HIGHER WILD KERNELS AND DIVISIBILITY IN THE K-THEORY OF NUMBER FIELDS

C. Weibel

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Abstract. The higher wild kernels are finite subgroups of the even K-groups of a number field F, generalizing Tate's wild kernel for K_2. Each wild kernel contains the subgroup of divisible elements, as a subgroup of index at most two. We determine when they are equal, *i.e.*, when the wild kernel is divisible in K-theory.

Introduction

In this paper, we generalize a theorem of Tate from K_2 to higher K-theory. If F is a number field, the classical wild kernel $K_2^w(F)$ is defined to be the kernel of all norm residue symbols $K_2(F) \to \mu(F_v)$ as F_v runs over the completions of F at all finite and real infinite places. This fits into the Moore exact sequence [Mil, p.157]:

$$0 \to K_2^w(F) \to K_2(F) \xrightarrow{\lambda_F} \oplus_v \mu(F_v) \to \mu(F) \to 0. \tag{0.1}$$

It is an unpublished result of Tate (see [BB, p.250]) that the subgroup $\operatorname{div} K_2(F)$ of divisible elements in the torsion group $K_2(F)$ is a subgroup of the wild kernel of index at most two. Hutchinson [Hu1, 4.4] has proven the more precise result that $\operatorname{div} K_2(F) \neq K_2^w(F)$ if and only F is *special*, a Galois-theoretic notion whose definition is given in 5.2 below, and explored in [Hu2].

Our main result concerns the higher wild kernel $K_{2i}^w(F)$, a subgroup of $K_{2i}(F)$ which we shall define shortly, and the subgroup $\operatorname{div} K_{2i}(F)$ of divisible elements of $K_{2i}(F)$.

Theorem A. *Let F be a number field.*

(1) *If i is odd and F is special, then* $\operatorname{div} K_{2i}(F)$ *is a subgroup of* $K_{2i}^w(F)$ *of index* 2.
(2) *If i is even, or if F is not special,* $\operatorname{div} K_{2i}(F) = K_{2i}^w(F)$.

Corollary A′. *The image of $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$ in $K_4(F)$ lies in* $\operatorname{div} K_4(F)$*, i.e., it vanishes in each bounded quotient $K_4(F)/m$.*

It is the 2-primary part of this theorem that is really new. Indeed, the odd torsion part of theorem A was established by Banaszak and Kolster (see [Ban]). We quickly review the proof for odd torsion in section 1 below; it goes back 25 years, to Schneider's theorem [Sch]. (For simplicity, we use the Voevodsky-Rost theorem [V03] to identify étale and algebraic K-theory, which Banaszak and Kolster did not).

In the event that F is an *non-exceptional* number field (*i.e.*, every cyclotomic extension of F is cyclic), the 2-primary part of theorem A also follows from Schneider's theorem (and Voevodsky's theorem [V]; see 1.1 below). This was first observed by Østvaer in [Ø, 4.1a]. More generally, if F is a totally imaginary number field, theorem A is proven in 4.7 and 5.5 below.

To define the higher wild kernels, recall that $\mu^{\otimes i}$ denotes the ith twist of the étale sheaf μ of all roots of unity; $\mu^{\otimes i}(F)$ is the (finite cyclic) subgroup fixed by the absolute Galois group of the field F. If v is a finite place of F, local duality yields isomorphisms $H^2(F_v, \mu_m^{\otimes i+1}) \cong \mu_m^{\otimes i}(F_v)$ which stabilize (at $\mu^{\otimes i}(F)$) for large m. There are *Dwyer-Friedlander maps* $\lambda_v : K_{2i}(F) \to K_{2i}(F_v) \to H^2(F_v, \mu^{\otimes i+1}) \cong \mu^{\otimes i}(F_v)$; see [DF], [Sou].

In contrast, when $F_v = \mathbb{R}$ we have $H^2(\mathbb{R}, \mu_m^{\otimes i+1}) \cong \mathbb{Z}/2$ for all even m. For each of the r_1 real places v of F we have maps $\lambda_v : K_{2i}(F) \to K_{2i}(\mathbb{R}) \to H^2(\mathbb{R}, \mathbb{Z}/2) \cong \mathbb{Z}/2$. If $i \not\equiv 1 \pmod 4$, these maps are zero, since $K_{2i}(\mathbb{R})/2$ is zero unless $i \equiv 1 \pmod 4$. (If $i \equiv 1 \pmod 4$ they yield a surjection $K_{2i}(F) \to (\mathbb{Z}/2)^{r_1}$; see [RW], [WK].)

Definition 0.2. The ith higher wild kernel $K_{2i}^w(F)$ is defined to be the kernel of the map $\lambda_F = \oplus\lambda_v : K_{2i}(F) \to (\mathbb{Z}/2)^{r_1} \oplus \coprod_{v \text{ finite}} \mu^{\otimes i}(F_v)$.

It is clear from the definition of the λ_v that $\operatorname{div} K_{2i}(F) \subseteq K_{2i}^w(F)$. These are finite groups, because it is also easy to see from (0.4) that $K_{2i}^w(F) \subseteq K_{2i}(\mathcal{O}_F)$.

Here is the analogue of sequence (0.1). If $i \equiv 1 \pmod 4$, let $K_{2i}^+(F)$ denote the kernel of the surjection $K_{2i}(F) \to (\mathbb{Z}/2)^{r_1}$. If $i \not\equiv 1 \pmod 4$, it is convenient to set $K_{2i}^+(F) = K_{2i}(F)$. The subgroup $K_{2i}^+(\mathcal{O}_S)$ of $K_{2i}(\mathcal{O}_S)$ is defined similarly.

Lemma 0.3. *If F is totally imaginary, or if $i \not\equiv 1 \pmod 4$, there is an exact sequence (of torsion abelian groups), analogous to (0.1):*

$$0 \to K_{2i}^w(F) \to K_{2i}(F) \xrightarrow{\lambda_F} \oplus_{v \text{ finite}} \mu^{\otimes i}(F_v) \to \mu^{\otimes i}(F) \to 0.$$

If $i \equiv 1 \pmod 4$, there is an exact sequence:

$$0 \to K_{2i}^w(F) \to K_{2i}^+(F) \xrightarrow{\lambda_F} \oplus_{v \text{ finite}} \mu^{\otimes i}(F_v) \to \mu^{\otimes i}(F) \to 0.$$

Proof. In each case, the ℓ-primary part of 0.3 follows from the Tate-Poitou sequence for the ℓ-primary cohomology groups $H^2(\ell) = \varinjlim_S H^2(\mathcal{O}_S, \mathbb{Z}_\ell(i+1))$. If F is totally imaginary, or if ℓ is odd, it suffices to use the observation that $K_{2i}(F)$ maps onto $H^2(\ell)$; see [DF, 8.9], [RW, 0.4] and [WK, 6.5]. Now fix $\ell = 2$. The same argument works for even i, since $H^2(\mathbb{R}, \mathbb{Z}_2(i+1)) = 0$ and $K_{2i}(F)$ maps onto $H^2(2)$ [WK]. For $i \equiv 3 \pmod 4$, use the fact that $K_{2i}(F)$ maps onto the kernel $\tilde{H}^2(2)$ of $H^2(2) \to (\mathbb{Z}/2)^{r_1}$; see [RW, 0.6]. For $i \equiv 1 \pmod 4$, it is $K_{2i}^+(F)$ which maps onto $\tilde{H}^2(2)$. □

Let $A\{\ell\}$ denote the ℓ-primary subgroup of an abelian group A.

Corollary 0.4. *For each prime ℓ, there is an exact sequence:*

$$0 \to K_{2i}^w(F)\{\ell\} \to K_{2i}^+(\mathcal{O}_F)\{\ell\} \to \oplus_{v|\ell}\, \mu_{\ell^\infty}^{\otimes i}(F_v) \to \mu_{\ell^\infty}^{\otimes i}(F) \to 0.$$

Indeed, (0.4) follows from the fact that $K_{2i}(\mathcal{O}_F)\{\ell\} \cong K_{2i}(\mathcal{O}_S)\{\ell\}$, where S denotes the set of finite places of F over ℓ, and a chase of the ℓ-primary part of the following diagram.

$$\begin{array}{ccccccccc} 0 \to K_{2i}^w(F) & \to & K_{2i}^+(F) & \to & \oplus_v\, \mu^{\otimes i}(F_v) & \to & \mu^{\otimes i}(F) \to 0 \\ \downarrow & & \downarrow = & & \downarrow & & \\ 0 \to K_{2i}^+(\mathcal{O}_S) & \to & K_{2i}^+(F) & \to & \oplus_{v\nmid\ell}\, \mu^{\otimes i}(\mathcal{O}_F/\mathfrak{p}_v) & \to & 0 \end{array}$$

The sequence (0.4) is sometimes used to define the ℓ-primary part of the higher wild kernel; see [Ban, thm. 4] and [Ø, 4.1a]. It also gives Tate's simple formula $\prod w_i(F_v)/w_i(F)$ for the index of $K_{2i}^w(F)$ in $K_{2i}(\mathcal{O}_F)$ when F is totally imaginary. (Here $w_i(F)$ denotes the order of the cyclic group $\mu^{\otimes i}(F)$)

When F has a real embedding, the connection between K-theory and étale cohomology becomes weaker. It is necessary to distinguish between the 2-primary subgroup of $K_{2i}(\mathcal{O}_S)$ and $H^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$; these are only isomorphic when $i \equiv 0, 1 \pmod 4$ by [RW, 0.6]. The cohomological analogue of theorem A concerns the Tate-Shafarevich groups $Ш^2(\mathcal{O}_S, \mu_m^{\otimes i+1})$ for $m = 2^\nu$, defined by the Tate-Poitou sequence:

$$(0.5)\quad 0 \to Ш^2(\mathcal{O}_S, \mu_m^{\otimes i+1}) \to H^2(\mathcal{O}_S, \mu_m^{\otimes i+1}) \xrightarrow{\beta} (\mathbb{Z}/2)^{r_1} \oplus \coprod_{\mathfrak{p}\in S} \mu_m^{\otimes i}(F_\mathfrak{p}) \to \mu_m^{\otimes i}(F) \to 0.$$

This group is independent of the choice of S, as long as it contains all primes over 2; the group $Ш^2(\mathcal{O}_S, \mathbb{Z}_2(i+1)) = \varprojlim Ш^2(\mathcal{O}_S, \mu_m^{\otimes i+1})$ is sometimes called the ith (2-primary) *higher étale wild kernel* of F; see [Kol, NQD, Hut3].

Passing to the inverse limit over m in the sequence (0.5) of finite groups yields a sequence for $Ш^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$. If i is odd, it looks exactly like (0.5), but if i is even the term $(\mathbb{Z}/2)^{r_1}$ vanishes in the limit and we have the analogue of (0.3):

$$(0.6)\quad 0 \to Ш^2(\mathcal{O}_S, \mathbb{Z}_2(i+1)) \to H^2(\mathcal{O}_S, \mathbb{Z}_2(i+1)) \xrightarrow{\beta} \coprod_{\mathfrak{p}\in S} \mu_{2^\infty}^{\otimes i}(F_\mathfrak{p}) \to \mu_{2^\infty}^{\otimes i}(F) \to 0.$$

Since the Dwyer-Friedlander maps are compatible with the Tate-Poitou maps, we see from lemma 0.3 that there is a canonical induced map $K_{2i}^w(F) \to Ш^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$.

