

TRANSFER FUNCTORS ON k -ALGEBRAS

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Dedicated to Wolmer Vasconcelos on his 65th birthday

ABSTRACT. We introduce and study the notion of a transfer functor for normal algebras over a field, i.e., a functor equipped with transfer maps for finite integral extensions. Any Hecke functor (such as a Galois module) induces a transfer functor, and conversely.

In this paper, we introduce the notion of a *transfer functor* F for normal algebras over a field k . It is a modification of Voevodsky's notion of a presheaf with transfers, taking advantage of the fact (0.2) that every correspondence may be normalized, and is inspired by the work of Suslin and Voevodsky [SV] [SV1]. The restriction of transfer functors to smooth algebras forms a special case of Voevodsky's theory of presheaves with transfer, which is developed in [SV1], [V] and [MVW].

If k has characteristic zero, a transfer functor consists of a covariant functor F , from the category $Norm_k$ of normal k -algebras of finite type to Abelian groups, together with contravariant “transfer” maps $j^T : F(A') \rightarrow F(A)$ associated to finite extensions $j : A' \supseteq A$, subject to certain axioms which go back to Mackey's study [Ma] of group representations. The axioms allow us to compose the covariant and contravariant maps.

The forgetful functor $F(A) = A$ and the units functor $U(A) = A^\times$ are the prototype transfer functors, upon which other examples are based. Their respective transfer maps are the classical trace and norm (see 6.1 and 6.2). Hecke functors for the absolute Galois group, such as Galois modules, form another family of examples (see 6.4). Transfer functors F satisfying the “homotopy invariance” condition $F(A) = F(A[t])$ form the building blocks of Voevodsky's triangulated category of motives [V].

An equivalent way to define transfer functors is to make them additive functors from a certain additive category Cor_k to abelian groups, where Cor_k is concocted so as to contain both $Norm_k$ and (in characteristic zero) the dual of the category of finite extensions $B \supset A$ of normal domains. This is the approach we shall follow. Thus for us a *transfer functor* will be an additive functor from Cor_k to abelian groups.

Except for the composition law, it is easy to define the category Cor_k . The objects of Cor_k are the normal k -algebras of finite type, and the algebra $B_1 \times B_2$ is the categorical product of B_1 and B_2 . Morphisms are integer combinations of elementary correspondences; we write $Cor_k(A, B)$ for the group of morphisms from A to B in Cor_k . When A and B are smooth, our $Cor_k(A, B)$ is the group $Cor_k(\text{Spec } B, \text{Spec } A)$ of [MVW].

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Definition 0.1. If B is a normal domain, an *elementary correspondence* from A to B is a prime ideal P in $A \otimes_k B$ such that $P \cap B = 0$ and the inclusion $B \subseteq (A \otimes_k B)/P$ is finite. If $B = \prod B_i$ is a finite product of normal domains, an elementary correspondence from A to B is a prime ideal P in $A \otimes_k B$ such that, for some i , $P \cap B_i = 0$ and the inclusion $B_i \subseteq A \otimes_k B/P$ is finite. When P is understood, we will say that the quotient $B' = A \otimes_k B/P$ is the elementary correspondence. A *finite correspondence* is a \mathbb{Z} -linear sum of elementary correspondences.

If $\text{char}(k) = 0$, $Cor_k(A, B)$ is the group of all finite correspondences from A to B . Note that $Cor_k(A, \prod B_i) = \oplus Cor_k(A, B_i)$ by construction.

If $\text{char}(k) = p > 0$, $Cor_k(A, B)$ is defined to be the subgroup of all finite correspondences which are *universally integral*. Being universally integral is a mild technical restriction; if $[P]$ is an elementary correspondence then $p^n[P]$ is universally integral for some n . (See 5.6 below.)

There are two distinguished kinds of elementary correspondences. To a ring homomorphism $f : A \rightarrow B$ (with B a domain) we associate the kernel P of $A \otimes_k B \rightarrow B$, and sometimes write $[f]$ for its class in $Cor_k(A, B)$; this association is compatible with composition ($[gf] = [g] \circ [f]$) and induces the embedding of the category $Norm_k$ into Cor_k that we mentioned above. To a finite inclusion $j : B' \supseteq B$ of domains, we associate the kernel of $B' \otimes_k B \rightarrow B'$. This will induce the embedding of the dual of the category of finite extensions into Cor_k . Therefore if F is a transfer functor and $\text{char}(k) = 0$, both $f : F(A) \rightarrow F(B)$ and $j^T : F(B') \rightarrow F(B)$ are defined.

If P is any elementary correspondence from A to B , and we write B_1 for the integral closure of the domain $A \otimes_k B/P$, then P will be the composition in Cor_k of the canonical homomorphism $f : A \rightarrow B_1$ and $j : B_1 \supseteq B$ (see 3.2.1). Thus, for any transfer functor F , the map $F(A) \rightarrow F(B)$ induced by P is just the composition:

$$(0.2) \quad F(A) \xrightarrow{f} F(B_1) \xrightarrow{j^T} F(B).$$

Theorem 0.3. *There is a unique associative composition $Cor(A, B) \otimes Cor(B, C) \rightarrow Cor(A, C)$ satisfying the following axioms.*

- (1) *If $j : B \subset B_1$ is a finite inclusion of normal domains, then the composition $j^T \circ j : B \rightarrow B_1 \rightarrow B$ is d times the identity of B , where d is the degree of j .*
- (2) *Let B' be an elementary correspondence from A to B such that $B \subseteq B'$ is flat, and let $f : B \rightarrow C$ be a homomorphism of normal domains. Then $[f] \circ [B']$ is the sum $\sum_i \lambda_i [P_i]$. Here P_1, \dots, P_r denote the inverse images in $A \otimes_k C$ of the minimal primes of $B' \otimes_B C$ and λ_i denotes the length of $(B' \otimes_B C)_{P_i}$.*
- (3) *If $S \subset B$ is multiplicatively closed, then $Cor_k(A, B) \rightarrow Cor_k(A, S^{-1}B)$ is the canonical injection sending $[P]$ to $[S^{-1}P]$.*

Sections 1 and 2 describe the restriction of a transfer functor to finite field extensions of k . This recovers a well-known construction, that of *Hecke functors* for the Galois group $G = Gal(\bar{k}/k)$. In this setting, finite correspondences are called *Hecke operators*, or *cohomological Mackey functors* in the topological literature. I am grateful to Gaunce Lewis for pointing me in this expositional direction, and for sharing his notes [Lew].

For expositional purposes, we defer the description of composition to §3 and the proof of theorem 0.3 to §4 when $\text{char}(k) = 0$; the modification needed in finite characteristic is given in section 5. Examples are given in section 6.

§1. G -CORRESPONDENCES AND HECKE FUNCTORS

Let G be a group. Our first goal is to construct an additive self-dual category Cor_G containing the category $G\text{-sets}_f$ of finite G -sets. This material is scattered through the literature of analytic number theory and topology, and is not new.

By a (formal) multi-valued function f from a set X to a set Y we mean a function from X to $\mathbb{Z}[Y]$, the free abelian group on the elements of Y . Multi-valued functions are composed in the usual way: if $f(x) = \sum n_i [y_i]$ then $(g \circ f)(x) = \sum n_i g(y_i)$. Composition is associative, with identity functions $1_X(x) = [x]$.

If X and Y are G -sets, f is called *equivariant* if $f(\gamma \cdot x) = \gamma \cdot f(x)$ for all $\gamma \in G$. Let $\text{Cor}_G(X, Y)$ denote the abelian group of equivariant multi-valued functions from X to Y . Since the composition of equivariant multi-valued functions is easily seen to be equivariant, we get a category.

Definition 1.1. Let Cor_G denote the category of finite G -sets and equivariant multi-valued functions. It is easy to see that this is an additive category, with disjoint union of G -sets as the direct sum operation.

A *Hecke functor* M is a contravariant additive functor from Cor_G to abelian groups, or more generally to any additive category \mathcal{A} .

Any G -map $f: X \rightarrow Y$ is an equivariant single-valued function. In this way, we may regard $\text{Hom}_G(X, Y)$ as a subset of $\text{Cor}_G(X, Y)$. Note that if f and g are single-valued G -maps then $g \circ f$ is the usual composition of G -maps. In this way, we get a natural embedding of the category of finite G -sets into Cor_G .

Correspondences 1.2. Here is an equivalent description of Cor_G . A G -correspondence from X to Y (both G -sets) is an element of the free abelian group on the set of finite G -orbits Γ in $X \times Y$. The group of all G -correspondences from X to Y is isomorphic to the group $\text{Cor}_G(X, Y)$ by the following rule. To each G -correspondence $\sum n_i \Gamma_i$, we associate the equivariant multi-valued function $f(x) = \sum n_i [y_{ij}]$, where the sum is over all y_{ij} such that $(x, y_{ij}) \in \Gamma_i$.

If f and g are G -correspondences from X to Y , and from Y to Z , respectively, we define their composition $g \circ f$ to be the G -correspondence associated to the composition of multi-valued functions. This definition ensures that the category Cor_G is equivalent to the category of finite G -sets and G -correspondences.

It will be useful to have an orbit-theoretic construction of composition. Given a G -map $p: T \rightarrow X \times Y$, we write $[T] = [T]_p$ for the sum $\sum n_i [\Gamma_i]$ over all orbits $\Gamma_i \in X \times Y$, where n_i is the cardinality of $p^{-1}(\gamma)$, $\gamma \in \Gamma_i$. Since $n_i = |p^{-1}\Gamma_i|/|\Gamma_i|$, the element $[T]$ is independent of the choice of γ .

We say that a G -correspondence $\sum n_i \Gamma_i$ is *effective* if all the n_i are non-negative. The composition of two effective G -correspondences Γ_1 and Γ_2 is:

$$(1.2.1) \quad \Gamma_2 \circ \Gamma_1 = [\Gamma_{12}], \quad \Gamma_{12} = (\Gamma_1 \times Z) \cap (X \times \Gamma_2).$$

By additivity, a Hecke functor is uniquely determined by its restriction to the full subcategory of Cor_G on the orbits G/H . This subcategory is called the *Hecke category* by Yoshida in [Y], following its popularization in [Sh]. Correspondences between orbits have the following interpretation.

Example 1.3. Let $H_1, H_2 \subseteq G$ be subgroups of finite index. Then $Cor_G(G/H_1, G/H_2)$ is the free abelian group on the set of all double cosets H_1aH_2 in G . This is because the set of G -orbits in $G/H_1 \times G/H_2$ is in 1–1 correspondence with the set of double cosets H_1aH_2 in G , by the rule that sends the orbit of (x_1H_1, x_2H_2) to $H_1(x_1^{-1}x_2)H_2$.

Historical Remark 1.3.1. In the 1959 paper [Sh1] (see [Sh, 3.1]), Shimura defined the *Hecke ring with respect to H* , using a partial multiplication \cdot on double cosets $H \setminus G/H$. If G/H is infinite, there is a commensurability restriction on the double cosets. If G/H is finite, the Hecke ring with respect to H is just the endomorphism ring $Cor_G(G/H, G/H)$, because it is easy to see from (3.1.1) of [Sh] that Shimura’s product ‘ \cdot ’ is the composition law in Cor_G .

Shimura’s definition was abstracted from a construction of Hecke [H], in which $G = SL_2(\mathbb{R})$ and Hecke only considered subgroups commensurate with $SL_2(\mathbb{Z})$. In fact, one of the main thrusts of [Sh] is to connect the action of the Hecke rings with the zeta-functions associated with modular forms.

Independently in 1971, Green introduced the notion of a (cohomological) G -functor in [Gr]. Yoshida proved in [Y] that cohomological G -functors are the same as Hecke functors, i.e., additive functors on Cor_G . Central to Green’s definition was the *Mackey axiom* for subgroups $H, K \subset L \subset G$ that in our language describes the composition $G/H \rightarrow G/L \rightarrow G/K$ as the sum over double cosets $H\gamma K$ of composites $G/H \cong G/H^\gamma \rightarrow G/(H^\gamma \cap K) \rightarrow G/K$. This formula is a special case of (1.2.1).

