

INVARIANTS OF REAL CURVES *

Dedicated to Paolo Salmon on his 60th birthday

by

CLAUDIO PEDRINI¹ AND CHARLES WEIBEL²

ABSTRACT. In this paper we study several algebraic invariants of a real curve X : the Picard group $\mathbf{Pic}(X)$, the Brauer group $\mathbf{Br}(X)$, the K -theory matrix invariant $SK_1(X)$, the étale cohomology groups $H^n(X, \mathbf{Z}/2)$, and the sheaves \mathcal{H}^n associated to these cohomology groups. We relate these to two topological invariants of the space $X(\mathbf{R})$ of real points of X : the number c of components and the number λ of loops. For singular X we compute $\mathbf{Pic}X/2\mathbf{Pic}X$ and construct a Mayer-Vietoris sequence relating $\mathbf{Br}(X)$ to $SK_1(X)/2SK_1(X)$ and use it to show that $\mathbf{Br}(X) = (\mathbf{Z}/2)^c$ and $SK_1(X)/2SK_1(X) = (\mathbf{Z}/2)^\lambda$.

§0. Introduction.

In this paper we study several algebraic invariants of a real curve X . By “real curve” we mean a quasiprojective 1-dimensional reduced scheme defined over \mathbf{R} . The invariants are: the Picard group $\mathbf{Pic}(X)$, the Brauer group $\mathbf{Br}(X)$, the K -theory matrix invariant $SK_1(X)$, the étale cohomology groups $H_{\text{ét}}^n(X, \mathbf{Z}/2)$, and the sheaves \mathcal{H}^n associated to these cohomology groups. We relate these to two topological invariants of the space $X(\mathbf{R})$ of real points (with the euclidean topology): the number c of connected components and the number λ of loops. We show that:

(0.1) $\mathbf{Br}(X) \cong (\mathbf{Z}/2)^c$. This was previously known in the smooth projective case by [Witt] and in the affine case by [DK]. Since $\mathbf{Br}(X) = H^0(X, \mathcal{H}^2)$, the smooth case also appears as a particular case of a general result in [CTP]: if X is a smooth quasi-projective variety over \mathbf{R} then $(\mathbf{Z}/2)^c = H^0(X, \mathcal{H}^n)$ for any $n > \dim X$.

(0.2) There is a natural surjection from $\mathbf{Pic}(X) \otimes \mathbf{Z}/2$ to $(\mathbf{Z}/2)^\lambda$. If X is smooth this is an isomorphism except when X is projective and $X(\mathbf{R}) = \emptyset$, when $\lambda=0$ but

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$\mathbf{Pic}(X) \otimes \mathbf{Z}/2 = \mathbf{Z}/2$. For general X we have $\mathbf{Pic}(X) \otimes \mathbf{Z}/2 \cong (\mathbf{Z}/2)^{\lambda+E}$, where E is the number of components of X which are projective and have either no real points or have at most a finite set of isolated real points (Th.1.10).

For smooth projective curves the calculation of $\mathbf{Pic}(X)$ goes back to Weichold's 1882 thesis [Whd]. This includes some subtleties when $X(\mathbf{R}) = \emptyset$ (prop. 1.1). For affine curves our calculation corrects an example of DeMeyer and Knus [DK, 3.5], where inequality is claimed.

(0.3) $SK_1(X) \cong (\mathbf{Z}/2)^\lambda \oplus D$. Here D is a divisible abelian group, which is uniquely divisible if X is affine. In particular, $SK_1(X) \otimes \mathbf{Z}/2 \cong \mathbf{Z}/2^\lambda$. This seems to be a new result. The interest in $SK_1(X)$ lies in the fact that for an affine curve $X = \mathbf{Spec}(A)$, $SK_1(X)$ is the quotient $SL_3(A)/E_3(A)$ of the 3×3 special linear group $SL_3(A)$ (matrices of determinant 1) by $E_3(A)$, the subgroup generated by the elementary matrices. Therefore computation of $SK_1(A) \otimes \mathbf{Z}/2$ reduces to a computation in a group of matrices. Our key technical result for this is the isomorphism $SK_1(X) \otimes \mathbf{Z}/2 \cong H^1(X, \mathcal{H}^2)$, which in the smooth case is implicit in [MS].

(0.4) Let \tilde{X} be the normalization of X , $Y = \mathbf{Sing}(X)$ and $\tilde{Y} = Y \times_X \tilde{X}$. Then there is a natural exact sequence:

$$0 \rightarrow \mathbf{Br}(X) \rightarrow \mathbf{Br}(\tilde{X}) \oplus \mathbf{Br}(Y) \rightarrow \mathbf{Br}(\tilde{Y}) \rightarrow SK_1(X) \otimes \mathbf{Z}/2 \rightarrow SK_1(\tilde{X}) \otimes \mathbf{Z}/2 \rightarrow 0.$$

The first three terms of this sequence were found by DeMeyer and Knus when X is affine [DK]. The surprise is the continuation of the DeMeyer-Knus sequence to part of the Mayer-Vietoris sequence in algebraic K -theory.

(0.5) We can also compute all of the étale cohomology groups $H_{et}^i(X, \mathbf{Z}/2)$ for a real curve (Cor. 1.11). For $i \geq 3$ the isomorphism $H_{et}^i(X, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^{c+\lambda}$ can be easily deduced from [Cox,1;2]. For $i=2$ we show that $H_{et}^i(X, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^{c+\lambda+E}$. In the case where X is either affine or smooth projective with $X(\mathbf{R}) \neq \emptyset$, this appeared in [CT] as consequence of Cox's results.

We have laid out this paper as follows. §1 contains results and computations on the Picard group of real curves, as well as the calculations (0.5) on étale cohomology. In §2 we consider a Mayer-Vietoris sequence which relates the sheaves \mathcal{H}^q with $\pi_* \mathcal{H}^q$, where $\pi : \tilde{X} \rightarrow X$ is a finite birational map. Then we apply this to the case of a curve X and its normalization. In §3 we consider the Brauer group, proving (0.1). In §4 we give some technical results about the sheaf \mathcal{K}_2/ℓ on a curve, using Chern classes and K -theory mod ℓ . These results are used in §5, where proofs of (0.3), and (0.4) are given. We give examples (see 5.7.1) computing $SK_1(X)$ of a singular curve X obtained from a smooth affine curve by glueing points.

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Notation. If A is an abelian group, we shall write ${}_\ell A$ for the ℓ -torsion subgroup $\{a \in A : \ell a = 0\}$. \mathbf{G}_m and μ_ℓ are the étale sheaves of units and ℓ^{th} roots of unity,

respectively. A “real curve” is a reduced quasiprojective 1-dimensional scheme defined over the real numbers \mathbf{R} ; if X is a real curve we write $X(\mathbf{R})$ for the associated topological space of real points of X , topologized using the Euclidean topology.

§1. Picard groups of real curves.

We first turn our attention to the Picard group $\mathbf{Pic}(X)$ of an irreducible smooth projective real curve X . If X happens to be defined over \mathbf{C} then of course $X(\mathbf{R}) = \emptyset$ and $\mathbf{Pic}(X) \cong \mathbf{Z} \oplus (\mathbf{R}/\mathbf{Z})^{2g}$. Otherwise X is geometrically irreducible and we set $X_{\mathbf{C}} = X \times_{\mathbf{R}} \mathbf{C}$.

Proposition 1.1. *Let X be geometrically irreducible, smooth, projective curve over \mathbf{R} with c real components and genus g . Then $\mathbf{Pic}(X)$ is a subgroup of $\mathbf{Pic}(X_{\mathbf{C}})$ and as an abelian group:*

$$\mathbf{Pic}(X) \cong \mathbf{Z} \times (\mathbf{R}/\mathbf{Z})^g \times \begin{cases} (\mathbf{Z}/2)^{c-1} & \text{if } c \neq 0 \\ 0 & \text{if } c = 0 \end{cases}.$$

Thus $\mathbf{Pic}(X) \otimes \mathbf{Z}/2 \simeq (\mathbf{Z}/2)^c$ if $c \neq 0$, while $\mathbf{Pic}(X) \otimes \mathbf{Z}/2 \simeq \mathbf{Z}/2$ if $c = 0$.

Proof. Let $A = \mathbf{Pic}^0(X_{\mathbf{C}})$ be the $2g$ -dimensional Picard variety of $X_{\mathbf{C}}$, so that as abelian groups $\mathbf{Pic}(X_{\mathbf{C}}) \cong \mathbf{Z} \times A$ and $A \cong (\mathbf{R}/\mathbf{Z})^{2g}$. The Galois group $G = \mathbf{Gal}(\mathbf{C}/\mathbf{R})$ acts naturally on A and on $\mathbf{Pic}(X_{\mathbf{C}})$, and there is an exact sequence of G -modules: $0 \rightarrow A \rightarrow \mathbf{Pic}X_{\mathbf{C}} \rightarrow \mathbf{Z} \rightarrow 0$. (The map from $\mathbf{Pic}X_{\mathbf{C}}$ to \mathbf{Z} is the degree map.)

By using the calculations of Weichold [Whd] (and Klein [K,§7] when $c = 0$) on periods of real integrals, Comessatti [C1] and [C2] and Witt [Witt] showed that the invariant subgroup of A is:

$$A^G \cong (\mathbf{R}/\mathbf{Z})^g \times \begin{cases} (\mathbf{Z}/2)^{c-1} & \text{if } c \neq 0 \\ 0 & \text{if } c = 0 \text{ and } g \text{ is even} \\ \mathbf{Z}/2 & \text{if } c = 0 \text{ and } g \text{ is odd} \end{cases}.$$

If $c \neq 0$ it is well-known that $\mathbf{Pic}(X) = (\mathbf{Pic}X_{\mathbf{C}})^G = \mathbf{Z} \times A^G$, and we are done.

To analyze the case $c=0$, consider the Leray spectral sequence associated to the morphism $f : X \rightarrow \mathbf{Spec}(\mathbf{R})$ and the sheaf \mathbf{G}_m on X_{et} . This yields an exact sequence

$$0 \rightarrow \mathbf{Pic}X \rightarrow (\mathbf{Pic}X_{\mathbf{C}})^G \rightarrow \mathbf{Br}\mathbf{R} \rightarrow H^2(X, \mathbf{G}_m),$$

where $\mathbf{Br}\mathbf{R} = \mathbf{Z}/2$ and $H^2(X, \mathbf{G}_m) = \mathbf{Br}X$ because X is a real curve. Because $X(\mathbf{R}) = \emptyset$, we have $\mathbf{Br}X = 0$ by [Witt] (or 3.6 below), yielding the exact sequence:

$$(1.1.1) \quad 0 \rightarrow \mathbf{Pic}X \rightarrow (\mathbf{Pic}X_{\mathbf{C}})^G \rightarrow \mathbf{Z}/2 \rightarrow 0.$$

So if $c=0$, $\mathbf{Pic}X$ has index 2 in $(\mathbf{Pic}X_{\mathbf{C}})^G$. The degree map $\mathbf{Pic}X \rightarrow \mathbf{Pic}X_{\mathbf{C}} \rightarrow \mathbf{Z}$ has image $2\mathbf{Z}$ and its kernel \mathbf{Pic}^0X embeds into A^G via the following map of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Pic}^0X & \longrightarrow & \mathbf{Pic}X & \longrightarrow & 2\mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^G & \longrightarrow & (\mathbf{Pic}X_{\mathbf{C}})^G & \longrightarrow & \mathbf{Z} & \longrightarrow & H^1(G, A) \end{array}$$

Let σ denote complex conjugation. Since $X(\mathbf{R}) = \emptyset$ every divisor on X is of the form $x + \sigma(x)$ for some divisor x on $X_{\mathbf{C}}$. Hence the usual transfer map from $A = \mathbf{Pic}^0X_{\mathbf{C}}$ to $\mathbf{Pic}^0\bar{X}$ which sends x to $x + \sigma(x)$ is onto. Therefore we may identify \mathbf{Pic}^0X in $A = \mathbf{Pic}^0X_{\mathbf{C}}$ with the image of the continuous map $\tau : A \rightarrow A$ sending a to $a + \sigma(a)$. A is a connected Lie group, so the image of τ is a connected Lie subgroup of A^G containing $2A^G$. If g is even, this gives $\mathbf{Pic}^0X = A^G$; if g is odd, τ cannot hit the subgroup $\mathbf{Z}/2$ of A^G , and we are done.

