ELEMENTS OF ∞-CATEGORIES

CHARLES A. WEIBEL

As the title suggests, this book is about ∞-categories. Although this terminology has recently become mainstream in certain areas of math, including homotopy theory and algebraic geometry, it has a long history. In this review, we will discuss its evolution out of more familiar mathematical ideas.

1. Categories

The long journey to ∞-categories began with the introduction of Category Theory by Eilenberg and Mac Lane [7] in 1945. Here is a quick, elementary reminder:

Category theory is a language for talking about abstract mathematics. When one wants to talk about a collection of things, like topological spaces, differentiable manifolds or groups, it is important to clarify the maps between them. For topogical spaces, the natural kind of map is a continuous function; for diferentiable manifolds, the natural map is a "smooth" function; for groups the maps are group homomorphisms. In the jargon of category theory, the things are called "objects" and the maps are called"morphisms." There are of course rules: the composition of any two morphisms $X \to Y$ and $Y \to Z$ must be a morphism $X \to Z$, composition is associative and each object has an identity map. Higher category theory is concerned with the structure of composition of morphisms.

Up to set theory, there is also a category **Cat** whose objects are "small" categories and whose morphisms are functors. Here "small" means that the sets of objects and morphisms have cardinality less than some fixed regular cardinal κ.

In mathematics, we often consider an invariant $F(a)$ of an object a in a category A as an object of a category B. A good invariant will send a morphism $a \longrightarrow a'$ to a morphism $F(a) \to F(a')$ in B, respecting composition and identity maps. We say that such an F is a *functor* from A to B.

For example, the fundamental group $\pi_1(X)$ is a functor from pointed topological spaces to groups, and the abelianization $G/[G,G]$ of a group G is a functor from groups to the category **Ab** of abelian groups. Functors can be composed in useful ways; the composition of $\pi_1(X)$ and $G/[G,G]$ is the homology $H_1(X)$ of a space X.

$$\begin{array}{ccccc} X & \dashrightarrow & \pi_1(X) & \dashrightarrow & H_1(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi_1(f)} & & \downarrow{\scriptstyle H_1(f)} \\ Y & \dashrightarrow & \pi_1(Y) & \dashrightarrow & H_1(Y). \end{array}$$

Date: March 1, 2023.
Weibel was supported by NSF grant 2001417.

Another example is the *fundamental groupoid* of a topological space X. The objects are the points of X, and $\mathrm{Hom}(x, y)$ is the set of homotopy classes of continous paths $I \to X$ from x to y; composition of paths defines the composition law. The term "groupoid" implies that $\mathrm{Hom}(x, x)$ is a group for every $x \in X$. In fact, $\mathrm{Hom}(x, x)$ is the fundamental group of the based space (X, x).

My favorite functors are "forgetful" functors from spaces, groups, etc., to **Sets**. They take each object to their underlying set, and homomorphisms to the functions between these sets.

The language of category theory was instrumental in the development of homotopy theory, homological algebra and algebraic geometry during the 1950s and 1960s. Much of this development was due to Grothendieck [9] and Kan [14].

2. Topological categories

Many of the ideas of "higher category theory" began in the 1960s with the the notion of a category A being *enriched* over **Cat**. This just means that the set of morphisms in A from a to a' is replaced by a category $\mathbf{Fun}_A(a, a')$, i.e., an object in **Cat**; the underlying morphism set $\mathrm{Hom}_A(a, a')$ of the ordinary category A is the set of objects of the category $\mathbf{Fun}_A(a, a')$. Such an enriched category is also called a (strict) 2-*category*, or a *bicategory*. The morphisms in $\mathbf{Fun}_A(a, a')$ are called *natural transformations*, or 2-cells. (We leave the list of axioms to the reader.)

A category can be enriched over any complete and cocomplete monoidal closed category such as **Ab**, **Top**[1] or simplicial sets. Homological algebra deals with **Ab**-categories (categories enriched over **Ab**). Categories enriched over **Top** are called *topological categories*, and form the most geometric model of ∞-categories.