Example 1.4. Let M be a left G -module. It is easy to see from the definitions that $M(G/H) = M^H$ is a Hecke functor; if $H \subset K$, the “transfer” map $M^H \rightarrow M^K$ associated to $G/H \rightarrow G/K$ is $m \mapsto \sum_{\gamma \in K/H} \gamma(m)$.

More generally, the Hecke algebras act on the group cohomology of M , i.e., that each $H^n(G/H) \mapsto H^n(H; M)$ is a Hecke functor. This was first shown by Rhie and Whaples, in a modular setting. (See [RW] and [Sh, §8].) This result has been independently discovered by many people, including Cline, Parshall and Scott [CPS], Green [Gr] and Dress [D].

Returning to the general theory, note that the transpose $\tau: X \times Y \rightarrow Y \times X$ gives a natural isomorphism between $Cor_G(X, Y)$ and $Cor_G(Y, X)$. By inspection of (1.2.1), the transpose commutes with the composition pairing: $\tau([\Gamma_2] \circ [\Gamma_1]) = \tau[\Gamma_{12}] = [\tau\Gamma_1] \circ [\tau\Gamma_2]$. Thus:

Corollary 1.5. *The transpose τ provides an isomorphism between Cor_G and its opposite category.*

Lemma 1.6. *For every effective G -correspondence Γ from X to Y there is a G -set T and G -maps $X \xleftarrow{p} T \xrightarrow{f} Y$ such that Γ factors in Cor_G as the composition*

$$X \xrightarrow{[\tau p]} T \xrightarrow{f} Y.$$

Proof. If $\Gamma \subset X \times Y$ is an orbit, we may take $T = \Gamma$ and let p and f be the projections. Since $\tau p \in \text{Cor}_G(X, \Gamma)$ is the multi-valued function $x \mapsto \sum_{(x,y) \in \Gamma} [(x, y)]$, it is clear that $f \circ \tau p = \Gamma$. The general case follows by taking disjoint unions of orbits.

Corollary 1.7. *Given G -maps $f_i: Y_i \rightarrow X$ ($i = 1, 2$), let $p_i: Y_1 \times_X Y_2 \rightarrow Y_i$ be the projections from the pullback. Then the following diagram commutes in Cor_G :*

$$\begin{array}{ccc} Y_1 \times_X Y_2 & \xrightarrow{[p_2]} & Y_2 \\ [\tau p_1] \uparrow & & \uparrow [\tau f_2] \\ Y_1 & \xrightarrow{[f_1]} & X. \end{array}$$

Proof. The pullback $T = Y_1 \times_X Y_2$ is naturally a G -subset of $Y_1 \times Y_2$. Its correspondence $[T] \in \text{Cor}_G(Y_1, Y_2)$ is effective, and equals the composition $Y_1 \rightarrow T \rightarrow Y_2$ by the lemma. For the composition $[\tau f_2] \circ [f_1]$ we obtain the set

$$\Gamma_{12} = \{(y_1, x, y_2) \in Y_1 \times X \times Y_2 \mid f_1(y_1) = x = f_2(y_2)\}.$$

Clearly, Γ_{12} is isomorphic to the pullback T , so this composition is also $[T]$.

We are now ready to connect the above construction with Mackey functors on finite G -sets. The following definition is due to Andreas Dress [D].

Definition 1.8. A *Mackey functor* $M = (M_*, M^*)$ from $G\text{-sets}_f$ to a category \mathcal{A} consists of a rule which associates to every finite G -set X an object $M(X)$ of \mathcal{A} such that: (a) M sends finite coproducts to products; (b) M extends to both a covariant functor M_* and a contravariant functor M^* , and (c) for every pair of G -maps $f_i: Y_i \rightarrow X$ the following diagram commutes in \mathcal{A} :

$$\begin{array}{ccc} M(Y_1 \times_X Y_2) & \xrightarrow{pr_{2*}} & M(Y_2) \\ p_1^* \uparrow & & \uparrow f_2^* \\ M(Y_1) & \xrightarrow{f_{1*}} & M(X). \end{array}$$

If \mathcal{A} is an additive category, a Mackey functor $M: G\text{-sets}_f \rightarrow \mathcal{A}$ is *additive* if for every pair of G -maps $f, g: X \rightarrow Y$ the fold map $\nabla: X \amalg X \rightarrow X$ satisfies:

- (1) $f_* + g_*$ is the composite $M(X) \xrightarrow{\nabla^*} M(X \amalg X) \xrightarrow{(f \amalg g)_*} M(Y)$;
- (2) $f^* + g^*$ is the composite $M(Y) \xrightarrow{(f \amalg g)^*} M(X \amalg X) \xrightarrow{\nabla^*} M(X)$.

Note that composition with an additive functor $F: \mathcal{A} \rightarrow \mathcal{A}'$ sends additive Mackey functors to additive Mackey functors. Combining the above results, we see that we have proven:

Proposition 1.9. *The embedding $G\text{-sets}_f \subset \text{Cor}_G$ and its transpose $G\text{-sets}_f^{\text{op}} \subset \text{Cor}_G$ form an additive Mackey functor.*

Moreover, there is a 1-1 correspondence between additive Mackey functors and Hecke functors $M: \text{Cor}_G \rightarrow \mathcal{A}$.

Remark 1.9.1. Lindner [Li] has constructed a category $Sp(\mathbf{G}\text{-sets}_f)$ of “spans” in $\mathbf{G}\text{-sets}_f$, and proved that it is universal in the sense that the category of Mackey functors from $\mathbf{G}\text{-sets}_f$ to \mathcal{A} is isomorphic to the category of all functors from $Sp(\mathbf{G}\text{-sets}_f)$ to \mathcal{A} which preserve finite products. Lewis has observed in [Lew] that the category Cor_G is a quotient category of $Sp(\mathbf{G}\text{-sets}_f)$; this gives another way to see that every Hecke functor is a Mackey functor.

§2. GALOIS THEORY AND HECKE OPERATORS

When the group G is the absolute Galois group of k , we may translate the language into field theory. The category Cor_G translates into the zero-dimensional subcategory of the category Cor_k of 0.1, and composition takes a particularly nice form here.

If E' and E'' are finite-dimensional algebras over a field k , then $E' \otimes_k E''$ is a finite-dimensional E'' -algebra. A well-known problem of the 1950's was to describe the decomposition $E' \otimes_k E'' = \prod E_i$ when E' and E'' are fields. This was essentially solved by Grothendieck's Galois theory [Mi, p.43], which states that there is an equivalence between the dual category of finite separable k -algebras and discrete finite G -sets, where G is the Galois group of \bar{k}/k . Under this correspondence, the finite separable field extension $E = \bar{k}^H$ corresponds to the G -set G/H .

Lemma 2.1. *Let K be a Galois extension of k with Galois group $G = Gal(K/k)$. If E' and E'' are subfields of K , finite over k , the decomposition $E' \otimes_k E'' \cong \prod E_i$ is determined as follows.*

Let H' and H'' be the subgroups of G corresponding to E' and E'' , and write the finite G -set $X = G/H' \times G/H''$ as a union of orbits, $X \cong \coprod G/H_i$. Then $E_i = \bar{k}^{H_i}$.

Proof. The product of separable extensions is separable, so $E' \otimes_k E''$ is finite separable, and hence a finite product of fields. To determine the decomposition, note that the coproduct $E' \otimes_k E''$ corresponds to the product G -set X under the Galois correspondence.

Example 2.1.1. Let K be the splitting field of $x^3 - 2$ over $k = \mathbb{Q}$, and set $E = \mathbb{Q}(\sqrt[3]{2})$. Then $E \otimes_{\mathbb{Q}} E \cong E \times K$. To see this, note that $Gal(K/k)$ is the dihedral group D_3 , and $E = K^H$ for a subgroup H of order 2. Since $G/H \times G/H \cong G/H \coprod G$ as a G -set, the result follows from lemma 2.1.

If E' and E'' are finite over k , we define the group of correspondence $Cor_k(E', E'')$ to be the free abelian group on the finite set of quotient fields (or equivalently, prime ideals) of the finite-dimensional algebra $E' \otimes_k E''$. This definition also makes sense if E' and E'' are finite-dimensional algebras. We shall sometimes write 1_E for the canonical element $[E]$ of $Cor_k(E, E)$, because it is the identity for the composition 2.2 below.

If both E' and E'' are separable over k , then $Cor_k(E', E'')$ is the free abelian group on the E_i described in lemma 2.1. In the inseparable case, let $i' : E'_s \subseteq E'$ and $i'' : E''_s \subseteq E''$ denote the respective maximal separable subfields; then $Cor_k(E', E'') \cong Cor_k(E'_s, E''_s)$, and 2.1 describes the generators of the right hand side.

More generally, suppose that E' and E'' are finitely generated field extensions of k . In this case, we define $Cor_k(E', E'')$ to be the free abelian group on the set of quotient fields of $E' \otimes_k E''$ which are finite over E'' .

To define composition of correspondences, it is useful to introduce the notation $[M]$ for the correspondence corresponding to an $E' \otimes_k E''$ -module M of finite length. By definition, $[M]$ is the sum $\sum_i \lambda_i [E_i]$, where λ_i is the E_i -primary length of M , i.e., the number of factors isomorphic to E_i in a composition series for M . If either E' or E'' is separable over k then $E' \otimes_k E'' = \prod E_i$ and $M = \oplus M_i$, so $\lambda_i = \dim_{E_i}(M_i)$.

Definition 2.2 (Composition). Suppose that E' , E'' and E''' are finite-dimensional over k . Let K_1 be a quotient field of $E' \otimes_k E''$, and K_2 a quotient field of $E'' \otimes_k E'''$. Then $K_1 \otimes_{E''} K_2$ is an $E' \otimes_k E'''$ module, and we define the composition $K_2 \circ K_1$ of K_1 and K_2 to be the class $[K_1 \otimes_{E''} K_2]$ in $Cor_k(E', E''')$.

The composition of arbitrary correspondences is defined so as to be bilinear. That is, $(\sum m_i K_{1i}) \circ (\sum n_j K_{2j}) = \sum m_i n_j K_{2j} \circ K_{1i}$.

If E' , E'' and E''' are arbitrary field extensions, and we assume that $K_1 = E' \otimes_k E'' / P_1$ and $K_2 = E'' \otimes_k E''' / P_2$ are finite field extensions of E'' and E''' , respectively, then $K_1 \otimes_{E''} K_2$ is finite over E''' and again we define $[K'] \circ [K'']$ to be the class $[K_1 \otimes_{E''} K_2]$ in $Cor_k(E', E''')$.

It is clear from the definition that if K is a correspondence from E to F then we have $1_F \circ K = K \circ 1_E = K$.

Example 2.2.1. Any field extension $i : E \rightarrow F$ determines an element $[i]$ in $Cor_k(E, F)$ corresponding to the quotient $E \otimes_k F \rightarrow F$ sending $x \otimes y$ to $i(x) \cdot y$. Since $[i_2 i_1] = [i_2] \circ [i_1]$, the composition of field extensions agrees with the composition in 2.2. If $[F : E]$ is finite, the analogous map $F \otimes_k E \rightarrow F$ ($x \otimes y \mapsto x \cdot i(y)$) determines an element $[i^T]$ of $Cor_k(F, E)$. Again, $[(i_2 i_1)^T] = [i_2^T] \circ [i_1^T]$, so the composition of finite extensions agrees with composition under the contravariant operation $j \mapsto j^T$.

If $j : F_0 \subset F$ is finite, the image F' of $E \otimes_k F_0$ in F is a subfield. If $d = [F : F']$ then $[j^T] \circ [i] = d[F']$ in $Cor_k(E, F_0)$. Setting $F_0 = i(E)$, we obtain the formula $[i^T] \circ [i] = d \cdot 1_E$ in $Cor_k(E, E)$, $d = [F : E]$.

Example 2.2.2. Let ℓ be a finite Galois extension of k . Then the endomorphism ring $Cor_k(\ell, \ell)$ is just the group ring $\mathbb{Z}[G]$ of $G = \text{Gal}(\ell/k)$. This follows from 2.2.1 since the automorphisms $g : \ell \rightarrow \ell$ in G form a \mathbb{Z} -basis of $Cor_k(\ell, \ell)$ by 2.1.

