

# PICARD GROUPS AND CLASS GROUPS OF MONOID SCHEMES

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ABSTRACT. We define and study the Picard group of a monoid scheme and the class group of a normal monoid scheme. To do so, we develop some ideal theory for (pointed abelian) noetherian monoids, including primary decomposition and discrete valuations. The normalization of a monoid turns out to be a monoid scheme, but not always a monoid.

## INTRODUCTION

The main purpose of this paper is to study the Picard group of a monoid scheme  $X$  and, if  $X$  is normal, the Weil divisor class group of  $X$ . Along the way, we develop the analogues for pointed abelian monoids of several notions in commutative ring theory such as associated prime ideals, discrete valuations and normalization.

Recall from [2] or [9] that a *monoid scheme* is a topological space  $X$  equipped with a sheaf  $\mathcal{A}$  of pointed abelian monoids such that locally  $X$  is  $\text{MSpec}(A)$ , the set of prime ideals of a pointed monoid  $A$  with its natural structure sheaf. The interest in monoid schemes stems from the fact that every toric variety (or rather its fan  $\Delta$ ) is associated to a monoid scheme  $X_\Delta$  in a natural way; see [2, 4.2].

To study the divisor class group of a normal monoid, we need to establish the connection between height one primes and discrete valuations of a normal monoid. To this end, the first few sections develop the theory of associated primes for pointed abelian noetherian monoids, in analogy with the theory for commutative rings [1].

The notion of normalization behaves differently for monoids than it does in ring theory. The normalization of a cancellative monoid is well known [5], but the surprise is that we can make sense of normalization for partially cancellative monoids, i.e., for arbitrary quotients of cancellative monoids. The normalization of  $A$  turns out to be not a monoid but a *monoid scheme*: the disjoint union over the minimal primes of the normalizations of the  $A/\mathfrak{p}$ . That is, we need to embed monoids (contravariantly) into the larger category of monoid schemes.

With these preliminaries, we define the Weil divisor class group of a normal monoid or monoid scheme in the same way as in algebraic geometry, and give its basic properties in Section 4. It turns out that the class group of a toric monoid scheme agrees with the divisor class group of the associated toric variety.

The *Picard group*  $\text{Pic}(X)$  of a monoid scheme  $X$  is the set of isomorphism classes of invertible sheaves on  $X$  (see Definition 5.1). In more detail, recall that if  $A$  is a pointed monoid then an *A-set* is a pointed set  $L$  on which  $A$  acts; the smash product  $L_1 \wedge_A L_2$  is again an  $A$ -set. Similarly, if  $(X, \mathcal{A})$  is a monoid scheme then a sheaf of  $\mathcal{A}$ -sets is a sheaf of pointed sets  $\mathcal{L}$ , equipped with a pairing  $\mathcal{A} \wedge \mathcal{L} \rightarrow \mathcal{L}$  making each stalk  $\mathcal{L}_x$  an  $\mathcal{A}_x$ -set. We say that  $\mathcal{L}$  is *invertible* if it is locally isomorphic to  $\mathcal{A}$

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in the Zariski topology; the smash product of invertible sheaves is again invertible, and the dual sheaf  $\mathcal{L}^{-1}$  satisfies  $\mathcal{L} \wedge_A \mathcal{L}^{-1} \cong \mathcal{A}$ . This gives  $\text{Pic}(X)$  the structure of an abelian group; see Section 5.

The group of  $A$ -set automorphisms of any monoid  $A$  is canonically isomorphic to  $A^\times$ . Thus a standard argument (see Lemma 5.3) shows that  $\text{Pic}(X) \cong H^1(X, \mathcal{A}^\times)$ . Using this, we show in 5.5 and 7.1 that (unlike algebraic geometry) we always have  $\text{Pic}(X) = \text{Pic}(X \times \mathbb{A}^1)$  and  $\text{Pic}(X) = \text{Pic}(X_{\text{sn}})$ , where  $X_{\text{sn}}$  is the seminormalization of  $X$ .

In Section 6 we show that  $\text{Pic}(X)$  is always a subgroup of  $\text{Cl}(X)$  when  $X$  is normal. It turns out that the Picard group of a toric monoid scheme agrees with the Picard group of the associated toric variety; see Theorem 6.6. This explains the calculation by Vezzani (our original inspiration) that the Picard group of the projective monoid scheme  $\mathbb{P}^n$  is  $\mathbb{Z}$ . We conclude this paper by providing some exact sequences relating  $\text{Pic}(X)$  to  $\text{Pic}(X_{\text{nor}})$ .

After this paper was written, we were informed by Bernhard Köck that many of the results in the first two sections were obtained by Ulrich Kobsa in [8], and by Franz Halter-Koch in [6]. Since these sources are couched in a different framework, we have elected to retain our sections in order to be self-contained.

### *Notation*

In this paper the term ‘monoid’ will always mean a pointed, abelian monoid unless otherwise stated. We write the product multiplicatively, so that the zero element  $0$  is the basepoint and  $1$  is the unit. We also write  $A^\times$  for the group of invertible elements in  $A$ .

We remind the reader of some basic facts concerning such monoids. Let  $A$  be a monoid. An ideal  $I \subseteq A$  is a subset of elements such that  $AI \subseteq I$ ; a nonzero ideal of the form  $Ax$  is called a *principal ideal*. The quotient or factor monoid  $A/I$  identifies all elements of the ideal  $I$  with  $0$ . If  $B$  is an unpointed monoid, we can form the (pointed) monoid  $B_+$  by adding a disjoint zero element.

An ideal  $\mathfrak{p} \subseteq A$  is *prime* when  $xy \in \mathfrak{p}$  means  $x \in \mathfrak{p}$  or  $y \in \mathfrak{p}$ . The set of all prime ideals form a topological space, denoted  $\text{MSpec}(A)$ , under the Zariski Topology. If the elements of  $A$  satisfy  $ab = ac$  implies  $b = c$ , we say that  $A$  is *cancellative*. If  $A$  is a quotient of a cancellative monoid, we say it is *partially cancellative*, or *pc*.

Let  $S \subseteq A$  be a multiplicatively closed subset. We can then form the monoid  $S^{-1}A$ , termed the localization of  $A$  at  $S$ , which is obtained from  $A$  by inverting all elements of  $S$ . When  $S = A \setminus \mathfrak{p}$ , we say that  $S^{-1}A$  is obtained from  $A$  by localizing at  $\mathfrak{p}$  and denote this special case by  $A_{\mathfrak{p}}$ . Note that  $\mathfrak{p}$  is the maximal ideal of  $A_{\mathfrak{p}}$ .

In the case where  $\mathfrak{p} = \{0\}$  is a prime ideal, we write  $A_0$  for  $A_{\mathfrak{p}}$ ; it is an abelian group with a disjoint basepoint, called the group completion of  $A$  and there is an inclusion  $A \hookrightarrow A_0$  precisely when  $A$  is cancellative. We shall write  $A_0^\times$  for the abelian group  $A_0 \setminus \{0\}$ ; it is the classical group completion of the unpointed monoid  $A \setminus \{0\}$ .

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## 1. PRIMARY DECOMPOSITION

The definitions and proofs in this section are direct translations from commutative ring theory and are given here for convenience.

We say that a monoid  $A$  is *noetherian* when it satisfies the ascending chain condition on ideals. This is equivalent to the condition that every ideal is finitely generated, by the usual proof: if  $I_1 \subseteq I_2 \subseteq \dots$  is an ascending chain of finitely generated ideals of  $A$ , then the union is finitely generated if and only if it equals some  $I_n$ .

**Remark 1.1.** When  $A^\times$  is a finitely generated abelian group, a monoid  $A$  is noetherian if and only if  $A$  satisfies the ACC on congruences, i.e., the descending chain condition on quotient monoids. We will not consider this (slightly) stronger chain condition.

