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Nilpotence and K-Theory

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Karoubi–Villamayor K-theory behaves well in the presence of nilpotent elements. We can mod out by nilpotent ideals without affecting the groups $KV_n(R)$. If we ignore the units in $KV_1(R)$, we can replace an ideal by its radical, or a ring by its seminormalization (if the singular points are nice enough), all without changing the K-theory. These results also hold for the homotopization $[K_0]$ of K_0.

Section 1 introduces homotopization as a tool for analyzing the Karoubi–Villamayor K-theory. Section 2 studies the effect of modding out by a nilpotent ideal on the groups KV_n, as well as the homotopization $[K_n]$ of the classical groups K_n, $n \leqslant 3$. In Section 3 we introduce functors SKV_n, based on SL, which agree with the Karoubi–Villamayor functors for $n \geqslant 2$. This allows us to ignore units, and carry out the remaining calculations.

1. HOMOTOPIZATION

We work in the category of associative rings without unit. A subcategory $\mathfrak{R}$ is "admissible" if, whenever a ring R is in $\mathfrak{R}$, so is the ring $R[t]$, as well as the four morphisms

$$t : R \to R[t],\ \text{“}t = 0\text{” and “}t = 1\text{”}: R[t] \to R, \text{ and “}t \mapsto st\text{”}: R[t] \to R[s, t].$$

References for this section are [6, 15, 17].

Let $F: \mathfrak{R} \to \mathfrak{A}$ be a functor. The *homotopization* $[F]$ of F is the functorial coequalizer (if it exists) of the natural transformations $F(t = 0)$ and $F(t = 1)$ from $F(R[t])$ to $F(R)$. If $F = [F]$, we call F a *homotopy functor* on $\mathfrak{R}$; this is equivalent to requiring $F(R) \to F(R[t])$ to be an isomorphism for all R in $\mathfrak{R}$. We call a ring R *F-regular* if $F(R) \to F(R[t_1, \ldots, t_n])$ is an isomorphism for all n; F is a homotopy functor on the full subcategory of F-regular rings.

Let $\mathfrak{R}(R)$ denote the smallest admissible subcategory of $\mathfrak{R}$ containing R, and fix $\mathfrak{R}_0 = \mathbf{Rings(Z)}$ as a universal model. We can think of $\mathfrak{R}(R)$ as the image of a

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functor from $\mathfrak{R}_0$ to $\mathfrak{R}$. If $F\colon \mathfrak{R} \to \mathfrak{A}$ is a functor, then (for each R in $\mathfrak{R}$) we can think of $F(R)$ as a functor from $\mathfrak{R}_0$ to $\mathfrak{A}$. With this abuse of notation it makes sense to refer to a natural transformation $F_1(R_1) \to F_2(R_2)$; such a transformation induces a map $[F_1]R_1 \to [F_2]R_2$.

From now on we shall assume that $\mathfrak{A}$ is a category of groups. We then have an explicit construction of $[F]$. Let $NF(R)$ be the kernel of $F(t = 0)$; the image of $F(t = 1)\colon NF(R) \to F(R)$ is a normal subgroup, and the quotient group is $[F]R$. We shall call an element x of $NF(R)$ a *contraction* of its image $x(1)$ in $F(R)$; thus $[F]$ is obtained from F by modding out the contractible elements.

We shall need the following result from [17]. We start with an exact sequence of natural transformations of group-valued functors:

$$F_0(R_0) \to F_1(R_1) \to F_2(R_2) \to F_3(R_3).$$

LEMMA 1.1. *The sequence*

$$[F_0]R_0 \to [F_1]R_1 \to [F_2]R_2 \to [F_3]R_3$$

is exact if $NF_3(R_3) = 0$. It is exact at $[F_2]R_2$ if and only if the images of $F_2(R_2)$ and $NF_3(R_3)$ in $F_3(R_3)$ intersect in the image of $NF_2(R_2)$.

We shall be explicitly concerned with the following examples. The Karoubi–Villamayor groups KV_n are homotopy functors ($n \geqslant 1$) and $KV_1 = [K_1] = [Gl]$. For convenience of presentation we shall refer to $[K_0]$ as KV_0. There is a natural transformation $[K_n] \to KV_n$, which is not an isomorphism in general [18].