The following result verifying axiom 0.3(2) is elementary, and immediate from 2.2.

Lemma 2.3. *If F' is a quotient field of $E \otimes_k F$, finite over F , then $[F'] \in Cor_k(E, F)$ is the composite $[j^T] \circ [i]$, where $i : E \subseteq F'$ and $j : F \subseteq F'$ are the evident inclusions.*

If $j_E : E \subset E'$ is a finite inclusion and $i : E \rightarrow F$ a field extension, then $E' \otimes_E F$ is a finite product $\prod A_\alpha$ of Artin local rings. If A_α has residue field F_α and length λ_α , then in $Cor_k(E', F)$ we have $[i] \circ [j^T] = \sum \lambda_\alpha [F_\alpha]$.

$$\begin{array}{ccc} E' & \xrightarrow{(\lambda_1, \dots)} & \prod F_\alpha \\ j_E^T \downarrow & & \downarrow j_F^T \\ E & \xrightarrow{i} & F \end{array}$$

Example 2.3.1. If E' is separable over E then $\lambda_\alpha = 1$ for all α and we obtain the analogue of the Mackey condition 1.7. If $F = E'$ is a Galois extension of E with group G , the formula simplifies to $[i] \circ [i^T] = \sum_{g \in G} [g]$ in $Cor_k(F, F)$. At the other extreme, if F is purely inseparable over E of degree p^e then $[i] \circ [i^T] = p^e \cdot 1_F$ in $Cor_k(F, F)$.

Lemma 2.4. *Composition is associative, and the (commutative) finite reduced k -algebras form the objects of an additive category Cor_k^0 , whose Hom-groups are given by Cor_k .*

If k is a perfect field, Cor_k^0 is equivalent to the opposite category of Cor_G , and a covariant additive functor on Cor_k^0 is the same thing as a Hecke functor on the Galois group $G = Gal(\bar{k}/k)$.

Proof. The verification of associativity and additivity is routine, using 2.2 and 2.3, and is left to the reader. Now assume that k is perfect. The Galois correspondence of 2.1 gives a natural isomorphism between $Cor_k(E', E'')$ and $Cor_G(G/H', G/H'')$. Therefore we may transport the composition of G -correspondences to this setting, and formula (1.2.1) translates exactly into formula 2.2. This reproves associativity, and shows that the $Cor_k(E', E'')$ form the Hom-sets of an additive category which is dual to Cor_G . \square

Example 2.4.1. Suppose that k is perfect, and that M is a Galois module, *i.e.*, a discrete module for $Gal(\bar{k}/k)$. Translating 1.4 into the current setting, we see that the étale cohomology groups $E \mapsto H_{et}^n(E, M)$ are Hecke functors, *i.e.*, additive covariant functors from Cor_k^0 to abelian groups.

If k is not perfect, purely inseparable extensions complicate composition in the category Cor_k^0 of finite reduced algebras. If $i : E_s \subseteq E$ and $j : F_s \subseteq F$ are the maximal separable subfields, then i^T and j make $Cor(E_s, F)$ and $Cor(E, F_s)$ into subgroups of finite (p -primary) index in $Cor(E, F)$.

Imperfect fields 2.5. Suppose that k is not perfect, and let G denote the absolute Galois group of k . Then the inclusion $Cor_G \subset Cor_k^0$ has a right adjoint, sending a field E to E_s , the separable closure of k in E . That is, $Cor_k(\ell, E_s) \cong Cor_k(\ell, E)$ for all E . Hence any Hecke functor M has a canonical Kan extension to an additive covariant functor on Cor_k^0 , determined by the formula $M(E) = M(E_s)$.

Example 2.5.1. If k is not perfect, and M is a Galois module, the canonical maps $H_{et}^n(E_s, M) \rightarrow H_{et}^n(E, M)$ are isomorphisms. Hence $E \mapsto H_{et}^n(E, M)$ is the Kan extension of a Hecke functor to Cor_k^0 .

Example 2.5.2. The underlying abelian group is a covariant additive functor on Cor_k^0 , where the transfer map $j^T : E' \rightarrow E$ is the trace map. This follows from standard properties of the trace, and is also immediate from 1.4 and 2.4 when k is perfect. If k is not perfect, it is not the Kan extension of a Hecke functor in the sense of 2.5.

Proposition 2.6. *The composition \circ of 2.2 is the unique associative composition satisfying axioms (1) and (2) of 0.3.*

Proof. The composition \circ is associative and satisfies the axioms by 2.4, 2.2.1 and 2.3.

Conversely, suppose given a composition \square satisfying the axioms. By axiom (2) and 2.3, we have $[j]\square c = [j] \circ c$ for every correspondence c and every homomorphism j .

We claim that $i^T \square [K']$ must also equal $[i^T] \circ [K']$ for every finite $i : E' \supset E$ and every $[K']$ in $Cor_k(K, E')$. To see this, choose a splitting field $j : E \subset F$ of E' over E . Axiom (2) shows that $[j] \square [i^T] = [j] \circ [i^T]$ has the form $\sum [f_\alpha]$ for various homomorphisms $f_\alpha : E' \rightarrow F$. By associativity,

$$[j] \circ ([i^T] \square [K']) = [j] \square [i^T] \square [K'] = \sum [f_\alpha] \square [K'] = \sum [f_\alpha] \circ [K'] = [j] \circ [i^T] \circ [K'].$$

Since $[j] \circ$ is a monomorphism in Cor_k by axiom (1), we must have $[i^T] \square [K'] = [i^T] \circ [K']$, as claimed.

By 2.3, every elementary correspondence F' in $Cor_k(E, F)$ factors as $[j^T] \circ [i]$, where $i : E \rightarrow F'$ and $j : F \rightarrow F'$ are the inclusions. The proposition now follows from the observation that $[F'] \square [E']$ must equal $[j^T] \square [i] \square [E'] = [j^T] \square ([i] \circ [E'])$, which we have seen is $[j^T] \circ [i] \circ [E'] = [F'] \circ [E']$. \square

Proposition 2.7. *If $i : K \subset L$ is a Galois extension with group G , and E is any field over k , then G acts on $Cor_k(E, L)$ and:*

- (1) $Cor_k(E, K) \cong Cor_k(E, L)^G$;
- (2) $i \circ i^T = \sum_{g \in G} g$ as endomorphisms of $Cor_k(E, L)$.

Proof. From 2.2.1 we have a homomorphism $G \rightarrow \text{Hom}_k(L, L) \rightarrow Cor_k(L, L)$. Composing with $Cor_k(E, L)$ is associative by 2.4, so G acts on $Cor_k(E, L)$. If $c \in Cor_k(E, L)$ then by 2.3.1 and 2.4 we have $[i] \circ [i^T] \circ c = \sum [g] \circ c$.

Because $[i^T] \circ [i]$ is multiplication by $[L : K]$, the canonical map $i : Cor_k(E, K) \rightarrow Cor_k(E, L)^G$ is an injection. To show that it is an isomorphism, it suffices to show that if L' is a quotient field of $E \otimes_k L$, finite over L , and $\{L_\alpha\}$ is its G -orbit in $Cor_k(E, L)$ then $\sum [L_\alpha]$ is in the image of $Cor_k(E, L)$. Let K' be the image of $E \otimes_k K$ in L' . Since it is a Galois extension of K' , $K' \otimes_{K'} L = \prod L_\alpha$ is a product of fields isomorphic to L' ; by definition, $i : Cor_k(E, K) \rightarrow Cor_k(E, L)$ sends $[K']$ to $\sum [L_\alpha]$. \square

Example 2.8. Let ℓ be a finite Galois extension of k and consider the representable functor $M(E) = Cor_k(\ell, E)$ on Cor_k^0 . From 2.2.2 it is clear that $M(\ell) = \mathbb{Z}[G/N]$, where N is the normal subgroup of $G = \text{Gal}(\bar{k}/k)$ corresponding to ℓ . If H is the subgroup of G fixing E elementwise, it follows from 2.7 that $M(E) \cong \mathbb{Z}[G/N]^H$, which is free abelian on the double cosets $H \backslash G/N$. Thus M may be identified via 2.4 (and 2.5) with the Hecke functor associated to the Galois module $M = \mathbb{Z}[G/N]$ in 1.4.

Example 2.9. Both the Milnor K -groups $K_n^M(F)$ and Quillen's K -groups $K_n(F)$ are not only Hecke functors but are also covariant additive functors on Cor_k^0 . This is perhaps the motivating example for the entire subject of transfer functors.

For Quillen's groups, this is easiest seen by observing that tensoring with a $E'-E''$ bimodule M is an exact functor from E' -modules to E'' -modules, and so induces a homomorphism $K_n(E') \rightarrow K_n(E'')$. The composition of $\otimes_{E'} M_1$ and $\otimes_{E''} M_2$ is induced by the exact functor $V \mapsto V \otimes_{E'} (M_1 \otimes_{E''} M_2)$, and since $M_1 \otimes_{E''} M_2 \cong \oplus A_i$ as an $E' \otimes_k E''$ -module, and $A_i \cong E_i^{\lambda_i}$ as an E''' -module, it follows that the functor K_n preserves composition of correspondences. The verification for Milnor K -groups is more tedious, but straightforward given 2.3.

Example 2.10. We claim that the E -module Ω_E of absolute Kähler differentials of E is a covariant additive functor on Cor_k^0 . Indeed, the restriction to finite separable algebras (*i.e.*, to Cor_G) is a Hecke functor by 2.5.2, because if E/k is separable then $\Omega_E \cong \Omega_k \otimes_k E$. If k is not perfect, then the claim is a simple exercise in algebra, since for any p -basis $\{x_i\}_{i \in I}$ of E the elements $\{dx_i\}_{i \in I}$ form a basis of Ω_E . (See [Mat, 26.5].)

§3. COMPOSITION OF CORRESPONDENCES

In this section, we describe the composition law for correspondences:

$$Cor_k(A, B) \otimes Cor_k(B, C) \xrightarrow{\circ} Cor_k(A, C),$$

and prove that it makes Cor_k into a category, when $\text{char}(k) = 0$. It suffices to describe the composition of elementary correspondences, represented by a prime ideal P of $A \otimes_k B$ and a prime ideal Q of $B \otimes_k C$. It will be an integral combination of the prime ideals Q_i described in the following lemma.

Lemma 3.1. *If $f : B \rightarrow C$ is a ring homomorphism, with C a normal domain, and $P \in \text{Spec}(A \otimes_k B)$ is an elementary correspondence from A to B , then each minimal prime $Q_i \subset A \otimes_k C$ over $P \otimes_B C$ is an elementary correspondence from A to C .*

Proof. By assumption, $B' = A \otimes_k B/P$ contains and is finite over B . By “going up,” $B' \otimes_B f(B)$ contains and is finite over $f(B)$. Since the quotient field E of C is flat over $f(B)$, it follows that $B' \otimes_B E$ contains both C and E . Hence $C' = B' \otimes_B C$ is finite over C , and contains C . Since the reduced algebra of C' is the product of the $C'_i = A \otimes C/Q_i$, they too are finite over C , and contain C . \square

In order to describe the transfer map j^T associated to a finite inclusion j , it is useful to introduce some notation. If $B' \subset B''$ is any finite inclusion of domains, it will be convenient to write $[B'' : B']$ for the degree $[K'' : K']$, where K'' and K' are the quotient fields of B'' and B' , respectively. If B' is an elementary correspondence from A to B , we will write $[B'']$ for the element $[B'' : B'] \cdot [B']$ of $Cor_k(A, B)$.

Definition 3.2. Let $j : B \subset B_1$ be a finite inclusion of normal domains. Then $j^T \circ : Cor_k(A, B_1) \rightarrow Cor_k(A, B)$ sends the class $[B'']$ of an elementary correspondence (from A to B_1) to the class $[B'']$ in $Cor_k(A, B)$.

That is, if B' is the image of $A \otimes_k B$ in B'' , then $j^T \circ [B''] = [B'' : B'] \cdot [B']$.