In order to establish a primary decomposition for ideals, we introduce some definitions. A proper ideal  $\mathfrak{q} \subseteq A$  is *primary* when  $xy \in \mathfrak{q}$  implies  $x \in \mathfrak{q}$  or  $y^n \in \mathfrak{q}$ . Alternatively,  $\mathfrak{q}$  is primary when every zero-divisor  $a \in A/\mathfrak{q}$  is nilpotent. The *radical* of an ideal  $I$  is  $\sqrt{I} = \{a \in A \mid a^n \in I\}$ ; it is a prime ideal when  $I$  is primary. An ideal  $I \subseteq A$  is said to be *irreducible* when  $I = J \cap K$ , with  $J, K \subseteq A$  ideals, implies  $I = J$  or  $I = K$ .

**Lemma 1.2.** *Every irreducible ideal of a noetherian monoid is primary.*

*Proof.* An ideal  $I \subseteq A$  is primary if and only if the zero ideal of  $A/I$  is primary. Therefore we need only show when  $(0)$  is irreducible, it is primary. Let  $xy = 0$  with  $y \neq 0$ ; we will show  $x^n = 0$ . Consider the ascending chain  $\text{ann}(x) \subseteq \text{ann}(x^2) \subseteq \dots$ , where  $\text{ann}(x) = \{a \in A \mid ax = 0\}$ . By the noetherian property, this chain must stabilize, say  $\text{ann}(x^n) = \text{ann}(x^{n+1}) = \dots$  for some  $n > 0$ .

We claim that  $0 = (x^n) \cap (y)$ . Let  $a \in (x^n) \cap (y)$ , say  $a = bx^n = cy$ . Then  $0 = c(xy) = ax = (bx^n)x = bx^{n+1}$ . Hence  $b \in \text{ann}(x^{n+1}) = \text{ann}(x^n)$  giving  $a = bx^n = 0$ . Since  $(0)$  is irreducible and  $y \neq 0$ , we must have  $x^n = 0$  proving  $(0)$  is primary.  $\square$

**Theorem 1.3** (Primary decomposition). *In a noetherian monoid every ideal  $I$  can be written as the finite intersection of irreducible primary ideals  $I = \bigcap_i \mathfrak{q}_i$ .*

*Proof.* Suppose the result is false. Since  $A$  is noetherian, the set of ideals which cannot be written as a finite intersection of irreducible ideals has a maximal element, say  $I$ . Since  $I$  is not irreducible, it can be written  $I = J \cap K$  where  $J, K$  are ideals of  $A$  containing  $I$ . By maximality, both  $J$  and  $K$  (and hence  $I$ ) can be written as a finite intersection of irreducible ideals. This is a contradiction, and the theorem follows via Lemma 1.2.  $\square$

**Remark 1.3.1.** We say that  $A$  is *reduced* if whenever  $a, b \in A$  satisfy  $a^2 = b^2$  and  $a^3 = b^3$  then  $a = b$ . This implies that  $A$  has no nilpotent elements, i.e., that  $\text{nil}(A) = \{a \in A \mid a^n = 0 \text{ for some } n\}$  vanishes. By Theorem 1.3,  $\text{nil}(A)$  is the intersection of all the prime ideals in  $A$ ; cf. [3, (1.1)]. These conditions are equivalent when  $A = C/I$  is the quotient of a cancellative monoid  $C$ ; in this case  $A_{\text{red}} = A/\text{nil}(A)$  is a reduced monoid (see [3, 1.6]).

Let  $A$  be a noetherian monoid and  $I \subseteq A$  an ideal. Given a minimal primary decomposition of  $I$ ,  $I = \bigcap_i \mathfrak{q}_i$ ,  $\text{Ass}(I)$  denotes the set of prime ideals occurring as

the radicals  $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ ; the  $\mathfrak{p}_i$  are called the associated primes of  $I$ . Although the primary decomposition need not be unique, the set  $\text{Ass}(I)$  of associated primes of  $I$  is independent of the minimal primary decomposition, by 1.6 below.

In order to show that the associated primes are ideal quotients, we recall the definition. Given ideals  $I, J$  of  $A$ , the *ideal quotient* of  $I$  by  $J$  is the ideal  $(I : J) = \{a \in A \mid ax \in I \text{ for all } x \in J\}$ . When  $I = aA$  is a principal ideal, we often write  $(a : J)$  for  $(aA : J)$ . The next two lemmas are simple, direct translations from ring theory (such as [1, 1.11, 4.4]) so their proofs are omitted.

**Lemma 1.4.** *Let  $\mathfrak{p}$  be prime ideal in a monoid  $A$ .*

- i) *If  $I_1, \dots, I_n$  are ideals such that  $\cap_i I_i \subseteq \mathfrak{p}$ , then  $I_i \subseteq \mathfrak{p}$  for some  $i$ . If in addition  $\mathfrak{p} = \cap_i I_i$ , then  $\mathfrak{p} = I_i$  for some  $i$ .*
- ii) *Let  $\mathfrak{q}$  be a  $\mathfrak{p}$ -primary ideal of  $A$ . If  $a \in A \setminus \mathfrak{q}$ , then  $(\mathfrak{q} : a)$  is  $\mathfrak{p}$ -primary.*

**Lemma 1.5.** *Let  $A$  be a noetherian monoid. For any ideal  $I$  of  $A$ ,  $(\sqrt{I})^n \subseteq I$  for some  $n$ .*

**Proposition 1.6.** *Let  $I \subseteq A$  be an ideal with minimal primary decomposition  $I = \cap_{i=1}^n \mathfrak{q}_i$ . Then  $\text{Ass}(I)$  is exactly the set of prime ideals which occur in the set of ideals  $\sqrt{(I : a)}$ , where  $a \in A$ . Hence  $\text{Ass}(I)$  is independent of the choice of primary decomposition.*

*In addition, the minimal elements in  $\text{Ass}(I)$  are exactly the set of prime ideals minimal over  $I$ .*

*Proof.* (Compare [1, 4.5, 4.6].) First note that

$$\sqrt{(I : a)} = \sqrt{(\cap_i \mathfrak{q}_i : a)} = \sqrt{\cap_i (\mathfrak{q}_i : a)} = \cap_i \sqrt{(\mathfrak{q}_i : a)}.$$

By Lemma 1.4(ii), this equals  $\cap_{a \notin \mathfrak{q}_i} \mathfrak{p}_i$ . If  $\sqrt{(I : a)}$  is prime, then it is  $\mathfrak{p}_i$  for some  $i$ , by Lemma 1.4(i). Conversely, by minimality of the primary decomposition, for each  $i$  there exists an  $a_i \notin \mathfrak{q}_i$  but  $a_i \in \cap_{j \neq i} \mathfrak{q}_j$ . Using Lemma 1.4(i) once more, we see  $\sqrt{(I : a_i)} = \mathfrak{p}_i$ .

