

K-THEORY OF CONES OF SMOOTH VARIETIES

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ABSTRACT. Let R be the homogeneous coordinate ring of a smooth projective variety X over a field k of characteristic 0. We calculate the K -theory of R in terms of the geometry of the projective embedding of X . In particular, if X is a curve then we calculate $K_0(R)$ and $K_1(R)$, and prove that $K_{-1}(R) = \oplus H^1(C, \mathcal{O}(n))$. The formula for $K_0(R)$ involves the Zariski cohomology of twisted Kähler differentials on the variety.

Let $R = k \oplus R_1 \oplus \cdots$ be the homogeneous coordinate ring of a smooth projective variety X over a field k of characteristic 0. In this paper we compute the lower K -theory ($K_i(R)$, $i \leq 1$) in terms of the Zariski cohomology groups $H^*(X, \mathcal{O}(t))$ and $H^*(X, \Omega_X^*(t))$, where $\mathcal{O}(1)$ is the ample line bundle of the embedding and Ω_X^* denotes the Kähler differentials of X relative to \mathbb{Q} . We also obtain computations of the higher K -groups $K_n(R)/K_n(k)$, especially for curves. A complete calculation for the conic $xy = z^2$ is given in Theorem 4.3. These calculations have become possible thanks to the new techniques introduced in [1], [2] and [4].

Here, for example, is part of Theorem 2.1; R^+ is the seminormalization of R .

Theorem. *Let R be the homogeneous coordinate ring of a smooth d -dimensional projective variety X in \mathbb{P}_k^N . Then $\text{Pic}(R) \cong (R^+/R)$ and*

$$K_0(R) \cong \mathbb{Z} \oplus \text{Pic}(R) \oplus \bigoplus_{i=1}^d \bigoplus_{t=1}^{\infty} H^i(X, \Omega_X^i(t)), \quad \text{and}$$

$$K_{-m}(R) \cong \bigoplus_{i=0}^{d-m} \bigoplus_{t=1}^{\infty} H^{m+i}(X, \Omega_X^i(t)), \quad m > 0.$$

We have $K_{-m}(R) = 0$ for $m > d$, and $K_{-d}(R) = \bigoplus_{t \geq 1} H^d(X, \mathcal{O}(t))$.

If k has finite transcendence degree over \mathbb{Q} then $K_0(R)/\mathbb{Z}$ and each $K_{-m}(R)$ are finite-dimensional k -vector spaces.

For example, if $X = \text{Proj}(R)$ is a smooth curve over k which is definable over a number field contained in k , we show that $\Omega_k^1 \otimes \mathcal{O}(t) \rightarrow \Omega_X^1(t)$ induces:

$$(0.1) \quad K_0(R) = \mathbb{Z} \oplus \text{Pic}(R) \oplus (\Omega_k^1 \otimes K_{-1}(R)), \quad K_{-1}(R) \cong \bigoplus_{t=1}^{\infty} H^1(X, \mathcal{O}_X(t)).$$

We also have $K_n^{(n+2)}(R) \cong \Omega_k^{n+1} \otimes K_{-1}(R)$ for all $n \geq 1$. (See Proposition 3.2(d).)

When R is normal, (0.1) implies that $K_0(R) = \mathbb{Z}$ holds if and only if either (a) k is algebraic over \mathbb{Q} , or (b) $K_{-1}(R) = 0$. Case (a) was discovered by Krishna and Srinivas [10, 1.2], while parts of case (b) were discovered in [25]. By Riemann-Roch, the vanishing of $K_{-1}(R)$ is equivalent to the vanishing of the vector spaces $H^0(X, \Omega_{X/k}^1(-t))$ for $t > 0$, which is a delicate arithmetic question (unless, for

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example, the embedding has degree $d \geq 2g - 2$). Note that case (b) clarifies Srinivas' theorem in [17] that when $k = \mathbb{C}$ and $H^1(X, \mathcal{O}(1)) \neq 0$ we have $K_0(R) \neq \mathbb{Z}$.