We will define composition with the element $[j^T]$ of $Cor_k(B_1, B)$ to be the map $j^T \circ$. When $A = B_1$ the map $j^T \circ : Cor_k(B_1, B_1) \rightarrow Cor_k(B_1, B)$ sends 1_{B_1} to $[j^T]$. If $j_1 : B_1 \subset B_2$ is another finite inclusion, it is easy to see that $j^T \circ (j_1^T \circ [B'']) = (j_1 j)^T \circ [B'']$ for all $[B'']$ in $Cor_k(A, B_2)$. In particular, $j^T \circ [j_1^T] = [(j_1 j)^T]$ in $Cor_k(B_2, B)$.

Any algebra map $f : A \rightarrow B$ induces a surjection $A \otimes_k B \rightarrow B$ and hence a canonical elementary correspondence $[f]$ from A to B .

Lemma 3.2.1. *Every elementary correspondence from A to B has a canonical factorization $j^T \circ [f]$, where $f : A \rightarrow B_1$ is homomorphism of normal k -algebras and $j : B \subseteq B_1$ is finite.*

Proof. Let $B' = A \otimes_k B/P$ be an elementary correspondence from A to B , and let B_1 be the normalization of B' . There is a canonical algebra map $f : A \rightarrow B' \subset B_1$, and the map j^T sends the class of the elementary correspondence $[f]$ to $[B_1]$. But since B' and B_1 have the same quotient field we have $[B_1 : B'] = 1$, and therefore $j^T \circ [f] = [B']$ in $Cor_k(A, B)$. \square

Let $C' = B \otimes_k C/Q$ be an elementary correspondence from B to C . In order to define the composition with $[C']$ from $Cor_k(A, B)$ to $Cor_k(A, C)$, it suffices by linearity to define $[C'] \circ [B']$ for all B' . Write $f : B \rightarrow C'$ for the algebra map implicit in the definition of C' .

Definition 3.3. Given elementary correspondences B' and C' , from A to B and from B to C , respectively. Let $\{Q_i\}$ denote the minimal primes of $B' \otimes_B C'$ and set $C'_i = B' \otimes_B C'/Q_i$. We define $[C'] \circ [B'] = \sum n_i [C'_i]$ in $Cor_k(A, C) \otimes \mathbb{Q}$, where the coefficients n_i are defined as follows.

Let K and K' denote the quotient fields of B and B' , respectively. Choose a finite normal field extension K'' of K' and let B'' denote the integral closure of B' in K'' . For each i , let $l(i)$ denote the number of minimal primes $\{Q_{ij}\}$ of $B'' \otimes_B C'$ over Q_i , and write $l = \sum l(i)$ for the total number of minimal prime ideals of the finite C -algebra $B'' \otimes_B C'$. Then

$$n_i = \frac{[B' : B] \cdot l(i)}{[C'_i : C'] \cdot l}.$$

The number n_i is independent of the choice of K'' by [SV, 5.13]. If $\text{char}(k) = 0$ then the n_i are always integers, and the composition $[C'] \circ [B']$ belongs to $Cor_k(A, C)$ by [SV, 5.11]. If $\text{char}(k) = p > 0$, then the n_i belong to $Cor_k(A, C) \otimes \mathbb{Z}[1/p]$ by *loc. cit.*

Example 3.3.1. When $B = C'$, *i.e.*, $j : C \subseteq B$ is a finite inclusion, we recover the transfer formula 3.2: $j^T \circ [B'] = [B']$. Indeed, we have $[C'_1 : C'] = [B' : B]$ and $l = l(1) = 1$ in the formula 3.3.

More generally, if we factor $[C'] = j^T \circ [f]$ as in 3.2.1, where $f : B \rightarrow C_1$ is a homomorphism and $j : C \subseteq C_1$ is finite, then it is clear from the construction in 3.3 that $[C'] \circ [B'] = (j^T \circ [f]) \circ [B'] = j^T \circ ([f] \circ [B'])$.

Paradigm 3.3.2. This is the special case where $A = B'$ and $C = C'$. Using 3.2.1, the correspondence $[C'_i]$ factors as $[j_i^T] \circ [f_i]$, where $f_i : B' \rightarrow C_i$ is the composition of $B' \rightarrow C'_i$ with the normalization $C'_i \subseteq C_i$, and $j_i : C \subseteq C_i$. Definition 3.3 yields the formula $[f] \circ [j^T] = \sum n_i [j_i^T] \circ [f_i]$. We may interpret this formula via the diagram:

$$\begin{array}{ccc} B' & \xrightarrow{(n_1 f_1, \dots)} & \prod C_i \\ j^T \downarrow & & \downarrow \prod j_i^T \\ B & \xrightarrow{f} & C. \end{array}$$

Example 3.3.3. If $f : A \rightarrow B$ and $g : B \rightarrow C$ are algebra maps, then $[g] \circ [f] = [gf]$. More generally, for every correspondence C' from B to C we have $[C'] \circ [f] = [C'] = d[C'_A]$, where C'_A is the image of $A \otimes_k C$ in C' (an elementary correspondence from A to C) and $d = [C' : C'_A]$.

These formulas come from the fact that (since $B = B'$) there is only one minimal prime of $B' \otimes_B C = C$, so $l(1) = l = 1$ and $C'_i = C$ in definition 3.3.

This formula is even associative: given an elementary correspondence $[A']$ from A_0 to A , then $[gf] \circ [A'] = [g] \circ ([f] \circ [A'])$ in $Cor_k(A_0, C)$. This is proven in [SV, 5.15], by showing that the coefficient m_j of $[C''_j]$ in $[gf] \circ [A']$ equals the sum of the products $n_i n'_j$, where $[f] \circ [A'] = \sum n_i [B'_i]$ and $[g] \circ [B'_i] = \sum n'_j [C''_{ij}]$.

Example 3.3.4. Axiom (3) of 0.3 holds. That is, if $S \subset B$ is multiplicatively closed, the composition 3.3 with $B \rightarrow S^{-1}B$ induces a canonical injection of $Cor_k(A, B)$ into $Cor_k(A, S^{-1}B)$, sending $[B']$ to $[S^{-1}B']$. This is a degenerate case of 3.3 because $S^{-1}B'$ and $S^{-1}B''$ are domains.

Passing to the limit over all S , we may also make sense of $Cor_k(A, E)$ when E is the quotient field of B . If $k(P)$ is the quotient field of A/P , then evidently $Cor_k(A, E) \cong \bigoplus_P Cor_k(k(P), E)$, the sum being taken over all primes P of A .

The following definition (taken from [SV, 5.6]) will allow us to avoid excess use of the adjective “normal” in describing the extension B'' of B and B' in definition 3.3.

Definition 3.4. Let $B \subset B_1$ be a finite extension, with quotient fields K and K_1 , respectively. We will say that B_1 is a *pseudo-Galois* extension of B if K_1 is a normal field extension of K , and if B_1 is the integral closure of B in K_1 . If K_1/K is separable and B_1 is flat over B , we will say that B_1 is *Galois* over B .

Example 3.4.1. Suppose that K is the quotient field of B , $K \subset K_1$ is a purely inseparable extension of degree p^r , and B_1 is the integral closure of B in K_1 . Then $j : B \subset B_1$ is pseudo-Galois and $[j] \circ [j^T] = p^r \cdot 1_{B_1}$, $[j^T] \circ [j] = p^r \cdot 1_B$. This follows directly from 3.3, since $B \rightarrow (B_1 \otimes_B B_1)_{red}$ is an isomorphism. It may also be deduced from 2.2.1 and 3.6 below.

Lemma 3.4.2. *Given a finite inclusion $j : B \subset B_1$ and $f : B \rightarrow C$, then for every $[B'_1] \in Cor_k(A, B_1)$ we have:*

$$[f] \circ (j^T \circ [B'_1]) = ([f] \circ [j^T]) \circ [B'_1].$$

Proof. Choose a pseudo-Galois extension B'' of B'_1 , and let B' denote the image of $A \otimes_k B$ in B'_1 . Let $\{Q_i\}$, $\{Q_\alpha\}$, $\{Q_\beta\}$ and $\{Q_\gamma\}$ denote the minimal primes of $B' \otimes_B C$, $B_1 \otimes_B C$, $B'_1 \otimes_B C$ and $B'' \otimes_B C$, respectively, and set $C_i = B' \otimes_B C/Q_i$, $C_\alpha = B_1 \otimes_B C/Q_\alpha$, $C_\beta = B'_1 \otimes_B C/Q_\beta$ and $C_\gamma = B'' \otimes_B C/Q_\gamma$. If there are l minimal primes in $B'' \otimes_B C$, then the coefficient of $[C_i]$ in the left-hand side is $l(i) \cdot [B'_1 : B]/l \cdot [C_i : C]$.

On the other hand, since $[C_\beta] = [C_\beta : C_i][C_i]$ in $Cor_k(A, C)$ for Q_β lying over Q_i , and $l(i) = \sum_{\beta/i} l(\beta)$, the right side is

$$\sum_{\alpha, \beta/\alpha} \frac{l(\alpha) \cdot [B_1 : B]}{l \cdot [C_\alpha : C]} \frac{l(\beta) \cdot [B'_1 : B_1]}{l(\alpha) \cdot [C_\beta : C_\alpha]} [C_\beta] = \sum_{\beta} \frac{l(\beta) \cdot [B'_1 : B]}{l \cdot [C_\beta : C]} [C_\beta] = \sum_i \frac{l(i) \cdot [B'_1 : B]}{l \cdot [C_i : C]} [C_i].$$

Now compare coefficients. \square

Consider the category $Norm_k$ of normal k -algebras of finite type and their algebra homomorphisms. Example 3.3.3 shows that the rule $f \mapsto [f]$ preserves composition. The following result shows that this rule defines a faithful functor, embedding $Norm_k$ as a subcategory of Cor_k if $\text{char}(k) = 0$, and of $Cor_k \otimes \mathbb{Z}[1/p]$ if $\text{char}(k) = p > 0$.

Theorem 3.5. *If $\text{char}(k) = 0$, the normal k -algebras of finite type form the objects of an additive category Cor_k , whose morphisms are the groups $Cor_k(A, B)$.*

If $\text{char}(k) = p > 0$, they form the objects of an additive category $Cor_k \otimes \mathbb{Z}[1/p]$, whose morphisms are the groups $Cor_k(A, B) \otimes \mathbb{Z}[1/p]$.

Proof. The only non-trivial point is to show that composition is associative. Let B' , C' and D' be elementary correspondences in $Cor_k(A, B)$, $Cor_k(B, C)$, and $Cor_k(C, D)$, respectively. Let C'_1 and D'_1 denote the normalizations of C' and D' , respectively, with $f : B \rightarrow C'_1$, $g : C \rightarrow D'_1$, $j_C : C \subseteq C'_1$ and $j_D : D \subseteq D'_1$ the natural maps. With this notation, 3.2.1 allows us to factor $[C']$ as $j_C^T \circ f$ and $[D']$ as $j_D^T \circ g$. See (3.5.1).

$$(3.5.1) \quad \begin{array}{ccccc} B & \xrightarrow{f} & C'_1 & \xrightarrow{(g_\alpha)} & \prod D_\alpha \\ & & j_C^T \downarrow & & \downarrow (j_\alpha^T) \\ & & C & \xrightarrow{g} & D'_1 & \xrightarrow{j_D^T} & D \end{array}$$

Using 3.3 and 3.2.1, choose $j_\alpha : D'_1 \subseteq D_\alpha$, λ_α and $g_\alpha : C'_1 \rightarrow D_\alpha$ so that $[g] \circ j_C^T = \sum \lambda_\alpha j_\alpha^T \circ [g_\alpha]$ in $Cor_k(C'_1, D'_1)$. By the construction in 3.3, we have

$$[D'] \circ [C'] = j_D^T \circ \sum \lambda_\alpha j_\alpha^T \circ [g_\alpha f] = \sum \lambda_\alpha (j_\alpha j_D)^T \circ [g_\alpha f].$$

Using 3.3.1, 3.3.3 and 3.4.2, we have the desired result, *viz.*:

$$\begin{aligned} ([D'] \circ [C']) \circ [B'] &= \left(\sum \lambda_\alpha (j_\alpha j_D)^T [g_\alpha f] \right) \circ [B'] \\ &= \sum \lambda_\alpha (j_\alpha j_D)^T ([g_\alpha f] \circ [B']) \\ &= j_D^T \circ \left\{ \sum \lambda_\alpha (j_\alpha^T \circ [g_\alpha]) \circ ([f] \circ [B']) \right\} \\ &= j_D^T \circ ([g] \circ j_C^T) \circ ([f] \circ [B']) \\ &= j_D^T \circ [g] \circ (j_C^T \circ ([f] \circ [B'])) \\ &= [D'] \circ ([C'] \circ [B']). \quad \square \end{aligned}$$

Lemma 3.6. *When E and F are field extensions of k , the formula 3.3 for the composition $Cor_k(A, E) \otimes Cor_k(E, F) \rightarrow Cor_k(A, F)$ agrees with the formula given in 2.2.*

Proof. It suffices (by 3.2.1) to consider a finite inclusion $j : E \subseteq E'$ and a homomorphism $i : E \rightarrow F$ of fields over k . We may further assume that the extension E'/E is either purely inseparable or separable.