Theorem B. *If F is a real number field, $H^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$ injects into the motivic cohomology group $H^{2,i+1} = H^2_M(F, \mathbb{Z}_{(2)}(i+1))$. Moreover:*

1. *If i is odd and F is special, $\operatorname{div} H^{2,i+1}$ is a subgroup of $Ш^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$ of index 2;*
2. *If i is even, or if i is odd and F isn't special, $\operatorname{div} H^{2,i+1}F = Ш^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$.*

We introduce $\operatorname{div} H^{2,i+1}F$ in section 7 and prove theorem B in 7.6 and 7.8. If $i \not\equiv 2 \pmod 4$ theorem A is proven in 7.9; the final case $i \equiv 2 \pmod 4$ is handled in section 8 below, using the calculations in [RW].

Notation. For any field K, and any Galois module M, we write $K(M)$ for the extension field of K which is the fixed field for the kernel of $\mathrm{Gal}(\bar{K}/K) \to \mathrm{Aut}(M)$. Thus $\mathrm{Gal}(\bar{K}/K)$ acts trivially on M if and only if $K = K(M)$. We will apply this to $M = \mu_m^{\otimes i} = \mu_m \otimes \cdots \otimes \mu_m$.

For any abelian group A, $\operatorname{div} A$ denotes the subgroup of all divisible elements in A, *i.e.*, $\operatorname{div} A = \cap_n nA = \{a \in A : (\forall n)(\exists b \in A) a = nb\}$, and $A\{\ell\}$ denotes the ℓ-primary subgroup of A.

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As this paper was being completed in May 2004, K. Hutchinson kindly sent me a draft copy of his preprint [Hu3] in which he obtains essentially the same results, following the approach of T. Nguyen Quang Do [NQD]. Hutchinson's point of view and his methods are somewhat different than those in this article.

§1. Non-exceptional fields

The purpose of this section is to give an off-the-shelf proof of the following result, which was implicit in Schneider's 1979 paper [Sch]. For ℓ odd, this was observed by Banaszak and Kolster; see [Ba1]. For $\ell = 2$, it was first published by Østvaer in [Ø]. We will give another proof of Schneider's theorem in section 4 below.

Theorem 1.1. *Let F be a number field and $i \geq 1$.*
a) If ℓ is odd, the ℓ–Sylow subgroups of $K_{2i}^{w}(F)$ and $\operatorname{div} K_{2i}(F)$ agree.
b) If F is non-exceptional, then $\operatorname{div} K_{2i}(F) = K_{2i}^{w}(F)$.

Recall that a field F is said to be *non-exceptional* if the Galois groups $\mathrm{Gal}(E/F)$ are cyclic for every cyclotomic extension E of F. Any field F containing $\sqrt{-1}$ or $\sqrt{-2}$ is non-exceptional, and every non-exceptional number field is totally imaginary. The quadratic fields $\mathbb{Q}(\sqrt{d})$ are exceptional for every squarefree d except $-1, -2$.

Example 1.1.1. When $i = 2$, theorem 1.1 states that the classical wild kernel of $K_2(F)$ equals $\operatorname{div} K_2(F)$ when F is non-exceptional. It seems certain that Tate knew this result; a proof of this result was given recently by Hutchinson in [Hu1, 4.5].

Proof of theorem 1.1. It is known that the ℓ–Sylow subgroup of the finite group $K_{2i}(\mathcal{O}_F)$ is isomorphic to $H^2(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(i+1))$ if $i > 0$ and either ℓ is odd [WK, 6.2] or $\ell = 2$ and F is totally imaginary [WK, 6.5]. This is a consequence of the Voevodsky-Rost theorem [V03] for odd ℓ, and Voevodsky's theorem [V] for $\ell = 2$. Theorem 1.1 is now just a translation of Schneider's theorem [Sch, 4.8], which we restate as 1.4 below. □

Although Schneider's theorem was originally stated only for odd ℓ, the proof in Schneider's paper applies *verbatim* for $\ell = 2$ when F is non-exceptional. This is hard to see, because the proof in loc. cit. never mentions the running hypothesis on ℓ. In fact, the hypothesis on ℓ is actually used only once, when Schneider cites the following theorem of Neukirch [Neu1].

Fix a prime ℓ and a number field F, and let S be a finite set of places of F, including all infinite ones and all places over ℓ. Let M be a finite ℓ-primary Galois module for the absolute Galois group $G = \mathrm{Gal}(\bar{F}/F)$, and let M' denote its Pontrjagin dual module, $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}(1))$.

Neukirch's Theorem 1.2. *Suppose that the ℓ-primary abelian group underlying M is cyclic. If $\ell = 2$, assume in addition that F is non-exceptional. Then:*
a) $H^1(F, M) \to \oplus_{v \in S} H^1(F_v, M)$ is surjective;
b) $H^2(F, M') \to \oplus_{v \in S} H^2(F_v, M')$ is injective;

Proof. The assumption that ℓ is odd or F is non-exceptional implies that the image of $G \to \mathrm{Aut}(M')$ is cyclic. Hence assertion (a) is a special case of 6.4(b) of [Neu1]. (See 6.4(e) if ℓ is odd.) By [Neu1, 4.4] (a version of Tate-Poitou duality), parts (a) and (b) are equivalent. □

Corollary 1.3. *(Schneider) Let M denote the Galois module $\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)$ for $i \neq 1$. If $\ell = 2$, assume in addition that F is non-exceptional. Then the subgroup $\operatorname{div} H^1(F, M)$ of divisible elements is contained in the subgroup $H^1(\mathcal{O}_F, M)$, and in fact*

$$\operatorname{div} H^1(F, M) = \{a \in H^1(\mathcal{O}_F, M) : a_v \in \operatorname{div} H^1(F_v, M) \forall v \text{ over } \ell\}.$$

Proof. If ℓ is odd, this is 4.4 of [Sch]. Schneider's proof remains valid if $\ell = 2$ and F is non-exceptional because it uses Neukirch's theorem 1.2, restated as (2.7) in [Sch, p.187]. □

Schneider's Theorem 1.4. *If ℓ is odd, or if F is non-exceptional, there is an exact sequence for all $i \neq 1$:*

$$0 \to \operatorname{div} H^2(F, \mathbb{Z}_\ell(i)) \to H^2(\mathcal{O}_S, \mathbb{Z}_\ell(i)) \to \oplus_{v \in S}\ H^2(F_v, \mathbb{Z}_\ell(i)) \to \mathbb{Z}/w_{i-1}(F) \to 0.$$

We may identify the maps $H^2(F_v, \mathbb{Z}_\ell(i)) \to \mathbb{Z}/w_{i-1}(F)$ with the canonical surjections $\mathbb{Z}/w_{i-1}(F_v) \to \mathbb{Z}/w_{i-1}(F)$.

Proof. This is a paraphrase of the conclusion of theorem 4.8 in [Sch]. The hypothesis of 4.8 is satisfied, because $H^2(\mathcal{O}_F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) = 0$ if $i \neq 1$ and either ℓ is odd or $\ell = 2$ and F is non-exceptional [Li, 9.5]. This hypothesis allows Schneider to identify $H^2(\mathcal{O}_S, \mathbb{Z}_\ell(i))$ with the quotient of $H^1(\mathcal{O}_F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ by its maximal divisible subgroup.

Now fix a local field F_v. By local duality, the tower of groups $\{H^2(F_v, \mu_{\ell^\nu}^{\otimes i})\}_{\nu=1}^\infty$ is Pontrjagin dual to the increasing sequence of groups $H^0(F_v, \mu_{\ell^\nu}^{\otimes 1-i}) = \mu_{\ell^\nu}^{\otimes 1-i}(F_v)$. This sequence stabilizes at $\mathbb{Z}/w_{1-i}(F_v)$ for $\nu \gg 0$, and this group is the Pontrjagin dual of $\mu^{\otimes i-1}(F_v)$. Therefore we may identify the final map in Schneider's sequence with the sum of the canonical surjections $\mu^{\otimes i-1}(F_v) \to \mu^{\otimes i-1}(F)$. □

Schneider is also credited with proving the following result, in [Sch, 6.1]:

Theorem 1.5. *Suppose that ℓ is odd (or that $\ell = 2$ and F is non-exceptional). For each $\nu > 0$, set $E_\nu = F(\mu_{\ell^\nu}^{\otimes i})$ and $G_\nu = \operatorname{Gal}(E_\nu/F)$. Then the ℓ-primary part of the wild kernel is*

$$K_{2i}^w(F)\{\ell\} \cong \varprojlim \operatorname{Pic}(\mathcal{O}_{E_\nu}[\tfrac{1}{\ell}])(i)_{G_\nu}\{\ell\}.$$

However, [Sch, 6.1] needs decoding as Schneider phrased his result in different language; see [Kol, 1.7] [NQD, 1.1]. By Tate-Poitou duality, the wild kernel is dual to the kernel R_{-i} of $H^1(\mathcal{O}_S, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-i)) \to \oplus_{v \in S} H^1(F_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-i))$. Schneider then passes to the field $F_\infty = \cup F(\zeta_{\ell^\nu})$, where the problem is solvable using the Iwasawa module and class field theory, and then uses Galois descent for the pro-cyclic group $\operatorname{Gal}(F_\infty/F)$.

We omit the details, since we will prove a slightly stronger result in 4.3 (and 4.3.1) below, namely that the inverse limit on the right of 1.5 stabilizes for large ν.

§2. Galois coinvariants

Eventually, we are going to consider a finite Galois extension $F \subset E$ and study the transfer maps between their respective Tate-Poitou sequences for $M = \mu_m^{\otimes i+1}$. Recall that Tate-Poitou duality for any Galois module M (over F) ends in the map

$$\oplus_{v \in S} H^2(F_v, M) \to H^0(F, M')^{\#}.$$

Each component of this map is a surjection, because it is dual (under local duality $H^2(F_v, M) \cong H^0(F_v, M')^{\#}$) to the injection $M'(F) \to M'(F_v)$.

2.1. For the Galois module $M = \mu_m^{\otimes i+1}$ we have $M' = \mu_m^{\otimes -i}$ and the dual of $\mu_m^{\otimes -i}(F)$ is $\mu_m^{\otimes i}(F)$, so we may further identify these maps with the canonical surjections $\mu_m^{\otimes i}(F_v) \to \mu_m^{\otimes i}(F)$. In particular, if $\mathrm{Gal}(\bar{F}/F)$ acts trivially on $\mu_m^{\otimes i}$, the ending of the Tate-Poitou sequence is $\oplus_{v \in S}\ \mu_m^{\otimes i} \to \mu_m^{\otimes i}$.

