SCHUR-FINITENESS IN λ-RINGS

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Abstract. We introduce the notion of a Schur-finite element in a λ-ring.

Since the beginning of algebraic K-theory in [G57], the splitting principle has proven invaluable for working with λ-operations. Unfortunately, this principle does not seem to hold in some recent applications, such as the K-theory of motives. The main goal of this paper is to introduce the subring of Schur-finite elements of any λ-ring, and study its main properties, especially in connection with the virtual splitting principle.

A rich source of examples comes from Heinloth's theorem [Hl], that the Grothendieck group $K_0(\mathcal{A})$ of an idempotent-complete $\mathbb{Q}$-linear tensor category $\mathcal{A}$ is a λ-ring. For the category $\mathcal{M}^{\text{eff}}$ of effective Chow motives, we show that $K_0(Var) \to K_0(\mathcal{M}^{\text{eff}})$ is not an injection, answering a question of Grothendieck.

When $\mathcal{A}$ is the derived category of motives $\mathbf{DM}_{gm}$ over a field of characteristic 0, the notion of Schur-finiteness in $K_0(\mathbf{DM}_{gm})$ is compatible with the notion of a Schur-finite object in $\mathbf{DM}_{gm}$, introduced in [Mz].

We begin by briefly recalling the classical splitting principle in Section 1, and answering Grothendieck's question in Section 2. In section 3 we recall the Schur polynomials, the Jacobi-Trudi identities and the Pieri rule from the theory of symmetric functions. Finally, in Section 4, we define Schur-finite elements and show that they form a subring of any λ-ring. We also state the conjecture that every Schur-finite element is a virtual sum of line elements.

Notation. We will use the term λ-ring in the sense of [Ber, 2.4]; we warn the reader that our λ-rings are called *special λ-rings* by Grothendieck, Atiyah and others; see [G57] [AT] [A].

A $\mathbb{Q}$-linear category $\mathcal{A}$ is a category in which each hom-set is uniquely divisible (i.e., a $\mathbb{Q}$-module). By a $\mathbb{Q}$-linear tensor category (or $\mathbb{Q}$TC) we mean a $\mathbb{Q}$-linear category which is also symmetric monoidal and such that the tensor product is $\mathbb{Q}$-linear. We will be interested in $\mathbb{Q}$TC's which are idempotent-complete.

Date: October 26, 2012.

Weibel's research was supported by NSA and NSF grants.

1. Finite-dimensional λ-rings

Almost all λ-rings of historical interest are finite-dimensional. This includes the complex representation rings $R(G)$ and topological K-theory of compact spaces [AT, 1.5] as well as the algebraic K-theory of algebraic varieties [G57]. In this section we present this theory from the viewpoint we are adopting. Little in this section is new.

Recall that an element x in a λ-ring R is said to be *even* of finite degree n if $\lambda_t(x)$ is a polynomial of degree n, or equivalently that there is a λ-ring homomorphism from the ring Λ_n defined in 1.2 to R, sending a to x. We say that x is a *line element* if it is even of degree 1, i.e., if $\lambda^n(x) = 0$ for all $n > 1$.

We say that x is *odd* of degree n if $\sigma_t(x) = \lambda_{-t}(x)^{-1}$ is a polynomial of finite degree n. Since $\sigma_{-t}(x) = \lambda_t(-x)$, we see that x is odd just in case $-x$ is even. Therefore there is a λ-ring homomorphism from the ring Λ_{-n} defined in 1.2 to R sending b to x.

We say that an element x is *finite-dimensional* if it is the difference of two even elements, or equivalently if x is the sum of an even and an odd element. The subset of even elements in R is closed under addition and multiplication, and the subset of finite-dimensional elements forms a subring of R.

Example 1.1. If R is a binomial λ-ring, then r is even if and only if some $r(r-1)\cdots(r-n) = 0$, and odd if and only if some $r(r+1)\cdots(r+n) = 0$. The binomial rings $\prod_{i=1}^{n}\mathbb{Z}$ are finite dimensional. If R is connected then the subring of finite-dimensional elements is just $\mathbb{Z}$.

There is a well known family of universal finite-dimensional λ-rings $\{\Lambda_n\}$.

Definition 1.2. Following [AT], let Λ_n denote the free λ-ring generated by one element $a = a_1$ of finite degree n (i.e., subject to the relations that $\lambda^k(a) = 0$ for all $k > n$). By [Ber, 4.9], Λ_n is just the polynomial ring $\mathbb{Z}[a_1, ..., a_n]$ with $a_i = \lambda^i(a_1)$.

Similarly, we write Λ_{-n} for the free λ-ring generated by one element $b = b_1$, subject to the relations that $\sigma^k(b) = 0$ for all $k > n$. Using the antipode S, we see that there is a λ-ring isomorphism $\Lambda_{-n} \cong \Lambda_n$ sending b to $-a$, and hence that $\Lambda_{-n} \cong \mathbb{Z}[b_1, ..., b_n]$ with $b_k = \sigma^k(b)$.

Consider finite-dimensional elements in λ-rings R which are the difference of an even element of degree m and an odd element of degree n. The maps $\Lambda_m \to R$ and $\Lambda_{-n} \to R$ induce a λ-ring map from $\Lambda_m \otimes \Lambda_{-n}$ to R.

Lemma 1.3. *If an element x is both even and odd in a λ-ring, then x and all the $\lambda^i(x)$ are nilpotent. Thus $\lambda_t(x)$ is a unit of $R[t]$.*

Proof. If x is even and odd then $\lambda_t(x)$ and $\sigma_{-t}(x)$ are polynomials in $R[t]$ which are inverse to each other. It follows that the coefficients $\lambda^i(x)$ of the t^i are nilpotent for all $i > 0$. □

If R is a graded λ-ring, an element $\sum r_i$ is even (resp., odd, resp., finite-dimensional) if and only if each homogeneous term r_i is even (resp., odd, resp., finite-dimensional). This is because the operations λ^n multiply the degree of an element by n.

The forgetful functor from λ-rings to commutative rings has both a right and a left adjoint; see [Kn, pp. 20–21]; the right adjoint sends R to the ring $W(R)$ of big Witt vectors. It follows that the forgetful functor preserves all limits and colimits. For example, the product of λ-rings exists and is the product of the underlying rings with coordinate-wise λ-operations; the coproduct also exists and it the coproduct (tensor product) of the underlying rings with suitable λ-operations; see [Kn, p. 21].

Lemma 1.4. *The category of λ-rings has all limits and colimits.*

Proof. We have observed that all small products and coproducts exist. By [ML, V.2], it suffices to show that the equalizer and coequalizer of a pair $f_1, f_2 : R \rightrightarrows R'$ exists. It is immediate that the ring equalizer $\{r \in R : f_1(r) = f_2(r)\}$ is a λ-subring of R. The ring coequalizer is R'/I, where I is the ideal of R' generated by all elements $f_1(r)-f_2(r)$, $r \in R$. Given $y = f_1(r)-f_2(r)$, each $\lambda^i y = f_1(\lambda^i r)-f_2(\lambda^i r)$ is in I. If $x \in R'$, the universal formula for $\lambda^n(xy)$ shows that $xy \in I$. It follows that I is a λ-ideal and R'/I is a λ-ring. $\square$

For example, if $B \leftarrow A \rightarrow C$ is a diagram of λ-rings, the tensor product $B \otimes_A C$ has the structure of a λ-ring. Here is a typical, classical application of this construction, originally proven in [AT, 6.1].