Unfortunately, topological categories are hard to work with, because diagrams tend to only commute up to homotopies, good compositions require replacing maps with homotopy equivalent fibrations, and the bookkeeping needed to keep track of the homotopy data can be unwieldly. So the story didn't end here.

Looking ahead, the *fundamental* ∞*-groupoid* of a topological space X has the same space of objects as the fundamental groupoid, but now the topological space $\mathrm{Hom}(x, y)$ is the topological space of paths from x to y in X, and there is an entire topological space of compositions of $\gamma_0 : x \to y$ and $\gamma_1 : y \to z$, parametrized by maps from the 2-simplex Δ^2 to X: one composition for each continuous map $\sigma : \Delta^2 \to X$, whose 3 boundary edges consist of γ_0, γ_1 and their σ-composition $\gamma_2 : x \to z$; the map σ is a homotopy between the concatenation $\gamma_1 \circ \gamma_0$ and the σ-composition γ_2. The fundamental ∞-groupoid also contains higher-dimensional information; the 2-dimensional homotopies have 3-dimensional homotopies between them, i.e., continuous maps from the 3-simplex Δ^3 to X, and so on; see [17, 1.1.1.4]. We shall skip the technical details, hoping that the reader gets the general idea.

3. Homotopy everything spaces

In the topology community of the post-World War II era, there was a recognition that the space ΩX of loops $S^1 \to X$ (for a space X with a specified basepoint) had a composition $S^1 \to S^1 \vee S^1 \to X$ which was not strictly associative, yet it had more structure than just being associative "up to homotopy" (the relation needed

[1] the category of compactly generated, weakly Hausdorff spaces

to define $\pi_1(X)$). This recognition arose gradually in the 1950's, with H-spaces and models for the classifying space BG of a group G being motivating examples.

Stasheff's 1961 thesis [27] defined a nested sequence of homotopy associative coherence conditions that a composition $X \times X \to X$ in a space X might satisfy; an A_2-space is an H-space, a space satisfying the first n conditions is called an A_n-space, and n-fold loop spaces are A_n-spaces. This gave a bookkeeping device for keeping track of the higher homotopies of associativity involved in composition.

Boardman and Vogt [4] extended Stasheff's techniques to define E_∞ structures on a space X; this implied that X is an infinite loop space. This notion led to the development of "infinite loop spectra" and even E_∞-spectra in the 1970s, by Adams, May [20], and others. These ideas were modified much later to define A_∞-algebras; see [15] for a survey.

Similar considerations were also making an appearance in homological algebra. Chain complexes and their homology are the analogues of topological spaces and their homotopy groups. But there was a problem: chain complexes carried too much irrelevant information, while homology lost too much information. The notion of chain homotopy was an interpolation between these extremes.

The notion of a *triangulated category*, due to Verdier [28], was a formalism that tried to capture this middle ground. For example, the stable homotopy classes of spectra form a triangulated category. Chain complexes in an abelian category A can be given either the chain homotopy relation or the "quasi-isomorphism" relation; the homotopy category with respect to chain homotopy is triangulated, as is the homotopy category with respect to quasi-isomorphisms; the latter is called the derived category of A, $D(A)$. (More generally, a "homological" functor on a triangulated category $\mathcal{C}$ induces a localization of $\mathcal{C}$, which is a new triangulated category that plays the role of the homotopy category of $\mathcal{C}$.)

Here too, there was a recognition that the derived category lost too much information for some applications. The notion of a "t-structure" [2] was one attempt to find a middle ground. The stable ∞-category of chain complexes is another candidate for such a middle ground.

4. Simplicial constructions

Simplicial sets were first defined in 1950 by Eilenberg–Zilber [8], and have played a major role in algebraic topology ever since. Here is a rapid tour of the subject.

Let Δ be the category whose objects are the finite ordered sets $[n] = \{0 < \cdots < n\}$ and whose morphisms are order-preserving. A *simplicial set* is a contravariant functor $K : \Delta \to \mathbf{Sets}$, and the elements of $K(n)$ are called its n-cells. The representable simplicial set $\Delta[n]$ is $\mathrm{Hom}(-, [n])$.