Finally, if  $I \subseteq \mathfrak{p}$  then  $\cap \mathfrak{p}_i \subseteq \mathfrak{p}$ , so  $\mathfrak{p}$  contains some  $\mathfrak{p}_i$  by Lemma 1.4(i). If  $\mathfrak{p}$  is minimal over  $I$  then necessarily  $\mathfrak{p} = \mathfrak{p}_i$ .  $\square$

**Proposition 1.7.** *Let  $A$  be a noetherian monoid and  $I \subseteq A$  an ideal. Then the associated prime ideals of  $I$  are exactly the prime ideals occurring in the set of ideals  $(I : a)$  where  $a \in A$ .*

*Proof.* The ideals  $I_i = \cap_{j \neq i} \mathfrak{q}_j$  strictly contain  $I$  by minimality of the decomposition. Since  $\mathfrak{q}_i \cap I_i = I$ , any  $a \in I_i \setminus I$  is not contained in  $\mathfrak{q}_i$ , hence  $(I : a)$  is  $\mathfrak{p}_i$ -primary by Lemma 1.4(ii). Now, by Lemma 1.5 we have  $\mathfrak{p}_i^n \subseteq \mathfrak{q}_i$  for some  $n > 0$ , hence

$$\mathfrak{p}_i^n I_i \subseteq \mathfrak{q}_i I_i \subseteq \mathfrak{q}_i \cap I_i = I.$$

Choose  $n$  minimal so that  $\mathfrak{p}_i^n I_i \subseteq I$  (hence in  $I_i$ ) and pick  $a \in \mathfrak{p}_i^{n-1} I_i$  with  $a \notin I$ . Since  $\mathfrak{p}_i a \subseteq I$  we have  $\mathfrak{p}_i \subseteq (I : a)$ ; as  $(I : a)$  is  $\mathfrak{p}_i$ -primary, we have  $\mathfrak{p}_i = (I : a)$ . Conversely, if  $(I : a)$  is prime then it is an associated prime by Proposition 1.6.  $\square$

**Lemma 1.8.** *Let  $A$  be a noetherian monoid and  $I \subseteq A$  an ideal.*

- i) *If  $I$  is maximal among ideals of the form  $(0 : a)$ ,  $a \in A$ , then  $I$  is an associated prime of 0.*
- ii) *If  $a \in A$ , then  $a = 0$  in  $A$  if and only if  $a = 0$  in  $A_{\mathfrak{p}}$  for every prime  $\mathfrak{p}$  associated to 0.*

*Proof.* Suppose that  $I = (0 : a)$  is maximal, as in (i). If  $xy \in I$  but  $y \notin I$ , then  $axy = 0$  and  $ay \neq 0$ . Hence  $I \subseteq I \cup Ax \subseteq (0 : ay)$ ; by maximality,  $I = I \cup Ax$  and  $x \in I$ . Thus  $I$  is prime; by Proposition 1.6,  $I$  is associated to  $(0)$ .

Suppose that  $0 \neq a \in A$ , and set  $I = (0 : a)$ . By (i),  $I \subseteq \mathfrak{p}$  for some associated prime  $\mathfrak{p}$ . But then  $a \neq 0$  in  $A_{\mathfrak{p}}$ .  $\square$

## 2. NORMAL AND FACTORIAL MONOIDS

In this section, we establish the facts about normal monoids needed for the theory of divisors.

The vocabulary for integral extensions of monoids mimicks that for commutative rings. If  $A$  is a submonoid of  $B$ , we say that an element  $b \in B$  is *integral* over  $A$  when  $b^n \in A$  for some  $n > 0$ , and the *integral closure* of  $A$  in  $B$  is the submonoid of elements integral over  $A$ . If  $A$  is a cancellative monoid, we say that it is *normal* (or *integrally closed*) if it equals its integral closure in its group completion. (See [2, 1.6].)

**Example 2.1.** It is elementary that all factorial monoids are normal. The affine toric monoids of [2, 4.1] are normal, and so are arbitrary submonoids of a free abelian group closed under divisibility. By [2, 4.5], every finitely generated normal monoid is  $A \wedge U_*$  for an affine toric monoid  $A$  and a finite abelian group  $U$ .

We now present some basic facts concerning normal monoids which parallel results for commutative rings.

**Lemma 2.2.** *If  $A_1$  and  $A_2$  are normal monoids, so is  $A_1 \wedge A_2$*

*Proof.* The group completion of  $A_1 \wedge A_2$  is  $(A_1 \wedge A_2)^+ = A_1^+ \wedge A_2^+$ . If  $a_1 \wedge a_2 \in A_1^+ \wedge A_2^+$  is integral over  $A_1 \wedge A_2$ , then  $(a_1 \wedge a_2)^n = a_1^n \wedge a_2^n \in A_1 \wedge A_2$ , hence  $a_1^n \in A_1$ ,  $a_2^n \in A_2$ . By normality,  $a_1 \in A_1$  and  $a_2 \in A_2$ .  $\square$

**Lemma 2.3.** *Let  $A \subseteq B$  be monoids and  $S \subseteq A$  be multiplicatively closed. We have the following:*

- i) *If  $B$  is integral over  $A$ , then  $S^{-1}B$  is integral over  $S^{-1}A$ .*
- ii) *If  $B$  is the integral closure of  $A$  in a monoid  $C$ , then  $S^{-1}B$  is the integral closure of  $S^{-1}A$  in  $S^{-1}C$ .*
- iii) *If  $A$  is normal, then  $S^{-1}A$  is normal.*

*Proof.* Suppose that  $b$  is integral over  $A$ , i.e.,  $b^n \in A$  for some  $n > 0$ . Then  $b/s \in S^{-1}B$  is integral over  $S^{-1}A$  because  $(b/s)^n \in S^{-1}A$ . This proves (i). For (ii), it suffices by (i) to suppose that  $c/1 \in S^{-1}C$  is integral over  $S^{-1}A$  and show that  $c/1$  is in  $S^{-1}B$ . If  $(c/1)^n = a/s$  in  $S^{-1}A$  then  $c^n st = at$  in  $A$  for some  $t \in S$ . Thus  $cst$  is in  $B$ , and  $c/1 = (cst)/st$  is in  $S^{-1}B$ . It is immediate that ii) implies iii).  $\square$

**Lemma 2.4.** *Let  $A$  be a cancellative monoid and  $I \subseteq A$  a finitely generated ideal. If  $u \in A_0$  is such that  $uI \subseteq I$ , then  $u$  is integral over  $A$ .*

*Proof.* Let  $X = \{x_1, \dots, x_r\}$  be the set of generators of  $I$ . Since  $uI \subseteq I$ , there is a function  $\phi : X \rightarrow X$  such that  $ux \in A\phi(x)$  for each  $x \in X$ . Since  $X$  is finite there is an  $x \in X$  and an  $n$  so that  $\phi^n(x) = x$ . For this  $x$  and  $n$  there is an  $a \in A$  so that  $u^n x = ax$ . By cancellation,  $u^n = a$ .  $\square$

*Discrete valuation monoids.* Recall from [2, 8.1] that a *valuation monoid* is a cancellative monoid  $A$  such that for every non-zero  $\alpha$  in the group completion  $A_0$ , either  $\alpha \in A$  or  $\alpha^{-1} \in A$ . Passing to units, we see that  $A^\times$  is a subgroup of the abelian group  $A_0^\times$ , and the *value group* is the quotient  $A_0^\times/A^\times$ . The value group is a totally ordered abelian group ( $x \geq y$  if and only if  $x/y \in A$ ). Following [2, 8.3], we call  $A$  a *discrete valuation monoid*, or DV monoid for short, if the value group is infinite cyclic. In this case, a lifting  $\pi \in A$  of the positive generator of the value group generates the maximal ideal  $\mathfrak{m}$  of  $A$  and every  $a \in A$  can be written  $a = u\pi^n$  for some  $u \in A^\times$  and  $n \geq 0$ . Here  $\pi$  is called a *uniformizing parameter* for  $A$ .

It is easy to see that valuation monoids are normal, and that noetherian valuation monoids are discrete [2, 8.3.1]. We now show that one-dimensional, noetherian normal monoids are DV monoids.

**Proposition 2.5.** *Every one-dimensional, noetherian normal monoid is a discrete valuation monoid (and conversely).*

*Proof.* Suppose  $A$  is a one-dimensional noetherian normal monoid, and choose a nonzero  $x$  in the maximal ideal  $\mathfrak{m}$ . By primary decomposition 1.3,  $\sqrt{xA}$  must be  $\mathfrak{m}$  and (by Lemma 1.5) there is an  $n > 0$  with  $\mathfrak{m}^n \subseteq xA$ ,  $\mathfrak{m}^{n-1} \not\subseteq xA$ . Choose  $y \in \mathfrak{m}^{n-1}$  with  $y \notin xA$  and set  $\pi = x/y \in A_0$ . Since  $\pi^{-1} \notin A$  and  $A$  is normal,  $\pi^{-1}$  is not integral over  $A$ . By Lemma 2.4,  $\pi^{-1}\mathfrak{m} \not\subseteq \mathfrak{m}$ ; since  $\pi^{-1}\mathfrak{m} \subseteq A$  by construction, we have  $\pi^{-1}\mathfrak{m} = A$ , or  $\mathfrak{m} = \pi A$ .