Proposition 2.2. *Let Γ be a profinite group, Γ_1 a closed normal subgroup of finite index, and $G = \Gamma/\Gamma_1$. Let M be a discrete Γ-module, or a bounded complex of modules, such that $H^q(\Gamma, M) = 0$ for all $q > n$. Then the corestriction map induces an isomorphism $H^n(\Gamma_1, M)_G \xrightarrow{\cong} H^n(\Gamma, M)$.*

Proof. The result follows from the second quadrant Tate spectral sequence $E_2^{p,q} = H_{-p}(G, H^q(\Gamma_1, M)) \Rightarrow H^{p+q}(\Gamma, M)$, which converges because M has finite cohomological dimension. See [Se, Ch. I, App. 1] for example. □

Corollary 2.3. *Let E/F be a Galois extension of totally imaginary number fields, with Galois group G. For every finite Galois module M, and every G-invariant set of places S of F containing all ramified places and all places over $|M|$, the corestriction maps induce isomorphisms*

$$H^2(E, M)_G \xrightarrow{\cong} H^2(F, M), \qquad H^2(\mathcal{O}_{E,S}, M)_G \xrightarrow{\cong} H^2(\mathcal{O}_S, M).$$

Proof. It is well known that $H^q(F, M) = 0$ and $H^q(\mathcal{O}_S, M) = 0$ for $q > 2$; see [Se, II.4.4] or [Kahn, 3.1.1]. The hypotheses on S ensure that $\mathcal{O}_{E,S}$ is étale over $\mathcal{O}_S$. □

Similarly, for every Galois extension $F_v \subset E_w$ of nonarchimedean local fields, and every Galois module M, the corestriction map $H^2(E_w, M)_G \to H^2(F_v, M)$ is an isomorphism. This follows from lemma 2.2, since F_v has cohomological dimension 2.

In the specific case $M = \mu_m^{\otimes i+1}$, we saw before 3.1 that $H^2(F_v, M) \cong \mu_m^{\otimes i}(F_v)$. From this, we extract the following useful formula, which we will need in 2.7.

Corollary 2.4. *For any Galois extension $F_v \subset E_w$ of nonarchimedean local fields, and $Z = \mathrm{Gal}(E_w/F_v)$, the H^2–corestriction induces an isomorphism $\mu_m^{\otimes i}(E_w)_Z \xrightarrow{\cong} \mu_m^{\otimes i}(F_v)$.*

Application 2.5. Let E/F be a Galois extension of totally imaginary number fields, with Galois group G. Then the vertical corestriction maps are all onto in the Tate-Poitou diagram:

$$\begin{array}{ccccccc} H^2(E, \mu_m^{\otimes i+1}) & \longrightarrow & \oplus_w \mu_m^{\otimes i}(E_w) & \longrightarrow & \mu_m^{\otimes i}(E) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^2(F, \mu_m^{\otimes i+1}) & \longrightarrow & \oplus_v \mu_m^{\otimes i}(F_v) & \longrightarrow & \mu_m^{\otimes i}(F) & \longrightarrow & 0. \end{array}$$

Now suppose that we are given a finite Galois extension $F \subset E$ of number fields, with Galois group G. For each prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$, G acts on the set of primes over $\mathfrak{p}$; if $\mathfrak{q}$ is a prime of $\mathcal{O}_E$ over $\mathfrak{p}$, the subgroup fixing $\mathfrak{q}$ is the decomposition subgroup $Z_\mathfrak{p} \subset G$, and it is the Galois group of $E_\mathfrak{q}/F_\mathfrak{p}$. Moreover, $Z_\mathfrak{p}$ also acts faithfully on $\mu_m^{\otimes i}$.

Lemma 2.6. *Let $Z_\mathfrak{p} \subset G$ denote the decomposition subgroup of a prime $\mathfrak{q}$ of E over a prime $\mathfrak{p}$ of F. Then $\oplus_{\mathfrak{q}|\mathfrak{p}}\ \mu_m^{\otimes i}$ is the induced module $\mathrm{Ind}\,_{Z_\mathfrak{p}}^G(\mu_m^{\otimes i})$.*

Proof. This is straightforward; *cf.* the proof of [Hu1, 4.1]. □

Corollary 2.7. *Set $M = \oplus_{\mathfrak{q}|2}\ H^2(E_\mathfrak{q}, \mu_m^{\otimes i+1}) \cong \oplus_{\mathfrak{q}|2}\ \mu_m^{\otimes i}$. Then $M_G \cong \oplus_{\mathfrak{p}|2}\ \mu_m^{\otimes i}(F_\mathfrak{p})$, and $H_n(G, M) \cong \oplus_{\mathfrak{p}|2}\ H_n(Z_\mathfrak{p}, \mu_m^{\otimes i})$ for all n.*

Proof. By 2.6, $M \cong \oplus_\mathfrak{p}\ \mathrm{Ind}\,_{Z_\mathfrak{p}}^G(\mu_m^{\otimes i})$. Since induced modules are also coinduced modules [WH, 6.3.4], the second assertion follows from Shapiro's Lemma. The description of M_G follows from the observation 2.4 that $(\mu_m^{\otimes i})_{Z_\mathfrak{p}} \cong \mu_m^{\otimes i}(F_\mathfrak{p})$. □

Proposition 2.8. *If F is totally imaginary, and $G = \mathrm{Gal}(F(\mu_m^{\otimes i})/F)$, then the H^2–corestriction ${\mu_m^{\otimes i}}_G \xrightarrow{\cong} \mu_m^{\otimes i}(F)$ is an isomorphism for all m.*

Proof. Applying the right exact G-coinvariant to the top row of 2.5 yields the diagram with exact rows:

$$\begin{array}{ccccccc}
H^2(E, \mu_m^{\otimes i+1})_G & \longrightarrow & (\oplus_{\mathfrak{q}|2}\ \mu_m^{\otimes i}(E_\mathfrak{q}))_G & \longrightarrow & \mu_m^{\otimes i}(E)_G & \longrightarrow & 0 \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} & & \downarrow & & \\
H^2(F, \mu_m^{\otimes i+1}) & \longrightarrow & \oplus_{\mathfrak{p}|2}\ \mu_m^{\otimes i}(F_\mathfrak{p}) & \longrightarrow & \mu_m^{\otimes i}(F) & \longrightarrow & 0.
\end{array}$$

The left vertical map is an isomorphism by 2.3, and the middle vertical is an isomorphism by 2.7. The right vertical map is now an isomorphism by the 5–lemma. □

We remark that the usual corestriction map $\mu_m^{\otimes i}(E_\mathfrak{q})_G \to \mu_m^{\otimes i}(F_\mathfrak{p})$ is the norm, and need not be an isomorphism when $|\mathcal{O}_F/\mathfrak{p}| \equiv 3 \pmod 4$ and $m = 2^\nu$; see 3.5 below.

§3. Group coinvariants

In this section we consider a finite group G acting faithfully on the sequence (0.3) and show that sometimes the coinvariant sequence is exact. We will fix a number field E and assume that $\mathrm{Gal}(\bar{E}/E)$ acts trivially on the Galois module $\mu_m^{\otimes i}$. We have in mind the case $E = F(\mu_m^{\otimes i})$, where $F = E^G$, so that the action of G on $\mu_m^{\otimes i}$ is faithful.

Definition 3.1. Let T be a set of finite places in a number field E on which a finite group G acts, and assume that $\mathrm{Gal}(\bar{E}/E)$ acts trivially on the Galois module $\mu_m^{\otimes i}$. Let M^0 denote the kernel of the map $\oplus_{\mathfrak{q}\in T}\ \mu_m^{\otimes i} \to \mu_m^{\otimes i}$ in (0.23. We will study the exact group homology sequence

$$\text{(3.1.1)} \quad \begin{array}{l} \xrightarrow{\rho_2} H_2(G, \mu_m^{\otimes i}) \to H_1(G, M^0) \to H_1(G, \oplus_{\mathfrak{q}\in T}\ \mu_m^{\otimes i}) \xrightarrow{\rho_1} H_1(G, \mu_m^{\otimes i}) \to \\ \qquad \to M^0_G \to (\oplus_{\mathfrak{q}\in T}\ \mu_m^{\otimes i})_G \to {\mu_m^{\otimes i}}_G \to 0. \end{array}$$

Let M be a Galois module, such as $\mu_m^{\otimes i}$. The next result is a modification of the proof of [Hu1, 2.3 and 2.4], which is the special case $i = 1$.

Lemma 3.2. *Let G be a finite cyclic subgroup of* $\mathrm{Aut}(M)$, *where* $M \cong \mathbb{Z}/2^\nu$ *as an abelian group. Assume that* $G \neq \{\pm 1\}$. *Then* $H_p(G, M) = 0$ *for all* $p \neq 0$. *Equivalently, the canonical norm map is an isomorphism:* $M_G \xrightarrow{\simeq} M^G$.

Proof. Under the identification of $\mathrm{Aut}(M)$ with $\mathbb{Z}/m^\times$, any element of $\mathrm{Aut}(M)$ is multiplication by an integer relatively prime to m. Let t be a generator of G. As $\ell = 2$, the generator t of any subgroup of $\mathrm{Aut}(M)$ is multiplication by either 5^{2^a} or -5^{2^a} for $a \le \nu - 2$, and if $t \neq \pm 1$ then $t^{|G|}$ is multiplication by $1 + 2^\nu j$ with j odd. (This is an easy exercise.) It follows that the norm $1 + t + \cdots + t^{|G|-1}$ is represented by an integer N satisfying $N(t-1) = 2^\nu j$. Given this, the sequence

$$M \xrightarrow{t-1} M \xrightarrow{N} M \xrightarrow{t-1} M$$

is manifestly exact, *i.e.*, the group homology vanishes.

Remarks 3.2.1. a) If $G = \{\pm 1\}$ the norm map is zero; see 3.6 below.
b) This lemma also holds when $M \cong \mathbb{Z}/\ell^\nu$ for odd ℓ, even if $G = \{\pm 1\}$.

Application 3.3. Consider $E = F(\mu_m^{\otimes i})$ and $G = \mathrm{Gal}(E/F)$, where $m = 2^\nu$. By construction, G acts faithfully on $\mu_m^{\otimes i}$ so we may identify G with a subgroup of $\mathrm{Aut}(\mu_m^{\otimes i}) \cong \mathbb{Z}/m^\times$. Here are two cases when $(\mu_m^{\otimes i})_G \cong (\mu_m^{\otimes i})^G = \mu_m^{\otimes i}(F)$.
a) If F is non-exceptional then G is cyclic, and (by considering larger cyclotomic extensions) it is easy to see that $G \neq \{\pm 1\}$ when $m \ge 8$.
b) If i is even, we also have G cyclic and $G \neq \{\pm 1\}$. To see this, choose $\gamma \in \mathrm{Gal}(\bar{F}/F)$ and suppose that $\gamma(\zeta) = \zeta^a$ for every mth root of unity in $\bar{F}$. T hen under $\mathrm{Gal}(\bar{F}/F) \to G \subseteq \mathrm{Aut}(\mu_m^{\otimes i}) \cong \mathbb{Z}/m^\times$, we see that γ maps to multiplication by a^i, which is a square. Hence G is a subgroup of the squares in $\mathbb{Z}/m^\times$. But the squares in $\mathbb{Z}/m^\times$ form a cyclic group which does not contain -1. The assertion follows.

Lemma 3.4. *Let* $\mathbb{E} = \mathbb{F}_q(\mu_m^{\otimes i})$, *where* $m = 2^\nu$. *Then the norm* $\mu_m^{\otimes i}(\mathbb{E}) \to \mu_m^{\otimes i}(\mathbb{F}_q)$ *is onto if and only if* $q^i \not\equiv -1 \pmod{m}$.

Proof. (Cf. Hutchinson [Hu1, 3.1]) Set $M = \mu_m^{\otimes i}$. By construction, $G = \mathrm{Gal}(\mathbb{E}/\mathbb{F}_q)$ is a subgroup of $\mathrm{Aut}(M)$. Now G is generated by the Frobenius ϕ, which acts on M as multiplication by q^i. Since $M^G = \mu_m^{\otimes i}(\mathbb{F})$, we see that there are two possibilities: $q^i \not\equiv -1 \pmod{m}$, when the norm map is onto by lemma 3.2, and $q^i \equiv -1 \pmod{m}$, when ϕ acts as -1 and the norm map $1 + \phi$ is zero. □

Remark 3.4.1. If $m = \ell^\nu$ for an odd prime ℓ, the same proof (using 3.2.1) shows that the norm map $\mu_m^{\otimes i}(\mathbb{E}) \to \mu_m^{\otimes i}(\mathbb{F}_q)$ is always onto.