If R is commutative, $[K_1]R = [U]R \oplus [SK_1]R$. We have $[SK_1]R = [Sl]R$ and $[U]R =$ units of R modulo unipotents $= U(R/\text{nil } R)$.

If we think of the relative groups $K_n(R, I)$ as functors on (admissible) pairs of rings, we can form the homotopization $[K_n](R, I)$ as the coequalizer of maps from $K_n(R[t], I[t])$ to $K_n(R, I)$.

LEMMA 1.2. *Let I be an ideal in a ring R. Then $KV_1(I) = [K_1](R, I)$, and is independent of R. Thus excision holds for $[K_1]$.*

Proof. The subgroup $E(R, I)$ of $Gl(I)$ lies in the image of $NGl(I)$, as $y(1 + rte_{ij})y^{-1}$ are contractions of the generators ($r \in I$, $y \in E(R)$). Hence $K_1(R, I)$ lies between $Gl(I)$ and $[Gl]I$, and so $[K_1](R, I) = [Gl]I$. Done.

We shall take advantage of Lemma 1.2, in suppressing ambient rings for $[K_1]$ and $[SK_1]$. Recall from [6] that $KV_{n+1}(R) = KV_1(\Omega^n R)$, where $\Omega R = (t^2 - t)R[t]$.

LEMMA 1.3. *For R commutative (with or without unit) and $n \geqslant 1$,*

$$KV_{n+1}(R) = [SK_1]\Omega^n R = [Sl]\Omega^n R.$$

Proof. It is enough to show that $[U]\Omega^n R = 0$. But $[U]\Omega^n R =$

$U(\Omega^n(R/\mathrm{nil}\, R)) = 0$ as $R/\mathrm{nil}\, R$ is reduced and $\Omega^n R/\mathrm{nil}\, \Omega^n R \cong \Omega^n(R/\mathrm{nil}\, R)$. Done.

Proposition 1.4. *For R commutative with unit, $[K_0]R = H(R) \oplus [\tilde{K}_0]R$, and there is a short exact sequence*

$$0 \to [SK_0]R \to [\tilde{K}_0]R \to [\mathrm{Pic}]R \to 0.$$

Proof. From XII (7.10) of [1] we see that H is a homotopy functor, so the first statement follows from $K_0 = H \oplus \tilde{K}_0$ Now we have an exact sequence of natural transformations

$$0 \to SK_0(R) \to \tilde{K}_0(R) \to \mathrm{Pic}(R) \to 0$$

Applying Lemma 1.1 yields exactness at $[\tilde{K}_0]$ and [Pic], and gives a criterion for exactness at $[SK_0]$. Given $x \in N\tilde{K}_0(R)$ with $x(1)$ in $SK_0(R)$, we have to find $y \in NSK_0(R)$ with $y(1) = x(1)$. Let L be a rank 1 projective $R[t]$-module for which $\det(x) = L$ in $\mathrm{Pic}(R[t])$. Then $L(1) = \det(x(1)) = R$, and $y = x + 1 - [L]$ lies in $NSK_0(R)$. We are done, because $y(1) = x(1) + 1 - [L(1)] = x(1)$.

2. Nilpotent Rings

Recall the following result from IX(1.3) of [1]:

Lemma 2.1. *If I is a nilpotent ideal in a ring R, then the natural map $K_0(R) \to K_0(R/I)$ is an isomorphism. Moreover, $K_0(I) = 0$ and $K_1(R) \to K_1(R/I)$ is onto. The natural map $U(I) \to K_1(R, I)$ is onto, and is an isomorphism if R is commutative. In this case $SK_1(R, I) = 0$ also.*

The results of this section also hold for a commutative nil ring, as it is the direct limit of nilpotent rings. The point is that the property of being nilpotent is inherited by $I[t]$, $EI = tI[t]$, and $\Omega I = (t^2 - t)I[t]$, while more general properties are not. It is an open question as to whether or not $I[t]$ is nil when I is.

Proposition 2.2. *If I is nilpotent, $[U]I = 0$ and $KV_n(I) = 0$ for all $n \geq 0$.*

Proof. $[U]I = 0$ since $1 + at$ is a contraction of the unit $1 + a$, $a \in I$. The cases $n = 0, 1$ follow from Lemma 2.1 because $K_0 \to [K_0]$ and $[U]I \to [K_1](R, I)$ are onto. The result for higher groups follows from the observation that $\Omega^n I$ is nilpotent.

