the proof of (5.6) shows that the sequence of étale (or henselian) sheaves

$$0 \to \mathbb{Z} \to \pi_*\mathbb{Z} \oplus i_*\mathbb{Z} \to \pi_* j_*\mathbb{Z} \to 0$$

is exact. Upon taking the long exact cohomology sequences, we find we have proven:

Proposition 7.8. *Given a finite map $\pi\colon \tilde{X} \to X$ and a conducting subscheme Y, the following are long exact sequences:*

(i) *(the Units-Pic sequence)*

$$1 \to U(X) \to U(\tilde{X}) \times U(Y) \to U(\tilde{Y}) \to \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X}) \times \mathrm{Pic}(Y) \to \mathrm{Pic}(\tilde{Y});$$

(ii) *(the $H^0 - L\mathrm{Pic}$ sequence)*

$$0 \to H^0(X, \mathbb{Z}) \to H^0(\tilde{X}, \mathbb{Z}) \times H^0(Y, \mathbb{Z}) \to H^0(\tilde{Y}, \mathbb{Z}) \to L\mathrm{Pic}(X) \to L\mathrm{Pic}(\tilde{X}) \times L\mathrm{Pic}(Y) \to \mathrm{Pic}(\tilde{Y}).$$

Theorem 7.9. *If X is locally pseudo-geometric, and* $\dim(X)$ *is finite, then*

$$L\mathrm{Pic}(X) = \mathbb{Z}^r \text{ for some } r.$$

Proof. Copy the proof of (2.3), citing (7.4) and (7.8). The analogue of (2.1.1) follows from consideration of the ring map $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \times H^0(Y, \mathcal{O}_Y) \to H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$. □

The filtered colimit trick used in (2.3.1) is not generally available in this setting, except when the noetherian approximation techniques of [EGA, I.6 and IV.8] apply. Note that noetherian schemes are quasicompact [EGA, I.2.7.2] and quasiseparated [EGA, I.6.13 or IV.1.2.8]. These approximation techniques were used in [TT, C.9] to show that any quasicompact quasiseparated scheme X is the inverse limit of schemes X_α which are finitely presented over $\mathbb{Z}$. Since (7.9) applies to each X_α, and $L\mathrm{Pic}(X)$ is the filtered colimit of the $L\mathrm{Pic}(X_\alpha)$, this proves:

Corollary 7.9.1. *If X is a noetherian scheme, or more generally any quasicompact, quasiseparated scheme, then $L\mathrm{Pic}(X)$ is a torsionfree abelian group.*

Example 7.9.2. Let X be the nonseparated scheme obtained by glueing together infinitely many copies of the node along their common smooth loci. X is quasiseparated but not quasicompact [EGA, I.6.1.12 or IV.1.2.7] and $L\mathrm{Pic}(X)=\prod \mathbb{Z}$ by (7.1).

§8. Connections with negative K-theory

We conclude this paper by addressing some questions raised by this paper which involve negative K-theory, i.e., the groups $K_{-1}(A)$, $K_{-2}(A)$, etc. One circle of questions involves possible filtrations on negative K-theory, and another involves the relationship of $L\mathrm{Pic}(X)$ to the term $E_2^{10}=H^1_{\mathrm{hen}}(X,\mathbb{Z})$ in the Brown-Gersten spectral sequence of [TT].

I do not know of a natural construction of λ-operations or Adams operations on negative K-theory which is compatible with the K-theory product, except up to torsion. By this I mean such that for $x\in K_{-d}(A)$ we have

$$\psi^k\{t,x\}=\{\psi^k t,\psi^k x\}=k\{t,\psi^k x\} \quad \text{in} \quad K_{-d+1}(A[t,t^{-1}]).$$

Clearly, such a filtration would be shifted from the filtration induced on K_0 of a suitable Laurent polynomial ring, and can be defined without having a direct definition of the operations λ^k or ψ^k.