Proposition 1.5 (Splitting Principle)**.** *If x is any even element of finite degree n in a λ-ring R, there exists an inclusion $R \subseteq R'$ of λ-rings and line elements $\ell_1, ..., \ell_n$ in R' so that $x = \sum \ell_i$.*

Proof. Let Ω_n denote the tensor product of n copies of the λ-ring $\Lambda_1 = \mathbb{Z}[\ell]$; this is a λ-ring whose underlying ring is the polynomial ring $\mathbb{Z}[\ell_1, ..., \ell_n]$, and the λ-ring Λ_n of Definition 1.2 is the subring of symmetric polynomials in Ω_n; see [AT, §2]. Let R' be the pushout of the diagram $\Omega_n \leftarrow \Lambda_n \rightarrow R$. Since the image of x is $1 \otimes x = a \otimes 1 = (\sum \ell_i) \otimes 1$, it suffices to show that $R \rightarrow R'$ is an injection. This follows from the fact that Ω_n is free as a Λ_n-module. $\square$

Corollary 1.6. *If x is any finite-dimensional element of a λ-ring R, there is an inclusion $R \subseteq R'$ of λ-rings and line elements ℓ_i, ℓ'_j in R' so that*

$$x = (\sum \ell_i) - (\sum \ell'_j).$$

Scholium 1.7. For later use, we record an observation, whose proof is implicit in the proof of Proposition 4.2 of [AT]: $\lambda^m(\lambda^n x) = P_{m,n}(\lambda^1 x, \ldots, \lambda^{mn} x)$ is a sum of monomials, each containing a term $\lambda^i x$ for $i \geq n$. For example, $\lambda^2(\lambda^3 x) = \lambda^6 x - x\,\lambda^5 x + \lambda^4 x\,\lambda^2 x$ (see [Kn, p. 11]).

2. K_0 of tensor categories

The Grothendieck group of a $\mathbb{Q}$-linear tensor category provides numerous examples of λ-rings, and forms the original motivation for introducing the notion of Schur-finite elements in a λ-ring.

A $\mathbb{Q}$-linear tensor category is *exact* if it has a distinguished family of sequences, called *short exact sequences* and satisfying the axioms of [Q], and such that each $A \otimes -$ is an exact functor. In many applications $\mathcal{A}$ is *split exact*: the only short exact sequences are those which split. By $K_0(\mathcal{A})$ we mean the Grothendieck group as an exact category, i.e., the quotient of the free abelian group on the objects $[A]$ by the relation that $[B] = [A] + [C]$ for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

Let $\mathcal{A}$ be an idempotent-complete exact category which is a $\mathbb{Q}$TC for $\otimes$. For any object A in $\mathcal{A}$, the symmetric group Σ_n (and hence the group ring $\mathbb{Q}[\Sigma_n]$) acts on the n-fold tensor product $A^{\otimes n}$. If $\mathcal{A}$ is idempotent-complete, we define $\wedge^n A$ to be the direct summand of $A^{\otimes n}$ corresponding to the alternating idempotent $\sum (-1)^\sigma \sigma / n!$ of $\mathbb{Q}[\Sigma_n]$. Similarly, we can define the symmetric powers $\mathrm{Sym}^n(A)$. It turns out that $\lambda^n(A)$ only depends upon the element $[A]$ in $K_0(\mathcal{A})$, and that λ^n extends to a well defined operation on $K_0(\mathcal{A})$.

The following result was proven by F. Heinloth in [Hl, Lemma 4.1], but the result seems to have been in the air; see [Dav, p. 486], [LL03], [LL04, 5.1] and [B1, B2, B3]. A special case of this result was proven long ago by Swan in [Sw].

Theorem 2.1. *If $\mathcal{A}$ is any idempotent-complete exact $\mathbb{Q}TC$, $K_0(\mathcal{A})$ has the structure of a λ-ring. If A is any object of $\mathcal{A}$ then $\lambda^n([A]) = [\wedge^n A]$.*

Kimura [Kim] and O'Sullivan have introduced the notion of an object C being finite-dimensional in any $\mathbb{Q}$TC $\mathcal{A}$: C is the direct sum of an even object A (one for which some $\wedge^n A \cong 0$) and an odd object B (one for which some $\mathrm{Sym}^n(B) \cong 0$). It is immediate that $[C]$ is a finite-dimensional element in the λ-ring $K_0(\mathcal{A})$. Thus the two notions of finite dimensionality are related.

Example 2.2. Let $\mathcal{M}^{\mathrm{eff}}$ denote the category of $\mathbb{Q}$-linear pure effective Chow motives with respect to rational equivalence over a field k. Its objects are summands of smooth projective varieties over a field k and morphisms are given by Chow groups. Thus $K_0(\mathcal{M}^{\mathrm{eff}})$ is the group generated by the classes of objects, modulo the relation $[M_1 \oplus M_2] = [M_1] + [M_2]$. Since $\mathcal{M}^{\mathrm{eff}}$ is a $\mathbb{Q}$TC, $K_0(\mathcal{M}^{\mathrm{eff}})$ is a λ-ring.

By adjoining an inverse to the Lefschetz motive to $\mathcal{M}^{\mathrm{eff}}$, we obtain the category $\mathcal{M}$ of Chow motives (with respect to rational equivalence). This is also a $\mathbb{Q}$TC, so $K_0(\mathcal{M})$ is a λ-ring.

The category $\mathcal{M}^{\mathrm{eff}}$ embeds into the triangulated category $\mathbf{DM}_{gm}^{\mathrm{eff}}$ of effective geometric motives; see [MVW, 20.1]. Similarly, the category $\mathcal{M}$ embeds in the triangulated category $\mathbf{DM}_{gm}$ of geometric motives [MVW, 20.2]. Bondarko proved in [Bo, 6.4.3] that $K_0(\mathbf{DM}_{gm}^{\mathrm{eff}}) \cong K_0(\mathcal{M}^{\mathrm{eff}})$ and $K_0(\mathbf{DM}_{gm}) \cong K_0(\mathcal{M})$. Thus we may investigate λ-ring questions in these triangulated settings. (Recall that if $\mathbf{D}$ is any triangulated category, such as $\mathbf{DM}_{gm}^{\mathrm{eff}}$, then $K_0(\mathbf{D})$ is the free abelian group on the objects by the relation that $[B] = [A] + [C]$ for every distinguished triangle $A \to B \to C \to$ in $\mathbf{D}$.) As far as we know, it is possible that every element of $K_0(\mathbf{DM}_{gm})$ is finite-dimensional.