Corollary 3.5. *Let F be a number field and fix* $m = 2^\nu \ge 8$ *large enough that* $E = F(\mu_m^{\otimes i}) \neq F$. *Suppose that either i is even,* $q \equiv +1 \pmod 4$ *or F is non-exceptional.*

Then for every prime $\mathfrak{p}$ *of* $\mathcal{O}_F[1/2]$ *and every prime* $\mathfrak{q}$ *of* $\mathcal{O}_E$ *over* $\mathfrak{p}$, *the norm map* $\mu_m^{\otimes i} = \mu_m^{\otimes i}(\mathcal{O}_E/\mathfrak{q}) \to \mu_m^{\otimes i}(\mathcal{O}_F/\mathfrak{p})$ *is onto.*

Proof. Set $\mathcal{O}_F/\mathfrak{p} = \mathbb{F}_q$ and $\mathcal{O}_E/\mathfrak{q} = \mathbb{E}$. We first claim that $\mathbb{E} = \mathbb{F}_q(\mu_m^{\otimes i})$. Indeed, the hypothesis that $F \neq E$ implies that if $i = 2^\lambda s$ with s odd then $a = \nu - \lambda$ is positive and $F(\mu_m^{\otimes i}) = F(\zeta_{2^a})$. (See [RW, 1.9] for example.) Since $\zeta_{2^a} \in \mathbb{E}$, the claim follows.

If i is even, then q^i cannot be $-1 \pmod m$ since -1 is not a square in $\mathbb{Z}/m^\times$. Similarly, if $q \equiv +1 \pmod 4$ then $q^i \not\equiv -1 \pmod 4$. In either case, we are done by 3.4.

We may therefore assume that i is odd and that $\mu_m^{\otimes i}(\mathbb{F}_q) = \{\pm 1\}$. Since F is non-exceptional and $m \geq 8$, no automorphism of $\mathbb{E}$ maps to -1 under $\mathrm{Gal}(\mathbb{E}/\mathbb{F}_q) \to \mathrm{Aut}(\mu_m^{\otimes i}) \cong \mathbb{Z}/m^\times$. This is in particular true for elements of the decomposition group of $\mathfrak{q}$ such as (lifts of the) Frobenius ϕ or ϕ^i, *i.e.*, for q^i. □

Lemma 3.6. *Suppose that $M \cong \mathbb{Z}/2^\nu$ as an abelian group, and that $G \subset \mathrm{Aut}(M) = (\mathbb{Z}/2^\nu)^\times$ contains -1. Then for any decomposition $G \cong \{\pm 1\} \times C$ with C cyclic,*

$$H_p(G, M) \cong H_p(\{\pm 1\}, M_C) \cong \mathbb{Z}/2$$

for all p. In particular, $H_1(G, M) \cong \mathrm{Hom}(\mathbb{Z}/2, M_C)$ and $M_G \cong H_2(G, M) \cong M/2$.

Proof. C satisfies the hypotheses of lemma 3.2, so the Lyndon-Hochschild-Serre spectral sequence $H_p(\{\pm 1\}, H_q(C, M)) \Rightarrow H_{p+q}(G, M)$ degenerates to yield the lemma. The specific formulas are standard; see [WH, 6.2.2]. □

Corollary 3.7. *If $Z \subset G \subseteq \mathrm{Aut}(M)$ with $Z \neq G$, then $H_1(Z, M) \to H_1(G, M)$ is the zero map, and if $-1 \in Z$ then $H_2(Z, M) \xrightarrow{\cong} H_2(G, M)$.*

Proof. If $-1 \neq Z$, then Z is cyclic and the result follows from lemma 3.2. Thus we may assume that $-1 \in Z$, and hence that G is not cyclic. Since $\mathrm{Aut}(M) \cong \{\pm 1\} \times C'$ with C' cyclic, we may find cyclic subgroups $1 \subseteq C_0 \subset C \subset C_0$ with $Z = \{\pm 1\} \times C_0$ and $G = \{\pm 1\} \times C$. Since $C_0 \neq C$, $M_{C_0} = M/2^a M$ and $M_C = C/2^b M$ for $a > b$. It follows that the map $\mathrm{Hom}(\mathbb{Z}/2, M_{C_0}) \to \mathrm{Hom}(\mathbb{Z}/2, M_C)$ is zero, and that there are natural isomorphisms $H_2(Z, M) \cong H_2(G, M) \cong M/2M$. □

Proposition 3.8. *Assume that i is even or that F is non-exceptional, and $m = 2^\nu$. Set $G = \mathrm{Gal}(F(\mu_m^{\otimes i})/F)$. Then $H_1(G, M^0) = 0$ and there is an exact sequence:*

$$0 \to M_G^0 \to \oplus_{\mathfrak{p}|2}\, \mu_m^{\otimes i}(F_{\mathfrak{p}}) \to \mu_m^{\otimes i}(F) \to 0.$$

Proof. By 3.2 and 3.3 we have $H_1(G, \mu_m^{\otimes i}) = H_2(G, \mu_m^{\otimes i}) = 0$, as well as $(\mu_m^{\otimes i})_G \cong (\mu_m^{\otimes i})^G = \mu_m^{\otimes i}(F)$. Hence the sequence (3.1.1) breaks up to yield an isomorphism $H_1(G, M^0) \cong H_1(G, M)$, $M = \oplus_{\mathfrak{q}|2}\, \mu_m^{\otimes i}$, and an exact sequence involving M_G. Given 2.7, it suffices to show that for each $\mathfrak{p} \mid 2$ we have $H_1(Z_{\mathfrak{p}}, \mu_m^{\otimes i}) = 0$. But this follows from 3.2, because each decomposition group $Z_{\mathfrak{p}}$ is a subgroup of G, so it is cyclic and is image in $\mathbb{Z}/m^\times$ does not contain -1. □

Remark 3.8.1. Proposition 3.8 holds for any number field F if we replace 2 by an odd prime ℓ. We omit the routine verification, noting that 3.2.1(b) is the crucial step, and that the Galois group $\mathrm{Gal}(F(\mu_{\ell^\nu}^{\otimes i}/F)$ is always cyclic.

§4. Totally imaginary Number Fields, i even

By (0.4), the wild kernel is the intersection of the kernels of $K_{2i}(\mathcal{O}_F) \to \mu_m^{\otimes i}(F_v)$ as $m \to \infty$. On the other hand, $\operatorname{div} K_{2i}(F)$ is the intersection of the kernels of the maps $K_{2i}(\mathcal{O}_F) \to K_{2i}(F)/m$. In this section, we will compare these subgroups using the identification (for m odd or for m even and F totally imaginary) of $K_{2i}(\mathcal{O}_F)/m$ with $H^2(\mathcal{O}_S, \mu_m^{\otimes i+1})$; see [WK, 6.5].

We first dispose of a basic case; if $i = 0$ it is the Kummer sequence in class field theory. In our applications, T will be the primes over 2.

It will be convenient to adopt the notation that $\operatorname{Pic}(\mathcal{O}_T)(i)$ denotes the Galois module $\operatorname{Pic}(\mathcal{O}_T) \otimes \mu_N^{\otimes i}$, where $N = |\operatorname{Pic}(\mathcal{O}_T)|$. Thus $\operatorname{Pic}(\mathcal{O}_T) \otimes \mu_m^{\otimes i} \cong \operatorname{Pic}(\mathcal{O}_T)(i)/m$ for all m.

Proposition 4.1. *Suppose that a number field E satisfies $E = E(\mu_m^{\otimes i})$ for some prime power $m \neq 2$, and that T is a finite set of nonarchimedean places of E containing all places over m. Then there is a natural exact sequence*

$$0 \to \operatorname{Pic}(\mathcal{O}_T) \otimes \mu_m^{\otimes i} \to H^2(\mathcal{O}_T, \mu_m^{\otimes i+1}) \to \bigoplus_{\mathfrak{q} \in T} \mu_m^{\otimes i} \xrightarrow{\text{add}} \mu_m^{\otimes i} \to 0.$$

Proof. (Tate [Ta76, 6.2]) The case $i = 0$ is the Kummer sequence from class field theory; tensor it with the free $\mathbb{Z}/m$-module $\mu_m^{\otimes i}$ and use the natural $H^2(\mathcal{O}_T, \mu_m^{\otimes i+1}) \cong H^2(\mathcal{O}_T, \mu_m) \otimes \mu_m^{\otimes i}$ to get the desired sequence. □

Remark 4.1.1. If in addition m annihilates $K_{2i}(\mathcal{O}_E)\{2\}$, we may identify $K_{2i}(\mathcal{O}_E)\{2\}$ with $H^2(\mathcal{O}_E[\frac{1}{2}], \mu_m^{\otimes i+1})$. In this case, (0.4) implies that $\operatorname{Pic}(\mathcal{O}_E[\frac{1}{2}])(i)/m \cong K_{2i}^w(E)$.

Application 4.2. Suppose that G is a group of automorphisms of E fixing $F = E^G$. Let S be a set of places of F, containing all places over m, and T the places of E over S. By naturality, G acts on the sequence of 4.1. Recall from 3.1 that M^0 denotes the kernel of the map "add" in 4.1. Applying group homology and invoking 2.3 with $S = T/G$, we obtain the following exact sequence, complimentary to (3.1.1):

$$H_1(G, M^0) \to \operatorname{Pic}(\mathcal{O}_T)(i)_G/m \to H^2(\mathcal{O}_S, \mu_m^{\otimes i+1}) \to M^0_G \to 0$$

We now turn to an interpretation of the terms in this sequence involving M^0, restricting to the case $m = 2^\nu \geq 4$. Let m_0 be the largest power of 2 dividing the order of $K_{2i}(\mathcal{O}_F)$, so that the 2–Sylow subgroup of $K_{2i}(\mathcal{O}_F)$ is isomorphic to $K_{2i}(\mathcal{O}_F)/m_0$.

Corollary 4.3. *If F is non-exceptional, or if i is even (and F is totally imaginary), then $\operatorname{Pic}(\mathcal{O}_E[\frac{1}{2}])(i)_G/m \cong K_{2i}^w(F)\{2\}$ for all $m = 2^\nu \geq m_0$, where $E = F(\mu_m^{\otimes i})$.*

Proof. Let S be the places over 2 and set $M = \mu_m^{\otimes i}$. Since F is totally imaginary, 2.3 identifies $K_{2i}(\mathcal{O}_F[\frac{1}{2}])/m = H^2(\mathcal{O}_F[\frac{1}{2}], \mu_m^{\otimes i})$ with $K_{2i}(\mathcal{O}_E[\frac{1}{2}])_G/m = H^2(\mathcal{O}_E[\frac{1}{2}], \mu_m^{\otimes i})_G$. The result follows by plugging 3.8 into 4.2, and comparing to (0.4). □

Remark 4.3.1. Replacing 2 by an odd prime ℓ yields a similar result: for any number field F, $K_{2i}^w(F)\{\ell\}$ is isomorphic to $\operatorname{Pic}(\mathcal{O}_E[1/\ell])_G/m$ for $E = F(\mu_m^{\otimes i})$, when $m = \ell^\nu$ is large enough. The proof is the same, replacing 3.8 by 3.8.1.