The category $\Delta^{\mathrm{op}}\mathbf{Sets}$ of simplicial sets is a good model for $\mathbf{Top}$; there is a realization functor from $\Delta^{\mathrm{op}}\mathbf{Sets}$ to $\mathbf{Top}$ sending $\Delta[n]$ to the n-simplex Δ^n. Its adjoint functor is the singular complex of a topological space, $[n] \mapsto \mathbf{Maps}(\Delta^n, X)$, which is used to define the singular homology of X.

G. Segal observed that every category A defines a simplicial set, called its *nerve*: the objects of A are the 0-cells, and the composable sequences $a_0 \to a_1 \to \cdots \to a_n$ are its n-cells. This observation formed the starting point of Segal's "infinite loop space machine."

The combinatorics of simplicial sets have given rise to a large dictionary of terminology. Here is some terminology which will be relevant to our review of ∞-categories.

The *k-horn* $\Lambda^k[n]$ of the n-simplex $\Delta[n]$ is the subcomplex obtained by removing the k-th face. It is called an *inner horn* if it is not the first or last horn of $\Delta[n]$.

A *Kan complex* is a simplicial set with the right lifting property against all horns. That is, every $f : \Lambda^k[n] \to K$ extends to a map $\Delta[n] \to K$. Kan complexes were first introduced in 1956 by Dan Kan [13]. The singular complex of a topological space is a Kan complex, and so is any simplicial group.

A simplicial map $p : E \to B$ is a (Kan) *fibration* if it has the appropriate lifting property against all horns: every map $f : \Lambda^k[n] \to E$, $k < n$, such that pf factors through a map $\Delta^n \to B$, factors through a map $\Delta[n] \to E$. There are weaker notions; for example p is an *inner fibration* if it has the lifting property against all inner horns.

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{f} & E \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow p \\ \Delta[n] & \longrightarrow & B \end{array}$$

One successful axiomatic approach to the axiomatization of homotopy theory, due to Quillen [21], was the notion of a *model category.* Such a category has limits, colimits, and three distinguished families of maps called cofibrations, fibrations, and weak equivalences. The homotopy category is obtained by modding out by weak equivalences.

There is a notion of a (left or right) *Quillen functor* between model categories which is useful in manipulating ∞-categories; see [24, App. C.3]. An adjunction between left and right Quillen functors is called a *Quillen adjunction*, and induces an adjunction of homotopy categories.

The classic application is to topology: simplical spaces have a Quillen model structure whose fibrations are Kan fibrations; the realization $\Delta^{\text{op}}\mathbf{Sets} \to \mathbf{Top}$ is a left Quillen functor and the singular complex is a right Quillen functor; together they form a Quillen adjunction and induce an equivalence of homotopy categories. Thus almost all of homotopy theory can be handled using simplicial techniques.

The following definition is relevant for ∞-categories.

Definition 4.1. An *isofibration* between quasi-categories is an inner fibration $E \overset{p}{\to} B$ such that for any e in E, a weak equivalence $p(e) \to b$ in B can be lifted to a weak equivalence $e \to e'$ in E; see [24, 1.1.17].

The prototype of an isofibration is a functor $p : E \to B$ between categories such that for every e in E, every isomorphism $p(e) \xrightarrow{\simeq} b$ in B lifts to an isomorphism $e \xrightarrow{\simeq} e'$ in E.

5. Quasi-categories

Definition 5.1. A *quasi-category* is a simplicial set K which has the right lifting property against all inner horns.

Quasi-categories are the primary working model of ∞-categories, as much of the technical manipulations of ∞-categories work well in that setting. In practice, many authors use the term "∞-category" to mean a quasi-category; see [17]. The definition used in the book under review is given in Definition 5.2 below.

In addition to Kan complexes, categories (or rather their nerves) are quasi-categories. Any 2-category also defines a quasi-category. Quasi-categories were first introduced by Boardman and Vogt [4], who called them "weak Kan complexes." Joyal [12] defined a Quillen model structure on the category of simplicial sets, one for which quasi-categories are the fibrant objects.