Recall that a motive M in $\mathcal{M}^{\mathrm{eff}}$ is a **phantom motive** if $H^*(M) = 0$ for every Weil cohomology H.

Proposition 2.3. *Let M be an object of $\mathcal{M}^{\mathrm{eff}}$. Then if $[M] = 0$ in $K_0(\mathcal{M}^{\mathrm{eff}})$, then M is a phantom motive.*

Proof. Since $\mathcal{M}^{\mathrm{eff}}$ is an additive category, $[M] = 0$ implies that there is another object N of $\mathcal{M}^{\mathrm{eff}}$ such that $M \oplus N \cong N$. But every effective motive is a summand of the motive of a scheme, hence we may assume $N = M(X)$. If M is not a phantom motive, there is a Weil cohomology and an i such that $H^i(M) \neq 0$. But then $H^i(M) \oplus H^i(X) \cong H^i(X)$; since these are finite-dimensional vector spaces, this implies $H^i(M) = 0$, a contradiction. □

Remark 2.3.1. The referee points out that Proposition 2.3 also follows from the observation that each $H^i : \mathcal{M}^{\mathrm{eff}} \to \mathbf{Ab}$ satisfies $H^i(M \oplus N) = H^i(M) \oplus H^i(N)$, so it factors through a group homomorphism $H^i : K_0(\mathcal{M}^{\mathrm{eff}}) \to K_0(\mathbf{Ab}) = \mathbb{Z}$.

Here is an application of these ideas. Recall that any quasi-projective scheme X has a motive with compact supports in $\mathbf{DM}^{\mathrm{eff}}$, $M^c(X)$. If k has characteristic 0, this is an effective geometric motive, and if U is open in X with complement Z there is a triangle $M^c(Z) \to M^c(X) \to M^c(U)$; see [MVW, 16.15]. It follows that $[M^c(X)] = [M^c(U)] + [M^c(Z)]$ in $K_0(\mathcal{M}^{\mathrm{eff}})$. This was originally proven by Gillet and Soulé in [GS, Thm. 4] before the introduction of $\mathbf{DM}$, but see [GS, 3.2.4], [GN].

Definition 2.4. Let $K_0(Var)$ be the Grothendieck ring of varieties obtained by imposing the relation $[U] + [X \setminus U] = [X]$ for any open U in a variety X. By the above remarks, there is a well defined ring homomorphism $K_0(Var) \to K_0(\mathcal{M}^{\mathrm{eff}})$.

Grothendieck asked in [G64, p.174] if this morphism was far from being an isomorphism. We can now answer his question.

Theorem 2.5. *The homomorphism $K_0(Var) \to K_0(\mathcal{M}^{\mathrm{eff}})$ is not an injection.*

Remark 2.5.1. After this paper was posted in 2010, we were informed by J. Sebag that Grothendieck's question had also been answered in [LS, Remark 14].

For the proof, we need to introduce Kapranov's zeta-function [Kap]. If X is any quasi-projective variety, its symmetric power S^nX is the quotient of X^n by the action of the symmetric group. We define $\zeta_t(X) = \sum [S^nX]t^n$ as a power series with coefficients in $K_0(Var)$.

Lemma 2.6. *(*[Gul]*) The following diagram is commutative:*

$$\begin{array}{ccc} K_0(Var) & \xrightarrow{\zeta_t} & 1 + K_0(Var)[[t]] \\ {\scriptstyle M^c}\downarrow & & \downarrow{\scriptstyle M^c} \\ K_0(\mathcal{M}^{\mathit{eff}}) & \xrightarrow{\sigma_t} & 1 + K_0(\mathcal{M}^{\mathit{eff}})[[t]]. \end{array}$$

Proof. It suffices to show that $[M^c(S^nX)] = \mathrm{Sym}^n[M^c(X)]$ in $K_0(\mathcal{M}^{\mathrm{eff}})$ for any X. This is proven by del Baño and Navarro in [dBN, 5.3]. □

Definition 2.7. Following [LL04, 2.2], we say that a power series $f(t) = \sum r_n t^n \in R[[t]]$ is *determinantally rational* over a ring R if there exists an $m, n_0 > 0$ such that the $m \times m$ Hankel matrices $(r_{n+i+j})_{i,j=1}^m$ have determinant 0 for all $n > n_0$.

The name comes from the classical fact (Émile Borel [1894]) that when R is a field (or a domain) a power series is determinantally rational if and only if it is a rational function $p(t)/q(t)$. For later use, we observe that $\deg(q) < m$ and $\deg(p) < n_0$. (This is relation (α) in [1894].)

Clearly, if $f(t)$ is not determinantally rational over R and $R \subset R'$ then $f(t)$ cannot be determinantally rational over R'.

As observed in [LL04, 2.4], if f is a rational function in the sense that $gf = h$ for polynomials $g(t), h(t)$ with $g(0) = 1$ then f is determinantally rational. For example, if $x = a_b$ is a finite-dimensional element of a λ-ring R, with a even and b odd, then $\lambda_t(a)$ and $\lambda_t(-b) = \lambda_t(b)^{-1}$ are polynomials so $\lambda_t(x) = \lambda_t(a)\lambda_t(b)$ and $\sigma_t(x) = \lambda_t(x)^{-1}$ are rational functions. This was observed by André in [A05].

Proof of Theorem 2.5. Let X be the product $C \times D$ of two smooth projective curves of genus > 0, so that $p_g(X) > 0$. Larsen and Lunts showed in [LL04, 2.4, 3.9] that $\zeta_t(X)$ is not determinantally rational over $R = K_0(Var)$. On the other hand, Kimura proved in [Kim] that X is a finite-dimensional object in $\mathcal{M}^{\mathrm{eff}}$, so $\sigma_t(X) = \lambda_t(X)^{-1}$ is a determinantally rational function in $R' = K_0(\mathcal{M}^{\mathrm{eff}})$. It follows that $R \to R'$ cannot be an injection. □

3. Symmetric functions

We devote this section to a quick study of the ring Λ of symmetric functions, and especially the Schur polynomials s_π, referring the reader to [Macd] for more information. In the next section, we will use these polynomials to define the notion of Schur-finite elements in a λ-ring.