For our purposes, it is convenient to use Kratzer's variation $F^m K_0$ of the usual γ-filtration $F^m_\gamma K_0$ on K_0 [Kr, p. 248]. By definition, $F^0 K_0(A)=K_0(A)$ and $F^1 K_0(A)=\tilde{K}_0(A)$. In general, $F^m K_0(A)$ is the subgroup of $K_0(A)$ generated by the $\gamma^i(x)$ with $i\geq m$ and $x\in\tilde{K}_0(A)$. In fact, because of the equation $xy=\gamma^2(x+y)-\gamma^2(x)-\gamma^2(y)$ and [SGA6, X.5.3.2], we also have $F^2 K_0(A)=F^2_\gamma K_0(A)=SK_0(A)$. Kratzer proved in *loc. cit.* that his filtration coincides with the γ-filtration up to torsion, and that ψ^k is multiplication by k^m on the quotient $F^m/F^{m+1}K_0(A)$.

Definition 8.1. Fix $m>0$ and set $F^{-m}K_{-m}(A)=K_{-m}(A)$. For $i\geq -m$, we define $F^i K_{-m}(A)$ to be the image in $K_{-m}(A)=L^m K_0(A)$ of

$$F^{m+i}K_0(A[t_1,t_1^{-1},\dots,t_m,t_m^{-1}])$$

under the natural quotient map defining $L^m K_0$.

Question 8.2. Is $K_{-m}(A)=F^0 K_{-m}(A)$ for every m? This is true for $m=1$ and $m=2$ by (1.4.1), (1.7) and (2.4). For $m>2$ we have $K_{-m}(A)=F^{2-m}K_{-m}(A)$. An affirmative answer for all m, together with the following result, would imply that if A is noetherian then $K_{-m}(A)=0$ for $m>\dim(A)$. This would affirmatively solve the problem posed in [W, 2.9].

Proposition 8.3. *Let $R=A[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}, x_1, \ldots, x_n]$, where A is a commutative noetherian ring of Krull dimension d. Then $F^{d+1}K_0(R)=0$. In other words, $\gamma^k(x)=0$ for every $k>d$ and every $x\in\tilde{K}_0(R)$.*

Proof (Cf. [Sou, 2.2]). Write $x=[P]-[R^r]$, where P is a projective R-module of constant rank r. If $r>d$ then $P\cong Q\oplus R^{r-d}$ by [BLR], so $x=[Q]-[R^d]$ as well. Therefore, we may assume that $r\leqq d$. Now for $k>d$ we compute:

$$\begin{aligned}\gamma^k(x)&=\lambda^k(x+k-1)\\&=\lambda^k([P\oplus R^{k-r-1}])\\&=[\Lambda^k(P\oplus R^{k-r-1})]\\&=[0]=0. \quad \square\end{aligned}$$

Corollary 8.3.1. $F^{d+1-m}K_{-m}(A)=0$ *when $m>0$, A is noetherian and* $d=\dim(A)$.

Remark 8.3.2. This fits nicely with Soulé's result that $F^{d+1+m}K_m(A)=0$ for $m>0$ [Sou, Thm. 1], as well as with the classical result that $F^{d+1}K_0(A)=0$.

(8.4). A topic related to γ-filtrations is the natural decomposition of K-theory according to the eigenvectors of the Adams operations ψ^k. Recall that for i, $m\geqq 0$

$$K_m(A)^{(i)}=\{x\in K_m(A)\colon \psi^k(x)=k^i x \text{ for all } k\neq 0\}.$$

Although these subgroups need not span $K_m(A)$, and intersect nontrivially, it is true that $\mathbb{Q}\otimes K_m(A)$ is the direct sum of its subspaces $\mathbb{Q}\otimes K_m(A)^{(i)}$, and that $\mathbb{Q}\otimes F^r K_m(A)$ is the sum of those subspaces $\mathbb{Q}\otimes K_m(A)^{(i)}$ with $i\geqq r$ [Sou, 2.8]. The same is true for negative K-theory, provided we use the following definition of the action of the Adams operations on the groups $\mathbb{Q}\otimes K_{-m}(A)$. If $m>0$ and $x\in\mathbb{Q}\otimes K_{-m}(A)$, choose $\omega\in\mathbb{Q}\otimes K_0(A[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}])$ so that x is the image of ω, and define $\psi^k(x)$ to be the image in $K_{-m}(A)$ of $\psi^k(\omega)$. It is not hard to see that $\psi^k(x)$ is well-defined, and that ψ^k is a ring endomorphism of the $\mathbb{Z}$-graded ring $\mathbb{Q}\otimes K_*(A)$. When A is noetherian and $d=\dim(A)$, (8.3.1) implies that