We now turn to the subgroup $\operatorname{div} K_{2i}(F)$.

Lemma 4.4. *Let x be an arbitrary element of* $\mathrm{Pic}_+(\mathcal{O}_F)$. *For every m and $a \in \mathbb{Z}/m^\times$, there are infinitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$ that represent x in $\mathrm{Pic}_+(\mathcal{O}_F)$ and have norm $\equiv a \pmod{m}$.*

Proof. This is a special case of the generalized Dirichlet Density Theorem; see [Neu2, VII.13.2]. (Compare [Hu1], where the relevant details of the proof of Dirichlet Density are extracted.) □

Theorem 4.5. *Let $m = 2^\nu$ be large enough that $E = F(\mu_m^{\otimes i}) \neq F$. Then the corestriction map $H^2(\mathcal{O}_E[\frac{1}{2}], \mu_m^{\otimes i+1}) \to H^2(\mathcal{O}_F[\frac{1}{2}], \mu_m^{\otimes i+1})$ induces the exact sequence*

$$\mathrm{Pic}(\mathcal{O}_E[\tfrac{1}{2}])(i)_G/m \to H^2(\mathcal{O}_F[\tfrac{1}{2}], \mu_m^{\otimes i+1}) \to H^2(F, \mu_m^{\otimes i+1}) \to \oplus_{\mathfrak{p}\nmid 2}\, \mu_m^{\otimes i}(\mathcal{O}_F/\mathfrak{p}) \to 0.$$

If F is non-exceptional, or i even, the left map is an injection by 4.3 above.

Proof. Consider the diagram whose rows are the exact localization sequences:

$$\begin{array}{ccccccc}
E^\times \otimes \mu_m^{\otimes i} & \xrightarrow{d(i)} & \oplus_{\mathfrak{q}\nmid 2}\, \mu_m^{\otimes i} & \longrightarrow & H^2(\mathcal{O}_E[\tfrac{1}{2}], \mu_m^{\otimes i+1}) & \longrightarrow & H^2(E, \mu_m^{\otimes i+1}) \\
\downarrow & & \downarrow N & & \downarrow & & \downarrow \\
H^1(F, \mu_m^{\otimes i+1}) & \longrightarrow & \oplus_{\mathfrak{p}\nmid 2}\, \mu_m^{\otimes i}(\mathcal{O}_F/\mathfrak{p}) & \longrightarrow & H^2(\mathcal{O}_F[\tfrac{1}{2}], \mu_m^{\otimes i+1}) & \longrightarrow & H^2(F, \mu_m^{\otimes i+1}).
\end{array}$$

The cokernel of the upper left horizontal map $d(i)$ is $\mathrm{Pic}(\mathcal{O}_E[\frac{1}{2}]) \otimes \mu_m^{\otimes i}$. The cokernel on the lower right is the sum of the groups $H^1(\mathcal{O}_F/\mathfrak{p}, \mu_m^{\otimes i}) \cong H^0(\mathcal{O}_F/\mathfrak{p}, \mu_m^{\otimes -i})^\#$, which we have identified with $\mu_m^{\otimes i}(\mathcal{O}_F/\mathfrak{p})$.

If i is even, or F is non-exceptional, the second vertical map is onto by 3.5, and the result follows by a diagram chase. We may thus assume that i is odd. By 3.5 again, the second vertical map N is onto all terms $\mu_m^{\otimes i}(\mathbb{F}_q)$ with $q \equiv +1 \pmod 4$; we claim that the other terms come from $H^1(F, \mu_m^{\otimes i+1}) \oplus \mathrm{image}(N)$. To see this, suppose that $\mathcal{O}_F/\mathfrak{p}_0 = \mathbb{F}_q$ with $q \equiv 3 \pmod 4$. Since $\mu_m^{\otimes i}(\mathcal{O}_F/\mathfrak{p}_0) = \{\pm 1\}$, it suffices to lift the element x which is $[-1]$ in the $\mathfrak{p}_0$ factor and $[+1]$ elsewhere.

By lemma 4.4, there is a prime ideal $\mathfrak{p}_1$ with $[\mathfrak{p}_1] = [\mathfrak{p}_0]$ in $\mathrm{Pic}(\mathcal{O}_F[\frac{1}{2}])$, and $|\mathcal{O}_F/\mathfrak{p}_1| \equiv 1 \pmod 4$. From the commutative diagram

$$\begin{array}{ccccc}
H^1(F, \mu_2) & \longrightarrow & \oplus_{\mathfrak{p}} \mu_2 & \longrightarrow & \mathrm{Pic}(\mathcal{O}_F[\tfrac{1}{2}])/2 \to 0 \\
\downarrow & & \downarrow & & \\
H^1(F, \mu_m^{\otimes i+1}) & \longrightarrow & \oplus_{\mathfrak{p}} \mu_m^{\otimes i}(\mathcal{O}_F/\mathfrak{p}) & &
\end{array}$$

it follows that x is equivalent modulo the image of $H^1(F, \mu_m^{\otimes i+1})$ to a term supported at $\mathfrak{p}_1$, and we have seen that this term is in the image of N. □

Remark 4.5.1. Again, the result still holds if we replace 2 by an odd prime ℓ. In fact, the proof is easier, as N is onto by 3.4.1.

Corollary 4.6. *If* $m = 2^\nu \geq m_0$, *then the image of* $\mathrm{Pic}(\mathcal{O}_E[\frac{1}{2}])(i)_G \to K_{2i}(\mathcal{O}_F)\{2\}$ *is the 2–Sylow subgroup of* $\mathrm{div}\, K_{2i}(F)$.

Proof. Consider the kernel N_m of $K_{2i}(\mathcal{O}_F) \to K_{2i}(F)/m$. Since the N_m form a descending chain of subgroups of the finite group $K_{2i}(\mathcal{O}_F)$, they stabilize at $\mathrm{div}\, K_{2i}(F)$ for large m. But N_m is the image of $\mathrm{Pic}(\mathcal{O}_E[\frac{1}{2}])(i)_G \to K_{2i}(\mathcal{O}_F)\{2\}$ by 4.5. □

Remark 4.6.1. If ℓ is odd, the image of $\mathrm{Pic}(\mathcal{O}_E[1/\ell])(i)_G \to K_{2i}(\mathcal{O}_F)\{\ell\}$ is the ℓ–Sylow subgroup of $\mathrm{div}\, K_{2i}(F)$ for any number field F. Indeed, the proof of 4.6 goes through, using 4.5.1. Combining this with 4.3.1 yields a proof of Schneider's theorem 1.1(a).

Proposition 4.7. *Let* F *be totally imaginary, and suppose that either* i *is even or that* F *is non-exceptional. Then* $\mathrm{div}\, K_{2i}(F) = K_{2i}^w(F)$.

The ℓ*–Sylow subgroup is isomorphic to* $\mathrm{Pic}(\mathcal{O}_{E_\nu}[\frac{1}{\ell}])(i)_{G_\nu}/\ell^\nu$ *for all large* ν, *where* $E_\nu = F(\mu_{\ell^\nu}^{\otimes i})$ *and* $G_\nu = \mathrm{Gal}(E_\nu/F)$.

Proof. Combine 4.3 and 4.6 to see that the 2–Sylow subgroups are the same. The ℓ–Sylow subgroups are the same for $\ell \neq 2$ by Schneider's theorem 1.1(a); see 4.6.1. The identification with coinvariants of Picard groups is given by 4.3 and 4.3.1; *cf.* 1.5 □

§5. Totally imaginary Number Fields, i odd

We now consider the case in which i is odd and F is exceptional, but totally imaginary. In this case, $\mu_m^{\otimes i}(F) \cong \mathbb{Z}/2$. We will be comparing the sequences (3.1.1) and 4.2 when $m \geq m_0 \geq 4$; by 2.7, 2.8, and 4.6, the sequence

$$(5.1) \qquad 0 \to \operatorname{div} K_{2i}(F)\{2\} \to K_{2i}(\mathcal{O}_F)\{2\} \to \oplus_{\mathfrak{p}|2}\ \mu_m^{\otimes i}(F_\mathfrak{p}) \to \mu_m^{\otimes i}(F) \to 0$$

is exact, except possibly at $K_{2i}(\mathcal{O}_F)\{2\}$; (5.1) is exact iff the map ρ_1 is onto in (3.1.1).

Since F is exceptional and $4|m$, $\sqrt{-1} \in F(\mu_m^{\otimes i})$; this is because the automorphism $\zeta \mapsto \zeta^{-1}$ of F acts nontrivially on $\mu_m^{\otimes i}$. It follows that $F(\mu_m^{\otimes i})$ is a cyclotomic extension $F(\zeta)$ for some 2-primary root of unity ζ.

The following definition is due to Hutchinson [Hu2]. Note that if $\zeta^{2^a} = 1$ and $u = \zeta + \zeta^{-1}$ is in F then $\zeta \in F(\sqrt{-1})$: ζ satisfies $\zeta^2 - u\zeta + 1 = 0$.

Definition 5.2. A number field F is *special* if F is exceptional and for every prime $\mathfrak{p}$ of F over 2 there is a 2-primary root of unity ζ such that $\zeta + \zeta^{-1}$ belongs to $F_\mathfrak{p}$ but not to F. That is, $\zeta \in F_\mathfrak{p}(\sqrt{-1})$ but $\zeta \notin F(\sqrt{-1})$.

Remark 5.2.1. Setting $E = F(\mu_m^{\otimes i})$, it follows from [Hu2, 2.2(1)] that F is special iff $Z_\mathfrak{p} \neq \operatorname{Gal}(E/F)$ for every prime $\mathfrak{p}$ over 2.

Example 5.3. (Hutchinson [Hu2, 2.7]) Suppose that $F = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. Then F is special if and only if either: (a) $d \equiv -1 \pmod 8$ and $d \neq -1$, or (b) $d \equiv \pm 2 \pmod{16}$ and $d \neq \pm 2$. Indeed, F is exceptional iff $d \neq -1, -2$, while $F_2 = \mathbb{Q}_2(\sqrt{d})$ is non-exceptional iff $d \equiv -1 \pmod 8$ or $d \equiv -2 \pmod{16}$; if F_2 is exceptional then $(\zeta_8 + \zeta_8^{-1})/2 = \sqrt{2} \in F_2$ iff $d \equiv 2 \pmod{16}$.

Lemma 5.4. *Let i be odd. If F is an exceptional, totally imaginary number field, then (for $m \geq m_0 \geq 4$): (5.1) is exact if and only F is not special.*

Proof. (Cf. [Hu1, 4.4]) We have remarked that the sequence (5.1) is exact iff ρ_1 is onto. If F is special, then each $H_1(Z_\mathfrak{p}, M) \to H_1(G, M)$ is zero by 3.7, and hence $\rho_1 = 0$.

If F is not special, then there is some $\mathfrak{p}'$ with $Z_{\mathfrak{p}'} = G$. Using 2.7, we see that the maps ρ_1 and ρ_2 are split surjections. Thus (3.1.1) yields $H_1(G, M^0) \cong \oplus_{\mathfrak{p}\neq\mathfrak{p}'} H_1(Z_\mathfrak{p}, \mu_m^{\otimes i})$ and (with 4.2) the exactness of sequence (5.1). □

Combining 5.4 with 3.2, 4.2 and (3.1.1), we obtain the following theorem. It was proven for $i = 1$ in [Hu1, 4.4].