Lemma 2.4 also implies that  $\pi^{-1}I \not\subseteq I$  for every ideal  $I$ . If  $I \neq A$  then  $I \subseteq \pi A$  so  $\pi^{-1}I \subseteq A$ . Since  $I = \pi^{-1}(\pi I) \subset \pi^{-1}I$ , we have an ascending chain of ideals which must terminate at  $\pi^{-n}I = A$  for some  $n$ . Taking  $I = aA$ , this shows that every element  $a \in A$  can be written  $u\pi^n$  for a unique  $n \geq 0$  and  $u \in A^\times$ . Hence every element of  $A_0$  can be written  $u\pi^n$  for a unique  $n \in \mathbb{Z}$  and  $u \in A^\times$ , and the valuation map  $\text{ord} : A_0 \rightarrow \mathbb{Z} \cup \{\infty\}$  defined by  $\text{ord}(u\pi^n) = n$  makes  $A$  a discrete valuation monoid.  $\square$

**Corollary 2.6.** *If  $\mathfrak{p}$  is a height one prime ideal of a noetherian normal monoid  $A$  then  $A_{\mathfrak{p}}$  is a discrete valuation monoid (DV monoid).*

*Proof.* The monoid  $A_{\mathfrak{p}}$  is one dimensional and normal by Lemma 2.3. Now use Proposition 2.5.  $\square$

**Lemma 2.7.** *If  $A$  is a noetherian normal monoid, and  $\mathfrak{p}$  is a prime ideal associated to a principal ideal, then  $\mathfrak{p}$  has height one and  $\mathfrak{p}_{\mathfrak{p}}$  is a principal ideal of  $A_{\mathfrak{p}}$ .*

*Proof.* Let  $a \in A$  and  $\mathfrak{p}$  a prime ideal associated to  $aA$  so that by Proposition 1.7,  $\mathfrak{p} = (a : b)$  for some  $b \in A \setminus aA$ . To show  $\mathfrak{p}_{\mathfrak{p}} \subseteq A_{\mathfrak{p}}$  is principal, we may first localize and assume that  $A$  has maximal ideal  $\mathfrak{p}$ . Let  $\mathfrak{p}^{-1} = \{u \in A_0 \mid u\mathfrak{p} \subseteq A\}$ . Since  $A \subseteq \mathfrak{p}^{-1}$ , we have  $\mathfrak{p} \subseteq \mathfrak{p}^{-1}\mathfrak{p} \subseteq A$ , and since  $\mathfrak{p}$  is maximal, we must have  $\mathfrak{p}^{-1}\mathfrak{p} = \mathfrak{p}$  or  $\mathfrak{p}^{-1}\mathfrak{p} = A$ .

If  $\mathfrak{p}^{-1}\mathfrak{p} = \mathfrak{p}$ , every element of  $\mathfrak{p}^{-1}$  must be integral over  $A$  by Lemma 2.4. Since  $A$  is normal,  $\mathfrak{p}^{-1} \subseteq A$ , hence  $\mathfrak{p}^{-1} = A$  and  $\mathfrak{p}b \subseteq aA$  implies  $b/a \in \mathfrak{p}^{-1} = A$ . This is only the case if  $b \in aA$ , since  $a$  is not a unit, contradicting the assumption. Therefore  $\mathfrak{p}^{-1}\mathfrak{p} = A$  and there exists  $u \in \mathfrak{p}^{-1}$  with  $u\mathfrak{p} = A$ , namely  $\mathfrak{p} = u^{-1}A$ .  $\square$

To finish the section we show that any noetherian normal monoid is the intersection of its localizations at height one primes. As with Corollary 2.6, this result parallels the situation in commutative rings.

**Theorem 2.8.** *A noetherian normal monoid  $A$  is the intersection of the  $A_{\mathfrak{p}}$  as  $\mathfrak{p}$  runs over all height one primes of  $A$ .*

*Proof.* That  $A$  is contained in the intersection is clear. Now, suppose  $a/b \in A_0 \setminus A$  so that  $a \notin bA$ . Any  $\mathfrak{p} \in \text{Ass}(b)$  has  $\mathfrak{p}_{\mathfrak{p}}$  principal by Lemma 2.7, hence height one, and  $a/b \notin A_{\mathfrak{p}}$  when  $a \notin bA_{\mathfrak{p}}$ . Therefore to find an associated prime  $\mathfrak{p}$  of  $bA$  with  $a \notin bA_{\mathfrak{p}}$  will complete the proof. But this is easy since  $a \in bA_{\mathfrak{p}}$  for every  $\mathfrak{p} \in \text{Ass}(b)$  if and only if  $a = 0$  in  $A_{\mathfrak{p}}/bA_{\mathfrak{p}} = (A/bA)_{\mathfrak{p}}$  for every  $\mathfrak{p} \in \text{Ass}(b)$ , which happens if and only if  $a = 0$  in  $A/bA$  by Lemma 1.8, which happens if and only if  $a \in bA$ . Since  $a \neq 0$ , such a prime must exist.  $\square$

### 3. NORMALIZATION

If  $A$  is a cancellative monoid, its normalization is the integral closure of  $A$  in its group completion  $A_0$ . In contrast, consider the problem of defining the normalization of a non-cancellative monoid  $A$ , which should be something which has a kind of universal property for morphisms  $A \rightarrow B$  with  $B$  normal.

We will restrict ourselves to the case when the monoid  $A$  is *partially cancellative* (or *pc*), i.e., a quotient  $A = C/I$  of a cancellative monoid  $C$  ([3, 1.3, 1.20]). One advantage is that  $A/\mathfrak{p}$  is cancellative for every prime ideal  $\mathfrak{p}$  of a pc monoid, and the normalization  $(A/\mathfrak{p})_{\text{nor}}$  of  $A/\mathfrak{p}$  exists.

**Lemma 3.1.** *If  $A$  is a pc monoid and  $f : A \rightarrow B$  is a morphism with  $B$  normal, then  $f$  factors through the normalization of  $A/\mathfrak{p}$ , where  $\mathfrak{p} = \ker(f)$ .*

*Proof.* The morphism  $A/\mathfrak{p} \rightarrow B$  of cancellative monoids induces a homomorphism  $f_0 : (A/\mathfrak{p})_0 \rightarrow B_0$  of their group completions. If  $a \in (A/\mathfrak{p})_0$  belongs to  $(A/\mathfrak{p})_{\text{nor}}$  then there is an  $n$  so that  $a^n \in A/\mathfrak{p}$ . Then  $b = f_0(a) \in B_0$  satisfies  $b^n \in B$ , so  $b \in B$ . Thus  $f_0$  restricts to a map  $(A/\mathfrak{p})_{\text{nor}} \rightarrow B$ .  $\square$

**Remark 3.1.1.** The non-pc monoid  $A = \langle x, y, z | xz = yz \rangle$  is non-cancellative, reduced (1.3.1) and even seminormal, yet has no obvious notion of normalization since  $0$  is a prime ideal. This shows the usefulness of restricting to pc monoids.

Thus the collection of maps  $A \rightarrow (A/\mathfrak{p})_{\text{nor}}$  has a kind of universal property. However, a strict universal property is not possible within the category of monoids because monoids are local. This is illustrated by the monoid  $A = \langle x_1, x_2 | x_1x_2 = 0 \rangle$ ; see Example 3.5 below. Following the example of algebraic geometry, we will pass to the category of (pc) monoid schemes, where the normalization exists.