$$\mathbb{Q}\otimes K_{-m}(A)=\mathbb{Q}\otimes K_{-m}(A)^{(1-m)}\oplus\mathbb{Q}\otimes K_{-m}(A)^{(2-m)}\oplus\ldots\oplus\mathbb{Q}\otimes K_{-m}(A)^{(d-m)},$$

and (8.2) asks if $\mathbb{Q}\otimes K_{-m}(A)^{(i)}=0$ for $i<0$.

Exercise 8.4.1. *Show that $\mathbb{Q}\otimes K_m(A)^{(i)}$ is a contracted functor for all $m\in\mathbb{Z}$, and that*

$$L\mathbb{Q}\otimes K_m(A)^{(i)}\cong\mathbb{Q}\otimes K_{m-1}(A)^{(i-1)}.$$

Another possible approach to studying the negative K-theory of a noetherian ring or scheme is via the Brown-Gersten spectral sequence of [TT]:

$$E_2^{pq}=H^p_{\text{hen}}(X,\mathcal{K}_{-q})\Rightarrow K_{-p-q}(X).$$

Here $\mathcal{K}_{-q}$ is the henselian sheaf associated to the presheaf $U\mapsto K_{-q}(U)$.

To illustrate this approach, suppose we can prove that $K_{-m}(R)=0$ for every $m>\dim(R)$ and every hensel local ring of some fixed scheme X. From the spectral sequence we could then deduce that $K_{-m}(X)=0$ for every $m>\dim(X)+\dim \operatorname{Sing}(X)$. For example, if X has isolated singularities then we only need consider the negative K-theory of a finite number of hensel local rings.

Example 8.5 (L. Reid). The negative K-theory of a hensel local ring need not vanish. Let A be a d-dimensional affine normal domain with exactly one singular point P, and with $K_{-d}(A)\neq 0$. Such examples were constructed in [Reid]. Let $\hat{A}$ denote the P-adic completion of A, and A^h the henselization of A at P. By [W, 1.6] we have

$$K_{-d}(A)\cong K_{-d}(A^h)\cong K_{-d}(\hat{A})$$

for $d>1$. (The argument does not work for $d=1$. In fact, *I* don't know of any hensel local ring for which K_{-1} is nonzero.)

Motivated by this example, we define $\tilde{K}_{-m}(X)$ to be the kernel of the global sections map $K_{-m}(X)\to H^0_{\text{hen}}(X,\mathscr{K}_{-m})$. Since the global sections map sends representatives of $L\operatorname{Pic}(X)$ to zero, the natural map $\tilde{K}_{-1}(X)\to L\operatorname{Pic}(X)$ is onto. We define $SK_{-1}(X)$ to be the kernel of this map, so that $L\operatorname{Pic}(X)\cong\tilde{K}_{-1}(X)/SK_{-1}(X)$. By definition, we also have $SK_{-1}(X)=\tilde{K}_{-1}(X)\cap LSK_0(X)$.

Now suppose that X is noetherian and that $\dim(X)$ is finite, so the Brown-Gersten spectral sequence of [TT] converges. There is a natural map from $\tilde{K}_{-1}(X)$ to the E_2^{10} term, which is $H^1_{\text{hen}}(X,\mathbb{Z})$. It is natural to ask if the isomorphism $L\operatorname{Pic}(X)\cong H^1_{\text{hen}}(X,\mathbb{Z})$ of (5.5) or (7.5) arises in this way.