Remark. Not all nilpotent rings are contractible (in the sense of [9]), so this is a new class of rings which are acyclic for Karoubi–Villamayor K-theory.

THEOREM 2.3. *Let I be a nilpotent ideal in a ring R. Then $R \to R/I$ is a Gl-fibration, and $KV_n(R) \to KV_n(R/I)$ is an isomorphism for all $n \geqslant 0$.*

Proof. The rings E^nI are nilpotent, so from Lemma 2.1 we deduce that $K_1(E^nR) \to K_1(E^nR/I)$ is onto, which implies that $Gl(E^nR) \to Gl(E^nR/I)$ is onto, i.e., $R \to R/I$ is a Gl-fibration. Proposition 2.2 and the resulting long exact sequence yield the result for $n \geqslant 1$. The case $n = 0$ follows from Lemma 2.1 and the functoriality of homotopization.

Again, when R is commutative, we can let I be any nil ideal. Hence R has the same Karoubi–Villamayor K-theory as its reduced ring. In contrast, $K_1(R)$ and $K_2(R)$ contain many elements that the groups for $R/\text{nil } R$ do not.

Similar phenomena hold for $[K_2]$. Here we let $K_2(R, I)$ denote either the Stein relativization [13] or its quotient, the relative groups in the long exact sequence for Quillen K-theory [8].

PROPOSITION 2.4. *Let I be a nil ideal in a commutative ring R. Then $[K_2](R, I) = 0$.*

Proof. The elements $\langle a, q \rangle$, $a \in R$, $q \in I$, are generators of $K_2(R, I)$ by [4], and have contractions $\langle a, qt \rangle$ in $K_2(R[t], I[t])$.

COROLLARY 2.5. $[K_2]R = [K_2]R/I$.

Proof. We have the exact sequence

$$K_2(R, I) \to K_2(R) \to K_2(R/I) \to SK_1(R, I).$$

By Lemma 2.1, the pair (R, I) is SK_1-regular, so the result follows from an application of Lemma 1.1.

We can get information on $[K_3]$ as well. The Gersten–Anderson spectral sequence [5] works for Stein's relative groups; we have ($p \geqslant 0$, $q \geqslant 1$)

$$E^1_{pq} = K_q(R[t_1, \ldots, t_p], I[t_1, \ldots, t_p]) \Rightarrow KV_{p-q}(R, I).$$

The boundary maps $\partial_i \colon E^1_{p+1,q} \to E^1_{p,q}$ $(0 \leqslant i \leqslant p + 1)$ are given by $t_{i-1} = 0$ for $i \leqslant p$ and $t_{p+1} \mapsto 1 - (t_1 + \cdots + t_p)$ for $i = p + 1$. In particular, the two maps $E^1_{1q} \to E^1_{0q}$ are evaluation at $t = 0, 1$, so $E^2_{0q} = [K_q](R, I)$.

PROPOSITION 2.6. *When I is nilpotent, R commutative, we have $E^2_{12} = 0$ and an isomorphism $\partial \colon E^2_{22} \to [K_3](R, I)$.*

Proof. The E^1_{p1} term is $K_1(I[t_1, \ldots, t_p]) = U(I[t_1, \ldots, t_p])$, and as a set is just $I[t_1, \ldots, t_p]$. Computing with the Moore complex [10] we see that

$$E^2_{p1} = [1 - (t_1 + \cdots + t_p)]\ U(E^pI)/\partial_{p+1}U(E^{p+1}I) = 0.$$

By Proposition 2.2 the sequence abuts to zero, so the terms E^2_{12}, $E^3_{22} = \ker(\partial)$ and $E^3_{02} = \operatorname{coker}(\partial)$ are all zero, whence the result.

QUESTION. If R is a commutative ring with $R/\text{nil}\, R$ regular, is the natural map $K_n(R) \to K_n(R/\text{nil}\, R)$ onto for all $n \geqslant 0$?

Classically, the map is onto for $n \leqslant 2$ without the regularity condition. As $KV_n(R) = K_n(R/\text{nil}\, R)$, the question is a special case of a question of Bass [2]: when are the edge maps $[K_n] \to KV_n$ isomorphisms? Strooker [18] has shown that the edge map $[K_2] \to KV_2$ is not onto for the ring $\mathbf{Z}[x, t, t^{-1}]/(x^2 - 4)$, so some restriction is necessary. We have the following partial result for $n = 3$.