**Definition 3.2.** Let  $A$  be a pc monoid. The *normalization*  $X_{\text{nor}}$  of  $X = \text{MSpec}(A)$  is the disjoint union of the monoid schemes  $\text{MSpec}((A/\mathfrak{p})_{\text{nor}})$  as  $\mathfrak{p}$  runs over the minimal primes of  $A$ . By abuse of notation, we will refer to  $X_{\text{nor}}$  as the normalization of  $A$ .

This notion is stable under localization: the normalization of  $U = \text{MSpec}(A[1/s])$  is an open subscheme of the normalization of  $\text{MSpec}(A)$ ; by Lemma 2.3, its components are  $\text{MSpec}$  of the normalizations of the  $(A/\mathfrak{p})[1/s]$  for those minimal primes  $\mathfrak{p}$  of  $A$  not containing  $s$ .

If  $X$  is a pc monoid scheme, covered by affine opens  $U_i$ , one can glue the normalizations  $\tilde{U}_i$  to obtain a normal monoid scheme  $X_{\text{nor}}$ , called the *normalization* of  $X$ .

**Remark 3.3.** The normalization  $X_{\text{nor}}$  is a normal monoid scheme: the stalks of  $\mathcal{A}_{\text{nor}}$  are normal monoids. It has the universal property that for every connected normal monoid scheme  $Z$ , every  $Z \rightarrow X$  dominant on a component factors uniquely through  $X_{\text{nor}} \rightarrow X$ . As this is exactly like [7, Ex. II.3.8], we omit the details.

Recall that the (categorical) product  $A \times B$  of two pointed monoids is the set-theoretic product with slotwise product and basepoint  $(0, 0)$ .

**Lemma 3.4.** *Let  $A$  be a pc monoid. The monoid of global sections  $H^0(X_{\text{nor}}, \mathcal{A}_{\text{nor}})$  of the normalization of  $A$  is the product of the pointed monoids  $(A/\mathfrak{p})_{\text{nor}}$  as  $\mathfrak{p}$  runs over the minimal primes of  $A$ .*

*Proof.* For any sheaf  $\mathcal{F}$  on a disjoint union  $X = \coprod X_i$ ,  $H^0(X, \mathcal{F}) = \prod H^0(X_i, \mathcal{F})$  by the sheaf axiom.  $\square$

**Example 3.5.** The normalization of  $A = \langle x_1, x_2 | x_1 x_2 = 0 \rangle$  is the disjoint union of the affine lines  $\langle x_i \rangle$ . The monoid of its global sections is  $\langle x_1 \rangle \times \langle x_2 \rangle$ , and is generated by  $(1, 0), (0, 1), (x_1, 1), (1, x_2)$ .

#### *Seminormalization*

Recall from [3, 1.7] that a reduced monoid  $A$  is *seminormal* if whenever  $b, c \in A$  satisfy  $b^3 = c^2$  there is an  $a \in A$  such that  $a^2 = b$  and  $a^3 = c$ . Any normal monoid is seminormal, and  $\langle x, y | xy = 0 \rangle$  is seminormal but not normal. The passage from monoids to seminormal monoids (and monoid schemes) was critical in [3] for understanding the behaviour of cyclic bar constructions under the resolution of singularities of a pc monoid scheme.

The *seminormalization* of a monoid  $A$  is a seminormal monoid  $A_{\text{sn}}$ , together with an injective map  $A_{\text{red}} \rightarrow A_{\text{sn}}$  such that every  $b \in A_{\text{sn}}$  has  $b^n \in A_{\text{red}}$  for all  $n \gg 0$ . It is unique up to isomorphism, and any monoid map  $A \rightarrow C$  with  $C$  seminormal factors uniquely through  $A_{\text{sn}}$ ; see [3, 1.11]. In particular, the seminormalization of  $A$  lies between  $A$  and its normalization, i.e.,  $\text{MSpec}(A)_{\text{nor}} \rightarrow \text{MSpec}(A)$  factors through  $\text{MSpec}(A_{\text{sn}})$ .

We shall restrict ourselves to the seminormalization of pc monoids (and monoid schemes). By [3, 1.15], if  $A$  is a pc monoid, the seminormalization of  $A$  exists and is a pc monoid. When  $A$  is cancellative,  $A_{\text{sn}}$  is easy to construct.

**Example 3.6.** When  $A$  is cancellative,  $A_{\text{sn}} = \{b \in A_0 : b^n \in A \text{ for } n \gg 0\}$ ; this is a submonoid of  $A_{\text{nor}}$ , and  $A_{\text{nor}} = (A_{\text{sn}})_{\text{nor}}$ . Since the normalization of a cancellative monoid induces a homeomorphism on the topological spaces  $\text{MSpec}$  [2, 1.6.1], so does the seminormalization.

If  $A$  has more than one minimal prime, then  $\text{MSpec}(A)_{\text{nor}} \rightarrow \text{MSpec}(A)$  cannot be a bijection. However, we do have the following result.

**Lemma 3.7.** *For every pc monoid  $A$ ,  $\text{MSpec}(A_{\text{sn}}) \rightarrow \text{MSpec}(A)$  is a homeomorphism of the underlying topological spaces.*

*Proof.* Write  $A = C/I$  for a cancellative monoid  $C$ , so  $\text{MSpec}(A)$  is the closed subspace of  $\text{MSpec}(C)$  defined by  $I$ . By [3, 1.14],  $A_{\text{sn}} = C_{\text{sn}}/(IC_{\text{sn}})$ . Thus  $\text{MSpec}(A_{\text{sn}})$  is the closed subspace of  $\text{MSpec}(C_{\text{sn}})$  defined by  $I$ . Since  $\text{MSpec}(C_{\text{sn}}) \rightarrow \text{MSpec}(C)$  is a homeomorphism (by 3.6), the result follows.  $\square$

The seminormalization of any pc monoid scheme exists and has a universal property (see [3, 1.21]). It may be constructed by glueing, since the seminormalization of  $A$  commutes with localization [3, 1.13]. Thus if  $X$  is a pc monoid scheme then there are canonical maps

$$X_{\text{nor}} \rightarrow X_{\text{sn}} \rightarrow X_{\text{red}} \rightarrow X,$$

and  $X_{\text{sn}} \rightarrow X$  is a homeomorphism by Lemma 3.7. We will return to this notion in Proposition 7.1.

#### 4. WEIL DIVISORS

Although the theory of Weil divisors is already interesting for normal monoids, it is useful to state it for normal monoid schemes.

Let  $X$  be a normal monoid scheme with generic monoid  $A_0$ . Corollary 2.6 states that the stalk  $\mathcal{A}_x$  is a DV monoid for every height one point  $x$  of  $X$ . When  $X$  is separated, a discrete valuation on  $A_0$  uniquely determines a point  $x$  [2, 8.9].

By a *Weil divisor* on  $X$  we mean an element of the free abelian group  $\text{Div}(X)$  generated by the height one points of  $X$ . We define the divisor of  $a \in A_0^\times$  to be the sum, taken over all height one points of  $A$ :

$$\text{div}(a) = \sum_x v_x(a)x.$$

When  $A$  is of finite type, there are only finitely many prime ideals in  $A$ , so this is a finite sum. Divisors of the form  $\text{div}(a)$  are called *principal divisors*. Since  $v_x(ab) = v_x(a) + v_x(b)$ , the function  $\text{div} : A_0^\times \rightarrow \text{Div}(X)$  is a group homomorphism, and the principal divisors form a subgroup of  $\text{Div}(X)$ .

**Definition 4.1.** The *Weil divisor class group* of  $X$ , written as  $\text{Cl}(X)$ , is the quotient of  $\text{Div}(X)$  by the subgroup of principal divisors.