If E' is purely inseparable over E , then $E' \otimes_E F$ is Artin local. If it has length λ and residue field F' , then $[E' : E] = \lambda[F' : F]$ and $l = l(1) = 1$, so we have $[i] \circ [E'] = \lambda[F']$, agreeing with 2.3.

Now suppose that E' is a separable field extension of E . Being a separable algebra over F , $E' \otimes_E F$ is a product of separable field extensions F'_α of F . Embedding E' in a Galois extension E'' of E , the Galois group $G = \text{Gal}(E''/E)$ acts transitively on the set of minimal primes (or factors) of both $E' \otimes_E F$ and $E'' \otimes_E F$. Fix a prime in each ring, and let G_1 and G_{11} be the corresponding stabilizer subgroups. If H is the subgroup of G corresponding to E' , then it is easy to see that $[G : H] = [E' : E]$, $[G_1 : H] = [F'_1 : F]$, $l = [G : G_{11}]$, and $l(i) = [G_1 : G_{11}]$ for all i . From this we deduce that $n_i = 1$ for all i , in agreement with 2.3.1. \square

Corollary 3.7. *Suppose that $f : B \rightarrow C$ is an injection of normal domains, and $C'_i = B' \otimes_B C/Q_i$ for some minimal prime Q_i . Then the coefficient n_i of $[C'_i]$ in $[f] \circ [B']$ equals the length of the Artinian algebra $(B' \otimes_B C)_{Q_i}$.*

In particular, $[f] \circ [B'] = \sum [C'_i]$ if $\text{char}(k) = 0$, or more generally if the quotient field of either B' or C is separable over the quotient field of B .

Proof. Let E, F and F_i denote the quotient fields of B, C and C'_i , respectively. Embedding $\text{Cor}_k(A, B)$ in $\text{Cor}_k(A, E)$ and $\text{Cor}_k(A, C)$ in $\text{Cor}_k(A, F)$, we see from 3.5, 3.6 and 2.2 that n_i equals the F_i -primary length of $B' \otimes_E F = (B' \otimes_B C) \otimes_C F$. \square

Proposition 3.8. *Suppose that $f : B \subset C$ is a Galois extension with Galois group G . Then G acts on $\text{Cor}_k(A, C)$ via $G \rightarrow \text{Hom}(C, C) \rightarrow \text{Cor}_k(C, C)$ and:*

- (1) f induces an isomorphism $\text{Cor}(A, B) \cong \text{Cor}(A, C)^G$.
- (2) $f \circ f^T = \sum_{g \in G} g$ as an endomorphism of $\text{Cor}(A, C)$.
- (3) $f : \text{Cor}(A, B) \rightarrow \text{Cor}(A, C)$ sends $[B']$ to $\sum [C'_i]$, using the notation of 3.3.

Proof. Item (3) implies (1) and (2) by the following argument. Given an elementary correspondence $C'_1 = A \otimes C/P_1$ from A to C , let P be the restriction of P_1 to $A \otimes B$ and set $B' = A \otimes B/P$. The group G acts transitively on the primes P_i over P and acts on $\text{Cor}(A, C)$, with g sending $[P_i]$ to $[gP_i]$. If G_1 is the stabilizer of P_1 then $f^T \circ [C'_1] = [G : G_1] \cdot [B']$ by 3.2. Items (1) and (2) follow since $[f] \circ [B'] = \sum A \otimes C/P_i$ by (3).

It remains to show that (3) holds, *i.e.*, that all coefficients $n_i = 1$. (Cf. [SV, 6.5(3)]). In the definition 3.3 of $[f] \circ [B']$, note that G permutes the Q'_i and the Q''_{ij} . If G_1 and G_{11} are the stabilizer subgroups of Q'_1 and Q''_{11} then it is easy to see that $|G| = [B' : B]$, $|G_1| = [C_1 : C]$, $l = [G : G_{11}]$, $l(i) = [G_1 : G_{11}]$ for all i . From this we deduce that $n_i = 1$ for all i , as claimed. \square

§4. UNIQUENESS OF COMPOSITION

We now turn to the verification of theorem 0.3, stated in the introduction.

Lemma 4.1. *If $j : B \subset C$ is a finite inclusion of degree $d = [C : B]$ then for every A :*

- (1) $[j]_\circ : \text{Cor}_k(A, B) \rightarrow \text{Cor}_k(A, C)$ is injective, and
- (2) $j^T \circ [j]_\circ : \text{Cor}_k(A, B) \rightarrow \text{Cor}_k(A, B)$ is multiplication by d .

Proof. Since (2) implies (1) by 3.3.1, it suffices by associativity 3.5 to observe that $j^T \circ [j] = d \cdot 1_B$, by 3.2. \square

We now have to show that axiom 0.3(2) holds.

Lemma 4.2. *Suppose that $j : B' \supset B$ is a flat pseudo-Galois extension, and that $f : B \rightarrow C$ is a homomorphism of normal domains. Let $\{Q_1, \dots, Q_l\}$ denote the minimal primes of $C' = B' \otimes_B C$ and set $C'_i = C'/Q_i$. Then the length λ of the Artinian ring C'_{Q_i} is independent of i , and $[f] \circ [j^T] = \lambda \sum [C'_i]$ in $\text{Cor}(B', C)$.*

Proof. The Galois group G acts transitively on the set of minimal prime ideals Q_i of $C' = B' \otimes_B C$, and induces isomorphisms between all the C'_i ; see [Bour, 5.2.3]. This proves that λ and $d = [C' : C]$ are independent of i . Since B' is flat over B , C' is flat over C and $C'_E = C' \otimes_C E$ is flat over the quotient field E of C . Since there are no embedded primes, $C'_E = \prod C'_{Q_i}$. Thus $[B' : B] = [C'_E : E] = \sum [C'_{Q_i} : C] = dl\lambda$. It follows from 3.3 that the coefficient of $[C'_i]$ in $[f] \circ [j^T]$ is $n_i = [B' : B]/dl = \lambda$. \square

Recall that if $B \subset B'$ is a finite inclusion of Dedekind domains, and \mathfrak{n} is a prime of B' lying over a prime \mathfrak{m} of B , the *ramification index* at \mathfrak{n} is that integer e such that $\mathfrak{m}B' = \mathfrak{n}^e$, i.e., e is the length of $(B'/\mathfrak{m}B')_{\mathfrak{n}}$.

Corollary 4.3. *Let $j : B \subset B'$ be a finite extension of normal domains and $\pi : B \rightarrow B/P$ the quotient by a height one prime ideal P of B such that B/P is integrally closed. If $\{P_i\}$ are the primes of B' lying over P , and e_i is the ramification index of B'_{P_i} over B_P , then $[\pi] \circ [j^T] = \sum e_i [B'/P_i]$ in $\text{Cor}_k(B', B/P)$.*

Proof. If B' is pseudo-Galois over B , this is just a restatement of 4.2. In the general case, choose a pseudo-Galois extension $j' : B' \subseteq B''$, let $\{P_{ij}\}$ denote the primes of B'' over the prime P_i of B' , with e_{ij} the ramification index of $B''_{P_{ij}}$ over B'_{P_i} , and let f_{ij} denote the degree $[B''/P_{ij} : B'/P_i]$. Since B''_P is flat over the Dedekind domain B'_P , $\sum_j e_{ij} f_{ij} = [B'' : B']$ for all i . Since $PB''_P = \prod_{i,j} P_{ij}^{e_i e_{ij}} B''_P$, we see by 4.2 that $\text{Cor}_k(B'', B) \xrightarrow{\pi} \text{Cor}_k(B'', B/P) \xrightarrow{j'} \text{Cor}_k(B', B/P)$ sends $[B''] = [(j'j)^T]$ to

$$[\pi] \circ [B''] \circ [j'] = \sum_{i,j} e_i e_{ij} [B''/P_{ij}] \circ [j'] = [B'' : B'] \sum_i e_i [B'/P_i].$$

By 4.1, $[B''] \circ [j'] = [j^T] \circ [j'^T] \circ [j'] = [B'' : B'] \cdot [j^T]$, so $[\pi] \circ [B''] \circ [j'] = [B'' : B'] [\pi] \circ [j^T]$ in $\text{Cor}_k(B', B/P)$. Since this group is torsionfree, we may divide by $[B'' : B']$ to get the result. \square

Proposition 4.4. *Let B be a normal domain and $\pi : B \rightarrow B/P$ the quotient at a height one prime ideal P such that B/P is integrally closed. If B' is an elementary correspondence from A to B , let $\{P_i\}$ be the primes of B' over P , and let λ_i denote the length of the Artin local ring $(B'/PB')_{P_i}$.*

Then $Cor_k(A, B) \rightarrow Cor_k(A, B/P)$ sends $[B']$ to $\sum \lambda_i [B'/P_i]$.

Note that the Artin local ring is R/I , where $R = B'_{P_i}$ and $I = PB'_{P_i}$. In fact, the length λ_i of R/I is also the multiplicity $e(I)$ of I . This is because R is flat over the discrete valuation domain B_P , so that $I^r/I^{r+1} = P^r/P^{r+1} \otimes_B R \cong R/I$ for all r .

Proof. Let B'_1 denote the integral closure of B' , with $j : B \subseteq B'_1$, and write $f : A \rightarrow B'_1$ for the induced map. Since $[B'] = [j^T] \circ [f]$ we must compute the coefficient n_i of $[B'/P_i]$ in $[\pi] \circ [j^T] \circ [f]$. If the set of primes of B'_1 over P_i is $\{P_{ij}\}$, we see from 4.3 that $n_i [B'/P_i] = \sum_j e_{ij} [B'_1/P_{ij}] \circ [f]$ in $Cor_k(A, B/P)$. If f_{ij} denotes the index $[B'/P_i : B'_1/P_{ij}]$, then $[B'_1/P_{ij}] \circ [f] = f_{ij} [B'/P_i]$ in $Cor_k(A, B/P)$, i.e., $n_i = \sum_j e_{ij} f_{ij}$. But the multiplicity λ_i of $I = PB'_{P_i}$ is $e(I, B''_{P_i}) = \sum_j e(I, B''_{P_{ij}}) = \sum_j e_{ij} f_{ij}$. \square

Lemma 4.5. *Let $f : B \rightarrow C$ be a homomorphism of normal domains, and let $h : C \hookrightarrow D$ be an injection. Given an elementary correspondence B' from A to B , write $\{P_i\}$ for the minimal primes of $C' = B' \otimes_B C$, and write $\{Q_\alpha\}$ for the minimal primes of $D' = B' \otimes_B D$, respectively. Suppose that $[f] \circ [B'] = \sum n_i [C'/P_i]$ in $Cor_k(A, C)$ and that $[hf] \circ [B'] = \sum m_\alpha [D'/Q_\alpha]$ in $Cor_k(A, D)$.*

Then $n_i = \text{length}(C'_{P_i})$ for all i if and only if $m_\alpha = \text{length}(D'_{Q_\alpha})$ for all α .