PROPOSITION 2.7. *If R is commutative and $R/\text{nil}\, R$ is K_1-regular, then there is an exact sequence*

$$KV_4(R) \to E^2_{22} \xrightarrow{\partial} [K_3]R \to KV_3(R) \to E^2_{12} \to 0.$$

Thus if $R/\text{nil}\, R$ is regular, $K_3(R) \to K_3(R/I)$ is onto if and only if $E^2_{12} = 0$.

Proof. It suffices to show that $E^2_{p1}(R) = 0$ for $p \neq 0$. We have a short exact sequence of simplicial groups:

$$0 \to E^1_{p1}(R, \text{nil}\, R) \to E^1_{p1}(R) \to E^1_{p1}(R/\text{nil}\, R) \to 0.$$

The left complex is acyclic by the proof of Proposition 2.6, and the right complex is constant by K_1-regularity. The result follows from the long exact homotopy sequence.

3. THE GROUPS SKV_n

We assume all rings, with or without units, are commutative, and define

$$SKV_n(R) = [SK_1]\Omega^{n-1}R$$

for $n \geqslant 1$. By Lemma 1.3 we see that these groups agree with the Karoubi–Villamayor groups $KV_n(R)$ for $n \geqslant 2$. A ring map $A \to B$ is an *Sl-fibration* when the maps $Sl(E^nA) \to Sl(E^nB)$ are onto for $n \geqslant 1$. As Sl is a "left exact Mayer–Vietoris functor," the results of [7] are applicable. The composition of Sl-fibrations is an Sl-fibration, and Gl-fibrations are Sl-fibrations. Theorem 2.3 and a simple calculation yield

LEMMA 3.1. *An Sl-fibration must be onto. A map $R \to R/I$ is a Gl-fibration if and only if it is an Sl-fibration and $I + \text{nil}\, R = \sqrt{I}$. It is an Sl-fibration if and only if $R \to R/\sqrt{I}$ is a Gl-fibration.*

LEMMA 3.2. *Let I be an ideal in a ring R. Then $I \to \sqrt{I}$ induces isomorphisms for* $[K_0]$, [Pic], SK_0, $[SK_0]$, *and* $[SK_1]$.

Proof. We give the proof for $[SK_1]$ only, as the other cases are similar. We have the exact functorial sequence [11]:

$$K_2(\sqrt{I}/I) \to SK_1(I) \to SK_1(\sqrt{I}) \to SK_1(\sqrt{I}/I).$$

By Lemma 2.1 the pair $(R/I, \sqrt{I}/I)$ is SK_1-regular. Applying Lemma 1.1, and using Propositions 2.2 and 2.4, we obtain the exact sequence

$$0 \to [SK_1]I \to [SK_1]\sqrt{I} \to 0.$$

Again, for convenience we let SKV_0 denote $[SK_0]$.

THEOREM 3.3. *Let I be an ideal in a commutative ring R. Then $SKV_n(I) \to SKV_n(\sqrt{I})$ is an isomorphism for all $n \geqslant 0$. The map $KV_n(I) \to KV_n(\sqrt{I})$ is an isomorphism for all $n \neq 1$. For $n = 1$ it is an injection with cokernel $U(\sqrt{I})/U(I + \text{nil } R)$.*

Proof. As $[U]I = U(I + \text{nil } R/\text{nil } R)$, it is easy to see that $[U]I \to [U]\sqrt{I}$ is an injection with the desired cokernel. The results for $n = 0, 1$ now follow from Lemma 3.2. The result for $n \geqslant 2$ now follows from Lemma 1.3 and the observation that $\Omega^n\sqrt{I}$ is the radical of $\Omega^n I$ in $R[t_1, ..., t_n]$. Done.

COROLLARY 3.4. *If $R \to R/I$ is an Sl-fibration, there is a long exact ideal sequence of SKV-groups, ending in*

$$SKV_1(R/I) \to SK_0(I) \to SK_0(R) \to SK_0(R/I).$$

Proof. From Lemma 3.1, $R \to R/\sqrt{I}$ is a *Gl*-fibration, and our sequence splits off from the resulting long exact sequence of Karoubi–Villamayor K-theory, with $\sqrt{I}$ in place of I. We need only use Theorem 3.3 and Lemmas 2.1, 3.2 to substitute I back in for $\sqrt{I}$. Done.