Theorem 5.5. *Let F be a totally imaginary number field, and let i be odd.*

1. *If F is not special, then* $\operatorname{div} K_{2i}(F) = K_{2i}^w(F)$.
2. *If F is special, then* $\operatorname{div} K_{2i}(F)$ *is a subgroup of index 2 in $K_{2i}^w(F)$, and there is a map $K_{2i}(\mathcal{O}_F) \to M_G^0$ which induces a surjection $K_{2i}^w(F) \to H_1(G, \mu_m^{\otimes i}) \cong \mathbb{Z}/2$.*

Example 5.6. (Hutchinson [Hu2, 3.1]) Suppose that $F = \mathbb{Q}(\sqrt{d})$, where $d < 0$ is square-free and $d \equiv 2 \pmod{16}$. Then $\{-1, -1\}$ is a nonzero element of $K_2^w(F)$, not in $\operatorname{div} K_2(F)$, and $K_2^w(F) \cong \operatorname{div} K_2(F) \oplus \mathbb{Z}/2$. For these fields, $K_2(\mathcal{O}_F)/K_2^w(F)$ has odd order.

§6. Totally positive cohomology

New features arise when F has a real embedding. Although we still have available all of section 3 (including 3.8 for even i), the descent result 2.3 fails, because when i is odd the composition of $H^2(E,\mu_m^{\otimes i})_G \to H^2(F,\mu_m^{\otimes i})$ with the surjection $H^2(F,\mu_m^{\otimes i}) \to H^2(\mathbb{R},\mu_m^{\otimes i}) \cong \mathbb{Z}/2$ must be zero. And when i is even, the exact sequence 4.1 must be adjusted to include real places in T. These facts follow from the following considerations:

6.1. When i is odd and $4|m$, the field $F(\mu_m^{\otimes i})$ is a cyclotomic extension containing $\sqrt{-1}$, by the argument after (5.1). However, when i is even, the field $F(\mu_m^{\otimes i})$ has many real embeddings; the automorphism $\zeta \mapsto \zeta^{-1}$ acts trivially on $F(\mu_m^{\otimes i})$ because it acts trivially on $\mu_m^{\otimes i}$.

Definition 6.2. Suppose that M is a 2-primary Galois module over the S-integers of a number field F with $r_1 > 0$ real embeddings $i_v : F \to \mathbb{R}$. Then M is a submodule of the induced module $\oplus_v (i_v)_* M$. The *totally positive* étale cohomology of M is defined to be $H^p_+(\mathcal{O}_S, M) = H^{p-1}(X, \oplus_v (i_v)_* M/M)$. It is shown in [CKPS] that $H^p_+(\mathcal{O}_S, M) = H^p_+(F, M) = 0$ for $p \neq 1, 2$. By construction, there is an exact sequence

$$\begin{aligned} 0 \to & H^1_+(\mathcal{O}_S, M) \to H^1(\mathcal{O}_S, M) \to \oplus_v H^p(\mathbb{R}, M) \to \\ & H^2_+(\mathcal{O}_S, M) \to H^2(\mathcal{O}_S, M) \to \oplus_v H^p(\mathbb{R}, M) \to 0. \end{aligned}$$

The Tate-Poitou sequence for $M = \mu_m^{\otimes i+1}$ is

$$H^2_+(\mathcal{O}_S, \mu_m^{\otimes i+1}) \to \coprod_{\mathfrak{p} \in S} \mu_m^{\otimes i}(F_\mathfrak{p}) \to \mu_m^{\otimes i}(F) \to 0. \tag{6.2.1}$$

Example 6.3. Set $M = \mathbb{Z}_2(i+1) = \varprojlim \mu_m^{\otimes i+1}$. Since the cohomology groups with coefficients $\mu_m^{\otimes i+1}$ are finite we have $H^2(\mathcal{O}_S, M) = \varprojlim H^2(\mathcal{O}_S, \mu_m^{\otimes i+1})$. Since $H^n(\mathbb{R}, M)$ is $\mathbb{Z}/2$ if $i-n$ is even, and zero if $i-n$ is odd, we have sequences:

$$\begin{aligned} (\mathbb{Z}/2)^{r_1} \to H^2_+(\mathcal{O}_S, M) \to H^2(\mathcal{O}_S, M) \to 0, & \qquad i \text{ even}, \\ 0 \to H^2_+(\mathcal{O}_S, M) \to H^2(\mathcal{O}_S, M) \to (\mathbb{Z}/2)^{r_1} \to 0, & \qquad i \text{ odd}. \end{aligned}$$

Apparently the H^*_+ construction first arose in a letter from Kato to Tate, circa 1973. The cohomological bound is due to Tate; see [Ta70]. In particular, lemma 2.2 applies to $\oplus_v (i_v)_* M/M[1]$, and we deduce the analogue of 2.3:

Lemma 6.4. *Let E/F be a Galois extension of number fields, with Galois group G. For every finite 2-primary Galois module M, and every G-invariant set of places S of F containing all ramified places and all places over $|M|$, the corestriction maps induce isomorphisms*

$$H^2_+(E, M)_G \xrightarrow{\cong} H^2_+(F, M), \qquad H^2_+(\mathcal{O}_{E,S}, M)_G \xrightarrow{\cong} H^2_+(\mathcal{O}_S, M).$$

Corollary 6.5. *When $E = F(\mu_m^{\otimes i})$, $H^2_+(\mathcal{O}_{E,S}, \mu_m^{\otimes i})_G \cong H^2_+(\mathcal{O}_S, \mu_m^{\otimes i})$.*

Example 6.6. It is not hard to see that: $H^0_+(\mathcal{O}_S, \mathbb{G}_m) = 0$; if $\mathcal{O}_{S,+}^\times$ is the group of totally positive units and Pic_+ is the narrow Picard group then $H^1_+(\mathcal{O}_S, \mathbb{G}_m)$ is $\mathrm{Pic}_+(\mathcal{O}_S) \oplus \mathbb{R}^{r_1} / \ln(\mathcal{O}_{S,+}^\times)$; and $H^2_+(\mathcal{O}_S, \mathbb{G}_m)$ is the positive Brauer group $\mathrm{Br}_+(\mathcal{O}_S)$, defined as the kernel of the surjection $\mathrm{Br}(\mathcal{O}_S) \to (\mathbb{Z}/2)^{r_1}$; see [RW, 7.3]. The classical Kummer sequence has the analogue

$$0 \to \mathrm{Pic}_+(\mathcal{O}_S)/m \to H^2_+(\mathcal{O}_S, \mu_m) \to \bigoplus_{\text{finite } \mathfrak{q}\in S} \mathbb{Z}/m \xrightarrow{\text{add}} \mathbb{Z}/m \to 0.$$

Let j be the signature defect of $\mathcal{O}_S$, *i.e.*, the rank of the cokernel of the signature map $\mathcal{O}_S^\times \to \{\pm 1\}^{r_1}$. The Kummer sequence also shows that $(\mathbb{Z}/2)^j$ is the kernel of $\mathrm{Pic}_+(\mathcal{O}_S) \to \mathrm{Pic}(\mathcal{O}_S)$. Moreover, if $E = F(\zeta_m)$ and $G = \mathrm{Gal}(F(\zeta_m)/F)$ then $H^2(\mathcal{O}_{E,S}, \mu_m)_G \cong H^2_+(\mathcal{O}_S, \mu_m)$ by 6.4. It follows from 6.2 that for $m = 2^\nu$ larger than the order of $\mathrm{Pic}(\mathcal{O}_S)\{2\}$ we have the sequence

$$(6.6.1) \qquad 0 \to (\mathbb{Z}/2)^j \to H^2(\mathcal{O}_{E,S}, \mu_m)_G \to H^2(\mathcal{O}_S, \mu_m) \to (\mathbb{Z}/2)^{r_1} \to 0.$$

The sequence (6.6.1) illustrates the failure of 2.3 for real number fields.

Lemma 6.7. *If $G = \mathrm{Gal}(F(\mu_m^{\otimes i})/F)$, then the H^2–corestriction $\mu_m^{\otimes i}{}_G \xrightarrow{\cong} \mu_m^{\otimes i}(F)$ is an isomorphism for all m.*

Proof. Copy the proof of lemma 2.8, with H^2_+ in place of H^2, using (6.2.1) and 6.5. □

As in the proof of 4.1, tensoring 6.6 with $\mu_m^{\otimes i}$ and using 6.4 yields:

Proposition 6.8. *Suppose that a real number field E satisfies $E = E(\mu_m^{\otimes i})$ for some $m = 2^\nu > 2$, and that T is a finite set of nonarchimedean places of E containing all places over 2. Then there is a natural exact sequence*

$$0 \to \mathrm{Pic}_+(\mathcal{O}_T) \otimes \mu_m^{\otimes i} \to H^2_+(\mathcal{O}_T, \mu_m^{\otimes i+1}) \to \bigoplus_{\mathfrak{q}\in T} \mu_m^{\otimes i} \xrightarrow{\text{add}} \mu_m^{\otimes i} \to 0.$$

Similarly, the discussion in 4.2 goes through, using Pic_+, H^2_+, 6.4 and 6.8 in place of Pic, H^2, 2.3 and 4.1, to get

$$(6.9) \qquad H_1(G, M^0) \to \mathrm{Pic}_+(\mathcal{O}_T)(i)_G/m \to H^2_+(\mathcal{O}_S, \mu_m^{\otimes i+1}) \to M^0_G \to 0.$$

Corollary 6.10. *Let i be even, F a number field, and $m = 2^\nu \geq m_0$. Setting $E = F(\mu_m^{\otimes i})$, we have $Ш^2(\mathcal{O}_F[\frac{1}{2}], \mu_m^{\otimes i+1}) \cong \mathrm{Pic}_+(\mathcal{O}_E[\frac{1}{2}])(i)_G/m$, and there are exact sequences:*

$$0 \to \mathrm{Pic}_+(\mathcal{O}_E[\tfrac{1}{2}])(i)_G/m \to H^2_+(\mathcal{O}_F[\tfrac{1}{2}], \mu_m^{\otimes i+1}) \to \oplus_{\mathfrak{p}|2}\, \mu_m^{\otimes i}(F_\mathfrak{p}) \to \mu_m^{\otimes i}(F) \to 0;$$

$$0 \to \mathrm{Pic}(\mathcal{O}_E[\tfrac{1}{2}])(i)_G/m \to H^2(\mathcal{O}_F[\tfrac{1}{2}], \mu_m^{\otimes i+1}) \to (\mathbb{Z}/2)^{r_1} \oplus \bigoplus_{\mathfrak{p}|2} \mu_m^{\otimes i}(F_\mathfrak{p}) \to \mu_m^{\otimes i}(F) \to 0.$$

Proof. Combining (6.9) with 3.8 yields the first sequence. The second follows from this by a diagram chase using

$$\begin{array}{ccccccc}(\mathbb{Z}/2)^{r_1} & \to & \mathrm{Pic}_+(\mathcal{O}_E[\frac{1}{2}])(i)_G/m & \to & \mathrm{Pic}(\mathcal{O}_E[\frac{1}{2}])(i)_G/m & \to & 0 \\ \| & & \downarrow & & \downarrow & & \\ (\mathbb{Z}/2)^{r_1} & \to & H^2_+(\mathcal{O}_F[\frac{1}{2}],\mu_m^{\otimes i+1}) & \to & H^2(\mathcal{O}_E[\frac{1}{2}],\mu_m^{\otimes i+1}) & \to & (\mathbb{Z}/2)^{r_1} \to 0.\end{array}$$

The description of $Ш^n(\mathcal{O}_F[\frac{1}{2}],\mu_m^{\otimes i+1})$ follows from the second sequence. □

Theorem 6.11. *Let $m = 2^\nu$ be large enough that $E = F(\mu_m^{\otimes i}) \neq F$. Then the corestriction map induces the exact sequences*

$$\mathrm{Pic}_+(\mathcal{O}_E[\tfrac{1}{2}])(i)_G/m \to H^2_+(\mathcal{O}_F[\tfrac{1}{2}],\mu_m^{\otimes i+1}) \to H^2_+(F,\mu_m^{\otimes i+1}) \to \oplus_{\mathfrak{p}\nmid 2}\ \mu_m^{\otimes i}(\mathcal{O}_E/\mathfrak{p}) \to 0;$$
$$\mathrm{Pic}(\mathcal{O}_E[\tfrac{1}{2}])(i)_G/m \to H^2(\mathcal{O}_F[\tfrac{1}{2}],\mu_m^{\otimes i+1}) \to H^2(F,\mu_m^{\otimes i+1}) \to \oplus_{\mathfrak{p}\nmid 2}\ \mu_m^{\otimes i}(\mathcal{O}_E/\mathfrak{p}) \to 0.$$

If i is even, the left maps are injections.