Proof. By 3.7, $[h] \circ [f] \circ [B']$ equals $\sum_\alpha n_{i(\alpha)} \lambda_\alpha [D'/Q_\alpha]$, where $P_i = P_{i(\alpha)}$ is the restriction of Q_α to C' and λ_α is the length of $(D'/P_i \otimes_C D)_{Q_\alpha}$. Localizing D , which is harmless by 3.3.4, we may assume that D is flat over C . In this case, tensoring a filtration of C'_{P_i} with D yields a filtration of $C'_{P_i} \otimes_C D$. Counting, we see that the length of $C'_{P_i} \otimes_C D$ is λ_α times the length of C'_{P_i} . Comparing coefficients of $[D'/Q_\alpha]$ yields the result. \square

We can now prove that axiom (2) holds for the composition \circ of 3.3.

Theorem 4.6. *Let B' be an elementary correspondence from A to B such that $B \subseteq B'$ is flat, and let $f : B \rightarrow C$ be a homomorphism of normal domains. Then $[f] \circ [B'] = \sum_i \lambda_i [C'/P_i]$, where: $C' = B' \otimes_B C$; $\{P_i\}$ are the minimal primes of C' ; and λ_i denotes the length of C'_{P_i} .*

Proof. This is clear if f is an injection, by 3.7. In the general case, let \bar{B} denote the integral closure of B/P , where P is the kernel of f . Applying 4.5 to $\bar{B} \subseteq C$, we may assume that $C = \bar{B}$. Replacing B by $B[1/b]$ for suitable $b \notin P$, allowed by 3.3.4, we may also assume that $\bar{B} = B/P$.

If R is any normal domain between B and its quotient field, of finite type over k , then $R' = B' \otimes_B R$ is an elementary correspondence from A to R (by flatness of B'), and axiom (3) implies that $[R']$ is the image of $[B']$ in $Cor_k(A, R)$.

Pick $x \in P$ nonzero, let R_0 denote the integral closure of $B[P/x]$, and pick a prime ideal $Q_0 \subset R_0$ minimal over xR_0 ; there is an $s \in R_0 - Q_0$ so that both $R = R_0[1/s]$ and R/Q are normal, where $Q = Q_0[1/s]$. By 4.4, the image of $[R']$ in $Cor_k(A, R/Q)$ is $\sum \text{length}(R'/QR')_{Q_i}$. The theorem now follows from 4.5. \square

Example 4.6.1. Suppose that B/P is normal and consider the effect of the projection $\pi : B \rightarrow B/P$ on a flat elementary correspondence B' , as in 4.6. If $\{P_i\}$ are the primes of B' over P , then the coefficient of $[B'/P_i]$ in $[\pi] \circ [B']$ is also the ratio of multiplicities $e(PB'_{P_i})/e(PB_P)$. Indeed, because B'/B is flat, this follows from 4.6 by the formula $e(PB'_{P_i}) = e(PB_P)\text{length}(B'/PB')_{P_i}$ of [BH, 4.6.9]. (Cf. [SV1, 3.5.4].)

Corollary 4.7. *Any transfer functor F satisfies the flat Mackey condition: if $j : B' \supset B$ is a finite flat extension of normal algebras and $f : B \rightarrow C$ is a homomorphism, then*

$$\begin{array}{ccc} F(B') & \xrightarrow{(\lambda_1 f_1, \dots)} & \prod F(C_i) \\ \downarrow & & \downarrow \\ F(B) & \xrightarrow{f} & F(C). \end{array}$$

commutes, where the product is over all minimal primes Q_i of $C' = B' \otimes_B C$, C_i is the normalization of C'/Q_i , $f_i : B' \rightarrow C_i$ and $\lambda_i = \text{length}(C'_{Q_i})$.

Theorem 4.8. *The composition \circ of 3.3 is the unique associative composition satisfying axioms (1), (2) and (3) of 0.3.*

Proof. Axioms (1) and (3) hold for \circ by 3.3.4, 4.6 and 4.4. Conversely, suppose given a composition law \square satisfying the axioms; we must show \square agrees with \circ .

Given a finite extension $j : B \subset B'$ of degree d , with quotient fields $i : B \subset E$ and $i' : B' \subset E'$, and $a \in \text{Cor}_k(A, B')$, axiom (3) implies that $[i] \square [j^T] \square a = [j^T_E] \square [i'] \square a$. We saw in 2.6 that $[j^T_E] \square = [j^T] \square$ is determined by axioms (1) and (2). It follows that $[j^T] \square a = [j^T] \circ a$.

Next, we claim that any elementary correspondence $[B']$ from A to B has a canonical factorization as the composition of $f : A \rightarrow B'_1$ and $j^T \in \text{Cor}_k(B'_1, B)$, as in 3.2.1. To see this we map $[B']$ and the composition to $\text{Cor}_k(A, E)$ where E is the quotient field of B . Here $[B']$ and $[j^T] \square [f]$ agree, by the argument of 2.6 which uses axioms (1) and (2). By (3) they must agree in $\text{Cor}_k(A, B)$, as claimed.

We are left to show that the composition $[f] \square [B']$ of $[B']$ with a homomorphism $f : B \rightarrow C$ equals $[f] \circ [B']$. Let P denote the kernel of f and suppose first that B'_P is flat over B_P . (This is the case for example when f is an injection.) Choose $S \subset (B - P)$ so that $S^{-1}B'$ is flat over $S^{-1}B$; the composition of $[f] \square [B']$ with $i : C \rightarrow S^{-1}C$ must equal $[S^{-1}f] \circ [S^{-1}B']$ by axioms (2) and (3). That is, $[i] \square [f] \square [B'] = [i] \circ [f] \circ [B']$. This implies by (3) that $[f] \square [B'] = [f] \circ [B']$.

In the general case, pick $x \in P$ nonzero and let R denote the integral closure of $B[P/x]$ in the quotient field E of B . For any height one prime Q of R minimal over xR there is an extension $j : C \subset D$ so that jf factors as $B \rightarrow R \rightarrow R/Q \rightarrow D$. Since $B \subset R \subset E$, the composition of $[B']$ with $B \rightarrow R$ is an elementary correspondence $[R']$, uniquely determined by axiom (3). Since R'_Q is flat over the discrete valuation domain R_Q , the composition of $[R']$ with $R \rightarrow D$ is uniquely determined by the flat case above. But the composition with $C \rightarrow D$ is an injection from $\text{Cor}_k(A, C)$ to $\text{Cor}_k(A, D)$, by the previous paragraph and 3.7. Hence the composition $[f] \square [B']$ must equal $[f] \circ [B']$, as required. \square

§5. CORRESPONDENCES IN POSITIVE CHARACTERISTIC

If k is a field of characteristic $p > 0$, the composition of two elementary correspondences need not have integer coefficients, as the following example shows. (The case $n = 1$ is due to Merkurjev [SV, p.77].)

Example 5.1. Let $a_1, a_2 \in k$ be linearly independent in $k^\times/k^{\times p}$ and set $q = p^n$, so that the field $\ell = k(\alpha_1, \alpha_2)$ has degree q^2 over k , where $\alpha_i = \sqrt[q]{a_i}$. Set $A = k[T_0, T_1, T_2]/(a_1T_1^q + a_2T_2^q = T_0^q)$ and $B = \ell[T_1, T_2]$. Both A and B are integrally closed, and the map $j : A \rightarrow B$ sending T_0 to $\alpha_1T_1 + \alpha_2T_2$ makes B a finite A -algebra. The field $L = \ell(T_1, T_2)$ of fractions of B is purely inseparable of degree q over the field K of fractions of A . At the maximal ideal $\mathfrak{m} = (T_0, T_1, T_2)A$, we have $A/\mathfrak{m} = k$ and $B/\mathfrak{m}B = \ell$, and the extension $\bar{j} : k \subset \ell$ has degree q^2 .

Consider the projections $\pi : A \rightarrow A/\mathfrak{m} = k$ and $\tilde{\pi} : B \rightarrow B/\mathfrak{m}B = \ell$. Since B is pseudo-Galois over A , definition 3.3 yields the formula $[\pi] \circ [j^T] = (1/p^n)[\bar{j}^T] \circ [\tilde{\pi}]$ in $Cor_k(B, k)$.

$$\begin{array}{ccc} B & \xrightarrow{(1/q)\tilde{\pi}} & \ell \\ j^T \downarrow & & \downarrow \bar{j}^T \\ A & \xrightarrow{\pi} & k. \end{array}$$

Remark 5.1.1. The value of the coefficient n in the formula $[\pi] \circ [j^T] = n[\bar{j}^T] \circ [\tilde{\pi}]$ is forced by the formulas $[j] \circ [j^T] = q \cdot 1_B$ and $[\bar{j}] \circ [\bar{j}^T] = q^2 \cdot 1_\ell$ of 3.4.1. Indeed, we have

$$q[\tilde{\pi}] = [\tilde{\pi}] \circ ([j] \circ [j^T]) = [\tilde{\pi}j] \circ [j^T] = [\bar{j}] \circ [\pi] \circ [j^T] = n[\bar{j}] \circ [\bar{j}^T] \circ [\tilde{\pi}] = nq^2[\tilde{\pi}].$$

The following definition is taken from [SV1, 3.3.9] [MVW, 1A.9]. Recall that a finite correspondence is just a \mathbb{Z} -linear combination of elementary correspondences. The composition of two correspondences is a $\mathbb{Z}[1/p]$ -linear combination of elementary correspondences by [SV, 5.11]. If P is a prime of B , we write $k(P)$ for the field B_P/PB_P .

Definition 5.2. A finite correspondence a from A to B is called *universally integral* when the composition $[\pi] \circ a$ has integer coefficients for every map $\pi : B \rightarrow k(P)$, where P ranges over all prime ideals of B . We write $Cor_k(A, B)$ for the set of all universally integral correspondences from A to B ; since these correspondences are clearly closed under addition, $Cor_k(A, B)$ is an abelian group.

Note that the condition for being universally integral could also be phrased more awkwardly in terms of the map from B to the integral closure of B/P .

Examples 5.2.1. a) Any homomorphism $f : A \rightarrow B$ is universally integral by 3.3.3. b) If B' is flat over B , $[B']$ is universally integral by axiom 0.3(2). c) If B is a Dedekind domain, every correspondence is universally integral by (b). d) An elementary correspondence $[B']$ is universally integral if the field extensions $k(P) \subseteq k(P'_i)$ are separable for every prime ideal P of B and every prime P'_i of B' lying over P ; see [SV, 5.11].

Nonexample 5.2.2. In example 5.1, the correspondence $[j^T]$ from B to A is not universally integral, but $p^n[j^T]$ is. This example also shows that transfer maps need not preserve universally integral correspondences; $j^T \circ$ does not send $Cor_k(B, B)$ into $Cor_k(B, A)$.

Lemma 5.3. *If R is regular, every finite correspondence is universally integral.*

Proof. (Cf. [SV, 5.18]) We show that $R \rightarrow k(P) = R_P/P$ preserves finite correspondences, by induction on the height of the prime ideal P . If $P=0$ this is axiom (3). Otherwise, pick an $x \in P - P^2$ and observe that R_P and R_P/xR_P are regular local rings. Localizing R , we may assume that xR is prime and hence R/xR is regular. Since $R \rightarrow R/xR$ preserves finite correspondences by 4.4, and $R/xR \rightarrow k(P)$ preserves them by induction, we are done. \square

Lemma 5.4. *The composition with any homomorphism $f : B \rightarrow C$ sends universally integral correspondences to universally integral correspondences, i.e., sends $Cor_k(A, B)$ to $Cor_k(A, C)$.*

Proof. Fix a universally integral correspondence $a \in Cor_k(A, B)$. If Q is a prime ideal of C and $\pi : C \rightarrow C_Q/Q$ is the projection, we must show that the composition $[\pi] \circ [f] \circ a$ has integer coefficients. Let P denote the restriction of Q to B , and \bar{a} the image of a in $Cor_k(A, B_P/P)$. Since \bar{a} has integer coefficients, and $B_P/P \rightarrow C_Q/Q$ is an injection, the image of \bar{a} (and hence a) in $Cor_k(A, C_Q/Q)$ has integer coefficients by 3.7. \square

Proposition 5.5. *The normal k -algebras of finite type form the objects of an additive category Cor_k , whose morphisms are the groups $Cor_k(A, B)$.*

Proof. Given theorem 3.5, we need only show that the composition of universally integral correspondences is universally integral. That is, the composition $b \circ a$ of $b \in Cor_k(B, C)$ and $a \in Cor_k(A, B)$ belongs to $Cor_k(A, C)$. Fix a prime ideal Q of C ; localizing C , we may suppose that C/Q is normal and need to show that $[\pi] \circ b \circ a$ has integer coefficients, where $\pi : C \rightarrow C/Q$.