Theorem 8.5. *If X is noetherian, and* $\dim(X)$ *is finite, then in the Brown-Gersten spectral sequence converging to $K_*(X)$:*

(a) $E_\infty^{10}=E_2^{10}=H^1_{\text{hen}}(X,\mathbb{Z})$;

(b) *The kernel of $\tilde{K}_{-1}(X)\to E_\infty^{10}$ is $SK_{-1}(X)$;*

(c) *The induced map from* $L\operatorname{Pic}(X)=\tilde{K}_{-1}(X)/SK_{-1}(X)$ *to* $E_2^{10}=H^1_{\text{hen}}(X,\mathbb{Z})$ *is an isomorphism.*

Proof. Consider the seminormalization ${}^+X\to X$. It maps $SK_{-1}(X)$ to $SK_{-1}({}^+X)$, and $H^1_{\text{hen}}(X,\mathbb{Z})\cong H^1_{\text{hen}}({}^+X,\mathbb{Z})$ by (7.6.1). We may therefore assume that X is seminormal. Given $\omega\in SK_{-1}(X)$, choose a lift $\sigma\in SK_0(X[t,t^{-1}])$. Because $K_0(X)$ is a contracted functor, split by multiplication by $t\in K_{+1}(\mathbb{Z}[t,t^{-1}])$, we can write

$$\sigma=\{t,\omega\}+v_++v_-,$$

where $v_\pm$ is the image in $K_0(X[t,t^{-1}])$ of an element of $N_{t^\pm}K_0(X)$. Since X is seminormal, i.e., $N\operatorname{Pic}(X)=0$, we have $\det(v^\pm)=0$. Consequently, $\det(\{t,\omega\})=\det(\sigma)=0$ in $\operatorname{Pic}(X[t,t^{-1}])$. On the other hand, it follows from naturality of the K-theory product and the construction of the Brown-Gersten spectral sequence in [TT] that

$$\begin{array}{ccccc}
\tilde{K}_{-1}(X) & \longrightarrow & E_\infty^{10}\subset E_2^{10} & = & H^1_{\text{hen}}(X,\mathbb{Z}) \\
\downarrow{\scriptstyle t} & & \downarrow & & \downarrow{\scriptstyle t} \\
\tilde{K}_0(X[t,t^{-1}]) & \xrightarrow{\det} & E_\infty^{1,-1}=E_2^{1,-1} & = & H^1_{\text{hen}}(X[t,t^{-1}],\mathbf{G}_m)
\end{array}$$

commutes. (It is well-known that the bottom horizontal map is the "det" map from $\tilde{K}_0(Y)$ to $\mathrm{Pic}(Y) \cong H^1_{\mathrm{hen}}(Y, \mathbf{G}_m)$, and that $E^{11}_\infty = E^{11}_2$.) The right vertical map is a split injection because by (7.5) it is the splitting map for Seq(Pic, X). Consequently, the top horizontal map sends $SK_{-1}(X)$ to zero, and factors through $L\mathrm{Pic}(X)$.

Finally, given ω in $E^{10}_2 = H^1_{\mathrm{hen}}(X, \mathbb{Z})$, set $L = \{t, \omega\}$ in $H^1_{\mathrm{hen}}(X[t, t^{-1}]) \cong \mathrm{Pic}(X[t, t^{-1}])$ and set $\lambda = [L] - 1$ in $K_0(X[t, t^{-1}])$. Let x be the image of λ in $K_{-1}(X)$; since L becomes trivial in every hensel local ring, x belongs to $\tilde{K}_{-1}(X)$. Choose $v_\pm \in N_{t^\pm} K_0(X)$ so that $\lambda = \{t, x\} + v_+ + v_-$ and note that $\det(\{t, x\}) = \det(\lambda) = \{t, \omega\}$. Inspection of the above diagram now reveals that x maps to ω. Thus the horizontal map $L\mathrm{Pic}(X) \to H^1_{\mathrm{hen}}(X, \mathbb{Z})$ is onto; running the argument backwards shows that it is also into, hence an isomorphism. □

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