The ring Λ is defined as the ring of symmetric "polynomials" in variables ξ_i. More precisely, it is the subring of the power series ring in $\{\xi_n\}$ generated by $e_1 = \sum \xi_n$ and the other *elementary symmetric power sums* $e_i \in \Lambda$; if we put $\xi_r = 0$ for $r > n$ then e_i is the i^{th} elementary symmetric polynomial in $\xi_1, \dots, \xi_n$; see [AT]. A major role is also played by the *homogeneous power sums* $h_n = \sum \xi_{i_1} \cdots \xi_{i_n}$ (where the sum being taken over $i_1 \leq \cdots \leq i_n$). Their generating functions $E(t) = \sum e_n t^n$ and $H(t) = \sum h_n t^n$ are $\prod(1 + \xi_i t)$ and $\prod(1 - \xi_i t)^{-1}$, so that $H(t)E(-t) = 1$. In fact, Λ is a graded polynomial ring in two relevant ways (with e_n and h_n in degree n):

$$\Lambda = \mathbb{Z}[e_1, ..., e_n, ...] = \mathbb{Z}[h_1, ..., h_n, ...].$$

Given a partition $\pi = (n_1, ..., n_r)$ of n (so that $\sum n_i = n$), we let $s_\pi \in \Lambda_n$ denote the Schur polynomial of π. The elements e_n and h_n of Λ are identified with $s_{(1,...,1)}$ and $s_{(n)}$, respectively. The Schur polynomials also form a $\mathbb{Z}$-basis of Λ by [Macd, 3.3]. By abuse, we will say that a partition π *contains* a partition $\lambda = (\lambda_1, ..., \lambda_s)$ if $n_i \geq \lambda_i$ and $r \geq s$, which is the same as saying that the Young diagram for π contains the Young diagram for λ.

Here is another description of Λ, taken from [Kn]: Λ is isomorphic to the direct sum R_* of the representation rings $R(\Sigma_n)$, made into a ring via the outer product $R(\Sigma_m) \otimes R(\Sigma_n) \to R(\Sigma_{m+n})$. Under this identification, $e_n \in \Lambda_n$ is identified with the class of the trivial simple representation V_n of Σ_n. More generally, s_π corresponds to the class $[V_\pi]$ in $R(\Sigma_n)$ of the irreducible representation corresponding to π. (See [Kn, III.3].)

Proposition 3.1. *Λ is a graded Hopf algebra, with coproduct Δ and antipode S determined by the formulas*

$$\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j, \quad S(e_n) = h_n \text{ and } S(h_n) = e_n.$$

Proof. The graded bialgebra structure is well known and due to Burroughs [Bu], who defined the coproduct on R_* as the map induced from the restriction maps $R(\Sigma_{m+n}) \to R(\Sigma_m) \otimes R(\Sigma_n)$, and established the formulas $\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j$. The fact that there is a ring involution S interchanging e_n and h_n is also well known; see [Macd, I(2.7)]. The fact that S is an antipode does not seem to be well known, but it is immediate from the formula $\sum (-1)^r e_r h_{n-r} = 0$ of [Macd, I(2.6)]. □

Remark 3.2. Atiyah shows in [A, 1.2] that Λ is isomorphic to the graded dual $R^* = \oplus \mathrm{Hom}(R(\Sigma_n), \mathbb{Z})$. That is, if $\{v_\pi\}$ is the dual basis in R^n to the basis $\{[V_\pi]\}$ of simple representations in R_n and the restriction of $[V_\pi]$ is $\sum c_\pi^{\mu\nu} [V_\mu] \otimes [V_\nu]$ then $v_\mu v_\nu = \sum_\pi c_\pi^{\mu\nu} v_\pi$ in R^*. Thus the product studied by Atiyah on the graded dual R^* is exactly the algebra structure dual to the coproduct Δ.

Let π' denote the conjugate partition to π. The *Jacobi-Trudi identities* $s_\pi = \det |h_{\pi_i+j-i}| = \det |e_{\pi'_i+j-i}|$ show that the antipode S interchanges s_π and $s_{\pi'}$. (Jacobi conjectured the identities, and his student Nicoló Trudi verified them in 1864; they were rediscovered by Giovanni Giambelli in 1903 and are sometimes called the *Giambelli identities*).

Let $I_{e,n}$ denote the ideal of Λ generated by the e_i with $i \geq n$. The quotient $\Lambda / I_{e,n}$ is the polynomial ring $\Lambda_{n-1} = \mathbb{Z}[e_1, ..., e_{n-1}]$. Let $I_{h,n}$ denote $S(I_{e,n})$, i.e., the ideal of Λ generated by the h_i with $i \geq n$.

Proposition 3.3. *The Schur polynomials s_π for partitions π containing (1^n) (i.e., with at least n rows) form a $\mathbb{Z}$-basis for the ideal $I_{e,n}$. The Schur polynomials with at most n rows form a $\mathbb{Z}$-basis of Λ_n.*

Similarly, the Schur polynomials s_π for partitions π containing (n) (i.e., with $\pi_1 \geq n$) form a $\mathbb{Z}$-basis for the ideal $I_{h,n}$.

Proof. We prove the assertions about $I_{e,n}$; the assertion about $I_{h,n}$ follows by applying the antipode S. By [Macd, I.3.2], the s_π which have fewer than n rows project onto a $\mathbb{Z}$-basis of $\Lambda_{n-1} = \Lambda / I_{e,n}$. Since the s_π form a $\mathbb{Z}$-basis of Λ, it suffices to show that every partition $\pi = (\pi_1, ..., \pi_r)$ with $r > n$ is in $I_{e,n}$. Expansion along the first row of the Jacobi-Trudi identity $s_\pi = \det |e_{\pi'_i+j-i}|$ shows that s_π is in the ideal $I_{e,r}$. □

Corollary 3.4. *The ideal $I_{h,m} \cap I_{e,n}$ of Λ has a $\mathbb{Z}$-basis consisting of the Schur polynomials s_π for partitions π containing the hook $(m, 1^{n-1}) = (m, 1, ..., 1)$.*

Definition 3.5. For any partition $\lambda = (\lambda_1, ..., \lambda_r)$, let I_λ denote the subgroup of Λ generated by the Schur polynomials s_π for which π contains λ, i.e., $\pi_i \geq \lambda_i$ for $i = 1, ..., r$. We have already encountered the special cases $I_{e,n} = I_{(1,...,1)}$ and $I_{h,n} = I_{(n)}$ in Proposition 3.3, and $I_{(m,1,...,1)} = I_{h,m} \cap I_{e,n}$ in Corollary 3.4.

Example 3.6. Consider the partition $\lambda = (2, 1)$. Since $I_\lambda = I_{h,2} \cap I_{e,2}$ by Corollary 3.4, Λ_λ is the fibre product of $\mathbb{Z}[a]$ and $\mathbb{Z}[b]$ over the common quotient $\mathbb{Z}[a]/(a^2) = \Lambda/(I_{(1,1)} + I_{(2)})$. The universal element of Λ_λ is $x = (a, b)$ and if we set $y = (0, b^2)$ then $\Lambda_{(2,1)} \cong \mathbb{Z}[x, y]/(y^2 - x^2 y)$. Since $\lambda^n(b) = b^n$ for all n, it is easy to check that $\lambda^{2i}(x) = y^i$ and $\lambda^{2i+1}(x) = xy^i$.

Lemma 3.7. *The I_λ are ideals of Λ, and $I_\lambda \cap I_\mu = I_{\lambda\cup\mu}$. Hence $\{I_\lambda\}$ is closed under intersection.*

Proof. The Pieri rule writes $h_p s_\pi$ as a sum of s_μ, where μ runs over partitions consisting of π and p other elements, no two in the same column. Thus I_λ is closed under multiplication by the h_p. As every element of Λ is a polynomial in the h_p, I_λ is an ideal.