Remark. We could have proved the corollary directly from the results of [7] and a splicing argument. Theorem 3.3 would then follow, as $\sqrt{I} \to \sqrt{I}/I$ is an *Sl*-fibration. We have chosen this approach to emphasize the relationship to Karoubi–Villamayor K-theory.

Here is an amusing application of these ideas.

COROLLARY 3.5. *Suppose $NK_0(R) = NK_0(R/I) = 0$. Then the sequence*

$$KV_1(R) \to KV_1(R/I) \to KV_0(I) \to KV_0(R) \to KV_0(R/I)$$

is exact. If $U(I) = U(\sqrt{I})$ *and* $R \to R/\sqrt{I}$ *is a Gl-fibration as well, it is the ending of a long exact ideal sequence of Karoubi–Villamayor K-theory.*

Proof. The first statement is a simple consequence of Lemma 1.1. Theorem 3.3 and the condition on units allow us to substitute I for $\sqrt{I}$ in the ideal sequence for $\sqrt{I}$. Done.

Note that if $I + \text{nil}\, R \neq \sqrt{I}$, then $R \to R/I$ is not a Gl-fibration, and $K_0(I)$ is not equal to $KV_0(I) = K_0(\sqrt{I})$. In this case there is no exact sequence

$$KV_1(R/I) \to K_0(I) \to K_0(R)$$

yielding the maps of Corollary 3.5 upon homotopization. As an example, we can take I to be any nonradical ideal of the integers but a 2-primary one.

We now consider Mayer–Vietoris sequences. Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A/I \arrow[r] & B/I \end{tikzcd} \tag{3.6}$$

be a Cartesian square. Let J_A, J_B denote the radicals of I in A and B.

Proposition 3.7. *In the square* (3.6), *suppose that* $A/J_A = B/J_B$ *and that* $N\,\text{Pic}(B/J_B) = 0$. *Then* $[K_0]A \cong [K_0]B$.

Proof. As $K_0(A/I) = K_0(B/I)$, the Mayer–Vietoris sequence for (3.6) shows that we have an isomorphism on H and SK_0, hence on $[SK_0]$. From Proposition 1.4 we have only to show $[\text{Pic}]A \cong [\text{Pic}]B$. The Mayer–Vietoris sequence shows we have an epimorphism on Pic, hence on [Pic]. By Lemma 1.1 the rows of

$$\begin{tikzcd} {[U]A/I} \arrow[r] \arrow[d] & {[\text{Pic}]I} \arrow[r] & {[\text{Pic}]A} \arrow[r] \arrow[d] & {[\text{Pic}]A/I} \arrow[d] \\ {[U]B/I} \arrow[r] & {[\text{Pic}]I} \arrow[r] & {[\text{Pic}]B} \arrow[r] & {[\text{Pic}]B/I} \end{tikzcd}$$

are exact. As the outside vertical arrows are isomorphisms, it follows that $[\text{Pic}]A \to [\text{Pic}]B$ is also monic, hence an isomorphism. Done.

Lemma 3.8. *If, in the square of* (3.6), $B \to B/I$ *is an Sl-fibration, so is* $A \to A/I$.

Proof. This follows from Proposition 2.10 of [7].

Theorem 3.9. *If* $B \to B/J_B$ *is a Gl-fibration, the Cartesian square* (3.6) *gives rise to a long exact Mayer–Vietoris sequence of SKV groups, ending in*

$$SKV_1(B/I) \to SK_0(A) \to SK_0(B) \oplus SK_0(A/I) \to SK_0(B/I).$$

Proof. By Lemmas 3.1, 3.8 the vertical arrows of (3.6) are Sl-fibrations. We can apply Corollary 3.4 to obtain long exact ideal sequences, and splice them together in the familiar way.

COROLLARY 3.10. *Suppose in the square* (3.6) *that* $A/J_A \rightarrow B/J_B$ *is an isomorphism and that* $B \rightarrow B/J_B$ *is a Gl-fibration. Then*

$$SKV_n(A) \rightarrow SKV_n(B)$$

is an isomorphism for all $n \geqslant 0$.