Because b is universally integral, $[\pi] \circ b = \sum n_i [Q_i]$ for certain primes Q_i in $B \otimes k(Q)$ and integers n_i . Each $[Q_i] \circ a$ has integer coefficients by 3.2 and 5.4, since it is the composition of a with $f_i : B \rightarrow B \otimes k(Q)/Q_i$, followed by the transfer $Cor_k(A, k(Q_i)) \rightarrow Cor_k(A, k(Q))$. Substituting, we see that $[\pi] \circ b \circ a$ has integer coefficients. \square

Proposition 5.6. *If B' is any elementary correspondence between normal k -algebras, and $\text{char}(k) = p > 0$, then some multiple $p^n[B']$ is universally integral.*

Proof. Let ν denote the number of generators of B' as a B -module, and let B'', l , etc. be as in definition 3.3. For every prime Q of B , the $k(Q)$ -vector space $B' \otimes_B k(Q)$ has dimension at most ν . Hence the denominators of the coefficients arising in 3.3 for the image of B' under $B \rightarrow k(Q)$ are powers of p bounded above by ν . If p^n is the largest power of p less than or equal to ν , then evidently $p^n[B']$ is universally integral. \square

Proposition 5.7. *The inclusion $Cor_k^0 \subset Cor_k$ has a right adjoint $A \mapsto \ell_A$, where ℓ_A is the integral closure of k in A .*

Proof. First note that if A is a domain then ℓ_A is finite-dimensional over k , because it injects into A/\mathfrak{m} for any maximal ideal \mathfrak{m} of A . A finite inclusion $i : A \subset B$ induces

$i_\ell : \ell_A \subseteq \ell_B$; the adjoint sends the finite correspondence $[i^T]$ to the element $[B : A\ell_B] \cdot [i_\ell^T]$ of $Cor_k(\ell_B, \ell_A)$. [Since $Cor_k(A, B)$ is a subgroup of all finite correspondences, the adjoint is defined.] Using the factorization 3.2.1 of elementary correspondences in Cor_k , it is easy to verify that ℓ_A depends functorially on A , and that the inclusion $\eta_A : \ell_A \subset A$ is a natural transformation.

We must show that the natural map $[\eta_A] \circ : Cor_k(E, \ell_A) \rightarrow Cor_k(E, A)$ is an isomorphism for all E and A . Since it is an injection by 3.7, it suffices to prove that it is a surjection. Given an elementary correspondence A' from E to a domain A , let ℓ' denote the subfield $E\ell_A$ of A' generated by the canonical subfields E and ℓ_A . Since ℓ' and A are linearly disjoint over ℓ_A , $\ell' A = \ell' \otimes_{\ell_A} A$ is a domain, finite over A . From 3.7 we see that $[\eta_A] \circ [\ell'] = [\ell' A]$. Since the natural surjection $E \otimes_k A \rightarrow (E \otimes_k \ell_A) \otimes_{\ell_A} A \rightarrow A'$ factors through $\ell' A$, we must have $\ell' A = A'$, as desired. \square

Base-change 5.8. If $k \subset \ell$ is a finite field extension, there is an obvious forgetful functor $Cor_\ell \rightarrow Cor_k$; an elementary correspondence of ℓ -algebras determines an elementary correspondence of the underlying k -algebras. Hence any transfer functor on normal k -algebras restricts to a transfer functor on normal ℓ -algebras.

Now suppose that ℓ is a separable field extension of k . If A is a k -algebra and B is an ℓ -algebra, then there is a canonical identification $Cor_k(A, B) \cong Cor_\ell(A \otimes_k \ell, B)$. Indeed, the separability of ℓ implies that $A \otimes_k \ell$ is normal; (see [BH, 2.2.23]), and a sum of primes in $A \otimes_k B$ being universally integral is the same over k as it is over ℓ .

Given this, it is easy to see that the forgetful functor is right adjoint to the additive functor $Cor_k \rightarrow Cor_\ell$ sending A to $A \otimes_k \ell$. Hence transfer functors over ℓ extend to transfer functors over k .

§6. EXAMPLES OF TRANSFER FUNCTORS

By definition, a transfer functor F is an additive covariant functor from Cor_k to abelian groups. In order to specify F , we need only construct F as a functor from normal k -algebras to abelian groups, and describe the transfer maps (in characteristic $p > 0$ we need slightly more). It suffices to work with integral domains, because F must commute with the categorical product in Cor_k : $F(A \times B) = F(A) \times F(B)$.

Trace maps 6.1. The underlying abelian group ($A \mapsto A$) is a transfer functor. If $A \subset B$ is a finite integral extension of normal domains, with fields of fractions $K \subset L$, then the trace map $L \rightarrow K$ induces a trace map $B \rightarrow A$. Indeed, the trace of $b \in B$ is integral over A and lies in K , so it is in A .

Norm maps 6.2. The group of units $U(A) = A^\times$ is a transfer functor. If $A \subset B$ is a finite extension of normal domains, with fields of fractions $K \subset L$, then the norm map $L^\times \rightarrow K^\times$ induces a norm map $B^\times \rightarrow A^\times$. Indeed, the norm of a unit $b \in B^\times$ and its inverse are both integral over A and lie in K , so they belong to A .

Constant transfer functors 6.3. The constant transfer functor associated to an abelian group M satisfies $F_M(A) = M$ for every domain A , and if $A \rightarrow B$ is a map of domains then $F_M(A) \rightarrow F_M(B)$ is the identity. If B' is an elementary correspondence from A to B then the map $F_M(B) \rightarrow F_M(A)$ is multiplication by the rank $[B' : B]$.

Representable transfer functors 6.4. Fix a k -algebra R . Then $A \mapsto Cor_k(R, A)$ is a transfer functor. The Yoneda Lemma, that $F(R) = \text{Hom}(Cor_k(R, -), F)$ for every transfer functor F , implies that $Cor_k(R, -)$ is a projective object in the abelian category of transfer functors, and that every transfer functor is a quotient of a direct sum of such representable transfer functors.

a) If $R = k$ then $Cor_k(k, A) = \mathbb{Z}$ for every domain A . If $A \subset B$ is a finite inclusion then the transfer map $Cor_k(k, B) \rightarrow Cor_k(k, A)$ is multiplication by the rank of B over A , $[B : A]$. In fact, this is just the constant transfer functor $F_{\mathbb{Z}}$ in the sense of 6.3.

b) If $R = k[x]$ then $Cor_k(k[x], A)$ is the free abelian group on the set of irreducible monic polynomials in $A[t]$. If $A \subset B$ then we may identify the transfer map $Cor_k(k[x], B) \rightarrow Cor_k(k[x], A)$ with the norm map applied to the monic polynomials.

Flasque transfer functors 6.4.1. If H is a contravariant additive functor from abelian groups to itself, and S is a fixed algebra, then $A \mapsto H Cor_k(A, S)$ is a transfer functor. In particular, for $S = k$ and $H = \text{Hom}(-, M)$, we see that $F(A) = \prod_{\mathfrak{m}} M$ is a transfer functor, where the product ranges over all maximal ideals \mathfrak{m} of A . ($F(A) = \bigoplus_{\mathfrak{m}} M$ is a transfer sub-functor.) If $A \supset B$ is finite, the component of $F(A) \rightarrow F(B)$ corresponding to a maximal ideal \mathfrak{n} of B is the sum of the components indexed by the finitely many maximal ideals \mathfrak{m}_i of A over \mathfrak{n} . This example is interesting because F is a flasque Zariski sheaf on each $\text{Spec}(A)$.

There are many variations, among which is the following. Set $E(A) = \prod_{\mathfrak{m}} A_{\mathfrak{m}}^\times$, where the product is again over all maximal ideals of A . If $A \supset B$ is finite, the component of $E(A) \rightarrow E(B)$ corresponding to $\mathfrak{n} \subset B$ is the product of the norms $A_{\mathfrak{m}_i}^\times \rightarrow B_{\mathfrak{n}}^\times$ of 6.2, over the maximal ideals \mathfrak{m}_i of A over \mathfrak{n} . If $f : B \rightarrow C$ is a homomorphism, $E(B) \rightarrow E(C)$ is defined because $\text{Spec}(C) \rightarrow \text{Spec}(B)$ preserves maximal ideals. The verification that E is a transfer functor is easy, given 6.2.

Hecke Functors 6.5. Recall from 1.1 that a Hecke functor for the Galois group $Gal(\bar{k}/k)$ is a contravariant additive functor from Cor_G to abelian groups. Since Cor_G is a subcategory of Cor_k by 3.6, 2.4 and 2.5, the restriction of any transfer functor to Cor_G is a Hecke functor.

Conversely, any Hecke functor M induces a transfer functor by the formula $M(A) = M(\ell_A^s)$, where $\ell_A^s \subseteq \ell_A$ is the separable closure of k in A . Indeed, this is just the composition of M with the adjunction $Cor_k \rightarrow Cor_k^0 \rightarrow Cor_G$ of 5.7.

A slightly more bizarre example occurs when k is not perfect. If ℓ_A is the integral closure of k in A , then $F(A) = \Omega_{\ell_A/k}$ is a transfer functor by 2.10 and 5.7.

Galois modules 6.6. Let G denote the Galois group of \bar{k}/k , and let M be a discrete G -module. If A is a domain, and ℓ_A is the integral closure of k in A , we set $M(A) = M^H$, where $H = Gal(\bar{k}/\ell_A)$. This is a transfer functor, because M is a Hecke functor by 1.4 and the formula in 6.5 comes from 2.5.1.

Étale Sheaves 6.7. Regarding a transfer functor F as a presheaf on normal schemes over k , we may sheafify to get an étale sheaf F_{et} . The associated covariant functor on normal algebras is a transfer functor. This is proven in [MVW, 6.17] (where smoothness is not needed), or in [V, 3.1.12 and 3.3.1].

For example, if E and F are the flasque Zariski sheaves of 6.4.1, then the transfer functors associated to the flasque étale sheaves E_{et} and F_{et} may be described as follows. Fix an algebraic closure \bar{k} of k , and set $F_{et}(A) = \prod_x M$, the product being taken over the set of all k -algebra maps $x : A \rightarrow \bar{k}$. The transfer formula for $F_{et}(A) \rightarrow F_{et}(B)$ is similar to that of 6.4.1. Similarly, $E_{et}(A) = \prod_x (A_x^{sh})^\times$, where A_x^{sh} denotes the strict henselization of A along x ; again the transfer map is induced from that of 6.4.1.

Étale Cohomology 6.7.1. The examples in 6.7 are special cases of a construction due to Deligne: the terms in the canonical flasque resolution of an étale sheaf with transfers F are themselves étale sheaves with transfers. (See [MVW, 6.20].) It follows that the étale cohomology of F , $A \mapsto H_{et}^*(\text{Spec}(A), F)$, is a transfer functor.

Picard groups 6.8. The Picard group is a well known functor from rings to abelian groups. By 6.2 and 6.7.1, it is also a transfer functor. This was first observed, in nascent form, by Roggenkamp and Scott [RS].