Remark 1.1.2: If $X(\mathbf{R}) = \emptyset$, the G -module structure of $\mathbf{Pic}(X_{\mathbf{C}})$ depends upon the parity of the genus g . Using Comessatti's calculations as in [Sil, Prop.3], we find that:

$$H^1(G, A) \cong \begin{cases} 0 & \text{if } g \text{ is even} \\ \mathbf{Z}/2 & \text{if } g \text{ is odd.} \end{cases}$$

Therefore as a subgroup of $\mathbf{Pic}(X_{\mathbf{C}})$ we have (cfr. [Sil, Prop.10])

$$\mathbf{Pic}(X_{\mathbf{C}})^G \cong \begin{cases} \mathbf{Z} \oplus (\mathbf{R}/\mathbf{Z})^g \oplus (\mathbf{Z}/2)^{c-1} & \text{if } c \neq 0 \\ \mathbf{Z} \oplus (\mathbf{R}/\mathbf{Z})^g & \text{if } c = 0 \text{ and } g \text{ is even} \\ 2\mathbf{Z} \oplus (\mathbf{R}/\mathbf{Z})^g \oplus (\mathbf{Z}/2) & \text{if } c = 0 \text{ and } g \text{ is odd.} \end{cases}$$

If g is even then $\mathbf{Pic}(X_{\mathbf{C}}) \cong A \times \mathbf{Z}$ as a G -module, but if g is odd then $\mathbf{Pic}X_{\mathbf{C}}$ is not split as a G -module. In contrast, the sequence (1.1.1) is not split when g is even, but is split when g is odd. In fact, when g is even, the map $(\mathbf{Pic}X_{\mathbf{C}})^G \rightarrow \mathbf{Z}/2$ in (1.1.1) is just the degree map modulo 2.

Historical Remark 1.1.3 In his 1882 Leipzig thesis [Whd], Weichold computed the period matrix Ω which gives the pairing

$$H_1(X_{\mathbf{C}}, \mathbf{Z}) \times H^0(X_{\mathbf{C}}, \Omega^1) \rightarrow \mathbf{C}.$$

The columns of Ω generate the lattice in \mathbf{C}^g giving the Jacobian variety of $X_{\mathbf{C}}$. Weichold's calculations were later refined by Klein [K]. Using the "pseudonormal" matrix of periods of real integrals Comessatti [C1] [C2] defined the so-called "real character" of a real abelian variety in the 1920's; by applying this to the case of the Jacobian A of a real curve of genus g he computed A^G . In the case $X(\mathbf{R}) \neq \emptyset$ this also yields a computation of $\mathbf{Pic}X$.

A modern translation of these results, using the action of the Galois group and cohomology may be found in [Sil]. The case $X(\mathbf{R}) = \emptyset$ has been considered by Gross and Harris in [GH, prop.2.2], where they also establish (1.1.1) and compute $\mathbf{Pic}X \otimes \mathbf{Z}/2$ [GH, prop.4.2]. We warn the reader that the calculation of $\mathbf{Pic}X/2\mathbf{Pic}X$ attributed to Witt in [DK,1.7] is false when $X(\mathbf{R}) = \emptyset$.

Example 1.2. In order to motivate the definition of loops below, consider the following classes of smooth affine curves X , each obtained from a smooth irreducible projective curve \bar{X} by removing points. Let c be the number of connected components of $X(\mathbf{R})$; each component is diffeomorphic to a circle.

Suppose first that we remove a single point p . Then there is an exact sequence

$$\mathbf{Z} \xrightarrow{i_p} \mathbf{Pic}\bar{X} \rightarrow \mathbf{Pic}X \rightarrow 0.$$

If p is a real point then geometrically $X(\mathbf{R}) = \bar{X}(\mathbf{R}) - \{p\}$ consists of $\lambda = c - 1$ smooth circles and one unbounded component, which is diffeomorphic to the real line. In this case the image of i_p is \mathbf{Z} , and $\mathbf{Pic}X \cong (\mathbf{R}/\mathbf{Z})^g \oplus (\mathbf{R}/\mathbf{Z})^{c-1}$. If p is a “complex point” (i.e., a scheme $p = \mathbf{Spec}\mathbf{C}$), then geometrically $X(\mathbf{R}) = \bar{X}(\mathbf{R})$ consists of $\lambda = c$ smooth circles. If X is defined over \mathbf{C} , then the image of i_p is \mathbf{Z} and $\mathbf{Pic}X = (\mathbf{R}/\mathbf{Z})^g$. Otherwise, X is geometrically irreducible and the image of i_p is $2\mathbf{Z}$. In this case $\mathbf{Pic}X \simeq (\mathbf{R}/\mathbf{Z})^g \oplus (\mathbf{Z}/2)^c$.

Now suppose that X is obtained from \bar{X} by removing a finite set $\{p_1, \dots, p_r\}$ of more than one point. The subgroup Γ of $\mathbf{Pic}(\bar{X})$ generated by the classes of the p_i 's can be fairly complicated, but as an abelian group the quotient $\mathbf{Pic}(X) = \mathbf{Pic}(\bar{X})/\Gamma$ will be the sum of a divisible abelian group and a finite number λ of copies of $\mathbf{Z}/2$. Geometrically, it is not hard to show that $X(\mathbf{R})$ consists of λ smooth circles and $c - \lambda$ unbounded components (each diffeomorphic to the real line).

Definition 1.3. *Let X be a curve defined over \mathbf{R} , possibly singular. The number of loops $\lambda(X)$ of $X(\mathbf{R})$ is the dimension of $H^1(X(\mathbf{R}), \mathbf{Z}/2)$.*

Topologically, each connected component of $X(\mathbf{R})$ is either contractible (no loops), a circle (one loop) or (when X is singular) homotopy equivalent to a bouquet of a finite number λ_i of circles (λ_i loops); $\lambda(X)$ is the sum over the set of components of $X(\mathbf{R})$ of the number of the loops on each component. If X is smooth and projective then $\lambda = c$ is just the number of connected components of $X(\mathbf{R})$. A singular example is the node $y^2 = x^2 + x^3$; the affine node has $\lambda = 1$ loop but the projective node has $\lambda = 2$ loops.

Combining Prop. 1.1 with the above definition and with (1.2) we get

Corollary 1.4. *Let X be a irreducible smooth real curve with λ loops. Then with one exception:*

$$\mathbf{Pic}(X) \otimes \mathbf{Z}/2 \cong (\mathbf{Z}/2)^\lambda.$$

The exception is an irreducible projective curve with $X(\mathbf{R}) = \emptyset$, when

$$\mathbf{Pic}(X) \otimes \mathbf{Z}/2 \cong \mathbf{Z}/2.$$

More generally, if X is any smooth real curve with λ loops, let E denote the number of irreducible algebraic components X_i of X which are projective and for which $X_i(\mathbf{R})$ is empty. Then:

$$\mathbf{Pic}(X) \otimes \mathbf{Z}/2 \cong (\mathbf{Z}/2)^{\lambda+E}.$$

In the remainder of this section we are going to extend the computation of $\mathbf{Pic}(X) \otimes \mathbf{Z}/2$ to any real curve.

Let $\pi : X' \rightarrow X$ be a finite morphism of curves, $Y = \mathbf{Sing}X$, $Y' = \pi^{-1}(Y)$. The cartesian square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \pi \\ Y & \longrightarrow & X \end{array}$$

yields the following Mayer-Vietoris sequence

$$0 \rightarrow U(X) \rightarrow U(X') \oplus U(Y) \xrightarrow{\Phi} U(Y') \rightarrow \mathbf{Pic}X \rightarrow \mathbf{Pic}X' \rightarrow 0$$

where $U(X) = H^0(X, \mathcal{O}_X^*)$ is multiplicative group of global units on X etc. For example $U(Y')$ consists of one copy of \mathbf{R}^* for every real point of Y' and one copy of \mathbf{C}^* for every complex point. Tensoring with $\mathbf{Z}/2$, the Mayer-Vietoris exact sequence yields the following one

$$(1.5) \quad H \oplus (U(Y) \otimes \mathbf{Z}/2) \rightarrow U(Y') \otimes \mathbf{Z}/2 \rightarrow \mathbf{Pic}X \otimes \mathbf{Z}/2 \rightarrow \mathbf{Pic}X' \otimes \mathbf{Z}/2 \rightarrow 0$$

where $H = (U(X') \otimes \mathbf{Z}/2) \oplus {}_2\mathbf{Pic}X$.

If $U(Y) \otimes \mathbf{Z}/2$ maps onto $U(Y') \otimes \mathbf{Z}/2$ then $\mathbf{Pic}X \otimes \mathbf{Z}/2 \cong \mathbf{Pic}X' \otimes \mathbf{Z}/2$. This is the case if $Y'(\mathbf{R}) = \emptyset$, or more generally when the map $Y'(\mathbf{R}) \rightarrow Y(\mathbf{R})$ is 1-1. This observation motivates the following

Definition 1.6. Let X be a real curve with normalization \tilde{X} and singular locus Y . We say that an intermediate curve $\tilde{X} \rightarrow X' \rightarrow X$ has property (P) if no two real points of $Y' = Y \times_X X'$ lie over the same (real) point of Y . If X'_1 and X'_2 have this property then so does $X'_1 \times_X X'_2$. Therefore there is a largest intermediate curve X' having property (P); we call X' the partial normalization of X . It may be obtained from \tilde{X} by glueing together the real points lying over each point of Y . From (1.5) it follows that $\mathbf{Pic}X \otimes \mathbf{Z}/2 \cong \mathbf{Pic}X' \otimes \mathbf{Z}/2$ if X' is the partial normalization of X .

Example 1.7. Let X be the affine real curve defined by $y^2 = x^2(x+1)(x+2)^2$. If $t = \frac{y}{x+2}$ then the partial normalization X' is the affine node $t^2 = x^2(x+1)$ and the normalization is the affine line $\tilde{X} = \mathbf{Spec}(\mathbf{R}[\frac{t}{x}])$. In this case Y consists of two real points, Y' consists of one real point and one complex point, and X' is obtained from \tilde{X} by glueing the two real points $\frac{t}{x} = \pm 1$ together into the real point of Y' .

Proposition 1.8. (Cox, cf.[CT]). *If X is any real curve then*

$$H^i(X, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^{c+\lambda} \text{ for all } i \geq 3.$$

Proof. The result may be deduced from [Cox, 1.2] as follows. Let X be a real variety, G the Galois group of \mathbf{C} over \mathbf{R} , EG the infinite sphere S^∞ . EG is a principal G -space with quotient $BG = \mathbf{RP}^\infty$, where G acts on EG by the antipodal map. Let $X(\mathbf{C}) \times_G EG$ be the quotient of $X(\mathbf{C}) \times EG$ by the diagonal action of G . By [Cox,1.1] there is a isomorphism for any $q \geq 0$:

$$H_{et}^q(X, \mathbf{Z}/2) \cong H^q(X(\mathbf{C}) \times_G EG, \mathbf{Z}/2).$$

Applying the Künneth formula, we also have isomorphisms for any $q \geq 1$:

$$H^q(X(\mathbf{R}) \times BG, \mathbf{Z}/2) \cong H^0(X(\mathbf{R}), \mathbf{Z}/2) \oplus H^1(X(\mathbf{R}), \mathbf{Z}/2) \cong (\mathbf{Z}/2)^c \oplus (\mathbf{Z}/2)^\lambda.$$

To compare these isomorphisms, we introduce the quotient space $X_\infty = X(\mathbf{C})/G$. X_∞ inherits the Euclidean topology from $X(\mathbf{C})$ and $X(\mathbf{R})$ is a closed subset both of X_∞ and $X(\mathbf{C})$. By [Bred, VII.1.1] the relative cohomology groups $H^*(X_\infty, X(\mathbf{R}), \mathbf{Z}/2)$ and $H^*(X(\mathbf{C}) \times_G EG, X(\mathbf{R}) \times BG, \mathbf{Z}/2)$ are isomorphic. Using this and the exact sequence of the pair $(X(\mathbf{C}) \times_G EG, X(\mathbf{R}) \times BG)$ one gets a long cohomology exact sequence [Cox, 1.2]

$$\cdots H^q(X_\infty, X(\mathbf{R}), \mathbf{Z}/2) \xrightarrow{\alpha^q} H^q(X(\mathbf{C}) \times_G EG, \mathbf{Z}/2) \rightarrow H^q(X(\mathbf{R}) \times BG, \mathbf{Z}/2) \rightarrow \cdots$$

Since X_∞ is a 2-dimensional complex, $H^q(X_\infty, X(\mathbf{R}), \mathbf{Z}/2) = 0$ for $q \geq 3$ and the proposition follows. ■

Lemma 1.9. *Let X be a real curve and let X_i be its irreducible algebraic components. Assume that for each i either X_i is affine or $X_i(\mathbf{R}) \neq \emptyset$. Then*

$$H_{et}^2(X, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^{c+\lambda}.$$