The category **QCat** of (small) quasi-categories is the main ∞-category analogue of **Cat**. For example, the analogue of the ordinary category of sets is the quasi-category $\mathcal{S}$ of *spaces*, which is the simplicial nerve of $\mathcal{K}an$, the subcategory of simplicial sets whose objects are Kan complexes. The analogue of the ordinary category of finite sets is the quasi-category $\mathcal{S}^{\text{fin}}$ of *finite spaces*, the smallest full sub-quasicategory of $\mathcal{S}$ containing the point and stable under finite colimits.

The *homotopy category* hA of a quasi-category A is the category with the same objects as A and homotopy classes of morphisms in A as its maps. An *isomorphism* in a quasi-category A is a 1-cell $f : a \to b$ which represents an isomorphism in hA. If A is (the nerve of) an ordinary category, then $A = hA$ and isomorphism has its usual meaning.

An ∞-category is *pointed* if it has an element '0' which is both initial and terminal. The existence of a point allows many familiar constructions to make sense, such as the adjunction of loops and supension functors (when they exist). All of the above examples have pointed versions.

One of the most useful kinds of ∞-category is a *stable* ∞-category. This is a pointed ∞-category which admits pullbacks and pushouts, and pullback squares coincide with pushout squares [16]. Stable ∞-categories have all finite limits and colimits, such as finite direct sums and products. In fact, if A is a stable ∞-category, then its homotopy category hA is a triangulated category; see [18, 1.12.9]. In addition, any stable ∞-category is canonically enriched over the ∞-category of spectra. Here are the most commonly used stable ∞-categories.

- There is an ∞-category $\mathcal{S}p$ of Spectra; it is a stable ∞-category. Any stable ∞-category is enriched over Spectra; a construction of this ∞-category is given in [18, 1.4.1].
- For every abelian category A, there is an ∞-category $Ch(A)$ of chain complexes in A and chain complex maps; its 2-cells are chain homotopies; see [18, 1.3.1]. This is also a stable ∞-category, and its homotopy category is equivalent to the classical derived category of A, $D(A)$.

Largely due to the influence of Lurie's 2009 book [17], most experts regard quasi–categories as the main examples of ∞-categories. However, there are several other related settings where the term "∞-category" is used. One of the main goals of the book under review is to axiomatize the terminology of just what an ∞-category is. The authors solve this problem by coining the term "∞-cosmos" and defining an ∞-category to be an object in an ∞-cosmos. The category **QCat** of quasi-categories is the prototype of an ∞-cosmos, and many model categories enriched over Joyal's model structure on simplicial sets also form an ∞-cosmos; see [24, 1.2.13]. I think of an ∞-cosmos as a higher analogue of **Cat**.

Definition 5.2. An ∞-*cosmos* is a category which is enriched over quasi-categories, contains a terminal object, small products, cotensors with simplicial sets, and is equipped with a notion of "isofibration," satisfying some axioms which this review will omit; see [24, 1.2.1].

An ∞-*category* is defined to be an object in an ∞-cosmos. When the ∞-cosmos is **QCat**, we recover the usage of a quasi-category being an ∞-category.

Being enriched over quasi-categories means that for every two ∞-categories A and B, $\mathbf{Fun}(A, B)$ is a quasi-category. A *cosmological functor* between ∞-categories is a simplicial functor preserving limits and isofibrations.

Here is a list of ∞-cosmoi, many having been discovered in the decade 2000–2010:

- **Cat** is a trivial example of an ∞-cosmos.
- The category **QCat** of small quasi-categories is the prototypical ∞-cosmos. Given two quasi-categories $\mathcal{A}$ and $\mathcal{B}$, $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ is the largest Kan complex contained in the quasi-category $\mathbf{Fun}(\mathcal{A}, \mathcal{B})$.
- Any 2-category $\mathcal{C}$ having "sufficient limits" defines an ∞-cosmos, where $\mathbf{Fun}(A, B)$ is the nerve of the category $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- *Segal categories* are bisimplicial sets satisfying a version of Segal's condition; they were introduced by Dwyer, Kan and Smith [5, 6], and studied in [26].