If $\mu = (\mu_1, ..., \mu_s)$ is another partition, then s_π is in $I_\lambda \cap I_\mu$ if and only if $\pi_i \geq \max\{\lambda_i, \mu_i\}$ Thus $I_\lambda \cap I_\mu = I_{\lambda\cup\mu}$. □

Remark 3.8. The ideal $I_\lambda + I_\mu$ need not be of the form I_ν for any ν. For example, $I = I_{(2)} + I_{(1,1)}$ contains every Schur polynomial except 1 and $s_1 = e_1$.

We conclude this section by connecting Λ with λ-rings. Recall from [Ber, 4.4], [G57, I.4] or [AT, §2] that the universal λ-ring on one generator $a = a_1$ is the polynomial ring $\mathbb{Z}[a_1, \dots, a_n, \dots]$, with $\lambda^n(a) = a_n$. Given an element x in a λ-ring R, there is a unique morphism $u_x : \Lambda \to R$ with $u_x(a) = x$. Following [A] and [Kn], we identify this universal λ-ring with Λ, where the a_i are identified with the $e_i \in \Lambda$.

The ring Λ is naturally isomorphic to the ring of natural operations on the category of λ-rings; an operation ϕ corresponds to $\phi(a) \in \Lambda$. Conversely, given $f \in \Lambda$, the formula $f(x) = u_x(f)$ defines a natural operation. The operation λ^n corresponds to e_n. The operation σ^n, defined by $\sigma^n(x) = (-1)^n \lambda^n(-x)$, corresponds to h_n; this may be seen by comparing the generating functions $H(t) = E(-t)^{-1}$ and $\sigma_t(x) = \lambda_{-t}(x)^{-1}$.

Proposition 3.9. *If ϕ is an element of Λ, and $\Delta(\phi) = \sum \phi'_i \otimes \phi''_i$ then the corresponding natural operation on λ-rings satisfies $\phi(x + y) = \sum \phi'_i(x)\phi''_i(y)$.*

Proof. Consider the set Λ' of all operations in Λ satisfying the condition of the proposition. Since Δ is a ring homomorphism, Λ' is a subring of Λ. Since $\Delta(e_n) = \sum e_i \otimes e_{n-i}$ and $\lambda^n(x + y) = \sum \lambda^i(x)\lambda^{n-i}(y)$, Λ' contains the generators e_n of Λ, and hence $\Lambda' = \Lambda$. □

The Littlewood-Richardson rule states that $\Delta([V_\pi])$ is a sum $\sum c_\pi^{\mu\nu}[V_\mu] \otimes [V_\nu]$, where $\mu \subseteq \pi$ and π is obtained from μ by concatenating ν in a certain way; see [Macd, §I.9]. By Proposition 3.9, we then have

Corollary 3.10. $s_\pi(x + y) = \sum c_\pi^{\mu\nu} s_\mu(x)s_\nu(y)$.

4. Schur-finite λ-rings

In this section we introduce the notion of a Schur-finite element in a λ-ring R, and show that these elements form a subring of R containing the subring of finite-dimensional elements. We conjecture that they are the elements for which the virtual splitting principle holds.

Definition 4.1. We say that an element x in a λ-ring R is *Schur-finite* if there exists a partition λ such that $s_\mu(x) = 0$ for every partition μ containing λ. That is, I_λ annihilates x. We call such a λ a *bound* for x.

By Remark 3.8, $x \in R$ may have no unique minimal bound λ. By Example 4.4 below, $s_\lambda(x) = 0$ does not imply that λ is a bound for x.

Proposition 4.2. *Each I_λ is a radical λ-ideal, and $\Lambda_\lambda = \Lambda/I_\lambda$ is a reduced λ-ring. Thus every Schur-finite $x \in R$ with bound λ determines a λ-ring map $\Lambda_\lambda \to R$ which sends the universal element of Λ_λ to x.*

Moreover, if λ is a rectangular partition then I_λ is a prime ideal, and Λ_λ is a subring of a polynomial ring in which a becomes finite-dimensional.

In general, Λ_λ is a subring of $\prod \Lambda_{\beta_i}$ with β_i rectangular and hence of a product of polynomial rings in which a becomes finite-dimensional.

Proposition 4.2 verifies Conjecture 3.9 of [KKT].

Proof. Fix a rectangular partition $\beta = ((n+1)^{m+1}) = (n+1, ..., n+1)$, and consider the universal λ-ring map

$$f : \Lambda \to \Lambda_m \otimes \Lambda_{-n} \cong \mathbb{Z}[a_1, ..., a_m, b_1, ..., b_n] \tag{4.2.1}$$

sending e_1 to the finite-dimensional element $x = a_1 + b_1$ (see Definition 1.2). We claim that the kernel of f is I_β. Since $\mathrm{Ker}(f)$ is a λ-ideal, this proves that I_β is a λ-ideal and that Λ/I_β embeds into the polynomial ring $\mathbb{Z}[a_1, ..., a_m, b_1, ..., b_n]$. Since any partition λ can be written as a union of rectangular partitions β_i, Lemma 3.7 implies that $I_\lambda = \cap I_{\beta_i}$ is also a λ-ideal.

By the Littlewood-Richardson rule 3.10, $f(s_\pi) = s_\pi(x) = \sum c_\pi^{\mu\nu} s_\mu(a_1) s_\nu(b_1)$, where μ and ν run over all partitions such that π is obtained from μ by concatenating ν in a certain way. We may additionally restrict the sum to μ with at most m rows and ν with $\nu_1 \le m$, since otherwise $s_\mu(a_1) = 0$ or $s_\nu(b_1) = 0$. By Proposition 3.3, the $s_\mu(a_1)$ run over a basis of Λ_m and the $s_\nu(b_1)$ run over a basis of Λ_{-n}.

If π contains β then $f(s_\pi) = s_\pi(x) = 0$, because in every term of the above expansion, either the length of μ is $> m$ or else $\nu_1 > n$. Thus $I_\beta \subseteq \mathrm{Ker}(f)$.