Proof. We use the notation of the proof of Cox's result. The map α^q factors through $H^q(X_\infty, \mathbf{Z}/2)$ via the commutative diagram:

$$\begin{array}{ccc} H^q(X_\infty, X(\mathbf{R}), \mathbf{Z}/2) & \xrightarrow{\cong} & H^q(X(\mathbf{C}) \times_G EG, X(\mathbf{R}) \times BG, \mathbf{Z}/2) \\ \downarrow & & \downarrow \alpha^q \\ H^q(X_\infty, \mathbf{Z}/2) & \xrightarrow{\beta^q} & H^q(X(\mathbf{C}) \times_G EG, \mathbf{Z}/2) \\ \downarrow & & \downarrow \\ H^q(X(\mathbf{R}), \mathbf{Z}/2) & \longrightarrow & H^q(X(\mathbf{R}) \times BG, \mathbf{Z}/2) \end{array}$$

If X is a curve then as cellular complexes we have $\dim(X_\infty)=2$ and $\dim(X(\mathbf{R}))=1$. Therefore we get an exact sequence:

$$H^2(X_\infty, \mathbf{Z}/2) \xrightarrow{\beta^q} H_{et}^2(X, \mathbf{Z}/2) \rightarrow H^2(X(\mathbf{R}) \times BG, \mathbf{Z}/2) \rightarrow 0.$$

We claim that $H_2(X_\infty, \mathbf{Z}/2) = 0$, and therefore that $H^2(X_\infty, \mathbf{Z}/2) = 0$. As in the proof of 1.8, this will yield our statement:

$$H_{et}^2(X, \mathbf{Z}/2) \cong H^2(X(\mathbf{R}) \times BG, \mathbf{Z}/2) = (\mathbf{Z}/2)^{c+\lambda}$$

To establish the claim, we may assume that X is irreducible. Indeed, $X_\infty = X(\mathbf{C})/G$ differs from the disjoint of the $X_i(\mathbf{C})/G$ by a finite set of points, so $H_2(X_\infty, \mathbf{Z}/2)$ is the direct sum of the $H_2(X_i(\mathbf{C})/G, \mathbf{Z}/2)$.

We now assume X is irreducible and write M for X_∞ considered as a 2-dimensional simplicial complex. Because of our assumptions, M is a pseudomanifold with boundary $X(\mathbf{R})$ in the sense of [Spa,p.150]. That is, every edge (1-simplex) is the boundary of at most two faces (2-simplices) of M and every pair of faces may be connected by a sequence of faces and edges.

The following counting argument shows that $H_2(M, \mathbf{Z}/2) = 0$. Because M is a pseudomanifold, the only possible simplicial 2-cycle (modulo 2) is the sum σ of the faces of M . If X is affine then M is not compact, so there are infinitely many faces and such a 2-cycle cannot exist. If M is compact, the boundary of σ is the sum of the edges in the boundary $X(\mathbf{R})$ of M . If $X(\mathbf{R}) \neq \emptyset$ then the 2-chain σ cannot be a 2-cycle, so $H_2(M, \mathbf{Z}/2) = 0$.

Theorem 1.10. *Let X be any real curve. Then*

$$\mathbf{Pic}X \otimes \mathbf{Z}/2 \cong (\mathbf{Z}/2)^{\lambda+E}.$$

Where λ is the number of loops in $X(\mathbf{R})$ and E is the number of irreducible algebraic components X_i of X which are proper and for which $X_i(\mathbf{R})$ is either empty or finite.

Proof. If X is smooth then the result follows from 1.4.

Assume X is singular. The numbers λ and E do not change if we pass from X to the partial normalization X' . Moreover $\mathbf{Pic}X \otimes \mathbf{Z}/2 \cong \mathbf{Pic}X' \otimes \mathbf{Z}/2$. By this process we may assume that there are no complex points in the normalization \tilde{X} lying over the singular points of X . Hence we may assume that X is a disjoint union of E components X_i which are smooth, proper and have $X_i(\mathbf{R}) = \emptyset$, together with a curve for which $E = 0$. Using 1.4 we are now left to show that if $E=0$ then $\mathbf{Pic}X \otimes \mathbf{Z}/2 \cong (\mathbf{Z}/2)^\lambda$. From Lemma 1.9 we get the isomorphism

$$H_{et}^2(X, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^{c+\lambda}.$$

We conclude using the exact Kummer sequence

$$0 \rightarrow \mathbf{Pic}X \otimes \mathbf{Z}/2 \rightarrow H_{et}^2(X, \mathbf{Z}/2) \rightarrow {}_2\mathbf{Br}X \rightarrow 0$$

and the isomorphism $\mathbf{Br}X \cong (\mathbf{Z}/2)^c$; see §3 for the proof of this isomorphism.

Example 1.10.1 If X is the projective curve defined by $XY(X^2 + Y^2) = 0$, then $\lambda = 2$ and $E = 1$. Thus $\mathbf{Pic}(X) \otimes \mathbf{Z}/2 \cong (\mathbf{Z}/2)^3$.

Corollary 1.11. *Let X be a real curve. Then:*

- a) $H_{et}^0(X, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^a$ where a is the number of connected components of X
- b) $H_{et}^1(X, \mathbf{Z}/2) \cong H^0(X, \mathcal{O}_X^*)/2 \oplus {}_2\mathbf{Pic}X$.
- c) $H_{et}^2(X, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^{c+\lambda+E}$
- d) $H_{et}^i(X, \mathbf{Z}/2) \cong (\mathbf{Z}/2)^{c+\lambda}$ for every $i \geq 3$.

Proof. a) is obvious and b) comes from Kummer theory. d) is 1.8. c) follows from 1.10 and the isomorphism $\mathbf{Br}X \simeq (\mathbf{Z}/2)^c$ of 3.6 below.

Remark 1.12 Let X be a irreducible affine real curve such that $X(\mathbf{R}) \neq \emptyset$ and X has no isolated singularities. Then 1.10 yields $\mathbf{Pic}X \otimes \mathbf{Z}/2 \simeq (\mathbf{Z}/2)^\lambda$. This seems to contradict the example in [DK,3.5], where X is the curve obtained from the curve $\tilde{X} : x^4 = y^2 - 1$ by glueing together the points $A = (1, 0)$ and $B = (-1, 0)$. X has no loops and $E=0$, yet in [DK] it is claimed that $\mathbf{Pic}X \otimes \mathbf{Z}/2 \cong \mathbf{Z}/2$. In fact their claim is false: the following correct computation yields $\mathbf{Pic}X \otimes \mathbf{Z}/2=0$.

The Mayer-Vietoris sequence for units and \mathbf{Pic} yields the exact sequence

$$0 \rightarrow U(X) \rightarrow U(\tilde{X}) \times \mathbf{R}^* \rightarrow \mathbf{R}^* \times \mathbf{R}^* \rightarrow \mathbf{Pic}X \rightarrow \mathbf{Pic}\tilde{X} \rightarrow 0$$

and the image of the map $U(\tilde{X}) \rightarrow \mathbf{R}^* \times \mathbf{R}^*$ is the diagonal copy of \mathbf{R}^* . Now $\mathbf{Pic}\tilde{X} \cong \mathbf{R}/\mathbf{Z}$, so $\mathbf{Pic}\tilde{X} \otimes \mathbf{Z}/2=0$ and ${}_2\mathbf{Pic}\tilde{X} \cong \mathbf{Z}/2$. Tensoring with $\mathbf{Z}/2$ we get the following exact sequence:

$$0 \rightarrow \mathbf{Z}/2 \rightarrow {}_2\mathbf{Pic}X \xrightarrow{\varphi} {}_2\mathbf{Pic}\tilde{X} \xrightarrow{\psi} \mathbf{R}^*/(\mathbf{R}^*)^2 \rightarrow \mathbf{Pic}X \otimes \mathbf{Z}/2 \rightarrow 0$$

(where $\mathbf{Z}/2$ is the 2-torsion in \mathbf{R}^*). To conclude it is enough to show that φ is the zero map and ψ is surjective. Let $\mathbf{R}[x, y]$ be the coordinate ring of \tilde{X} : we have $(x-1) = 2A$, $(x+1) = 2B$ as divisors on \tilde{X} . Hence $2A = 2B = 0$ in $\mathbf{Pic}\tilde{X}$. Let $t = x^2 - y$: t is a unit in $\mathbf{R}[x, y]$ and the divisor of $\frac{y}{t+1}$ is $A + B$, hence $A = B$ in $\mathbf{Pic}\tilde{X}$. Let L be the line through A : $a(X-1) + bY=0$ where $a \neq 0$ $b \neq 0$. Then $(L) = A + D$ in $\mathbf{Pic}\tilde{X}$ where D is a divisor not supported on A nor on B . Therefore A represents an element D in $\mathbf{Pic}X$ too. Let $f = \frac{L^2}{x-1}$ then $(f) = 2D$ so that $2D=0$ in $\mathbf{Pic}\tilde{X}$. Moreover $f(A) > 0$ and $f(B) < 0$. Therefore the image of $f \in U(\tilde{X})$ represents the class of $\{-1\}$ in $\mathbf{R}^*/(\mathbf{R}^*)^2$, hence ψ is surjective. Finally, $2D \neq 0$ in ${}_2\mathbf{Pic}X$ and $2D = 0$ in ${}_2\mathbf{Pic}\tilde{X}$; therefore φ is the zero map.

§2. Cohomological interlude.

Let $\pi : \tilde{X} \rightarrow X$ be a proper birational map of schemes which are essentially of finite type over a field k . Then there is a reduced closed subscheme Y of X with

inverse image $\tilde{Y} = \pi^{-1}(Y)$ such that $U = X - Y$ and $\tilde{U} = \tilde{X} - \tilde{Y}$ are isomorphic. If we write i and j for the inclusions $Y \subset X$ and $\tilde{Y} \subset \tilde{X}$, we have a square

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{j} & \tilde{X} \\ \downarrow \pi_{\tilde{Y}} & & \downarrow \pi \\ Y & \xrightarrow{i} & X. \end{array}$$

Proposition 2.1. *For every torsion abelian sheaf T on the big étale site of X there is a Mayer-Vietoris sequence in étale cohomology:*

$$\cdots H_{\text{ét}}^q(X, T) \rightarrow H_{\text{ét}}^q(\tilde{X}, T) \oplus H_{\text{ét}}^q(Y, T) \rightarrow H_{\text{ét}}^q(\tilde{Y}, T) \xrightarrow{\delta} H_{\text{ét}}^{q+1}(X, T) \cdots$$

Proof. Writing $\eta : U \subset X$ and $\gamma : \tilde{U} \subset \tilde{X}$ for the inclusions of $U = X - Y$ and $\tilde{U} = \tilde{X} - \tilde{Y}$, we have the canonical exact sequences of étale sheaves on X and \tilde{X} :

$$\begin{aligned} 0 \rightarrow \eta_!(T|U) \rightarrow T_X \rightarrow i_*(T|Y) \rightarrow 0 \\ 0 \rightarrow \gamma_!(T|\tilde{U}) \rightarrow T_{\tilde{X}} \rightarrow j_*(T|\tilde{Y}) \rightarrow 0. \end{aligned}$$

There is also a map of the cohomology long exact sequences:

$$\begin{array}{ccccccc} \cdots H^{q+1}(Y, T) & \rightarrow & H^q(X, \eta_!(T|U)) & \rightarrow & H^q(X, T) & \rightarrow & H^q(Y, T) \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{q+1}(\tilde{Y}, T) & \rightarrow & H^q(\tilde{X}, \gamma_!(T|\tilde{U})) & \rightarrow & H^q(\tilde{X}, T) & \rightarrow & H^q(\tilde{Y}, T) \cdots \end{array}$$

where $H^*(Y, T) = H^*(X, i_*(T|Y))$, $H^*(\tilde{X}, j_*(T|\tilde{Y})) = H^*(\tilde{Y}, T)$. By a standard diagram chase it suffices to show that when $\tilde{U} \cong U$ the maps from $H^*(X, \eta_!(T|U))$ to $H^*(\tilde{X}, \gamma_!(T|\tilde{U}))$ are isomorphisms. If X is complete, this is just the fact that the cohomology groups with compact support $H_c^q(U, T)$ are independent of the embedding $U \subset X$ [M, V.3.1]. The proof in general is similar. First one observes that the sheaves $\pi_*\gamma_!(T|\tilde{U})$ and $\eta_!(T|U)$ are isomorphic because they are both supported on U and they have the same stalks at every point of U . Hence $H^*(X, \eta_!(T|U)) \cong H^*(X, \pi_*\gamma_!(T|\tilde{U}))$.