**Lemma 4.2.** *If  $X$  is a normal monoid scheme of finite type, there is an exact sequence*

$$1 \rightarrow \mathcal{A}(X)^\times \rightarrow A_0^\times \xrightarrow{\text{div}} \text{Div}(X) \rightarrow \text{Cl}(X) \rightarrow 0.$$

*Proof.* We may suppose that  $X$  is connected. It suffices to show that if  $a \in A_0^\times$  has  $\text{div}(a) = 0$  then  $a \in \mathcal{A}(X)^\times$ . This follows from Theorem 2.8: when  $X = \text{MSpec}(A)$ ,  $A$  is the intersection of the  $A_x$ .  $\square$

**Example 4.2.1.** (Cf. [7, II.6.5.2]) Let  $A$  be the submonoid of  $\mathbb{Z}_*^2$  generated by  $x = (1, 0)$ ,  $y = (1, 2)$  and  $z = (1, 1)$ , and set  $X = \text{MSpec}(A)$ . (This is the toric monoid scheme  $xy = z^2$ .) Then  $A$  has exactly two prime ideals of height one:  $p_1 = (x, z)$  and  $p_2 = (y, z)$ . Since  $\text{div}(x) = 2p_1$  and  $\text{div}(z) = p_1 + p_2$ , we see that  $\text{Cl}(X) = \mathbb{Z}/2$ .

**Example 4.2.2.** If  $X$  is the non-separated monoid scheme obtained by gluing together  $n + 1$  copies of  $\mathbb{A}^1$  along the common (open) generic point, then  $\text{Cl}(X) = \mathbb{Z}^n$ , as we see from Lemma 4.2.

If  $U$  is an open subscheme of  $X$ , with complement  $Z$ , the standard argument [7, II.6.5] shows that there is a surjection  $\text{Cl}(X) \rightarrow \text{Cl}(U)$ , that it is an isomorphism

if  $Z$  has codimension  $\geq 2$ , and that if  $Z$  is the closure of a height one point  $z$  then there is an exact sequence

$$\mathbb{Z} \xrightarrow{z} \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0.$$

**Proposition 4.3.**  $\text{Cl}(X_1 \times X_2) = \text{Cl}(X_1) \oplus \text{Cl}(X_2)$ .

*Proof.* By [2, 3.1], the product monoid scheme exists, and its underlying topological space is the product. Thus a codimension one point of  $X_1 \times X_2$  is either of the form  $x_1 \times X_2$  or  $X_1 \times x_2$ . Hence  $\text{Div}(X_1 \times X_2) \cong \text{Div}(X_1) \oplus \text{Div}(X_2)$ . It follows from Lemma 2.2 that  $X_1 \times X_2$  is normal, and the pointed monoid at its generic point is the smash product of the pointed monoids  $A_1$  and  $A_2$  of  $X_1$  and  $X_2$  at their generic points. If  $a_i \in A_i$  then the principal divisor of  $a_1 \wedge a_2$  is  $\text{div}(a_1) + \text{div}(a_2)$ . Thus

$$\text{Cl}(X_1 \times X_2) = \frac{\text{Div}(X_1) \oplus \text{Div}(X_2)}{\text{div}(A_1) \oplus \text{div}(A_2)} \cong \text{Cl}(X_1) \oplus \text{Cl}(X_2). \quad \square$$

**Example 4.4.** By [2, 4.5], any connected separated normal monoid scheme  $X$  decomposes as the product of a toric monoid scheme  $X_\Delta$  and  $\text{MSpec}(U_*)$  for some finite abelian group  $U$ . ( $U$  is the group of units of  $X$ .) Since  $U_*$  has no height one primes,  $\text{Div}(X) = \text{Div}(X_\Delta)$  and the Weil class group of  $X$  is  $\text{Cl}(X_\Delta)$ , the Weil class group of the associated toric monoid scheme.

By construction [2, 4.2], the points of  $X_\Delta$  correspond to the cones of the fan  $\Delta$  and the height one points of  $X_\Delta$  correspond to the edges in the fan. Thus our Weil divisors correspond naturally to what Fulton calls a “ $T$ -Weil divisor” on the associated toric variety  $X_k$  (over a field  $k$ ) in [4, 3.3]. Since the group completion  $A_0$  is the free abelian group  $M$  associated to  $\Delta$ , it follows from [4, 3.4] that our Weil divisor class group  $\text{Cl}(X_\Delta)$  is isomorphic to the Weil divisor class group  $\text{Cl}(X_k)$  of associated toric variety.

## 5. INVERTIBLE SHEAVES

Let  $X$  be a monoid scheme with structure sheaf  $\mathcal{A}$ . An *invertible sheaf* on  $X$  is a sheaf  $\mathcal{L}$  of  $\mathcal{A}$ -sets which is locally isomorphic to  $\mathcal{A}$  in the Zariski topology. If  $\mathcal{L}_1, \mathcal{L}_2$  are invertible sheaves, their smash product is the sheafification of the presheaf  $U \mapsto \mathcal{L}_1(U) \wedge_{\mathcal{A}(U)} \mathcal{L}_2(U)$ ; it is again an invertible sheaf. Similarly,  $\mathcal{L}^{-1}$  is the sheafification of  $U \mapsto \text{Hom}_{\mathcal{A}}(\mathcal{L}(U), \mathcal{A}(U))$ , and evaluation  $\mathcal{L} \wedge_{\mathcal{A}} \mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{A}$  is an isomorphism. Thus the set of isomorphism classes of invertible sheaves on  $X$  is a group under the smash product.

**Definition 5.1.** The *Picard group*  $\text{Pic}(X)$  is the group of isomorphism classes of invertible sheaves on  $X$ .

Since a monoid  $A$  has a unique maximal ideal (the non-units), an invertible sheaf on  $\text{MSpec}(A)$  is just an  $A$ -set isomorphic to  $A$ . This proves:

**Lemma 5.2.** *For every affine monoid scheme  $X = \text{MSpec}(A)$ ,  $\text{Pic}(X) = 0$ .*

For any monoid  $A$ , the group of  $A$ -set automorphisms of  $A$  is canonically isomorphic to  $A^\times$ . Since the subsheaf  $\Gamma$  of generators of an invertible sheaf  $\mathcal{L}$  is a torsor for  $\mathcal{A}^\times$ , and  $\mathcal{L} = \mathcal{A} \wedge_{\mathcal{A}^\times} \Gamma$ , this proves:

**Lemma 5.3.**  $\text{Pic}(X) \cong H^1(X, \mathcal{A}^\times)$ .

Recall that a morphism  $f : Y \rightarrow X$  of monoid schemes is *affine* if  $f^{-1}(U)$  is affine for every affine open  $U$  in  $X$ ; see [2, 6.2].

**Proposition 5.4.** *If  $f : Y \rightarrow X$  is an affine morphism of monoid schemes, then the direct image  $f_*$  is an exact functor from sheaves (of abelian groups) on  $Y$  to sheaves on  $X$ . In particular,  $H^*(Y, \mathcal{L}) \cong H^*(X, p_*\mathcal{L})$  for every sheaf  $\mathcal{L}$  on  $Y$ .*

*Proof.* Suppose that  $0 \rightarrow \mathcal{L}' \rightarrow \mathcal{L} \rightarrow \mathcal{L}'' \rightarrow 0$  is an exact sequence of sheaves on  $Y$ . Fix an affine open  $U = \text{MSpec}(A)$  of  $X$  with closed point  $x \in X$ . Then  $f^{-1}(U) = \text{MSpec}(B)$  for some monoid  $B$ . If  $y \in Y$  is the unique closed point of  $\text{MSpec}(B)$  the stalk sequence  $0 \rightarrow \mathcal{L}'_y \rightarrow \mathcal{L}_y \rightarrow \mathcal{L}''_y \rightarrow 0$  is exact. Since this is the stalk sequence at  $x$  of  $0 \rightarrow f_*\mathcal{L}' \rightarrow f_*\mathcal{L} \rightarrow f_*\mathcal{L}'' \rightarrow 0$ , the direct image sequence is exact.  $\square$