For the converse, we use the reverse lexicographical ordering of partitions [Macd, p. 5]. For each π not containing β, set $\mu_\pi = (\pi_1, \ldots, \pi_m)$; this is the maximal μ (for this ordering) such that $c_\pi^{\mu\nu} \neq 0$ (with $\nu_\pi = \pi - \mu_\pi$). Given $t = \sum_{\beta \not\subseteq \pi} d_\pi s_\pi$, choose μ maximal subject to $\mu = \mu_\pi$ for some π with $d_\pi \neq 0$; choose π maximal with $\mu = \mu_\pi$ and $d_\pi \neq 0$, and set $\nu = \nu_\pi$. Then the coefficient of $s_\mu(a_1)s_\nu(b_1)$ in $f(t)$ is $d_\pi c_\pi^{\mu\nu} \neq 0$. Thus $\mathrm{Ker}(f) \subseteq I_\beta$. □

Corollary 4.3. *$\Lambda_{(2,2)}$ is the subring $\mathbb{Z} + x\mathbb{Z}[a, b]$ of $\mathbb{Z}[a, b]$, where $x = a + b$. Moreover $\lambda^{n+1}(x) = xb^n$.*

Proof. By Proposition 4.2 with $m = n = 1$, $\Lambda_{(2,2)}$ is the subring of $\mathbb{Z}[a, b]$ generated by $x = a + b$ and the $\lambda^n(x)$; see (4.2.1). Since a and $-b$ are line elements and

$$\lambda^{n+1}(x) = a\lambda^n(b) + \lambda^{n+1}(b) = ab^n + b^{n+1} = xb^n,$$

we have $\Lambda_{(2,2)} = \mathbb{Z}[x, xb, xb^2, \ldots, xb^n, \ldots] = \mathbb{Z} + x\mathbb{Z}[a, b]$. □

Remark 4.3.1. The ring $\Lambda_{(2,2)}$ was studied in [KKT, 3.8], where it was shown that $\Lambda_{(2,2)}$ embeds into $\mathbb{Z}[x, y]$ sending e_n to xy^{n-1}. This is the same as the embedding in Corollary 4.3, up to the change of coordinates $(x, y) = (a + b, b)$.

Example 4.4. Let I be the ideal of $\Lambda_{(2,2)}$ generated by the $\lambda^{2i}(x)$ $(i > 0)$ and set $R = \Lambda_{(2,2)}/I$. Then R is a λ-ring and x is a non-nilpotent element such that $\lambda^{2i}(x) = 0$ but $\lambda^{2i+1}(x) \neq 0$. In particular, $\lambda^2(x) = 0$ yet $\lambda^3(x) \neq 0$.

Using the embedding of Corollary 4.3 and that $\lambda^{n+1}(x) = xb^n$ from its proof, we see that I contains $x(xb^{2i-1})$ and $(xb)(xb^{2i-1})$ and hence the ideal J of $\mathbb{Z}[a,b]$ generated by x^2b. In fact, I is additively generated by J and the $\{xb^{2i-1}\}$. It follows that R has basis $\{1, x^n, xb^{2n} | n \geq 1\}$. Since $\lambda^n(\lambda^{2i}(x))$ is equivalent to an integer multiple of $\lambda^{2in}(x) = xb^{2in-1}$ modulo J (by 1.7), it lies in I. Hence I is a λ-ideal of $\Lambda_{(2,2)}$.

There is no λ-ring extension $R \subset R'$ in which $x = \ell_1 - \ell_2$ for line elements ℓ_i, because we would have $\lambda^3(x) = \lambda^3(x + \ell_2) = 0$. On the other hand, there is a λ-ring extension $R \subset R'$ in which $x = \ell_1 + \ell_2 - \ell_3 - \ell_4$ for line elements ℓ_i; $R' = \Lambda_2 \otimes \Lambda_{-2}/I'$, where I' is the ideal generated by $\lambda^2 x$ and $\lambda^4 x$.

Lemma 4.5. *If x and y are Schur-finite, so is $x + y$.*

Proof. Given a partition λ, there is a partition π_0 such that whenever π contains π_0, one of the partitions μ and ν appearing in the Littlewood-Richardson rule 3.10 must contain λ. If x and y are both killed by all Schur polynomials indexed by partitions containing λ, we must therefore have $s_\pi(x + y) = 0$. □

Corollary 4.6. *Finite-dimensional elements are Schur-finite.*

Proof. Proposition 3.3 shows that even and odd elements are Schur-finite. □

Example 4.7. If R is a binomial ring containing $\mathbb{Q}$, then every Schur-finite element is finite-dimensional. This follows from Example 1.1 and [Macd, Ex. I.3.4], which says that $s_\pi(r)$ is a rational number times a product of terms $r - c(x)$, where the $c(x)$ are integers.

Example 4.8. The universal element x of the λ-ring $\Lambda_{(2,1)}$ is Schur-finite but not finite-dimensional. To see this, recall from Example 3.6 that $\Lambda_{(2,1)} \cong \mathbb{Z}[x,y]/(y^2 - x^2y)$. Because $\Lambda_{(2,1)}$ is graded, if x were finite-dimensional it would be the sum of an even and odd element in the degree 1 part $\{nx\}$ of $\Lambda_{(2,1)}$. If $n \in \mathbb{N}$ and $n \neq 0$, nx cannot be even because the second coordinate of $\lambda^k(nx)$ is $\binom{-n}{k} b^k$ by 1.2. And nx cannot be odd, because the first coordinate of $\sigma^k(nx)$ is $(-1)^k \binom{-n}{k} a^k$.

Lemma 4.9. *Let $R \subset R'$ be an inclusion of λ-rings. If $x \in R$ then x is Schur-finite in R', if and only if x is Schur-finite in R. In particular, if x is finite-dimensional in R', then x is Schur-finite in R.*

Proof. Since $s_\pi(x)$ may be computed in either R or R', the set of partitions π for which $s_\pi(x) = 0$ is the same for R and R'. The final assertion follows from Lemma 4.6. □

Lemma 4.10. *If π is a partition of n, $s_{\pi'}(-x) = (-1)^n s_\pi(x)$.*

Proof. Write s_π as a homogeneous polynomial $f(e_1, e_2, ...)$ of degree n, where e_i has degree i. Applying the antipode S in Λ, we have $s_{\pi'} = f(h_1, h_2, ...)$. It follows that $s_{\pi'}(-x) = f(\sigma^1, \sigma^2, ...)(-x)$. Since $\sigma^i(-x) = (-1)^i \lambda^i(x)$, and f is homogeneous, we have

$$s_{\pi'}(-x) = f(-\lambda^1, +\lambda^2, ...)(x) = (-1)^n f(\lambda^1, \lambda^2, ...)(x) = s_\pi(x). \qquad \square$$

Remark 4.10.1. If a is a line element then $s_\pi(ax) = a^n s_\pi(x)$. From Lemma 4.10, we have $s_\pi(-ax) = (-a)^n s_{\pi'}(x)$.

Theorem 4.11. *The Schur-finite elements form a λ-subring of any λ-ring, containing the subring of finite-dimensional elements.*

Proof. The Schur-finite elements are closed under addition by Lemma 4.5. Since π contains λ just in case π' contains λ', Lemma 4.10 implies that $-x$ is Schur-finite whenever x is. Hence the Schur-finite elements form a subgroup of R. It suffices to show that if x and y are Schur-finite in R, then xy and all $\lambda^i(x)$ are Schur-finite.

Let x be Schur-finite with rectangular bound μ, so there is a map from the λ-ring Λ_μ to R sending the generator e to x. Embed Λ_μ in $R' = \mathbb{Z}[a_1,\dots,b_1,\dots]$ using Proposition 4.2. Since every element of R' is finite-dimensional, $\lambda^n(e)$ is finite-dimensional in R', and hence Schur-finite in Λ_μ by Lemma 4.9. It follows that the image $\lambda^n(x)$ of $\lambda^n(e)$ in R is also Schur-finite.