Then one considers the Leray spectral sequence

$$H^p(X, R^q\pi_*\gamma_!(T|\tilde{U})) \Rightarrow H^{p+q}(\tilde{X}, \gamma_!(T|\tilde{U})).$$

According to the proper base change theorem [M, VI 2.5] the stalk of the sheaf $R^q\pi_*\gamma_!(T|\tilde{U})$ at a geometric point \bar{x} of X may be computed on the geometric fibres of π . Since the fiber consists of one point if $x \in U$ and is 0 if $x \notin U$, we get $R^q\pi_*(\gamma_!(T|\tilde{U}))=0$ for $q > 0$. Thus the spectral sequence degenerates yielding the isomorphism

$$H^p(X, \pi_*\gamma_!(T|\tilde{U})) \cong H^p(\tilde{X}, \gamma_!(T|\tilde{U})).$$

The sequence of 2.1 is natural, so we may sheafify in the Zariski topology of X . The resulting sequence of sheaves on X is

$$(2.2) \quad \cdots \mathcal{H}_X^q \rightarrow \mathbf{R}^q \pi_*(T) \oplus i_* \mathcal{H}_Y^q \rightarrow \mathbf{R}^q (\pi j)_*(T) \rightarrow \mathcal{H}_X^{q+1} \cdots$$

Here \mathcal{H}_X^q and \mathcal{H}_Y^q are the sheaves on X_{Zar} and Y_{Zar} associated to the presheaf $U \mapsto H_{\text{et}}^q(U, T)$ and $\mathbf{R}^q \pi_*(T)$ is the sheaf associated to the presheaf $U \mapsto H_{\text{et}}^q(\pi^{-1}(U), T)$. By inspection, $\mathbf{R}^0 \pi_*(T) \cong \pi_* \mathcal{H}_{\tilde{X}}^0$. Thus $\mathcal{H}_X^1 \cong \mathcal{O}_X^* \otimes \mathbf{Z}/\ell$ and the sequence (2.2) begins with the sequence

$$0 \rightarrow \mathcal{H}_X^0 \rightarrow \pi_* \mathcal{H}_{\tilde{X}}^0 \oplus i_* \mathcal{H}_Y^0 \rightarrow (\pi j)_* \mathcal{H}_{\tilde{Y}}^0 \xrightarrow{\delta} \mathcal{H}_X^1.$$

Example 2.3. Let T be the étale sheaf μ_ℓ of ℓ^{th} roots of unity, where $\text{char}(k) \neq \ell$. Then \mathcal{H}_X^0 is the Zariski sheaf μ_ℓ . Since $\mathbf{Pic}(\mathcal{O}_{X,x}) = 0$, Kummer theory quickly yields isomorphisms for every point x in X :

$$(\mathcal{H}_X^1)_x \cong \mathcal{O}_{X,x}^* \otimes \mathbf{Z}/\ell \quad \text{and} \quad (\mathcal{H}_X^2)_x \cong {}_\ell \mathbf{Br}(\mathcal{O}_{X,x}).$$

Here ${}_\ell \mathbf{Br}(\mathcal{O}_{X,x})$ is the ℓ -torsion in the Brauer group, which equals the ℓ -torsion in $H_{\text{et}}^2(\mathbf{Spec} \mathcal{O}_{X,x}, \mathbf{G}_m)$. Thus $\mathcal{H}_X^1 \cong \mathcal{O}_X^* \otimes \mathbf{Z}/\ell$ and the sequence (2.2) begins

$$0 \rightarrow \mu_\ell(X) \rightarrow \mu_\ell(\tilde{X}) \oplus \mu_\ell(Y) \rightarrow \mu_\ell(\tilde{Y}) \xrightarrow{\delta} \mathcal{O}_X^* \otimes \mathbf{Z}/\ell.$$

The boundary map δ can be nonzero, since \tilde{Y} may have more roots of unity than either \tilde{X} or Y .

Returning to (2.2), we remark that in general the sheaves $\mathbf{R}^q \pi_*(T)$ can be fairly complicated. The stalk of $\mathbf{R}^q \pi_*(T)$ at $x \in X$ is $H_{\text{et}}^q(V, T)$, where $V = \pi^{-1}(\mathbf{Spec}(\mathcal{O}_{X,x}))$. Since the stalk of the direct image sheaf $\pi_*(\mathcal{H}_{\tilde{X}}^q)$ at $x \in X$ is $H_{\text{Zar}}^0(V, \mathcal{H}_V^q)$, there is a canonical map

$$\eta^q : \mathbf{R}^q \pi_*(T) \rightarrow \pi_* \mathcal{H}_{\tilde{X}}^q.$$

This is not an isomorphism in general, being merely an edge map in the Leray spectral sequence for $X_{\text{et}} \rightarrow X_{\text{Zar}}$:

$$E_2^{pq} = R^p \pi_*(\mathcal{H}_{\tilde{X}}^q) \Rightarrow \mathbf{R}^{p+q} \pi_*(T),$$

which stalkwise is the Leray spectral sequence

$$E_2^{pq} = H_{\text{Zar}}^p(V, \mathcal{H}_V^q) \Rightarrow H_{\text{et}}^{p+q}(V, T).$$

Example 2.4. This spectral sequence does not always degenerate, even if k is algebraically closed and $T = \mu_\ell$.

(a) Let $\pi : \tilde{X} \rightarrow X$ be the blow up of the affine cone X over a smooth projective curve C , with $Y = \{y\}$ the vertex, and $\tilde{Y} \cong C$. For $p \neq 0$, $R^p\pi_*(\mathcal{H}^q)$ is the skyscraper sheaf $i_*H^p(V, \mathcal{H}^q)$ where $V = \pi^{-1}(\mathbf{Spec}\mathcal{O}_{X,y})$. For $p = q = 1$, we know from [BO,7.7] that

$$H^1(V, \mathcal{H}^1) \cong \mathbf{Pic}(V) \otimes \mathbf{Z}/\ell \cong \mathbf{Pic}(C) \otimes \mathbf{Z}/\ell \cong \mathbf{Z}/\ell.$$

On the other hand, [BO, (0.3)] states that $H^p(V, \mathcal{H}^q) = 0$ for $p > q$, so $\mathbf{R}^1\pi_*\mu_\ell \cong \pi_*\mathcal{H}^1 = \pi_*\mathcal{O}_{\tilde{X}} \otimes \mathbf{Z}/\ell$ and we have the exact sequence

$$0 \rightarrow i_*(\mathbf{Z}/\ell) \rightarrow \mathbf{R}^2\pi_*(\mu_\ell) \rightarrow \pi_*\mathcal{H}^2 \rightarrow 0.$$

(b) Let \tilde{X} be the triangle $xy(x+y-1) = 0$ in the affine plane, and X the simple node defined by $t^2 = s^2 + s^3$. There is a finite cover $\pi : \tilde{X} \rightarrow X$ ramified at the node point $Y = \{y\}$. In this case it is easy to see that $H_{\mathbf{Zar}}^1(V, \mu_\ell) \cong \mathbf{Z}/\ell$, so this time there is an exact sequence

$$0 \rightarrow i_*(\mathbf{Z}/\ell) \rightarrow \mathbf{R}^1\pi_*(\mu_\ell) \rightarrow \pi_*\mathcal{H}^1 \rightarrow 0.$$

In order to proceed further we need to identify the sheaf $R^q\pi_*(T)$ with $\pi_*\mathcal{H}_{\tilde{X}}^q$. Example 2.4 (a) shows this need not be the case when π is not finite; example 2.4 (b) shows that even assuming π finite and $T \cong \mu_\ell^{\otimes n}$ is not enough. However for $q=1,2$ we have the following two results.

Lemma 2.5. *If π is finite and $T = \mu_\ell$, where $\text{char } k \neq \ell$, then the map of sheaves*

$$\mathbf{R}^1\pi_*(\mu_\ell) \rightarrow \mathbf{R}^1(\pi j)_*(\mu_\ell) \text{ is onto.}$$

Proof. We show that the map $\mathbf{R}^1\pi_*(\mu_\ell) \rightarrow \mathbf{R}^1(\pi j)_*(\mu_\ell)$ is stalkwise onto, hence onto. For $x \in Y$, $V = \pi^{-1}(\mathcal{O}_{X,x})$ is $\mathbf{Spec}(A)$ for some semilocal ring A . Set $V|Y = V \times_X Y = \pi^{-1}(\mathcal{O}_{Y,x})$. Then $V|Y$ is $\mathbf{Spec}(A/I)$ for some ideal I of A . The map of stalks at $x \in Y$ is $H_{\text{et}}^1(V, \mu_\ell) \rightarrow H^1(V|Y, \mu_\ell)$. Since A is semilocal, $\mathbf{Pic}(A) = \mathbf{Pic}(A/I) = 0$ and $A^* \rightarrow (A/I)^*$ is onto. By Kummer theory, $H_{\text{et}}^1(V, \mu_\ell) = A^* \otimes \mathbf{Z}/\ell$ and $H_{\text{et}}^1(V|Y, \mu_\ell) = (A/I)^* \otimes \mathbf{Z}/\ell$.

Lemma 2.6. *If π is finite, $\dim(Y) = 0$ and $T = \mu_\ell^{\otimes 2}$ ($\text{char } k \neq \ell$) then the map $\mathbf{R}^2\pi_*(\mu_\ell^{\otimes 2}) \rightarrow \mathbf{R}^2(\pi j)_*(\mu_\ell^{\otimes 2})$ is onto.*

Proof. Keeping the notation of the previous lemma, the map of stalks at $x \in Y$ is $H_{\text{et}}^2(V, \mu_\ell^{\otimes 2}) \rightarrow H^2(V|Y, \mu_\ell^{\otimes 2})$, the bottom arrow in the commutative diagram:

$$\begin{array}{ccc} H^1(V, \mu_\ell) \otimes H^1(V, \mu_\ell) & \longrightarrow & H^1(V|Y, \mu_\ell) \otimes H^1(V|Y, \mu_\ell) \\ \downarrow \cup & & \downarrow \cup \\ H^2(V, \mu_\ell^{\otimes 2}) & \longrightarrow & H^2(V|Y, \mu_\ell^{\otimes 2}). \end{array}$$

The vertical arrows are given by cup-product. The top horizontal arrow is the 2-fold tensor of the surjection of 2.5, so it is onto. Since the reduced scheme of $V|Y$ is a finite disjoint union of spectra of fields $k(y)$, $y \in \tilde{Y}$, $H^1(V|Y, \mu_\ell)$ is the direct sum $\oplus(k(y)^* \otimes \mathbf{Z}/\ell)$ and the right vertical arrow is the sum of the homomorphisms:

$$k(y)^* \otimes k(y)^* \otimes \mathbf{Z}/\ell \rightarrow K_2(k(y)) \otimes \mathbf{Z}/\ell \rightarrow H^2(k(y), \mu_\ell^{\otimes 2}).$$

Now the left hand map is surjective because K_2 of a field is generated by symbols and the right hand map is surjective by Merkurjev-Suslin [MS]. Hence the bottom arrow is also a surjection.

The next proposition gives a sufficient condition for the map $\eta^q : \mathbf{R}^q \pi_*(T) \rightarrow \pi_* \mathcal{H}_{\tilde{X}}^q$ to be an isomorphism.

Proposition 2.7. *If \tilde{X} is regular, π is finite and $T = \mu_\ell^{\otimes n}$ ($n \geq 0$) then*

$$\mathbf{R}^q \pi_*(\mu_\ell^{\otimes n}) \cong \pi_* \mathcal{H}_{\tilde{X}}^q \text{ for all } q.$$

Proof. It is enough to show that $R^p \pi_*(\mathcal{H}_{\tilde{X}}^q) = 0$ for $p \neq 0$. The stalk of this sheaf at $x \in X$ is $H^p(V, \mathcal{H}^q) = \varinjlim H^p(U, \mathcal{H}^q)$, where $V = \pi^{-1}(\mathbf{Spec}(\mathcal{O}_{X,x}))$ and the direct limit runs over all open U in \tilde{X} containing V . By [BO,4.2.2 and 4.7], $H^*(V, \mathcal{H}^q)$ is the cohomology of the complex:

$$(2.7.1) \quad 0 \rightarrow \bigoplus_{V^0} H_{et}^q(v, \mu_\ell^{\otimes n}) \rightarrow \bigoplus_{V^1} H_{et}^{q-1}(v, \mu_\ell^{\otimes n-1}) \rightarrow \cdots \rightarrow \bigoplus_{V^q} H_{et}^0(v, \mu_\ell^{\otimes n-q}) \rightarrow 0.$$

Here \bigoplus_{V^i} means the direct sum is taken over the points $v \in V$ of codimension i .