Here is an application, showing one way in which monoid schemes differ from schemes. Let  $T$  denote the free (pointed) monoid on generator  $t$ , and let  $\mathbb{A}^1$  denote  $\text{MSpec}(T)$ . Then  $A \wedge T$  is the analogue of a polynomial ring over  $A$ , and  $X \times \mathbb{A}^1$  is the monoid scheme which is locally  $\text{MSpec}(A) \times \mathbb{A}^1 = \text{MSpec}(A \wedge T)$ ; see [2, 3.1]. Thus  $p : X \times \mathbb{A}^1 \rightarrow X$  is affine, and  $f_*\mathcal{A}_Y^\times = \mathcal{A}_X^\times$ . From Proposition 5.4 we deduce

**Corollary 5.5.** *For every monoid scheme  $X$ ,  $\text{Pic}(X) \cong \text{Pic}(X \times \mathbb{A}^1)$ .*

## 6. CARTIER DIVISORS

Let  $(X, \mathcal{A})$  be a cancellative monoid scheme. We write  $A_0$  for the stalk of  $\mathcal{A}$  at the generic point of  $X$ , and  $\mathcal{A}_0$  for the associated constant sheaf. A *Cartier divisor* on  $X$  is a global section of the sheaf of groups  $\mathcal{A}_0^\times/\mathcal{A}^\times$ . On each affine open  $U$ , it is given by an  $a_U \in A_0^\times$  up to a unit in  $\mathcal{A}(U)^\times$ , and we have the usual representation as  $\{(U, a_U)\}$  with  $a_U/a_V$  in  $\mathcal{A}(U \cap V)^\times$ . We write  $\text{Cart}(X)$  for the group of Cartier divisors on  $X$ . The *principal* Cartier divisors, i.e., those represented by some  $a \in A_0^\times$ , form a subgroup of  $\text{Cart}(X)$ .

**Proposition 6.1.** *Let  $X$  be a cancellative monoid scheme. Then the map  $D \mapsto \mathcal{L}(D)$  defines an isomorphism between the group of Cartier divisors modulo principal divisors and  $\text{Pic}(X)$ .*

*Proof.* Consider the short exact sequence of sheaves of abelian groups

$$1 \rightarrow \mathcal{A}^\times \rightarrow \mathcal{A}_0^\times \rightarrow \mathcal{A}_0^\times/\mathcal{A}^\times \rightarrow 1.$$

Since  $\mathcal{A}_0^\times$  is constant and  $X$  is irreducible we have  $H^1(X, \mathcal{A}_0^\times) = 0$  [7, III.2.5]. By Lemma 5.3, the cohomology sequence becomes:

$$0 \rightarrow \mathcal{A}(X)^\times \rightarrow \mathcal{A}_0^\times \xrightarrow{\text{div}} \text{Cart}(X) \xrightarrow{\delta} \text{Pic}(X) \rightarrow 0. \quad \square$$

**Example 6.2.** If  $D$  is a Cartier divisor on a cancellative monoid scheme  $X$ , represented by  $\{(U, a_U)\}$ , we define a subsheaf  $\mathcal{L}(D)$  of the constant sheaf  $\mathcal{A}_0$  by letting its restriction to  $U$  be generated by  $a_U^{-1}$ . This is well defined because  $a_U^{-1}$  and  $a_V^{-1}$  generate the same subsheaf on  $U \cap V$ . The usual argument [7, II.6.13] shows that  $D \mapsto \mathcal{L}(D)$  defines an isomorphism from  $\text{Cart}(X)$  to the group of invertible subsheaves of  $\mathcal{A}_0^\times$ . By inspection, the map  $\delta$  in 6.1 sends  $D$  to  $\mathcal{L}(D)$ .

**Lemma 6.3.** *If  $X$  is a normal monoid scheme of finite type,  $\text{Pic}(X)$  is a subgroup of  $\text{Cl}(X)$ .*

*Proof.* Every Cartier divisor  $D = \{(U, a_U)\}$  determines a Weil divisor; the restriction of  $D$  to  $U$  is the divisor of  $a_U$ . It is easy to see that this makes the Cartier divisors into a subgroup of the Weil divisor class group  $D(X)$ , under which principal Cartier divisors are identified with principal Weil divisors. This proves the result.  $\square$

**Theorem 6.4.** *Let  $X$  be a separated connected monoid scheme. If  $X$  is locally factorial then every Weil divisor is a Cartier divisor, and  $\text{Pic}(X) = \text{Cl}(X)$ .*

*Proof.* By Example 2.1,  $X$  is normal since factorial monoids are normal. Thus  $\text{Pic}(X)$  is a subgroup of  $\text{Cl}(X)$ , and it suffices to show that every Weil divisor  $D = \sum n_i x_i$  is a Cartier divisor. For each affine open  $U$ , and each point  $x_i$  in  $U$ , let  $p_i$  be the generator of the prime ideals associated to  $x_i$ ; then the divisor of  $a_U = \prod p_i^{n_i}$  is the restriction of  $D$  to  $U$ , and  $D = \{(U, a_U)\}$ .  $\square$

**Lemma 6.5.** *For the projective space monoid scheme  $\mathbb{P}^n$  we have*

$$\text{Pic}(\mathbb{P}^n) = \text{Cl}(\mathbb{P}^n) = \mathbb{Z}.$$

**Remark 6.5.1.** This calculation formed the starting point of our investigation. We learned it from Vezzani (personal communication).

*Proof.* Since  $\mathbb{P}^n$  is locally factorial,  $\text{Pic}(\mathbb{P}^n) = \text{Cl}(\mathbb{P}^n)$ . By definition,  $\mathbb{P}^n$  is  $\text{MProj}$  of the free abelian monoid on  $\{x_0, \dots, x_n\}$ , and  $A_0$  is the free abelian group with the  $x_i/x_0$  as basis ( $i = 1, \dots, n$ ). On the other hand,  $\text{Div}(\mathbb{P}^n)$  is the free abelian group on the generic points  $[x_i]$  of the  $V(x_i)$ . Since  $\text{div}(x_i/x_0) = [x_i] - [x_0]$ , the result follows.  $\square$

Let  $\Delta$  be a fan,  $X$  the toric monoid scheme associated to  $\Delta$  by [2, 4.2], and  $X_k$  the usual toric variety associated to  $\Delta$  over some field  $k$ . ( $X_k$  is the  $k$ -realization  $X_k$  of  $X$ .) As pointed out in Example 4.4, our Weil divisors correspond to the  $T$ -Weil divisors of the toric variety  $X_k$  and  $\text{Cl}(X) \cong \text{Cl}(X_k)$ . Moreover, our Cartier divisors on  $X$  correspond to the  $T$ -Cartier divisors of [4, 3.3]). Given this dictionary, the following result is established by Fulton in [4, 3.4].

**Theorem 6.6.** *Let  $X$  and  $X_k$  denote the toric monoid scheme and toric variety (over  $k$ ) associated to a given fan. Then  $\text{Pic}(X) \cong \text{Pic}(X_k)$ .*

*Moreover,  $\text{Pic}(X)$  is free abelian if  $\Delta$  contains a cone of maximal dimension.*

## 7. Pic OF PC MONOID SCHEMES

In this section, we derive some results about the Picard group of pc monoid schemes. When  $X$  is a pc monoid scheme, we can form the reduced monoid scheme  $X_{\text{red}} = (X, \mathcal{A}_{\text{red}})$  using Remark 1.3.1: the stalk of  $\mathcal{A}_{\text{red}}$  at  $x$  is  $\mathcal{A}_x/\text{nil}(\mathcal{A}_x)$ . Since  $\mathcal{A}^\times = \mathcal{A}_{\text{red}}^\times$ , the map  $X_{\text{red}} \rightarrow X$  induces an isomorphism  $\text{Pic}(X) \cong \text{Pic}(X_{\text{red}})$ .

We will use the constructions of normalization and seminormalization given in Section 3.