Let x and y be Schur-finite with rectangular bounds μ and ν, and let $\Lambda_\mu \to R$ and $\Lambda_\nu \to R$ be the λ-ring maps sending the generators e_μ and e_ν to x and y. Since the induced map $\Lambda_\mu \otimes \Lambda_\nu \to R$ sends $e_\mu \otimes e_\nu$ to xy, we only need to show that $e_\mu \otimes e_\nu$ is Schur-finite. But $\Lambda_\mu \otimes \Lambda_\nu \subset \mathbb{Z}[a_1,\dots,b_1,\dots] \otimes \mathbb{Z}[a_1,\dots,b_1,\dots]$, and in the larger ring every element is finite-dimensional, including the tensor product. By Lemma 4.9, $e_\mu \otimes e_\nu$ is Schur-finite in $\Lambda_\mu \otimes \Lambda_\nu$. □

Conjecture 4.12 (Virtual Splitting principle)**.** *Let x be a Schur-finite element of a λ-ring R. Then R is contained in a larger λ-ring R' such that x is finite-dimensional in R'. By Corollary 1.6, this is equivalent to the assertion that R is contained in some λ-ring R'' in which x is a virtual sum of line elements: there are line elements ℓ_i, ℓ'_j in R'' so that*

$$x = (\sum \ell_i) - (\sum \ell'_j).$$

Example 4.13. The virtual splitting principle holds in the universal case, where $R_0 = \Lambda_\beta$, by Proposition 4.2. Indeed, we know that x is $a_1 + b_1$ in $R'_0 = \mathbb{Z}[a_1,\dots,b_1,\dots]$, where a_1 and b_1 are finite-dimensional; see (4.2.1). Corollary 1.6 implies that x is a difference of sums of line elements in a larger λ-ring R''_0.

Unfortunately, although the induced map $f : R \to R \otimes_{R_0} R''_0$ sends a Schur-finite element x to a difference of sums of line elements, the map f need not be an injection. For example, this fails for the ring R of Example 4.4.

Proposition 4.14. *If a λ-ring R is a domain, R is contained in a λ-ring R' such that every Schur-finite element of R is a a difference of sums of line elements in R'.*

Proof. Let E denote the algebraic closure of the fraction field of R and set $R' = W(E)$; R is contained in R' by $R \xrightarrow{\lambda_t} W(R) \subset W(E)$. If $x \in R$ is Schur-finite then, as we shall see in Proposition 5.3, $\lambda_t(x)$ is determinentally rational in $E[[t]]$ and hence a rational function p/q in $E(t)$ (see 2.7). Factoring p and q in $E[t]$, we have

$$\lambda_t(x) = \prod(1-\alpha_i t)/\prod(1-\beta_j t)$$

for suitable elements α_i, β_j of E. Since the underlying abelian group of $W(E)$ is $(1+tE[[t]],\times)$ and the $\ell_i = (1-\alpha_i t)$ and $\ell'_j = (1-\beta_j t)$ are line elements in $W(E)$, we are done. □

The proof shows that a bound π on x determines a bound on the degrees of $p(t)$ and $q(t)$ and hence on the number of line elements ℓ_i and ℓ'_j in the virtual sum.

Corollary 4.15. *The virtual splitting principle holds for reduced λ-rings.*

Proof. Let R be a reduced ring. If P is a minimal prime of R then the localization R_P is a domain and R embeds into the product $\prod E_P$ of the algebraic closures of the fields of fractions of the R_P. If in addition R is a λ-ring then R embeds into the λ-ring $R' = \prod W(E_P)$. If x is Schur-finite in R with bound π then $\lambda_t(R)$ is determinantly rational and each factor of $\lambda_t(x)$ is a rational function in $E_P(t)$; the bound π determines a bound N on the degrees of the numerator and denominator in each component. By Theorem 4.14, there are line elements $\ell_1, \dots, \ell_N, \ell'_1, \dots, \ell'_N$ in each component so that $x = (\sum \ell_i) - (\sum \ell'_j)$ in R'. □

As more partial evidence for Conjecture 4.12, we show that the virtual splitting principle holds for elements bounded by the hook $(2,1)$.

Theorem 4.16. *Let x be a Schur-finite element in a λ-ring R. If x has bound $(2,1)$, then R is contained in a λ-ring R' in which x is a virtual sum $\ell_1 + \ell_2 - a$ of line elements.*

Proof. The polynomial ring $R[a]$ becomes a λ-ring once we declare a to be a line element. Set $y = x + a$, and let I be the ideal of $R[a]$ generated by $\lambda^3(y)$.

For all $n \geq 2$, the equation $s_{2,1^{n-1}}(x) = 0$ yields $\lambda^{n+1}(x) = x\lambda^n(x) = x^{n-1}\lambda^2(x)$ in R, and therefore $\lambda^{n+1}(y) = (a+x)x^{n-2}\lambda^2(x) = x^{n-2}\lambda^3(y)$. It follows from Scholium 1.7 that $\lambda^m(\lambda^3 y) \in I$ for all $m \geq 1$ and hence that

$$\lambda^n(f \cdot \lambda^3 y) = P_n(\lambda^1(f), \dots, \lambda^n(f); \lambda^1(\lambda^3 y), \dots, \lambda^n(\lambda^3 y))$$

is in I for all $f \in R[a]$. Thus I is a λ-ideal of $R[a]$, $A = R[a]/I$ is a λ-ring, and the image of y in A is even of degree 2. By the Splitting Principle 1.5, the image of $x = y - a$ is a virtual sum $\ell_1 + \ell_2 - a$ of line elements in some λ-ring containing A.

To conclude, it suffices to show that R injects into $A = R[a]/I$. If $r \in R$ vanishes in A then $r = f\lambda^3(y)$ for some $f = f(a)$ in $R[a]$. We may take f to have minimal degree $d \geq 0$. Writing $f(a) = c\,a^d + g(a)$, with $c \in R$ and $\deg(g) < d$, the coefficient of a^{d+1} in $f\lambda^3(y)$, namely $c\,\lambda^2(x)$, must be zero. But then $c\lambda^3 y = 0$, and $r = g\,\lambda^3 y$, contradicting the minimality of f. □

Remark 4.17. The rank of a Schur-finite object with bound π cannot be well defined unless π is a rectangular partition. This is because any rectangular partition $\mu = (m+1)^{n+1}$ contained in π yields a map $R \to R'$ sending x to an element of rank $m-n$. If π is not rectangular there are different maximal rectangular sub-partitions with different values of $m - n$.

For example, let x be the element of Theorem 4.16. By Lemma 4.10, $-x$ also has bound $(2,1)$. Applying Theorem 4.16 to $-x$ shows that R is also contained in a λ-ring R'' in which x is a virtual sum $a - \ell_1 - \ell_2$ of line elements. Therefore x has rank 1 in R', and has rank -1 in R''.