We have to show that the complex (2.7.1) is exact (except at the left, where the kernel is $H^0(V, \mathcal{H}^q)$). Since π is finite, $S = \pi^{-1}(x)$ is a finite set of points in \tilde{X} . We proceed as on pp.190-191 of [BO]. Let $H_{Z^p}^q(V, n)$ denote the direct limit over all open U and closed Z in \tilde{X} such that $V \subseteq U$ and $\text{codim}(Z) \geq p$

$$H_{Z^p}^q(V, n) = \varinjlim_{U,Z} H_{Z \cap U}^q(U, \mu_\ell^{\otimes n})$$

Then there is an exact sequence (analogous to [BO,4.3]):

$$\cdots \bigoplus_{V^p} H^{q-p}(v, \mu_\ell^{\otimes n-p}) \xrightarrow{\delta} H_{Z^{p+1}}^{q+1}(V, n) \xrightarrow{i} H_{Z^p}^{q+1}(V, n) \xrightarrow{j} \bigoplus_{V^p} H^{q-p+1}(v, \mu_\ell^{\otimes n-p}) \cdots$$

such that the differential in (2.7.1) is obtained by composing δ with j . Therefore it suffices to show that the maps i are zero. Using the direct limit definition of $H_{Z^p}^q(V, n)$ and Poincaré Duality [BO, 1.3.5], the problem reduces to proving the following:

CLAIM. Given $Z' \subseteq \tilde{X}$ of codimension $p + 1$ and a finite subset S of \tilde{X} there is a $Z \subseteq \tilde{X}$ of codimension p containing Z' and an affine neighborhood U of S in \tilde{X} such that the map

$$H_i(Z', \mu_\ell^{\otimes n}) \rightarrow H_i(Z, \mu_\ell^{\otimes n}) \rightarrow H_i(Z \cap U, \mu_\ell^{\otimes n}) \quad \text{is zero.}$$

To obtain a proof of our Claim, simply copy the proof of the Claim on p.191 of [BO], replacing all occurrences of x by S . This completes the proof of Proposition 2.7. ■

Remark 2.7.2 If \tilde{Y} is also regular, then 2.7 also gives $\mathbf{R}^q(\pi j)_*(\mu_\ell^{\otimes n}) \cong (\pi j)_*\mathcal{H}_{\tilde{Y}}^q$. However even if \tilde{X} is regular and π is finite, the map $\mathbf{R}^q(\pi j)_*(\mu_\ell^{\otimes n}) \rightarrow (\pi j)_*\mathcal{H}^q$ need not be an isomorphism in general. This follows from Example 2.4 (b), letting X be obtained by glueing the triangle $\tilde{Y}: XY(X + Y - 1) = 0$ in the affine plane $\tilde{X} = \mathbf{A}_k^2$ to the simple node Y by the map $\pi: \tilde{Y} \rightarrow Y$ considered in 2.4 (b).

§3. Brauer groups of curves.

Let X be a curve over a field k , with normalization $\pi: \tilde{X} \rightarrow X$. To study the Brauer group $\mathbf{Br}(X)$ of X , we will apply the machinery of the last section to the sheaf $T = \mu_\ell$. The Leray spectral sequence for $X_{et} \rightarrow X_{Zar}$ degenerates to yield the exact sequence:

$$0 \rightarrow H_{Zar}^1(X, \mathcal{H}^1) \rightarrow H_{et}^2(X, \mu_\ell) \rightarrow H_{Zar}^0(X, \mathcal{H}^2) \rightarrow 0.$$

Since $H_{Zar}^1(X, -)$ is a right exact functor, and $\mathcal{H}^1(\mu_\ell) \cong \mathcal{O}_X^* \otimes \mathbf{Z}/\ell$ by 2.3, it follows that $H^1(X, \mathcal{H}^1(\mu_\ell)) \cong \mathbf{Pic}(X) \otimes \mathbf{Z}/\ell$. Since $\dim X=1$ there is an isomorphism $H_{et}^2(X, \mathbf{G}_m) \cong \mathbf{Br}(X)$ ([M,IV.2]). On the other hand, Kummer theory yields the exact sequence:

$$0 \rightarrow \mathbf{Pic}(X) \otimes \mathbf{Z}/\ell \rightarrow H_{et}^2(X, \mu_\ell) \rightarrow \ell\mathbf{Br}(X) \rightarrow 0.$$

Comparing these sequences as in [CTR, lemma 1.1] we obtain:

Lemma 3.1. ([BV, 3.5]) *For every curve X and ℓ prime to $\text{char}(k)$,*

$$H_{Zar}^0(X, \mathcal{H}^2(\mu_\ell)) \cong \ell\mathbf{Br}(X).$$

Remark 3.1.1 By [BO,7.7] and [CTR,1.1] we also have an isomorphism

$$H^0(Z, \mathcal{H}^2) \cong \ell\mathbf{Br}(Z)$$

for every regular scheme Z which is quasiprojective over k .

As X is a curve, \tilde{X} is regular and π is finite with $\dim(Y) = 0$. Hence much of the last section may be simplified. For example, $\mathbf{R}^q(\pi j)_*(\mu_\ell) \cong (\pi j)_*\mathcal{H}_{\tilde{Y}}^q$ by 2.7.2. This is a skyscraper sheaf on Y whose stalk at $y \in Y$ is the sum (over all $\tilde{y} \in \tilde{Y}$ lying over y) of the groups $H^q(\tilde{y}, \mu_\ell)$. For $q = 2$ we also have

$$H^0(X, \mathbf{R}^2(\pi j)_*\mu_\ell) = H^0(\tilde{Y}, \mathcal{H}^2) = \ell\mathbf{Br}(\tilde{Y}) = \bigoplus_{\tilde{y} \in \tilde{Y}} \ell\mathbf{Br}(k(\tilde{y})).$$

From (2.2), 2.5 and 2.7 we have an exact sequence of sheaves on X_{Zar} :

$$(3.2) \quad 0 \rightarrow \mathcal{H}_X^2(\mu_\ell) \rightarrow \pi_*\mathcal{H}_{\tilde{X}}^2(\mu_\ell) \oplus i_*\mathcal{H}_Y^2(\mu_\ell) \rightarrow (\pi j)_*\mathcal{H}_{\tilde{Y}}^2(\mu_\ell).$$

Now apply the left exact functor $H^0(X, -)$ to (3.2). Combining with 3.1 we obtain:

Proposition 3.3. *If X is any curve over k and $\text{char}(k) \neq \ell$, there is a natural exact sequence*

$$0 \rightarrow \ell\mathbf{Br}(X) \rightarrow \ell\mathbf{Br}(\tilde{X}) \oplus \ell\mathbf{Br}(Y) \rightarrow \ell\mathbf{Br}(\tilde{Y}).$$

Here $\pi : \tilde{X} \rightarrow X$ is the normalization, $Y = \mathbf{Sing}(X)$ and $\tilde{Y} = Y \times_X \tilde{X}$.

Application 3.4. Suppose that X is a real curve, i.e., a curve defined over \mathbf{R} . We know that $\mathbf{Br}(X) = {}_2\mathbf{Br}(X)$ because $\mathbf{Br}(X_{\mathbf{C}}) = 0$ [GB III,(1.2)]. Therefore (3.3) yields the DeMeyer-Knus exact sequence of elementary abelian 2-groups

$$0 \rightarrow \mathbf{Br}(X) \rightarrow \mathbf{Br}(\tilde{X}) \oplus \mathbf{Br}(Y) \rightarrow \mathbf{Br}(\tilde{Y}).$$

This sequence was originally obtained for affine curves by DeMeyer and Knus in [DK]. The following improvement on (3.2) will be used in §5 to get a long exact sequence in cohomology, whose H^0 -terms are just the DeMeyer-Knus sequence.

Proposition 3.5. *Let X be any curve over $k = \mathbf{R}$. Then the map*

$$(3.5.1) \quad \pi_*\mathcal{H}_{\tilde{X}}^2(\mu_\ell) \rightarrow (\pi j)_*\mathcal{H}_{\tilde{Y}}^2(\mu_\ell)$$

is onto. Therefore the following sequence is exact:

$$(3.5.2) \quad 0 \rightarrow \mathcal{H}_X^2(\mu_\ell) \rightarrow \pi_*\mathcal{H}_{\tilde{X}}^2(\mu_\ell) \oplus i_*\mathcal{H}_Y^2(\mu_\ell) \rightarrow (\pi j)_*\mathcal{H}_{\tilde{Y}}^2(\mu_\ell) \rightarrow 0.$$

Proof. We will show that the map (3.5.1) is onto stalkwise; (3.5.2) is a consequence of this and (3.2). Let y be any point in Y and B the semilocal ring of \tilde{X} at the set of points $\tilde{y} \in \tilde{Y}$ lying over y . Then the stalk of (3.5.1) at y is the map

$$H^0(\mathbf{Spec}B, \mathcal{H}^2(\mu_\ell)) \rightarrow \bigoplus_{\tilde{y}} \ell\mathbf{Br}(k(\tilde{y})).$$

If ℓ is odd, then $\ell \mathbf{Br}(k(\tilde{y})) = 0$ for any \tilde{y} because $\mathbf{Br}\mathbf{R} = \mathbf{Z}/2$ and $\mathbf{Br}\mathbf{C} = 0$. Hence the map (3.5.1) is onto.

Now suppose that $\ell = 2$. Since $-1 \in \mathbf{R}$ the sheaves μ_2 and $\mu_2^{\otimes 2}$ are isomorphic (non-canonically) to $\mathbf{Z}/2$. By 2.6 and 2.7 the map (3.5.1) is onto.

Finally, suppose that ℓ is even. Then $\bigoplus_{\tilde{y}} {}_2\mathbf{Br} k(\tilde{y}) \cong \bigoplus_{\tilde{y}} \ell \mathbf{Br} k(\tilde{y})$. We claim that the inclusion of μ_2 into μ_ℓ induces an isomorphism between $\mathcal{H}^2(\mu_2)$ and $\mathcal{H}^2(\mu_\ell)$, and therefore that the map (3.5.1) is onto by the case $\ell = 2$. To see this, we compare the Bloch-Ogus resolutions [BO, (0.1)] of the sheaves. Because $\mathbf{Br}\mathbf{R}(X)$ is 2-torsion, it is isomorphic to both $H^2(\mathbf{R}(X), \mu_2)$ and $H^2(\mathbf{R}(X), \mu_\ell)$. Similarly, the inclusion induces an isomorphism between $H^1(\tilde{y}, \mathbf{Z}/2) = \mathbf{Z}/2$ and

$$H^1(\tilde{y}, \mu_\ell) = \text{Hom}(\mathbf{Z}/2, \mathbf{Z}/\ell) = \mathbf{Z}/2$$

for any real closed point \tilde{y} in $\mathbf{Spec}B$. Therefore the Bloch-Ogus resolution for the sheaves $\mathcal{H}^2(\mu_2)$ and $\mathcal{H}^2(\mu_\ell)$ on $\mathbf{Spec}B$ are isomorphic, as claimed. ■

The following result extends the main theorem of [DK] to all algebraic curves defined over \mathbf{R} . For smooth real curves, it is essentially Satz III' of Witt's 1934 paper [Witt]. We write $X(\mathbf{R})$ for the set of real points of X with the Euclidean topology. The number c of connected components of $X(\mathbf{R})$ is a topological invariant; in terms of cohomology we have $c = \dim H^0(X(\mathbf{R}), \mathbf{Z}/2)$.

Proposition 3.6. *If X is a curve over \mathbf{R} , and $X(\mathbf{R})$ has c connected components, then $\mathbf{Br}(X) \cong (\mathbf{Z}/2)^c$.*

Proof. Topologically, the scheme X is obtained from \tilde{X} by glueing \tilde{Y} along the map $\tilde{Y} \rightarrow Y$ (this may be checked locally). Hence $X(\mathbf{R})$ is obtained from $\tilde{X}(\mathbf{R})$ and $Y(\mathbf{R})$ by identifying along $\tilde{Y}(\mathbf{R})$.