**Proposition 7.1.** *If  $X$  is a pc monoid scheme, the canonical map  $X_{\text{sn}} \rightarrow X$  induces an isomorphism  $\text{Pic}(X) \cong \text{Pic}(X_{\text{sn}})$ .*

*Proof.* Since  $X_{\text{red}}$  and  $X$  have the same underlying space, it suffices by Lemma 5.3 to assume that  $X$  is reduced and show that the inclusion  $\mathcal{A}^\times \rightarrow \mathcal{A}_{\text{sn}}^\times$  is an isomorphism. It suffices to work stalkwise, so we are reduced to showing that if  $A$  is reduced then  $A^\times \rightarrow A_{\text{sn}}^\times$  is an isomorphism. If  $b \in A_{\text{sn}}^\times$  then both  $b^n$  and  $(1/b)^n$  are in  $A$  for large  $n$ , and hence both  $b = b^{n+1}b^{-n}$  and  $b^{-1} = b^n(1/b)^{1+n}$  are in  $A$ , so  $b \in A^\times$ .  $\square$

**Lemma 7.2.** *Let  $X$  be a cancellative seminormal monoid scheme and  $p : X_{\text{nor}} \rightarrow X$  its normalization. If  $\mathcal{H}$  denotes the sheaf  $p_*(\mathcal{A}_{\text{nor}}^\times)/\mathcal{A}^\times$  on  $X$ , there is an exact sequence*

$$1 \rightarrow \mathcal{A}(X)^\times \rightarrow \mathcal{A}_{\text{nor}}(X_{\text{nor}})^\times \rightarrow H^0(X, \mathcal{H}) \rightarrow \text{Pic}(X) \xrightarrow{p^*} \text{Pic}(X_{\text{nor}}) \rightarrow H^1(X, \mathcal{H}).$$

*Proof.* At each point  $x \in X$ , the stalk  $A = \mathcal{A}_x$  is a submonoid of its normalization  $A_{\text{nor}} = p_*(\mathcal{A}_{\text{nor}})_x$  (by Lemma 2.3) and we have an exact sequence of sheaves on  $X$ :

$$1 \rightarrow \mathcal{A}^\times \rightarrow p_*(\mathcal{A}_{\text{nor}}^\times) \rightarrow \mathcal{H} \rightarrow 1.$$

Since  $p$  is affine, Proposition 5.4 implies that  $\mathcal{A}_{\text{nor}}(X_{\text{nor}})^\times = H^0(X, p_*\mathcal{A}_{\text{nor}}^\times)$  and  $\text{Pic}(X_{\text{nor}}) = H^1(X, p_*\mathcal{A}_{\text{nor}}^\times)$ , and the associated cohomology sequence is the displayed sequence.  $\square$

Here are two examples showing that  $\text{Pic}(X) \rightarrow \text{Pic}(X_{\text{nor}})$  need not be an isomorphism when  $X$  is seminormal and cancellative.

**Example 7.3.** Let  $A_+$  (resp.,  $A_-$ ) be the submonoid of the free monoid  $B = \langle x, y \rangle$  generated by  $\{x, y^2, xy\}$  (resp.,  $\{x, y^{-2}, xy^{-1}\}$ ). These are seminormal but not normal. If  $X$  is the monoid scheme obtained by gluing the  $U_{\pm} = \text{MSpec}(A_{\pm})$  together along  $\text{MSpec}(\langle x, y^2, y^{-2} \rangle)$  then it is easy to see that  $\text{Pic}(X) = \mathbb{Z}$ , with a generator represented by  $(U_+, y^2)$  and  $(U_-, 1)$ . The normalization  $X_{\text{nor}}$  is the toric monoid scheme  $\mathbb{A}^1 \times \mathbb{P}^1$ , and  $\text{Pic}(X_{\text{nor}}) \cong \mathbb{Z}$ , with a generator represented by  $(U_+, y)$  and  $(U_-, 1)$ . Thus  $\text{Pic}(X) \rightarrow \text{Pic}(X_{\text{nor}})$  is an injection with cokernel  $\mathbb{Z}/2$ .

**Example 7.4.** Let  $U$  be an abelian group and  $A_x$  the submonoid of  $B = U_* \wedge \langle x \rangle$  consisting of  $0, 1$  and all terms  $ux^n$  with  $u \in U$  and  $n > 0$ . Then  $A_x$  is seminormal and  $B$  is its normalization. Let  $X$  be obtained by gluing  $\text{MSpec}(A_x)$  and  $\text{MSpec}(A_{1/x})$  together along their common generic point,  $\text{MSpec}(U_* \wedge \langle x, 1/x \rangle)$ . The normalization of  $X$  is  $X_{\text{nor}} = \text{MSpec}(U_*) \times \mathbb{P}^1$ , and  $\text{Pic}(X_{\text{nor}}) = \mathbb{Z}$  by Example 4.4 and Lemma 6.5. Because  $p_*(\mathcal{A}_{\text{nor}}^\times)/\mathcal{A}^\times$  is a skyscraper sheaf with stalk  $U$  at the two closed points, we see from Lemma 7.2 that  $\text{Pic}(X) = \mathbb{Z} \times U$ . Thus  $\text{Pic}(X) \rightarrow \text{Pic}(X_{\text{nor}})$  is a surjection with kernel  $U$ .

Finally, we consider the case when  $X$  is reduced pc monoid scheme which is not cancellative. We may suppose that  $X$  is of finite type, so that the stalk at a closed point is an affine open  $\text{MSpec}(A)$  with minimal points  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ ,  $r > 1$ . Then the closure  $X'$  of  $\mathfrak{p}_1$  is a cancellative seminormal monoid scheme. Let  $X''$  denote the closure of the remaining minimal points of  $X$ , and set  $X''' = X' \cap X''$ . Then we have the exact Mayer-Vietoris sequence of sheaves on  $X$ :

$$0 \rightarrow \mathcal{A}_X \rightarrow p'_* \mathcal{A}_{X'} \times p''_* \mathcal{A}_{X''} \rightarrow p'''_* \mathcal{A}_{X'''} \rightarrow 1.$$

Because the immersions are affine, Proposition 5.4 yields the exact sequence (7.5)

$$1 \rightarrow \mathcal{A}(X)^\times \rightarrow \mathcal{A}_{X'}^\times \times \mathcal{A}_{X''}^\times \rightarrow \mathcal{A}_{X'''}^\times \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(X') \times \text{Pic}(X'') \rightarrow \text{Pic}(X''').$$

The Picard group may then be determined by induction on  $r$  and  $\dim(X)$ .

**Example 7.6.** If  $X$  is obtained by gluing together  $X_1, \dots, X_n$  at a common generic point, then (7.5) yields  $\text{Pic}(X) = \oplus \text{Pic}(X_i)$ .

## REFERENCES

- [1] M. Atiyah and I. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.
- [2] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, *Toric Varieties, Monoid Schemes and cdh descent*, J. reine angew. Math. 678 (2013), to appear. arXiv:1106.1389.
- [3] G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, *The K-theory of toric varieties in positive characteristic*, J. Top., to appear. arXiv:1207.2891
- [4] W. Fulton, *Introduction to Toric varieties*, Annals of Math. Studies 13, Princeton Univ. Press, 1993.
- [5] R. Gilmer, *Commutative Semigroup Rings*, Univ. Chicago Press, 1984.
- [6] F. Halter-Koch, *Ideal systems*, Monographs in Pure Appl. Math. 211, Marcel Dekker, 1998.
- [7] R. Hartshorne, *Algebraic Geometry*, Springer Verlag, 1977.
- [8] U. Kobsa, *Kommutative Algebra für Monoide*, Diplomarbeit, Univ. Regensburg, 1996. Available at <http://www.kobsa.me.uk/Articles/Thesis.htm>
- [9] A. Vezzani, *Deitmar's versus Toën-Vaquié's schemes over  $\mathbb{F}_1$* , Math. Z. 271 (2012), 911–926.
- [10] C. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994.
- [11] C. Weibel, *The K-book, An Introduction to Algebraic K-theory*, Grad. Studies in Math. 145, AMS, 2013.

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