COROLLARY 3.11. *Suppose* $A/J_A = B/J_B$ *is regular in the square* (3.6). *Then* $KV_n(A) \rightarrow KV_n(B)$ *is an isomorphism for all* $n \neq 1$. *If* A *is a subring of* B, *then it is an injection for* $n = 1$ *with cokernel* $U(B)/U(A)\ U(\text{nil } B)$.

Proof. We need only verify the statement about units. As $A \cap \text{nil } B = \text{nil } A$, $[U]A = U(A)/U(\text{nil } A)$ is a subgroup of $[U]B = U(B)/U(\text{nil } B)$ with the desired quotient.

As an application, let B be the ring of integers in some global field, and let A be a subring with the same quotient field. If the primes of the conductor do not split and their residue fields do not extend, A has the same Karoubi–Villamayor K-theory as B (except for units). For instance, $[SK_1]A = 0$ for $A = \mathbf{Z}[2i]$ or one of the rings considered in Theorem 3.1 of [14], although the classical SK_1 is nonzero:

PROPOSITION 3.12. $SK_1(\mathbf{Z}[2i]) = \mathbf{Z}/2$, *the nontrivial element being* $\left[{2 \atop 1+2i}\right]$. *A contraction of this symbol is provided by replacing* 2 *by* $2t$ *in either or both occurrences.*

Proof. The conductor is $I = 2B$, $B = \mathbf{Z}[i]$, and $B/I = \mathbf{Z}/2[\epsilon]$. The ideal sequences and Van der Kallen's computation of $K_2(B/I)$ show that $SK_1(A, I) = SK_1(A)$ and $SK_1(B, I) = 0$. By Swan's excision sequence [14], it is clear that the described Mennicke symbol is the only possible nonzero one; by Lemma 3.13 it is nonzero. Done.

LEMMA 3.13. *There is a nontrivial map* $SK_1(A, I) \rightarrow \{\pm 1\}$ *given by sending* $\left[{b \atop a}\right]$ *to* $\left({b \atop a}\right)_2$.

Proof. The proof of Lemma 3.2 of [14] goes through for $p = 2$, once we observe that if $b \in I$, $c \equiv 1 \bmod 2b$, we have $(c, b) = 1$ (Swan's notation). This follows from (A.17) of [3] unless $b = \pm 2, \pm 2i$, when it follows from the remark after (A.18) of [3]. Hence the map is well defined. When $a = 1 + 2i$, $b = 2$, we have $Na = 5$ and $b^2 \equiv -1 \bmod a$, so by definition [3, p. 86] we have $\left({b \atop a}\right)_2 = -1$.

Here is another application of Corollary 3.11.

PROPOSITION 3.14. *Let A be a commutative Noetherian ring, B its seminormalization, and $0 \neq I$ the conductor from B to A. If $A/\sqrt{I}$ is regular, then $KV_n(A) \cong KV_n(B)$, except possibly for units when $n = 1$.*

Proof. We have only to show $A/J_A \cong B/J_B$. By [16] there is a bijection between the primes of A/I and B/I, and the residue fields are identical. As A/J_A is normal, it is the product of a finite number of domains. Since B/J_B is an integral extension with the same total quotient ring, it must be A/J_A. Done.

COROLLARY 3.15. *Let Y be an affine curve over a field. Then (except for units) Y has the same Karoubi–Villamayor K-theory as its seminormalization.*

COROLLARY 3.16. *Let Y be a connected affine curve over a field k, whose normalization X is n affine lines. Suppose in addition that the m singular points of Y and the M points of X lying over them are all k-rational. Then*

$$KV_n(Y) = K_n(k) \oplus dK_{n+1}(k)$$

for all $n \geq -1$, where $d = M - m - n + 1$.

Proof. If Y is seminormal, it is a curve of "Type I," and the computation is contained in [12]. The only data that change upon passage from Y to its seminormalization are the values of m and M. But each singular point we lose has only one point lying over it, so $M - m$ (and hence d) does not change. Corollary 3.15 yields the result, except for units. But as Y is connected, the arguments of [12] show that the seminormalization has no more units than k does. As $U(Y)$ is a subgroup, it must be $U(k)$ as well, and we are done.

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