  Hirschowitz and Simpson [11] defined a model structure on the category of pre-categories for which the fibrant objects are the Segal categories.
- *Complete Segal spaces*, due to Rezk [23], are bisimplicial sets satisfying a version of Segal's condition for simplicial spaces; they are the fibrant objects in Rezk's Quillen model structure on the category of bisimplicial sets. Bergner [3] showed that the model structures on complete Segal spaces and Segal categories are Quillen–equivalent.
- The stable ∞-categories (and exact functors) in an ∞-cosmos also form an ∞-cosmos. See [24, 6.3.16].

Warning: Sometimes the new vocabulary causes confusion. Here are two examples.

- In a pointed ∞-category, the *kernel* and *cokernel* of a morphism are the topologist's homotopy fiber and cofiber.
- In an ordinary category A, the familiar limit of a functor $J \to A$ (when it exists) is right adjoint to the constant diagram functor. In an ∞-category, the limit is also defined to be the right adjoint of the constant diagram functor, but turns out to be the topologists' *homotopy limit* of a J-diagram of simplicial complexes. In this ∞-categorical setting, the "pushout" of $B \leftarrow A \to C$ is not the classical pushout, but rather the homotopy pushout of the diagram of complexes.

We conclude by mentioning two approaches to higher category theory which are not ∞-cosmoi:

(1) The theory of relative categories, due to Barwick and Kan [1];
(2) The theory of *Derivators*, due to Grothendieck [10], has been modified by Raptis [22] to give another model of ∞-categories.

6. The book

We now turn to the 759-page book under review. It is certainly a comprehensive, foundational text, laying out a model-independent approach to higher category theory. This makes it a valuable resource for experts. However, the book is written in a terse style, and is not for those not yet initiated into higher category theory.

In particular, I think that it is too advanced for a graduate course, as it relies on several appendices, many of which would warrant a course in their own right.

Part I is the heart of the book, defining ∞-categories and ∞-cosmoi as well as their associated homotopy categories and topics like adjunctions, limits and colimits. To quote from the Introduction, "the aim of Part I is to develop a substantial portion of the theory of ∞-categories from first principles, as rapidly and painlessly as possible—at least assuming that the reader finds classical abstract nonsense to be relatively innocuous." However, following the proofs frequently requires familiarity with the more classical material found in the Appendices.

Rather than picking a specific model of ∞-category and providing structural results in that model, the authors define an "∞-cosmos" to be a category $\mathcal{K}$ that is enriched over quasi–categories, as we did in the previous section. Just as ordinary categories and functors form the objects and morphisms of **Cat**, an ∞-category is defined to be an object A of $\mathcal{K}$ and an ∞-functor $A \to B$ is defined to be a morphism in $\mathcal{K}$.

Because the text ties many models together, the book contains both "synthetically proven results"—whose statements and proofs hold in any model of ∞-categories—and "analytically proven results"—whose statements and proofs depend on the features of a particular model. I appreciate the many synthetic proofs of results that had previously been proven only for quasi-categories.

Part II develops the calculus of "modules;" ∞-categories A, B are the analogues of classical rings, and an "A-B bimodule" is an ∞-category E on which A acts on the left and B on the right. This analogy is exploited to develop the category theory of ∞-categories (!) and (left and right) Kan extensions as well.

Part III discusses the difference between ordinary functors between ∞-cosmoi and "biequivalences" between ∞-cosmoi. A cosmological functor $F : A \to B$ is a *biequivalence* if it is essentially surjective and the maps $\mathbf{Fun}(A, B) \to \mathbf{Fun}(FA, FB)$ are equivalences of quasi-categories. Biequivalences preserve a lengthy list of ∞-categorical structures [24, 10.3.6], as well as preserving both synthetic and analytic proofs of theorems. Part III also includes a proof of the "Fundamental Theorem of ∞-categories" [24, 12.2.17]: in any ∞-cosmos, a functor is an equivalence if and only if it is fully faithful and essentially surjective.

The 6 Appendices A–F, which cover over 200 pages of the book, deal with higher category theory (enriched categories, 2-categories and Quillen model categories) and concrete constructions (simplicial sets, models of ∞-cosmoi, and quasi-categories).