Suppose first that every component of $\tilde{X}(\mathbf{R})$ is contractible, and form a bipartite graph Γ as follows. The $m + n$ vertices of Γ are the m connected components of $\tilde{X}(\mathbf{R})$, together with the n points of $Y(\mathbf{R})$. There is one edge for each of the q points $\tilde{y} \in \tilde{Y}(\mathbf{R})$, connecting $\pi(\tilde{y}) \in Y(\mathbf{R})$ to the appropriate component of $\tilde{X}(\mathbf{R})$. By construction, Γ is homotopy equivalent to $X(\mathbf{R})$. On the other hand, the simplicial cohomology of Γ is computed by the exact sequence:

$$0 \rightarrow H^0(\Gamma, \mathbf{Z}/2) \rightarrow (\mathbf{Z}/2)^m \oplus (\mathbf{Z}/2)^n \xrightarrow{\delta} (\mathbf{Z}/2)^q \rightarrow H^1(\Gamma, \mathbf{Z}/2) \rightarrow 0,$$

where δ is an incidence matrix. But $\mathbf{Br}(\tilde{X}) \cong (\mathbf{Z}/2)^m$ by [W] or [CT] and it is well-known that $\mathbf{Br}(Y) = (\mathbf{Z}/2)^n$, $\mathbf{Br}(\tilde{Y}) = (\mathbf{Z}/2)^q$. Since the incidence matrix δ is the natural map $\mathbf{Br}(\tilde{X}) \oplus \mathbf{Br}(Y) \rightarrow \mathbf{Br}(\tilde{Y})$, we obtain the desired isomorphism

$$\mathbf{Br}(X) \cong H^0(\Gamma, \mathbf{Z}/2) \cong H^0(X(\mathbf{R}), \mathbf{Z}/2).$$

In the general case, suppose that s of the connected components of $\tilde{X}(\mathbf{R})$ are homeomorphic to a circle. (The other components are homeomorphic to a line.)

Choose a set S of smooth real points of X (not in Y), one on each circular component of $\tilde{X}(\mathbf{R})$. Then $X(\mathbf{R})$ and $X(\mathbf{R}) - S(\mathbf{R})$ both have c components, so it suffices to prove that $\mathbf{Br}(X) \cong \mathbf{Br}(X - S)$. Since $\tilde{X}(\mathbf{R})$ and $\tilde{X}(\mathbf{R}) - S(\mathbf{R})$ have the same number of components, this follows from the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Br}(X) & \longrightarrow & \mathbf{Br}(\tilde{X}) \oplus \mathbf{Br}(Y) & \xrightarrow{\delta} & \mathbf{Br}(\tilde{Y}) \\ & & \downarrow & & \downarrow \cong & & \downarrow = \\ 0 & \longrightarrow & \mathbf{Br}(X - S) & \longrightarrow & \mathbf{Br}(\tilde{X} - S) \oplus \mathbf{Br}(Y) & \longrightarrow & \mathbf{Br}(\tilde{Y}) \end{array}$$

which commutes and has exact rows by (3.5). ■

Remark 3.6.1 The cokernel of δ is $H^1(\Gamma, \mathbf{Z}/2)$, whose dimension is the number of loops in $X(\mathbf{R}) - S(\mathbf{R})$. Thus if $\tilde{X}(\mathbf{R})$ has s circular components and $X(\mathbf{R})$ has λ loops, the cokernel of δ is $(\mathbf{Z}/2)^{\lambda-s}$. We will return to this point in §5.

Remark 3.6.2 By selecting a set S so that $X - S$ is affine, we could reduce the argument to the result of [DK, pp.230-232].

§4. Chern classes and curves.

We now wish to relate algebraic K -theory to étale cohomology using the natural Chern class map ([Sh],[Sou],[Gi],[J])

$$c_{ij} : K_n(X) \rightarrow H_{et}^j(X, \mu_\ell^{\otimes i}), \quad n + j = 2i.$$

Here X be a scheme essentially of finite type over a field k and ℓ be an integer (typically a prime power) such that $\frac{1}{\ell} \in k$. Let \mathcal{K}_n and \mathcal{K}_n/ℓ denote the Zariski sheaves on X associated to the presheaves sending U to $K_n(U)$ and $K_n(U) \otimes \mathbf{Z}/\ell$. Note that $\mathcal{K}_n/\ell = \mathcal{K}_n \otimes \mathbf{Z}/\ell$. The Chern classes sheafify to yield maps

$$c_{ij} : \mathcal{K}_n \rightarrow \mathcal{K}_n/\ell \rightarrow \mathcal{H}^j(\mu_\ell^{\otimes i}).$$

Example 4.1 The map $c_{11} : K_1(X) \rightarrow H^1(X, \mu_\ell)$ is the determinant map up to sign by [Sh], and it sheafifies to yield the canonical isomorphism $c_{11} : \mathcal{K}_1/\ell \cong \mathcal{O}_X^*/\ell \cong \mathcal{H}^1(\mu_\ell)$. The stalk of \mathcal{K}_2 at $x \in X$ is $K_2(\mathcal{O}_{X,x})$ so again by [Sh] the map $c_{22} : \mathcal{K}_2/\ell \rightarrow \mathcal{H}^2(\mu_\ell^{\otimes 2})$ is the Galois symbol.

Lemma 4.2. *If $Y = \mathbf{Spec}A$ where A is an Artinian k -algebra and $\frac{1}{\ell} \in k$ then*

$$K_*(Y) \otimes \mathbf{Z}/\ell \cong K_*(Y_{red}) \otimes \mathbf{Z}/\ell \quad \text{and} \quad H^*(Y, \mu_\ell^{\otimes i}) \cong H^*(Y_{red}, \mu_\ell^{\otimes i}).$$

Consequently we may identify the Chern classes c_{ij} on Y and Y_{red} .

Proof. Let J be the Jacobson radical of A , so that $Y_{red} = \mathbf{Spec}(A/J)$. Because A contains a field, $A \rightarrow A/J$ splits and $K_*(A) \cong K_*(A/J) \times K_*(A, J)$. Since $\frac{1}{\ell} \in A$, $K_*(A, J)$ is a $\mathbf{Z}[\frac{1}{\ell}]$ -module and $K_*(A, J) \otimes \mathbf{Z}/\ell = 0$ by [W1,1.4]. This establishes the K -theory result $K_*(A) \otimes \mathbf{Z}/\ell \cong K_*(A/J) \otimes \mathbf{Z}/\ell$. The cohomology result follows from the observation that the canonical isomorphism of sites $Y_{et} \cong (Y_{red})_{et}$ identifies the sheaves μ_ℓ on Y and on Y_{red} .

Lemma 4.3. *Let X be a smooth variety over a field k , with $\frac{1}{\ell} \in k$. Then $c_{22} : \mathcal{K}_2/\ell \cong \mathcal{H}^2(\mu_\ell^{\otimes 2})$.*

Proof. By [Gray,Cor.6], the sheaf \mathcal{K}_2/ℓ has the flasque resolution:

$$0 \rightarrow \mathcal{K}_2/\ell \rightarrow \eta_*(K_2k(X)/\ell) \rightarrow \coprod_{x \in X^1} i_{x*}(k(X)^*/\ell) \rightarrow \coprod_{x \in X^o} i_{x*}(\mathbf{Z}/\ell) \rightarrow 0,$$

where $\eta : \{\xi\} \rightarrow X$, is the inclusion of the generic point ξ of X . Given this, the proof on p.168 of [CTR] applies verbatim to yield the lemma, (even though k is supposed to be separably closed in *op.cit.*).

Remark 4.3.1 Lemma 4.3 is implicit in [MS,§18]. If we assume that k contains all ℓ^{th} -roots of unity, then [CTR,1.1] gives a natural isomorphism $H^0(X, \mathcal{K}_2/\ell) \cong_\ell \mathbf{Br}(X) \otimes \mu_\ell(k)$. By [S,4.4], [BO,7.7] there is an exact sequence

$$0 \rightarrow H^1(X, \mathcal{K}_2) \otimes \mathbf{Z}/\ell \rightarrow H^1(X, \mathcal{K}_2/\ell) \rightarrow {}_\ell CH^2(X) \rightarrow 0$$

and an isomorphism $H^2(X, \mathcal{K}_2/\ell) \cong CH^2(X) \otimes \mathbf{Z}/\ell$ for every smooth variety X .

Now let X be a curve with normalization $\pi : \tilde{X} \rightarrow X$. As in §2, there is a square with $Y = \text{Sing}(X)$:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{j} & \tilde{X} \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow{i} & X. \end{array}$$

Proposition 4.4. *If $\frac{1}{\ell} \in k$ there is an exact sequence of Zariski sheaves on X , with \mathcal{F} supported on Y :*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{K}_2/\ell \rightarrow \pi_*\mathcal{K}_2/\ell \times i_*\mathcal{K}_2/\ell \rightarrow (\pi j)_*\mathcal{K}_2/\ell \rightarrow 0.$$

Proof. Since the kernel and cokernel of $\mathcal{K}_2/\ell \rightarrow \pi_*\mathcal{K}_2/\ell$ are supported on Y , it is enough to check exactness at a point $x \in Y$. Write $A = \mathcal{O}_{X,x}$ and let B be the (semilocal) coordinate ring of $\pi^{-1}(\text{Spec}A)$. If I denotes the conductor ideal from B to A , A/I and B/I are Artinian rings. Moreover $Y = \text{Spec}(A/I)_{\text{red}}$ and $\tilde{Y} = \text{Spec}(B/I)_{\text{red}}$. By Lemma 4.2 the stalks of $i_*\mathcal{K}_2/\ell$ and $(\pi j)_*\mathcal{K}_2/\ell$ at x are $K_2(A/I) \otimes \mathbf{Z}/\ell$ and $K_2(B/I) \otimes \mathbf{Z}/\ell$, respectively. Furthermore B is regular so it follows from Quillen's proof of Gersten's conjecture ([Q] or [Gray]) that the stalk of $\pi_*\mathcal{K}_n/\ell$ at $x \in Y$ is:

$$H^0(\text{Spec}B, \mathcal{K}_n/\ell) = K_n(B) \otimes \mathbf{Z}/\ell.$$

for any n . Therefore it suffices to show that the following sequence is exact.

$$(4.4.1) \quad K_2(A) \otimes \mathbf{Z}/\ell \rightarrow K_2(B) \otimes \mathbf{Z}/\ell \times K_2(A/I) \otimes \mathbf{Z}/\ell \rightarrow K_2(B/I) \otimes \mathbf{Z}/\ell \rightarrow 0.$$

To prove this we use K -theory with coefficients mod ℓ . The groups $K_n(A; \mathbf{Z}/\ell)$ fit into Universal Coefficients sequences [W1, 2.1]:

$$(4.4.2) \quad 0 \rightarrow K_n(A) \otimes \mathbf{Z}/\ell \rightarrow K_n(A; \mathbf{Z}/\ell) \rightarrow \text{Tor}(K_{n-1}(A), \mathbf{Z}/\ell) \rightarrow 0.$$

Since $\frac{1}{\ell} \in k$ we know (from [W1, (1.3)]) that “excision holds” in the sense that associated to a cartesian square of rings

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/I & \longrightarrow & B/I \end{array}$$

there is a long exact sequence:

$$\cdots \rightarrow K_2(A; \mathbf{Z}/\ell) \rightarrow K_2(B; \mathbf{Z}/\ell) \times K_2(A/I; \mathbf{Z}/\ell) \rightarrow K_2(B/I; \mathbf{Z}/\ell) \rightarrow \cdots$$

Now we consider the following diagram:

$$(4.4.3) \quad \begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ & K_2(A) \otimes \mathbf{Z}/\ell & \rightarrow & (K_2(B) \otimes \mathbf{Z}/\ell) \times (K_2(A/I) \otimes \mathbf{Z}/\ell) & \rightarrow & K_2(B/I) \otimes \mathbf{Z}/\ell & \rightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ & K_2(A; \mathbf{Z}/\ell) & \rightarrow & K_2(B; \mathbf{Z}/\ell) \times K_2(A/I; \mathbf{Z}/\ell) & \rightarrow & K_2(B/I; \mathbf{Z}/\ell) & \rightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & {}_{\ell}A^* & \rightarrow & {}_{\ell}B^* \times {}_{\ell}(A/I)^* & \rightarrow & {}_{\ell}(B/I)^* & \rightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

where the columns are the exact Universal Coefficient sequences, so that the diagram commutes. The bottom sequence is the $\text{Tor}(-, \mathbf{Z}/\ell)$ sequence for

$$0 \rightarrow A^* \rightarrow B^* \times (A/I)^* \rightarrow (B/I)^* \rightarrow 0$$

(where $K_1(A) = A^*$ etc.) so it is exact. A diagram chase shows that the top sequence

$$K_2(A) \otimes \mathbf{Z}/\ell \rightarrow (K_2(B) \otimes \mathbf{Z}/\ell) \times (K_2(A/I) \otimes \mathbf{Z}/\ell) \rightarrow K_2(B/I) \otimes \mathbf{Z}/\ell$$

is exact. So to prove (4.4.1) we need only show that the map $K_2(B) \rightarrow K_2(B/I)$ is surjective. This follows from the fact that both B and B/I are semilocal, so that $K_2(B)$ and $K_2(B/I)$ are generated by Steinberg symbol $\{b, c\}$ with b and c units. Since units of B/I lift to units of B the Steinberg symbols lift.