Proof. Adding the subscript '+' to the groups in the proof of 4.5 readily proves exactness of the first sequence; the injectivity of Pic_+ in H^2_+ for even i is given by 6.10. Exactness of the second sequence, as well as injectivity of Pic in H^2 for even i, follows from this by chasing the following diagram, where $\mathcal{O}_S = \mathcal{O}_F[\frac{1}{2}]$.

$$\begin{array}{ccccccc}(\mathbb{Z}/2)^{r_1} & \longrightarrow & H^2_+(\mathcal{O}_S,\mu_m^{\otimes i+1}) & \longrightarrow & H^2(\mathcal{O}_S,\mu_m^{\otimes i+1}) & \longrightarrow & (\mathbb{Z}/2)^{r_1} \to 0 \\ \downarrow = & & \downarrow & & \downarrow & & \downarrow = \\ (\mathbb{Z}/2)^{r_1} & \longrightarrow & H^2_+(F,\mu_m^{\otimes i+1}) & \longrightarrow & H^2(F,\mu_m^{\otimes i+1}) & \longrightarrow & (\mathbb{Z}/2)^{r_1} \to 0.\end{array}$$ □

§7. The motivic wild kernel for real number fields

The 2-local motivic cohomology group $H^{2,i+1}_M(F) = H^2_M(F,\mathbb{Z}_{(2)}(i+1))$ has all the properties ascribed to the 2–Sylow subgroup of $K_{2i}(F)$ in sections 1 and 4. Its subgroup of divisible elements is the intersection of the kernels of the maps to the groups $H^2_M(F,\mathbb{Z}/m(i+1))$ for $m = 2^\nu$; by Voevodsky's theorem, the target groups are isomorphic to the étale cohomology groups $H^2(F,\mu_m^{\otimes i+1})$.

Lemma 7.1. *Let $\mathbb{F}$ be a finite field. Then for all $i > 0$,*

$$H^n_M(\mathbb{F},\mathbb{Z}_{(2)}(i)) \cong \begin{cases} K_{2i-1}(\mathbb{F}), & n = 1; \\ 0, & \textit{else}. \end{cases}$$

Proof. First note that by construction, $H^n_M(\mathbb{F},A(i)) = 0$ for $n > i$ and all coefficients A, so we may assume that $n \le i$. Since $H^n_M(\mathbb{F},\mathbb{Q}(i))$ is a summand of $K_{2i-n}(\mathbb{F})\otimes\mathbb{Q} = 0$ by [Bl], each $H^{n,i}_M = H^n_M(\mathbb{F},\mathbb{Z}_{(2)}(i))$ is a 2-primary torsion group. Since $H^n_M(\mathbb{F},\mathbb{Z}/2(i)) \cong H^n_{et}(\mathbb{F},\mathbb{Z}/2)$ vanishes for $n \neq 0, 1$, the universal coefficient theorem implies that $H^{n,i}_M = 0$ for $n \neq 1, 2$ and that $H^{2,i}_M$ is divisible. The motivic-to-K-theory spectral sequence now degenerates to yield $H^{1,i}_M \cong K_{2i-1}(\mathbb{F})$ and $H^{2,i+1}_M \cong K_{2i}(\mathbb{F}) = 0$. □

Lemma 7.2. *For $i \geq n \geq 2$, $H^n_M(\mathcal{O}_S, \mathbb{Z}_{(2)}(i)) \cong H^n_{et}(\mathcal{O}_S, \mathbb{Z}_2(i))$.*

Proof. As in the proof of 7.1, $H^n_M(\mathcal{O}_S, \mathbb{Q}(i)) = 0$ so $H^n_M(\mathcal{O}_S, \mathbb{Z}_{(2)}(i))$ is a 2-primary torsion group for all $n \geq 2$. Since $H^{n+1}(\mathcal{O}_S, \mu_m^{\otimes i}) \cong (\mathbb{Z}/2)^{r_1}$ for all $n \geq 2$ and $m = 2^\nu$, we see that $H^n_M(\mathcal{O}_S, \mathbb{Z}_{(2)}(i))$ is a finite group of exponent 2 if $n \geq 3$, and finite if $n = 2$. The result now follows from universal coefficients. □

Corollary 7.3. *If $i > 0$ is even, then there is an exact sequence for all large $m = 2^\nu$:*

$$0 \to H^2_M(\mathcal{O}_S, \mathbb{Z}_{(2)}(i+1)) \to H^2(\mathcal{O}_S, \mu_m^{\otimes i+1}) \to (\mathbb{Z}/2)^{r_1} \to 0.$$

Proof. Since $H^3(\mathcal{O}_S, \mathbb{Z}_{(2)}(i+1)) \cong \oplus H^3(\mathbb{R}, \mathbb{Z}_{(2)}(i+1)) \cong \mathbb{Z}/2$ for even i, and the group $H^2_M(\mathcal{O}_S, \mathbb{Z}_{(2)}(i+1))$ is finite, this follows from universal coefficients. □

Lemma 7.4. *For $i \geq 1$, $H^2_M(\mathcal{O}_S, \mathbb{Z}_{(2)}(i+1)) \to H^2_M(F, \mathbb{Z}_{(2)}(i+1))$ is an injection with cokernel $\oplus_{\mathfrak{p} \neq S} K_{2i-1}(\mathcal{O}_F/\mathfrak{p})\{2\}$, and $H^n_M(\mathcal{O}_S, \mathbb{Z}_{(2)}(i)) \cong H^n_M(F, \mathbb{Z}_{(2)}(i))$ for all $n \geq 3$.*

Proof. If $i = 0$, all groups are zero, so we may assume that $i \geq 1$. Set $M = \mathbb{Z}_{(2)}(i+1)$ and consider the localization sequence in motivic cohomology [Le], which by 7.1 simplifies to:

$$\begin{aligned} 0 &\to H^2_M(\mathcal{O}_S, M) \to H^2_M(F, M) \to \oplus K_{2i-1}\mathcal{O}_F/\mathfrak{p})\{2\} \\ &\to H^3_M(\mathcal{O}_S, M) \to H^3_M(F, M) \to 0 \to \cdots \\ \to 0 &\to H^n_M(\mathcal{O}_S, M) \to H^n_M(F, M) \to 0 \to \cdots . \end{aligned}$$

Now the composite $K_{2i}(F)\{2\} \to H^2_M(F, M) \to \oplus K_{2i-1}(\mathcal{O}_F/\mathfrak{p})\{2\}$ is onto; see [WK, 1.6]. The result follows. □

Recall that $H^{2,i+1}_M$ is an abbreviation for $H^2_M(F, \mathbb{Z}_{(2)}(i))$. Let $\operatorname{div} H^{2,i+1}_M(F)$ denote the subgroup of divisible elements in $H^{2,i+1}_M$.

Corollary 7.5. *If $m \geq m_0$, the image of* $\operatorname{Pic}(\mathcal{O}_E[\frac{1}{2}])(i)_G \to H^{2,i+1}_M$ *is* $\operatorname{div} H^{2,i+1}_M(F)$. *If i is even,* $\operatorname{Pic}(\mathcal{O}_E[\frac{1}{2}])(i)_G/m \cong \operatorname{div} H^{2,i+1}(F)$.

Moreover, $\operatorname{div} H^{2,i+1}_M(F) \subseteq Ш^2(\mathcal{O}_S, \mathbb{Z}_2(i+1)) \subseteq H^2_M(\mathcal{O}_S, \mathbb{Z}_{(2)}(i+1))$.

Proof. By lemma 7.4, $\operatorname{div} H^{2,i+1}_M$ is a subgroup of $H^{2,i+1}_M(\mathcal{O}_F) = H^2_M(\mathcal{O}_F[\frac{1}{2}], \mathbb{Z}_{(2)}(i+1))$, which is a finite group. Thus the sequence of kernels N_m of $H^{2,i+1}_M(\mathcal{O}_F) \to H^2(F, \mu_m^{\otimes i+1})$ stabilizes for large m, at $\operatorname{div} H^{2,i+1}_M$. But $H^{2,i+1}_M(\mathcal{O}_F)$ injects into $H^2(\mathcal{O}_F[\frac{1}{2}], \mu_m^{\otimes i+1})$ for large m, so N_m is the image of $\operatorname{Pic}(\mathcal{O}_E[\frac{1}{2}])(i)_G \to H^2(\mathcal{O}_F[\frac{1}{2}], \mu_m^{\otimes i+1})$ by 6.11.

To see that $\operatorname{div} H^{2,i+1}_M$ lies in $Ш^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$, it suffices to show that it lies in the kernel of each map $H^2_M(\mathcal{O}_S, \mathbb{Z}_{(2)}(i+1)) \to H^2(\mathcal{O}_S, \mu_m^{\otimes i+1}) \to H^2(F_v, \mu_m^{\otimes i+1})$. But this is clear since $\mathcal{O}_S \to F_v$ factors through F. □

Combining 6.10, 7.3 and 7.5 yields the following proposition, which together with (0.6) proves the "i even" half of theorem B(2):

Proposition 7.6. *Let F be a real number field. If i is even then*

$$0 \to \operatorname{div} H_M^{2,i+1}(F) \to H_M^2(\mathcal{O}_F[\tfrac{1}{2}], \mathbb{Z}_{(2)}(i+1)) \to (\mathbb{Z}/2)^{r_1} \oplus \coprod_{\mathfrak{p}|2} \mu_m^{\otimes i}(F_\mathfrak{p}) \to \mu_m^{\otimes i}(F) \to 0.$$

Now let i be odd. Combining (3.1.1), (6.2.1), (6.9) and 7.5, we obtain the analogue of (5.1), namely sequences

$$\begin{aligned} &0 \to \operatorname{div} H_M^{2,i+1} \to H_+^2(\mathcal{O}_F[\tfrac{1}{2}], \mu_m^{\otimes i+1}) \to \oplus_{\mathfrak{p}|2}\ \mu_m^{\otimes i}(F_\mathfrak{p}) \to \mu_m^{\otimes i}(F) \to 0 \\ &0 \to \operatorname{div} H_M^{2,i+1} \to H^2(\mathcal{O}_F[\tfrac{1}{2}], \mu_m^{\otimes i+1}) \to (\mathbb{Z}/2)^{r_1} \oplus \coprod_{\mathfrak{p}|2} \mu_m^{\otimes i}(F_\mathfrak{p}) \to \mu_m^{\otimes i}(F) \to 0 \end{aligned} \tag{7.7}$$

which are exact except possibly at H_+^2 and H^2; the homology at this point is the cokernel of the map ρ_1 in (3.1.1). Note that the target of ρ_1 is $\mathbb{Z}/2$ by 3.6. Of course, we see from (0.5) that the homology is $\text{Ш}^2(\mathcal{O}_S, \mu_m^{\otimes i+1})/\operatorname{div} H_M^{2,i+1}$. Substituting (6.9) for 4.2, the proof of lemma 5.4 goes through to prove:

Proposition 7.8. *Let i be odd. Then (for $m \geq m_0 \geq 4$):*

(1) *If F is not special, the sequences (7.7) are exact.*
(2) *If F is special, the homology of (7.7) at H^2 is $\text{Ш}^2(\mathcal{O}_S, \mu_m^{\otimes i+1})/\operatorname{div} H_M^{2,i+1} \cong \mathbb{Z}/2$.*

This completes the proof of the "i odd" half of theorem B.