That being said, I learned a lot from the book under review, and I believe that any reader familiar with simplicial sets can appreciate the rapid development in Chapters 1 and 2.

References

[1] C. Barwick and D. Kan, Relative categories: Another model for the homotopy theory of homotopy theories, 2011, arXiv:1011.1691

[2] A.A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, Astérisque 100, 1982.

[3] J. Bergner, A model category structure on the category of simplicial categories, Trans. Amer. Math. Soc. 359 (2007), 2043–2058.

[4] J.M. Boardman and R. Vogt, Homotopy invariant Algebraic structures on Topological spaces, Lecture Notes in Math. 347, Springer, 1973.

[5] W. Dwyer and D. Kan, Simplicial localizations of categories, J. Pure Appl. Algebra 17 (1980), 267–284.

[6] W. Dwyer, D. Kan and J. Smith, Homotopy commutative diagrams and their realizations, J. Pure Appl. Alg. 57 (1989), 5–24.

[7] S. Eilenberg and S. Mac Lane, General theory of natural equivalences, Trans. AMS 58 (1945), 231—294.

[8] S. Eilenberg and J. Zilber, Semisimplicial complexes and singular homology, Annals Math. 51 (1950), 499–513.

[9] A. Grothendieck, Sur quelques points d'algébre homologique, Tohoku Math. J. 9 (1957), 119—221.

[10] A. Grothendieck, Les Dérivateurs, Manuscript, 1991. French Transcription is available at webusers.imj-prg.fr/georges.maltsiniotis/groth/Derivateurs.html

[11] A. Hirschowitz and C. Simpson, Descente pour les n-champs, arXiv:math/9807049, 1998.

[12] A. Joyal, Quasi-categories and Kan complexes, J. Pure Appl. Alg. 175 (2002), 207–222.

[13] D. Kan, Abstract homotopy III, Proc. National Academy of Sciences 42 (1956), 419–421.

[14] D. Kan, Adjoin functors, Trans. AMS 87 (1958), 294–329.

[15] B. Keller, Introduction to A-infinity algebras and modules, Homology Homotopy Appl. 3 (2001), 1-–35.

[16] J. Lurie, Derived Algebraic Geometry I: Stable infinity categories, 2006, arXiv:math 0608228.

[17] J. Lurie, Higher Topos Theory, Annals Math. Studies 170, Princeton Univ. Press, 2009.

[18] J. Lurie, Higher Algebra, 2017, https://www.math.ias.edu/ lurie/papers/HA.pdf

[19] S. Mac Lane, Categories for the Working Mathematician, Second edition, Springer–Verlag, 1998

[20] J.P. May, The geometry of iterated loop spaces, Lecture Notes in Mathematics, Vol. 271, Springer-Verlag, 1972.

[21] D, Quillen, Homotopical algebra, Lecture Notes in Mathematics, Vol. 43, Springer-Verlag, 1967.

[22] G. Raptis, Higher homotopy categories, higher derivators, and K-theory, Forum Math. Sigma 10 (2022), Paper No. e54, 36 pp.

[23] C. Rezk, A model for the homotopy theory of homotopy theory, Trans. Amer. Math. Soc. 353 (2001), 973–1007.

[24] E. Riehl and D. Verity, Elements of ∞-Category Theory, Cambridge Univ. Press, 2022.

[25] G. Segal, Categories and cohomology theories, Topology 13 (1974), 293–312.

[26] C. Simpson, Simplicial localizations of categories, J. Pure Appl. Algebra 17 (1980), 267–284.

[27] J.D. Stasheff, Homotopy associativity of H-spaces I, II, Trans. Amer. Math. Soc. 108 (1963), 275–292 and 293—312.

[28] J.–L. Verdier, Catégories dérivées: quelques résultats (état 0), pp. 262-311 in *Cohomologie étale*, Lecture Notes in Math., 569, Springer, 1977.

Math. Dept., Rutgers University, New Brunswick, NJ 08901, USA
Email address: weibel@math.rutgers.edu
URL: http://math.rutgers.edu/~weibel