Corollary 4.5. *Under the same assumption as in (4.4) assume furthermore that k contains a primitive ℓ^{th} -root of unity. Then the following sequence of Zariski sheaves on X is exact.*

$$0 \rightarrow \mathcal{K}_2/\ell \rightarrow \pi_*\mathcal{K}_2/\ell \times i_*\mathcal{K}_2/\ell \rightarrow (\pi j)_*\mathcal{K}_2/\ell \rightarrow 0.$$

Proof. According to (4.4) it is enough to show $\mathcal{F}_x = 0$ for any point x in Y . From the exactness of sequence (4.4.1) and commutativity of the diagram (4.4.3) we get

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{F}_x & \rightarrow & K_2(A) \otimes \mathbf{Z}/\ell & \rightarrow & (K_2(B) \times K_2(A/I)) \otimes \mathbf{Z}/\ell \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ H & \rightarrow & K_3(B/I; \mathbf{Z}/\ell) & \rightarrow & K_2(A; \mathbf{Z}/\ell) & \rightarrow & K_2(B; \mathbf{Z}/\ell) \times K_2(A/I; \mathbf{Z}/\ell) \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \rightarrow & {}_\ell A^* & \rightarrow & {}_\ell B^* \times {}_\ell(A/I)^* \rightarrow \cdots \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

where $H = K_3(B; \mathbf{Z}/\ell) \times K_3(A/I; \mathbf{Z}/\ell)$. So it is enough to show that the map

$$\varphi : K_3(B; \mathbf{Z}/\ell) \rightarrow K_3(B/I; \mathbf{Z}/\ell)$$

is onto. By 4.2 and (4.4.2) $K_3(B/I; \mathbf{Z}/\ell) = K_3((B/I)_{\text{red}}; \mathbf{Z}/\ell)$ so we may assume that B/I is reduced, i.e., that $B/I = F$ is a finite product of fields. By [L,4.12] there is a short exact sequence

$$K_3^M(F) \rightarrow K_3(F; \mathbf{Z}/\ell) \xrightarrow{c_{21}} H^1(F, \mu_\ell^{\otimes 2}) \rightarrow 0,$$

where $K_3^M(F)$ is the Milnor K -group generated by products $\{a, b, c\}$ with $a, b, c \in F^*$. Since B^* maps onto $F^* = (B/I)^*$, every element of $K_3^M(F)$ can be lifted to an element of $K_3(B)$, hence to an element of $K_3(B; \mathbf{Z}/\ell)$, by (4.4.2). Therefore it suffices to show that each element $x \in K_3(F; \mathbf{Z}/\ell)$ lifts to an element $y \in K_3(B; \mathbf{Z}/\ell)$ in the sense that $c_{21}(\varphi(y)) = c_{21}(x)$, where φ is the map $K_3(B; \mathbf{Z}/\ell) \rightarrow K_3(F; \mathbf{Z}/\ell)$.

For this we use a simple method which is taken from [Sou, IV.1.5]: fix a primitive ℓ^{th} -root of unity ξ in B , and let $\beta \in K_2(B; \mathbf{Z}/\ell)$ be an element mapping to $\xi \in {}_\ell B^*$ under the map defined in (4.4.2). Since $H^1(F, \mu_\ell^{\otimes 2}) \cong F^* \otimes \mu_\ell$ we can write $c_{21}(x) = \bar{a} \otimes \xi$, for some $a \in B^*$. Using the K -theory product (as in [S,p.13]) $K_1(B) \times K_2(B; \mathbf{Z}/\ell) \rightarrow K_3(B; \mathbf{Z}/\ell)$, the elements a and β give an element

$y = \{a, \beta\}$ in $K_3(B; \mathbf{Z}/\ell)$. Then the element $y = \{a, \beta\}$ in $K_3(B; \mathbf{Z}/\ell)$. The map φ maps the element y to $\{\bar{a}, \beta\}$ in $K_3(F; \mathbf{Z}/\ell)$ and $c_{21}(\{\bar{a}, \beta\}) = \bar{a} \otimes \xi$ by the product formula [Sou,Th.1]. Hence $c_{21}(\varphi(y)) = c_{21}(x)$.

Remark 4.5.1. We do not know if it is possible to remove the hypothesis that $\xi \in B$ when B contains a field. An easy transfer argument shows that the result is still true if ℓ is prime and $\xi \notin B$.

§5. SK_1 of curves.

In section 3 we related the ℓ -torsion in the Brauer group $\mathbf{Br}(X)$ of a curve X to $H^0(X, \mathcal{H}^2(\mu_\ell))$. In this section we relate the K -theory group $SK_1(X)$ to $H^1(X, \mathcal{H}^2(\mu_\ell^{\otimes 2}))$.

Recall that $SK_1(X)$ is defined to be the kernel of the determinant map $K_1(X) \rightarrow H^0(X, \mathcal{O}_X^*)$. In fact $K_1(X)$ is the direct sum of $SK_1(X)$ and $H^0(X, \mathcal{O}_X^*)$. Let \mathcal{K}_2 and \mathcal{K}_2/ℓ be the sheaves defined in §4. There is a Chern class map (see [Gi]) from $SK_1(X)$ to $H^1(X, \mathcal{K}_2)$. Our first result is:

Theorem 5.1. *Let X be a curve over a field k with $\frac{1}{\ell} \in k$. Then*

$$SK_1(X) \otimes \mathbf{Z}/\ell \cong H^1(X, \mathcal{K}_2/\ell) \cong H^1(X, \mathcal{H}^2(\mu_\ell^{\otimes 2})).$$

The second isomorphism is induced from the Chern class $c_{22} : \mathcal{K}_2/\ell \rightarrow \mathcal{H}^2(\mu_\ell^{\otimes 2})$.

Proof. When X is a curve it is well known (see [W2,p.814]) that $SK_1(X) \cong H^1(X, \mathcal{K}_2)$. Since $\dim X=1$, $H_{\text{Zar}}^1(X, -)$ is a right exact functor; hence the sequence

$$H^1(X, \mathcal{K}_2) \xrightarrow{\ell} H^1(X, \mathcal{K}_2) \rightarrow H^1(X, \mathcal{K}_2/\ell) \rightarrow 0$$

is exact. This yields the first isomorphism. If X is a smooth curve then $c_{22} : \mathcal{K}_2/\ell \cong \mathcal{H}^2(\mu_\ell^{\otimes 2})$ by Lemma 4.3, yielding the second isomorphism in the smooth case. In general, we compare the Mayer-Vietoris sequences for K -theory and étale cohomology, associated to $\pi : \tilde{X} \rightarrow X$, where \tilde{X} is the normalization (cfr. §2).

Let \mathcal{G} be the image of the map $(\pi j)_* \mathcal{H}_{\tilde{Y}}^1(\mu_\ell^{\otimes 2}) \rightarrow \mathcal{H}_X^2(\mu_\ell^{\otimes 2})$ of (2.2). By naturality of c_{22} , 2.6, 2.7 and 4.4 we have a commutative diagram with exact rows:

$$(5.1.1) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{K}_2/\ell & \rightarrow & \pi_* \mathcal{K}_2/\ell \times i_* \mathcal{K}_2/\ell \rightarrow (\pi j)_* \mathcal{K}_2/\ell \rightarrow 0 \\ & & \downarrow & & \downarrow c_{22} & & \cong \downarrow \\ 0 & \rightarrow & \mathcal{G} & \rightarrow & \mathcal{H}_X^2 & \rightarrow & \pi_* \mathcal{H}_{\tilde{X}}^2 \times i_* \mathcal{H}_{\tilde{Y}}^2 \rightarrow (\pi j)_* \mathcal{H}_{\tilde{Y}}^2 \rightarrow 0 \end{array}$$

It follows that the kernel and cokernel of $\mathcal{K}_2/\ell \rightarrow \mathcal{H}_X^2$ are supported on Y , and therefore that $H^1(X, \mathcal{K}_2/\ell) \cong H^1(X, \mathcal{H}_X^2)$, as claimed.

We can in fact make a slightly stronger assertion when k contains ℓ^{th} -roots of unity.

Corollary 5.2. *Let X be a curve over a field k . Assume that $\frac{1}{\ell} \in k$ and that k contains a primitive ℓ^{th} -roots of unity. Then $\mathcal{F} = \mathcal{G} = 0$ in (5.1.1) and c_{22} is a isomorphism:*

$$\mathcal{K}_2/\ell \cong \mathcal{H}_X^2(\mu_\ell^{\otimes 2})$$

Proof. It is enough to check that $\mathcal{F}_x = \mathcal{G}_x = 0$ for every $x \in Y$; from Corollary 4.5 we get $\mathcal{F} = 0$. On the other hand, \mathcal{G}_x is the cokernel of the map from

$$(\pi_* \mathcal{H}_{\tilde{X}}^1)_x \times (i_* \mathcal{H}_Y^1)_x = H_{et}^1(B, \mu_\ell^{\otimes 2}) \times H_{et}^1(A/I, \mu_\ell^{\otimes 2}) \cong (B^* \times (A/I)^*) \otimes \mu_\ell(k)$$

to $(\pi j)_* \mathcal{H}^1 = H^1(B/I, \mu_\ell^{\otimes 2}) \cong (B/I)^* \otimes \mu_\ell(k)$. Since B^* maps onto $(B/I)^*$, this map is a surjection and $\mathcal{G}_x = 0$.

Theorem 5.3. *Let X be a curve over a field k with $\frac{1}{\ell} \in k$. Assume k contains a primitive ℓ^{th} -root of unity. Then there is a natural exact sequence*

$$\begin{aligned} 0 \rightarrow \mathbf{Br}(X) \rightarrow \mathbf{Br}(\tilde{X}) \oplus \mathbf{Br}(Y) \rightarrow \mathbf{Br}(\tilde{Y}) \rightarrow \\ \rightarrow SK_1(X) \otimes \mathbf{Z}/\ell \rightarrow SK_1(\tilde{X}) \otimes \mathbf{Z}/\ell \rightarrow 0 \end{aligned}$$

Proof. This is the cohomology sequence of the short exact sequence in (3.5.2) of Zariski sheaves on X . The first three terms are the exact sequence of Proposition 3.3, and we have used 5.1 to identify $SK_1(X) \otimes \mathbf{Z}/\ell$ with $H^1(X, \mathcal{H}^2(\mu_\ell^{\otimes 2})) = H^1(X, \mathcal{H}^2(\mu_\ell))$.

Application 5.4 (Real Curves) Suppose that X is a curve over \mathbf{R} . We first observe that if $X(\mathbf{R}) = \emptyset$ then $H^1(X, \mathcal{H}^2) = SK_1(X)/2 = 0$. If X is smooth the Bloch-Ogus resolution for \mathcal{H}^2 yields $\mathcal{H}^2 \cong i_*(H_{et}^2(\mathbf{R}(X), \mathbf{Z}/2) = 0$ by [CTP, 1.2.1]. If X is not smooth apply 5.3 to X and \tilde{X} to get $SK_1(X)/2 \cong SK_1(\tilde{X})/2 = 0$.

If $X(\mathbf{R}) \neq \emptyset$ then the sequence (5.3) begins with $\mathbf{Br}(X) \cong (\mathbf{Z}/2)^c$ as described in 3.4 and 3.6. We showed in 3.6.1 that if $X(\mathbf{R})$ has λ loops and $\tilde{X}(\mathbf{R})$ has s loops (circular components) then the cokernel of the second map is $(\mathbf{Z}/2)^{\lambda-s}$. Thus 5.3 yields the exact sequence

$$(5.4.1) \quad 0 \rightarrow (\mathbf{Z}/2)^{\lambda-s} \rightarrow SK_1(X) \otimes \mathbf{Z}/2 \rightarrow SK_1(\tilde{X}) \otimes \mathbf{Z}/2 \rightarrow 0.$$

We are going to show that in fact $SK_1(X) \otimes \mathbf{Z}/2 \cong (\mathbf{Z}/2)^\lambda$.

To do this, we use the topological K -theory KO^* associated to real vector bundles [Hu]. For any quasiprojective scheme X over \mathbf{R} there is a natural map from $SK_1(X)$ to

$$SKO^{-1}(X(\mathbf{R})) = [X(\mathbf{R}), SO],$$

the homotopy classes of maps from $X(\mathbf{R})$ to the special orthogonal group SO . Since $[S^{-1}, SO] = \pi_1(SO) = \mathbf{Z}/2$, it follows that $SKO^{-1}(X(\mathbf{R})) \cong (\mathbf{Z}/2)^\lambda$, where λ is the number of loops of $X(\mathbf{R})$, defined in 1.3.