5. Rationality of $\lambda_t(x)$

Let R be a λ-ring and $x \in R$. One central question is to determine when the power series $\lambda_t(x)$ is a rational function. (See [A05], [LL04], [Hl], [Gul], [B1, B2], [KKT] for example.) Following [LL04, 2.1], we make this rigorous by restricting to power series in $R[[t]]$ congruent to 1 modulo t and define a (globally) rational function to be a power series $f(t)$ such that there exist polynomials $p, q \in R[t]$ with $p(0) = q(0) = 1$ such that $p(t) = f(t)q(t)$.

As noted in 2.7, it is well known that if x is a finite-dimensional element then $\lambda_t(x)$ is a rational function. Larsen and Lunts observed in [LL04] that the property of being a rational function is not preserved by passing to subrings and proposed replacing 'rational function' by 'determinantally rational function' (see 2.7). We propose an even weaker condition, which we now define.

Given a power series $f(t) = \sum r_n t^n \in R[[t]]$ and a partition π, we form the the Jacobi-Trudi matrix (a_{ij}) with $a_{i,j} = r_{\pi'_i+i-j}$ and define $s_\pi(f) \in R$ to be its determinant. (If π has m columns, π' has m rows and (a_{ij}) is an $m \times m$ matrix over R.) The terminology comes from the fact that the commutative ring homomorphism

$\rho : \Lambda \to R$, defined by $\rho(x_n) = r_n$, satisfies $\rho(s_\pi) = \det(a_{i,j})$ by the Jacobi-Trudi identities.

Definition 5.1. Let R be a commutative ring. We say that a power series $f(t) = \sum r_n t^n \in R[[t]]$ is *Schur-rational* over R if there exists a partition μ such that $s_\pi(f) = 0$ for every partition π containing μ.

If μ is the rectangular partition m^{m+n} then $(a_{i,j})$ is the matrix $(r_{n+i-j})_{i,j=1}^m$ in Definition 2.7 up to row permutation. It follows that if $f(t)$ is Schur-rational then it is determinentally rational. The converse fails, as we show in Example 5.2.

It is easy to see that a (globally) rational function is Schur-rational. Thus being Schur-rational is a property of f intermediate between being rational and being determinentally rational.

Example 5.2. Let R_m be the quotient of Λ by the ideal generated by all m-fold products $x_{i_1} \cdots x_{i_m}$ where $|i_j - i_k| < 2m$ for all j, k. Then $f(t) = \sum x_n t^n$ is determinentally rational. On the other hand, $f(t)$ is not Schur-rational because for each λ with l columns there are partitions $\pi \supset \lambda$ such that $\pi' = (\pi'_1, \dots, \pi'_l)$ is lacunary, so that $s_\pi(f)$ is non-zero in R_m, because $s_\pi(f)$ is an alternating sum of monomimals and the diagonal monomial $\prod r_{\pi_i}$ is nonzero and occurs exactly once.

The notion of Schur-rationality is connected to Schur-finiteness.

Proposition 5.3. *An element x in a λ-ring is Schur-finite if and only if the power series $\lambda_t(x)$ is Schur-rational.*

In particular, if x is Schur-finite then $\lambda_t(x)$ is determinantally rational.

The "if" part of this proposition was proven in [KKT, 3.10] for λ-rings of the form $K_0(\mathcal{A})$, using categorical methods.

Proof. By definition, the power series $\lambda_t(x)$ is Schur-rational if and only if there is a partition μ so that for every π containing μ, the determinant $\det(\lambda^{\pi'_i+i-j}(x))$ is zero. Since this determinant is $s_\pi(x)$ by the Jacobi-Trudi identity, this is equivalent to x being Schur-finite (definition 4.1). ☐

We conclude by connecting our notion of Schur-finiteness to the notion of a Schur-finite object in a $\mathbb{Q}$-linear tensor category $\mathcal{A}$, given in [Mz]). By definition, an object A is *Schur-finite* if some $S_\lambda(A) \cong 0$ in $\mathcal{A}$. By [Mz, 1.4], this implies that $S_\pi(A) = 0$ for all π containing λ. It is evident that if A is a Schur-finite object of $\mathcal{A}$ then $[A]$ is a Schur-finite element of $K_0(\mathcal{A})$. However, the converse need not hold. For example, if $\mathcal{A}$ contains infinite direct sums then $K_0(\mathcal{A}) = 0$ by the Eilenberg swindle, so $[A]$ is always Schur-finite.

Here are two examples of Schur-finite objects whose class in $K_0(\mathcal{A})$ is finite-dimensional even though they are not finite-dimensional objects.

Example 5.4. Let $\mathcal{A}$ denote the abelian category of positively graded modules over the graded ring $A = \mathbb{Q}[\varepsilon]/(\varepsilon^2)$. It is well known that $\mathcal{A}$ is a tensor category under $\otimes_{\mathbb{Q}}$, with the λ-ring $K_0(\mathcal{A}) \cong \Lambda_{-1} = \mathbb{Z}[b]$; $1 = [\mathbb{Q}]$ and $b = [\mathbb{Q}[1]]$. The graded object A is Schur-finite but not finite-dimensional in $\mathcal{A}$ by [Mz, 1.12]. However, $[A]$ is a finite-dimensional element in $K_0(\mathcal{A})$ because $[A] = [\mathbb{Q}] + [\mathbb{Q}[1]]$.

Example 5.5 (O'Sullivan). Let X a Kummer surface; then there is an open subvariety U of X, whose complement Z is a finite set of points, such that $M(U)$ is Schur-finite but not finite-dimensional in the Kimura-O'Sullivan sense in the category $\mathcal{M}$ of motives [Mz, 3.3]. However, it follows from the distinguished triangle

$$M(Z)(2)[3] \to M(U) \to M(X) \to M(Z)(2)[4]$$

that $[M(U)] = [M(Z)(2)[3]] + [M(X)]$ in $K_0(\mathbf{DM}_{gm})$ and hence in $K_0(\mathcal{M})$. Since both $M(X)$ and $M(Z)(2)[3]$ are finite-dimensional, $[M(U)]$ is a finite-dimensional element of $K_0(\mathcal{M})$.

Proposition 5.6. *Let M be a classical motive. If M is Schur-finite in $\mathcal{M}$, then $\lambda_t([M])$ is determinantally rational. If $\lambda_t([M])$ is determinantally rational, then there exists a partition λ such that $S_\lambda(M)$ is a phantom motive.*

Proof. If M is Schur-finite, then there is a λ such that $0 = [S_\pi M] = s_\pi([M])$ for all $\pi \supseteq \lambda$. Thus $[M]$ is Schur-finite in $K_0(\mathcal{M})$ or equivalently, by 5.3, $\lambda_t([M])$ is Schur-rational, and hence determinantally rational.

If $\lambda_t([M])$ is determinantally rational, then by Definition 2.7 there is a rectangular λ so that $0 = s_\lambda([M]) = [S_\lambda M]$ in $K_0(\mathcal{M}^{\text{eff}})$. By Proposition 2.3, $S_\lambda(M)$ is a phantom motive. □

Acknowledgments. The authors would like to thank Anders Buch, Alessio Del Padrone and Christophe Soulé for valuable discussions.

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