Here is a proof avoiding 6.7.1. Define C by the short exact sequence of transfer functors $0 \rightarrow A^\times \rightarrow E_{et}(A) \rightarrow C(A) \rightarrow 0$, where the middle term is the transfer functor described in 6.7. Let C_{et} denote the étale sheaf associated to C ; it is a transfer functor by 6.7. The long exact cohomology sequence yields an exact sequence, proving that Pic is a transfer functor:

$$0 \rightarrow A^\times \rightarrow \prod_x (A_x^{sh})^\times \rightarrow C_{et}(A) \rightarrow \text{Pic}(A) \rightarrow 0.$$

It is instructive to describe the transfer map $i_* : \text{Pic}(B) \rightarrow \text{Pic}(A)$ associated to a finite integral extension $i : B \supset A$ of normal domains. If we regard $L \in \text{Pic}(B)$ as a locally free B -module, there are $s_1, \dots, s_n \in A$ and units $b_{ij} \in B[1/s_i s_j]^\times$ so that L is obtained by patching: there are isomorphisms $L[1/s_i] \cong B[1/s_i]$ so that the composite isomorphisms of $B[1/s_i s_j]$ are multiplication by b_{ij} . We define $i_*(L)$ to be the A -module

obtained by patching the modules $A[1/s_i]$ using the norms of the b_{ij} . It is not hard to see that this is independent of the choices made.

Alternatively, note that B is flat over A (say of rank d) except on a set Y of codimension ≥ 2 , because at every height one prime A_P is a discrete valuation ring. Hence $\wedge_A^d(L)$ is locally free of rank one except on Y , and its double dual $\wedge_A^d(L)^{**}$ is locally free; we have $i_*(L) = \wedge_A^d(L)^{**}$.

Homotopy invariant transfer functors 6.9. We say that F is *homotopy invariant* if $k \rightarrow k[t]$ induces an isomorphism $F(A) \cong F(A[t])$ for all A . The units and constant transfer functors (6.2 and 6.3) are homotopy invariant but examples 6.1 and 6.4 are not.

For any transfer functor F and any n , $C_n F(A) = F(A[T_n])$ is a transfer functor, where $k[T_n]$ denotes the polynomial ring $k[t_0, \dots, t_n]/(\sum t_i = 1)$. Moreover, any algebra map $k[T_m] \rightarrow k[T_n]$ induces a morphism of transfer functors $C_m F \rightarrow C_n F$. The coequalizer $[F]A$ of $F(A[t]) \rightrightarrows F(A)$ (induced by $t \mapsto 0, 1$) is homotopy invariant, and in fact $F \rightarrow [F]$ is universal with respect to this property.

Applying this construction to the simplicial ring $k[T_\bullet]$ yields a chain complex $C_\bullet F$ in the abelian category of transfer functors. Hence the homology $A \mapsto H_* C_\bullet F(A)$ and the cohomology $A \mapsto H^* \text{Hom}(C_\bullet F(A), \mathbb{Z}/m)$ are transfer functors. They are also homotopy invariant by [W81, 2.4].

Zariski sheaves 6.9.1. If F is homotopy invariant, then the Zariski sheaf F_{zar} is also a transfer functor by [V, 3.1.12] or [MVW, 21.1].

Betti Cohomology 6.10. If $k = \mathbb{C}$ then the Betti cohomology $H^*(X, \mathbb{Z}/m)$ of the topological space X of \mathbb{C} -points of $\text{Spec}(A)$ are transfer functors. This is a special case of 6.7.1, since the Betti cohomology agrees with the étale cohomology groups $H_{\text{ét}}^n(\text{Spec } A, \mathbb{Z}/m)$.

This is essentially the main result 7.8 of [SV], since the singular homology $A \mapsto H_*^{\text{sing}}(\text{Spec}(A), \mathbb{Z}/m)$ of [SV] is isomorphic to the cohomology of the cochain complex $S^*(A) = \text{Hom}(C_\bullet \text{Cor}_k(A, k), \mathbb{Z}/m)$ of 6.9 by [SV, 6.7–6.8]. Since S^* is a complex of transfer functors by 6.4.1, its cohomology also consists of transfer functors.

Base change 6.11. If $k \subset \ell$ is a finite field extension, we saw in 5.8 that the restriction of any transfer functor on normal k -algebras is also a transfer functor on normal ℓ -algebras. If ℓ is a separable field extension of k , the extension $\tilde{F}(A) = F(A \otimes_k \ell)$ of a transfer functor F over ℓ is also a transfer functor over k by 5.8. For Galois modules, considered as transfer functors by 6.6, these operations are just the classical restriction and extension of modules over a group.

The following lemma could have been used for most of the above examples.

Lemma 6.12. *Let F be a functor from normal k -algebras to abelian groups which preserves products and injections. Suppose that F also has a functorial transfer for finite extensions which commutes with localization and satisfies the flat Mackey condition 4.7. If $\text{char}(k) = p > 0$, we assume that each $F(A)$ is uniquely p -divisible. Then F extends to a transfer functor.*

Proof. Any elementary correspondence has a canonical factorization $A \rightarrow B_1 \supseteq B$ by 3.2.1; we define $F(A) \rightarrow F(B)$ by (0.2). To see that composition in Cor_k is preserved

by F , it suffices to compare the composition of the transfer $F(B_1) \rightarrow F(B)$ with the map induced by $f : B \rightarrow C$. (This uses 3.5.) Localizing C , we may suppose that each $C_i = C'/Q_i$ is normal and flat over C . As in the proof of 4.6, there is a normal domain R between B and its quotient field, and an extension $C \subseteq D$, such that $B \rightarrow C \rightarrow D$ factors through R , and $R_1 = B_1 \otimes_B R$ is flat over R . Localizing D , we may assume that $B_1 \otimes_B D = \coprod D_\alpha$ is normal, so we have a commutative diagram

$$\begin{array}{ccccccc}
 F(B_1) & \longrightarrow & F(R_1) & \longrightarrow & \prod F(D_\alpha) & \longleftarrow & \prod F(C_i) & \longleftarrow & F(B_1) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 F(B) & \longrightarrow & F(R) & \longrightarrow & F(D) & \longleftarrow & F(C) & \longleftarrow & F(B).
 \end{array}$$

The two bottom composites $F(B) \rightarrow F(D)$ are the same, and the map $F(C) \rightarrow F(D)$ is an injection because $C \rightarrow D$ is. The middle two squares commute by the flat Mackey condition. The upper two composites $F(B_1) \rightarrow \prod F(D_\alpha)$ are the same by 3.2.1 applied to the correspondence from B_1 to D . The left square commutes because transfer commutes with localization (compare B and R to a common localization). The result now follows by a diagram chase. \square

Following [SV, 6.1], a functor F from normal algebras to abelian groups is called a *qfh-sheaf* if it defines a Zariski sheaf on each normal affine scheme $\text{Spec}(B)$ and if $F(A) \rightarrow F(B)^G$ is an isomorphism for every pseudo-Galois extension $A \subset B$ with Galois group G . Since any constant sheaf is a *qfh-sheaf*, it is clear that *qfh-sheaves* need not be p -divisible in characteristic p .

Corollary 6.13. *Let F be a *qfh-sheaf*. If $\text{char}(k) = p > 0$, assume that the groups $F(A)$ are uniquely p -divisible. Then F has transfer maps, making it into a transfer functor.*

Proof. Given $A' \supset A$, we define the transfer map $F(A') \rightarrow F(A)$ by embedding A' in a pseudo-Galois extension A'' with group G , and sending $a \in F(A')$ to

$$[k(A') : k(A)]_{\text{insep}} \sum_{h \in \text{Hom}_A(A', A'')} h_*(a),$$

regarded as an element of $F(A'')^G \cong F(A)$. Suslin and Voevodsky verify the (flat) Mackey condition 3.3.2 in [SV, 5.17]. \square

Let $\tau(A)$ denote the torsion submodule of the absolute Kähler differentials Ω_A where A is a noetherian domain. If A is regular, then $\tau(A) = 0$ because Ω_A is projective. If A is normal, then the annihilator of $\tau(A)$ has height at least two. If $pd_A \Omega_A < \infty$ and A is separable over k , this is actually equivalent to the normality of A by [Vas]. The following lemma uses the trick introduced in the proof of theorem 4.6.

Lemma 6.14. *Suppose that $\mathbb{Q} \subset A$. If $f : A \rightarrow B$ is a homomorphism of domains, then $\Omega_A \rightarrow \Omega_B$ sends $\tau(A)$ into $\tau(B)$.*

Proof. We may assume that B is a field E . In characteristic zero, any field extension $E \subset F$ induces an injection of Ω_E into Ω_F , so we may pass to arbitrary extensions of

E . Let P be the kernel of f and x nonzero in P ; passing first to $A[P/x]$, normalizing and localizing at a height one prime, we get a DVR R between A and its quotient field such that (after extending E) f factors through R . But $\tau(A)$ already vanishes in the torsionfree module Ω_R . \square

Lemma 6.15. *Let A be a finite-dimensional augmented local algebra over a field K , so that there is a canonical splitting $\Omega_A \cong \Omega_K \oplus \Omega_{A,K}$. Then the transfer map $\Omega_A \rightarrow \Omega_K$ is zero on $\Omega_{A,K}$, and is multiplication by $\dim_K(A)$ on the summand Ω_K .*

Proof. Choose a K -basis of A compatible with the filtration $A \supset I \supseteq I^2 \supseteq \cdots \supseteq I^n = 0$, where I is the kernel of the augmentation $A \rightarrow K$. For any $a \in A$, the matrix (a_{ij}) representing multiplication by a is lower triangular, and strictly lower triangular if $a \in I$. Since the trace of adb is $\sum_{i,j} a_{ij} db_{ji}$ and $\Omega_{A,K}$ is generated over K by the terms adb with either a or b in I , the result is immediate. \square

Corollary 6.16. *Let A be a finite-dimensional local algebra over a field K , with residue field L separable over K . Then the transfer $\Omega_A \rightarrow \Omega_K$ is $\text{length}(A)$ times the projection $\Omega_A \rightarrow \Omega_L$, followed by the usual transfer $\Omega_L \rightarrow \Omega_K$.*

Theorem 6.17. *In characteristic zero, the Kähler differentials induce a transfer functor $A \mapsto \Omega_A/\tau(A)$.*

Proof. This is a functor on normal k -algebras by 6.14, and preserves products and injections in characteristic zero. If $A \supset B$ is a finite inclusion, the extension of quotient fields $E \supset F$ induces a transfer $\Omega_E \rightarrow \Omega_F$ as in 2.10, sending dx to dy , $y = \text{trace}(x)$. It is easy to see that this transfer sends the image $\Omega_A/\tau(A)$ of Ω_A into the image $\Omega_B/\tau(B)$ of Ω_B , yielding a transfer maps constructed so as to commute with localization.

Finally, the flat Mackey condition is easy to check. Let $A \supset B$ be finite and flat, with quotient field extension $E \supset F$. If $B \rightarrow F'$ is an injection, with F' a field, the diagram

$$\begin{array}{ccccc} \Omega_A/\tau(A) & \longrightarrow & \Omega_E & \longrightarrow & \Omega_{E \otimes_F F'} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_B/\tau(B) & \longrightarrow & \Omega_F & \longrightarrow & \Omega_{F'} \end{array}$$

commutes, which by 3.7 is the Mackey condition. Given this, it suffices to consider the maps $B \rightarrow B/P$, where P is a prime ideal of B . The transfer $\Omega_A \rightarrow \Omega_B$ sends $P\Omega_A$ to $P\Omega_B$ and $d(PA)$ to $d(P)$, so it induces a transfer from $\Omega_{A/PA}/\tau(A)$ to $\Omega_{B/P}/\tau(B)$. We may localize B (and A) at P to assume that A/PA is a finite-dimensional algebra over the field $K = B/P$. If $\{P_i\}$ are the primes of A over P , then $A/PA = \prod A_i$, where each A_i is a finite-dimensional nilpotent extension of the field $K_i = A/P_i$. By 6.16, the transfer on Ω_{A_i} is the projection onto the submodule Ω_{K_i} , followed by $\lambda_i = \text{length}(A_i)$ times the usual transfer $\Omega_{K_i} \rightarrow \Omega_K$. \square

$$\begin{array}{ccccc} \Omega_A & \longrightarrow & \Omega_{A/PA} & \xrightarrow{\text{project}} & \Omega_{K_i} \\ \downarrow & & \downarrow & & \downarrow \lambda_i \cdot \text{trace} \\ \Omega_B/\tau(B) & \longrightarrow & \Omega_{B/P} & \xrightarrow{=} & \Omega_K \end{array}$$

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