Consider the map from $K_0(X) = \mathbf{Z} \times \mathbf{Pic}(X)$ to $K_1(X) = H^0(X, \mathcal{O}_X^*) \times SK_1(X)$ obtained by multiplication by $[-1] \in K_1(\mathbf{R})$. It is compatible with the map from $KO(X)$ to $KO^{-1}(X)$ obtained by multiplication by the canonical class η in

$$KO^{-1}(S^0) = [S^0, \mathcal{O}] \cong \mathbf{Z}/2.$$

This yields a commutative diagram:

$$(5.4.2) \quad \begin{array}{ccc} \mathbf{Pic}(X) & \longrightarrow & \mathbf{Pic}(X(\mathbf{R})) \cong (\mathbf{Z}/2)^\lambda \\ [-1] \downarrow & & \cong \downarrow \eta \\ SK_1(X) & \longrightarrow & SKO^{-1}(X(\mathbf{R})). \end{array}$$

Since $X(\mathbf{R})$ is a 1-dimensional complex, Bott periodicity implies that the right vertical map is an isomorphism. The top map is a surjection by Theorem 1.10. Hence the bottom map in (5.4.2) is a surjection. The following theorem says that it is an isomorphism modulo 2.

Theorem 5.5. *Let X be any curve, and let λ be the number of loops in $X(\mathbf{R})$. Then the natural map*

$$SK_1(X) \otimes \mathbf{Z}/2 \rightarrow SKO^{-1}(X(\mathbf{R})) \cong (\mathbf{Z}/2)^\lambda$$

is an isomorphism.

Proof. We have already seen that the map is surjective. From (5.4.1) we see that it suffices to prove the result for smooth curves. When X is a smooth curve over \mathbf{R} , the localization sequence in K -theory is a sequence of $K_*(\mathbf{R})$ -modules, so there is a commutative diagram with exact rows:

$$(5.5.1) \quad \begin{array}{ccccccc} \mathbf{R}(X)^* & \longrightarrow & \coprod_{x \in X^1} \mathbf{Z} & \longrightarrow & \mathbf{Pic}(X) & \longrightarrow & 0 \\ \downarrow [-1] & & \downarrow [-1] & & \downarrow [-1] & & \\ K_2 \mathbf{R}(X) & \longrightarrow & \coprod_{x \in X^1} k(x)^* & \longrightarrow & SK_1(X) & \longrightarrow & 0 \end{array}$$

Tensoring with $\mathbf{Z}/2$ yields the exact diagram

$$(5.5.2) \quad \begin{array}{ccccccc} \coprod_x \mathbf{Z}/2 & \longrightarrow & \mathbf{Pic}(X) \otimes \mathbf{Z}/2 & \longrightarrow & 0 \\ \downarrow [-1] & & \downarrow [-1] & & \\ \coprod_x k(x)^*/k(x)^{*2} & \longrightarrow & SK_1(X) \otimes \mathbf{Z}/2 & \longrightarrow & 0 \end{array}$$

In the lower left corner, $k(x)^*/k(x)^{*2}$ is $\mathbf{R}^*/\mathbf{R}^{*2} \cong \mathbf{Z}/2$ if $x \in X(\mathbf{R})$ and $\mathbf{C}^*/\mathbf{C}^{*2} = 0$ otherwise. If $X(\mathbf{R}) = \emptyset$ this implies that $SK_1(X) \otimes \mathbf{Z}/2 = 0$, the conclusion of 5.5 already made in 5.4. When $X(\mathbf{R}) \neq \emptyset$, the left vertical map of (5.5.2) is onto,

proving that the map $\mathbf{Pic}(X) \otimes \mathbf{Z}/2 \rightarrow SK_1(X) \otimes \mathbf{Z}/2$ is onto. This establishes 5.5 because from (5.4.2) the composition

$$(\mathbf{Z}/2)^\lambda \cong \mathbf{Pic}(X) \otimes \mathbf{Z}/2 \rightarrow SK_1(X) \otimes \mathbf{Z}/2 \rightarrow SKO^{-1}(X(\mathbf{R})) \cong (\mathbf{Z}/2)^\lambda$$

is a isomorphism.

Remark 5.5.3 Here is an alternative proof, using a recent result of Scheiderer ([Sch],[CT]). Let $\mathbf{Spec}_r(X)$ denote the real spectrum of X and $\varphi : \mathbf{Spec}_r(X) \rightarrow X_{\text{Zar}}$ the support map. Scheiderer proves that if X is smooth then $\mathcal{H}^n \cong \varphi_*(\mathbf{Z}/2)$ for $n \geq 2$ and that $R^i\varphi^*(\mathbf{Z}/2) = 0$ for $i \neq 0$. Consequently for all $i \geq 0$ and $n \geq 2$

$$H^i(X, \mathcal{H}^n) \cong H^i(\mathbf{Spec}_r X, \mathbf{Z}/2) \cong H^i(X(\mathbf{R}), \mathbf{Z}/2).$$

In particular, using 5.1 we have the result for smooth X :

$$SK_1(X) \otimes \mathbf{Z}/2 \cong H^1(X, \mathcal{H}^2) \cong H^1(X(\mathbf{R}), \mathbf{Z}/2) \cong (\mathbf{Z}/2)^\lambda.$$

Remark 5.5.4 If X is smooth projective and $X(\mathbf{R}) \neq \emptyset$, then the isomorphism $SK_1(X) \otimes \mathbf{Z}/2 \cong (\mathbf{Z}/2)^c$ follows from [CTR,prop.4.9.b)]

Lemma 5.6. *Let X be an irreducible smooth curve over \mathbf{C} . If X is projective then $SK_1(X) = \mathbf{C}^* \times V(X)$ where $V(X)$ is uniquely divisible. If X is affine then $SK_1(X)$ is either 0 or it is uniquely divisible. In either case $K_2(X)$ is a divisible abelian group.*

Proof. The sequence (4.4) of [S] is (for $\ell \in \mathbf{Z}$):

$$0 \rightarrow H^0(X, \mathcal{K}_2) \otimes \mathbf{Z}/\ell \rightarrow H_{\text{et}}^2(X, \mu_\ell) \otimes \mu_\ell \rightarrow {}_\ell SK_1(X) \rightarrow 0.$$

Suppose first that X is projective. The assertion about $SK_1(X)$ is proven in [R,1.1]. Since $H_{\text{et}}^2(X, \mu_\ell) \cong \mathbf{Z}/\ell$ and ${}_\ell SK_1(X) \cong \mu_\ell$, this shows that $H^0(X, \mathcal{K}_2)$ is divisible.

If X is affine then X has étale cohomological dimension 1. Therefore the above sequence immediately shows that $SK_1(X)$ is torsionfree and that $H^0(X, \mathcal{K}_2)$ is divisible. Since $SK_1(X) \otimes \mathbf{Z}/\ell = 0$ by 5.1, $SK_1(X)$ is divisible. Finally, we use the localization sequence

$$\coprod_{x \in X^1} K_2(\mathbf{C}) \rightarrow K_2(X) \rightarrow H^0(X, \mathcal{K}_2) \rightarrow 0$$

and divisibility of $K_2(\mathbf{C})$ to deduce that $K_2(X)$ is divisible.

Proposition 5.7. *If X is a curve defined over \mathbf{R} , then there is a uniquely divisible abelian group $D = D(X)$ and a natural decomposition*

$$SK_1(X) \cong (\mathbf{Z}/2)^\lambda \oplus D \oplus (\mathbf{C}^*)^r,$$

where r is the number of components of X which are projective and defined over \mathbf{C} .

Proof. Let $\tilde{X} \rightarrow X$ be the normalization of X and suppose that \tilde{X} has r_1 real and r_2 complex projective components. Because $\tilde{Y}_{\mathbf{C}} \rightarrow Y_{\mathbf{C}}$ is étale, excision holds for K_1 ; this follows from [GR] in the affine case, and the general case easily reduces to the affine case. Therefore there is an exact Mayer-Vietoris sequence

$$K_2(\tilde{X}_{\mathbf{C}}) \oplus K_2(Y_{\mathbf{C}}) \rightarrow K_2(\tilde{Y}_{\mathbf{C}}) \rightarrow SK_1(X_{\mathbf{C}}) \rightarrow SK_1(\tilde{X}_{\mathbf{C}}) \rightarrow 0.$$

Since $K_2(\tilde{Y}_{\mathbf{C}}) = \coprod K_2(\mathbf{C})$ and $K_2(\mathbf{C})$ is uniquely divisible, lemma 5.6 shows that $SK_1(X_{\mathbf{C}})$ is the direct sum of a uniquely divisible group and $SK_1(\tilde{X}_{\mathbf{C}})$. Using lemma 5.6 again, we get that $SK_1(X_{\mathbf{C}}) \cong V(X_{\mathbf{C}}) \oplus (\mathbf{C}^*)^h$, for some uniquely divisible group $V(X_{\mathbf{C}})$, where h is the number of components of $\tilde{X}_{\mathbf{C}}$ which are projective. Now let D' be the image of the transfer map from $SK_1(X_{\mathbf{C}})$ to $SK_1(X)$. Since $SK_1(X) \rightarrow SK_1(X_{\mathbf{C}}) \rightarrow SK_1(X)$ is multiplication by 2, we get $2D' \subseteq 2SK_1(X) \subseteq D'$. As D' is divisible, $2SK_1(X) = D'$. By 5.5, $SK_1(X) \cong (\mathbf{Z}/2)^\lambda \oplus D'$. If X has a projective component which is defined over \mathbf{R} then the transfer map sends \mathbf{C}^* to the group of positive reals which has no torsion. If X has a projective component defined over \mathbf{C} then $X_{\mathbf{C}}$ has 2 components and the transfer map maps the corresponding summand $\mathbf{C}^* \oplus \mathbf{C}^*$ onto \mathbf{C}^* yielding torsion. Hence D' is the sum of a uniquely divisible group D and possibly some \mathbf{C}^* 's, one for every projective component of X defined over \mathbf{C} . Therefore $D' \cong D \oplus (\mathbf{C}^*)^r$ and, by 5.5, $SK_1(X) = (\mathbf{Z}/2)^\lambda \oplus D' = (\mathbf{Z}/2)^\lambda \oplus D \oplus (\mathbf{C}^*)^r$ where D is uniquely divisible.

Example 5.8 a) Let $\tilde{X} = \mathbf{Spec}(\mathbf{R}[t, t^{-1}])$ and let X be the curve obtained from \tilde{X} by gluing the points $t = 1$ and $t = \alpha$ where $\alpha \neq 0, 1$. Excision holds (by [GR, Thm.2]) so we get a mayer-Vietoris exact sequence

$$K_2(\tilde{X}) \xrightarrow{\Phi} K_2(\mathbf{R}) \rightarrow SK_1(X) \rightarrow 0.$$

Hence $SK_1(X) \cong K_2(\mathbf{R})/Im\Phi$. In [RW, prop.4.10] $Im\Phi$ has been computed for different values of α : if $\alpha = -1$ then $SK_1(X) = K_2(\mathbf{R})/\{-1, -1\} = K_2(\mathbf{C})^+$ where $K_2(\mathbf{C}) = K_2(\mathbf{C})^+ \oplus K_2(\mathbf{C})^-$. Hence $SK_1(X)/2 = 0$, and $X(\mathbf{R})$ has no loops. If $\alpha < 0$, $\alpha \neq -1$ then $SK_1(X) = K_2(\mathbf{R})/\{\mathbf{R}^*, \alpha\}$; in this case we also have $SK_1(X)/2 = 0$. If $\alpha > 0$, $\alpha \neq 1$ then $SK_1(X) = \mathbf{Z}/2 \oplus V$, for some divisible group V . In the first two cases the real loop has been punctured, while in the last case the curve X has 1 loop.

b) Let $\tilde{X} = \mathbf{Spec}(\mathbf{R}[X, Y]/(X^2 + Y^2 - 1))$. For each θ let X be the curve obtained by identifying the real points $(1, 0)$ and $(\cos \theta, \sin \theta)$ of \tilde{X} . In this case $X(\mathbf{R})$ has always two loops so that $SK_1(X)/2 = (\mathbf{Z}/2)^2$. However the computation in [RW, Ex. 4.11] show that the divisible part of $SK_1(X)$ depends on the choice of θ . We always have $SK_1(X) = (\mathbf{Z}/2)^2 \oplus V$; if $e^{i\theta}$ is a root of unity then $V = K_2(\mathbf{C})^+$, otherwise V is just a quotient of $K_2(\mathbf{C})^+$.

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Dip. di Matematica
Università di Genova
Via L.B.Alberti, 4
Genova, Italy

Math.Dept.
Rutgers University
New Brunswick
08903 USA