§8. K-THEORY WILD KERNELS

We now turn to algebraic K-theory. Rognes and Weibel [RW, 0.6][WK, 7.9] show that if $1/2 \in \mathcal{O}_S$ and $i \equiv 0, 1 \pmod 4$ then $K_{2i}(\mathcal{O}_S)\{2\} \cong H^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$, while if $i \equiv 3 \pmod 4$ then there is an exact sequence

$$0 \to K_{2i}(\mathcal{O}_S)\{2\} \to H^2(\mathcal{O}_S, \mathbb{Z}_2(i+1)) \to (\mathbb{Z}/2)^{r_1} \to 0.$$

Proposition 8.1. *If $i \not\equiv 2 \pmod 4$ then the maps $K_{2i}(\mathcal{O}_S)\{2\} \to H^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$ induce isomorphisms $K_{2i}^w(F)\{2\} \cong Ш^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$ for each S.*

The case $i \not\equiv 2 \pmod 4$ of Theorem A follows immediately from 8.1 and theorem B.

Proof. When $i \equiv 0 \pmod 4$, the isomorphism $K_{2i}(\mathcal{O}_S) \cong H^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$ is compatible with the maps to $\mu^{\otimes i}(F_v)$. The proposition follows immediately from the comparison of (0.4) and (0.6).

When i is odd, we compare sequences (0.4) and (0.5), letting m be large enough that the 2–Sylow subgroup of $H^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$ equals $H^2(\mathcal{O}_S, \mu_m^{\otimes i+1})$, $\mu_m^{\otimes i+1}(F) = \mu^{\otimes i+1}(F)$ and $\mu_m^{\otimes i+1}(F_v) = \mu^{\otimes i+1}(F_v)$ for each $v \mid 2$.

When $i \equiv 3 \pmod 4$, we see from 6.3 that $K_{2i}(\mathcal{O}_S)\{2\} \cong H_+^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$. When $i \equiv 1 \pmod 4$, the comparison of (0.4) and (0.5) shows that $K_{2i}^+(\mathcal{O}_S)\{2\} \cong H_+^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$. Since $Ш^2(\mathcal{O}_S, \mathbb{Z}_2(i+1))$ is the kernel of $H_+^2(\mathcal{O}_S, \mathbb{Z}_2(i+1)) \to \coprod \mu^{\otimes i}(F_v)$, this suffices to prove the proposition in both cases. □

When $i \equiv 2 \pmod 4$, there are elements of $K_{2i}(F)$ not detected by $H^2(\mathcal{O}_S, \mu_m^{\otimes i+1})$. This includes the image of the Milnor K-group $K_4^M(F) \cong (\mathbb{Z}/2)^{r_1}$ in $K_4(\mathcal{O}_S) \subset K_4(F)$; see [RW]. Rognes and Weibel [RW][WK, 7.9] show that if $1/2 \in \mathcal{O}_S$ then there is an exact sequence

$$0 \to (\mathbb{Z}/2)^{\rho} \to K_{2i}(\mathcal{O}_S)\{2\} \to H^2(\mathcal{O}_S, \mathbb{Z}_2(i+1)) \to 0. \tag{8.2}$$

The number ρ is not yet understood, but satisfies $j \le \rho < r_1$, where j is the *signature defect* of $\mathcal{O}_S$, defined as the dimension of the cokernel of $H^1(\mathcal{O}_S, \mathbb{Z}/2) \to (\mathbb{Z}/2)^{r_1}$.

The motivic-to-K-theory spectral sequence yields the exact sequence

$$H^1(F, \mu_m^{\otimes i+1}) \xrightarrow{d_2} H^4(F, \mu_m^{\otimes i+2}) \to K_{2i}(F)/m \to H^2(F, \mu_m^{\otimes i+1}) \to 0. \tag{8.3}$$

Proposition 8.4. *If $i \equiv 2 \pmod 4$, $m = 2^\nu$ and $E = E(\mu_m^{\otimes i})$, the edge maps $(\mathbb{Z}/2)^{r_1} \cong H^4(E, \mu_m^{\otimes i+2}) \to K_{2i}(E)/m$ are zero.*

In particular, the maps $K_4^M(E) \to K_4(E)/m$ are zero.

Proof. The last sentence is the particular case $i = 2$, since $K_4^M(E) \cong H^4(E, \mu_m^{\otimes 4})$. It suffices to show that the differential $d_2 : H^1(E, \mu_m^{\otimes i+1}) \to H^4(E, \mu_m^{\otimes i+2})$ is onto in the motivic-to-K-theory spectral sequence. If $m = 2$, d_2 is onto because we may identify it with the signature map $E^\times/2 \to (\mathbb{Z}/2)^{r_1}$; the details are given in [RW, 7.4–5]. This shows that the bottom right horizontal map is onto in the commutative diagram:

$$\begin{array}{ccccc}
E^\times \otimes \mu_m^{\otimes i} & \xrightarrow{\cong} & H^1(E, \mu_m^{\otimes i+1}) & \xrightarrow{d_2} & H^4(E, \mu_m^{\otimes i+2}) \cong (\mathbb{Z}/2)^{r_1} \\
\text{onto}\downarrow & & \downarrow & & \downarrow\cong \\
E^\times \otimes \mu_2^{\otimes i} & \xrightarrow{\cong} & H^1(E, \mu_2^{\otimes i+1}) & \underset{m=2}{\xrightarrow{\text{onto}}} & H^4(E, \mu_2^{\otimes i+2}).
\end{array}$$

Since $H^4(E,\mu_m^{\otimes i+2}) \cong (\mu_m^{\otimes i+2}/2)^{r_1}$, the right vertical map is an isomorphism. The two left horizontal maps are isomorphisms because $\mathrm{Gal}(\bar{E}/E)$ acts trivially on $\mu_m^{\otimes i}$. A diagram chase establishes the surjectivity of the differential d_2, so the result follows from (8.3). □

Lemma 8.5. *If $F = F(\mu_m^{\otimes i})$ and $E = F(\mu_{2m}^{\otimes i})$, where F is exceptional, i is even and $m = 2^\nu$, then there is a root of unity ζ so that $E = F(u)$, $u = \zeta + \zeta^{-1}$, and $c = \zeta^2 + \zeta^{-2} \in F$.*

Proof. If $E = F$ we are done. Otherwise, let a be minimal such that $F(\sqrt{-1})$ does not contain a primitive 2^ath root of unity ζ (so $\zeta^2 \in F(\sqrt{-1})$). Then $\nu = a + b - 1$, where $i = 2^b j$ (j odd); see [WK]. Let ξ be a primitive $2^{\nu+1}$st root of unity and set $G = \mathrm{Gal}(F(\xi)/F)$. Since $\sqrt{-1} \notin F$, we may identify G with the unique subgroup of $\mathrm{Aut}(\mu_{2m})$ of order $|G| = 2^{b+2}$ containing $\tau(\xi) = \xi^{-1}$, namely $\langle\tau,\sigma\rangle$, where $\sigma(\xi) = \xi^s$ for $s = 1 + 2^{a-1}$. Since τ and σ fix $c = \zeta^2 + \zeta^{-2}$, $c \in F$. Since τ and σ^2 act trivially on $\mu_{2m}^{\otimes i}$, they generate $\mathrm{Gal}(F(\xi)/E)$. It follows that E contains $u = \zeta + \zeta^{-1}$ but $u \notin F$. □

Lemma 8.6. *If $E = F(\mu_m^{\otimes i})$, i even, then $H^4(E,\mu_m^{\otimes i+2}) \cong \mathrm{Ind}_1^G H^4(F,\mu_m^{\otimes i+2})$, and $H^4(E,\mu_m^{\otimes i+2})_G \cong H^4(F,\mu_m^{\otimes i+2})$.*

In particular, $K_4^M(E) \cong \mathrm{Ind}_1^G K_4^M(F)$ and $K_4^M(E)_G \cong K_4^M(F)$.

Proof. Since the isomorphism $H^4(F,\mu_m^{\otimes i+2}) \cong \prod_{r_1} H^4(\mathbb{R},\mu_m^{\otimes i+2})$ is natural in F and m, it suffices to compare the real embeddings of E and F.

By induction on m, we may assume that $[E:F] = 2$. By 8.5, there is a root of unity ζ so that $E = F(\zeta + \zeta^{-1})$, and $c = \zeta^2 + \zeta^{-2} \in F$. But every real embedding of F sends c to $2\cos(\theta)$ for some θ, and sends $c+2$ to a positive real number, because $|2\cos(\theta)| < 2$. Since $u^2 = c+2$, every real embedding of F induces two real embeddings of E, conjugate under $\mathrm{Gal}(E/F)$. □

Theorem 8.7. *The kernel $(\mathbb{Z}/2)^\rho$ of $K_{2i}(\mathcal{O}_S)\{2\} \to H^2(\mathcal{O}_S,\mathbb{Z}_2(i+1))$ in (8.2) is a subgroup of* $\mathrm{div}\, K_{2i}(F)$.

In particular, the image of $K_4^M(F) \to K_4(F)$ lies in $\mathrm{div}\, K_4(F)$.

Proof. Every element of the kernel lifts to $H^4(E,\mu_m^{\otimes i+2})$ by 8.6. Since it is m-divisible in $K_{2i}(E)$ by 8.4, it is m-divisible in $K_{2i}(F)$. □

Proof of theorem A when i is even. Comparing (0.4), (0.6), (8.2) and 8.7 yields the commutatitive diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{Z}/2)^\rho & \longrightarrow & \mathrm{div}\, K_{2i}(F) & \longrightarrow & \mathrm{div}\, H^2(F,\mathbb{Z}_{(2)}(i+1)) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & (\mathbb{Z}/2)^\rho & \longrightarrow & K_{2i}^w(F) & \longrightarrow & Ш^2(\mathcal{O}_S,\mathbb{Z}_2(i+1)) & \longrightarrow & 0.
\end{array}$$

The right vertical map is an isomorphism by theorem B. Theorem A, which asserts that the middle vertical map is an isomorphism, now follows from the 5-lemma. □

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Department of Mathematics, Rutgers University, U.S.A.
E-mail address: `weibel@math.rutgers.edu`