Still assuming that X is a curve, suppose in addition that k is a number field; then $\Omega_k^1 = 0$ and hence $K_0(R) = \mathbb{Z} \oplus (R^+/R)$. We also establish (in 1.17 and 2.11) the previously unknown calculations that

$$(0.2) \quad K_1(R) = k^\times \oplus \left[\bigoplus_{t=1}^{\infty} H^0(X, \Omega_{X/k}^1(t)) \right] / \Omega_{R/k}^1, \quad K_2(R) = K_2(k) \oplus \text{tors } \Omega_{R/k}^1,$$

$$(0.3) \quad K_n(R) = K_n(k) \oplus HC_{n-1}(R)/HC_{n-1}(k), \quad n \geq 3.$$

The K_1 formula (0.2) is a clarification of a result of Srinivas [19]. When k is not algebraic over \mathbb{Q} , formulas (0.1), (0.2) and (0.3) need to be altered to involve the arithmetic Gauss-Manin connection; see Proposition 3.5 and Example 3.6.

For any smooth d -dimensional variety X , $K_0(R)/\mathbb{Z}$ is the direct sum of the eigenspaces $K_0^{(i)}(R)$ of the Adams operation, $1 \leq i \leq d + 1 = \dim R$, and we give a formula for these eigenspaces. For example, the top eigenspace, $K_0^{(d+1)}(R)$, may be identified with the Chow group of smooth zero-cycles in $\text{Spec}(R)$; we show that

$$K_0^{(d+1)}(R) \cong \bigoplus_{t=1}^{\infty} H^d(X, \Omega_X^d(t)).$$

As pointed out in [10], the normal domain $R_k = k[x, y, z]/(x^n + y^n + z^n)$ has $K_0(R_{\mathbb{Q}}) = \mathbb{Z}$ but if $n \geq 4$ then $H^1(X, \mathcal{O}(1))$ is nonzero while $K_0(R_{\mathbb{C}})/\mathbb{Z}$ is a very big \mathbb{C} -vector space; by (0.1), it is the direct sum of the $\Omega_{\mathbb{C}}^1 \otimes H^1(X, \mathcal{O}(t))$, $t \geq 1$.

We also obtain reasonably nice formulas for the eigenspaces $K_n^{(i)}(R)$ when $n > 0$ and $i \geq n$; see Theorem 1.13. To illustrate the range of our cohomological results, consider $K_1(R)$ when X is a smooth curve and R is normal; we have $K_1(R) = k^\times \oplus K_1^{(2)}(R) \oplus K_1^{(3)}(R)$, where

$$(0.4) \quad K_1^{(2)}(R) \cong \left(\bigoplus_{t=1}^{\infty} H^0(X, \Omega_X^1(t)) \right) / \Omega_R^1, \quad \text{and}$$

$$K_1^{(3)}(R) = \bigoplus_{t=1}^{\infty} \text{coker} \{ \Omega_k^1 \otimes H^0(X, \Omega_{X/k}^1(t)) \xrightarrow{\nabla} \Omega_k^2 \otimes H^1(X, \mathcal{O}_X(t)) \}.$$

The map ∇ in (0.4) is a twisted Gauss-Manin connection (see Lemma 3.4). In Section 3, we prove that if $n \geq 1$ then $K_n^{(n+1)}(R)$ contains $\Omega_k^{n-1} \otimes_{\mathbb{Q}} k^{d+g-1}$ as a direct summand provided that either

- (a) X has genus g and is embedded in \mathbb{P}_k^N by a complete linear system of degree d , with $d \geq 2g - 1$, or
- (b) X is induced by base change to k from a curve defined over a number field contained in k .

(See Theorem 3.8 and Example 3.9.) In particular $K_1^{(2)}(R) \neq 0$, and in general, $K_n^{(n+1)}(R) \neq 0$ if $n - 1 \leq \text{tr. deg}(k/\mathbb{Q})$. Observe that the case $n = 1$ improves the result of Srinivas in [19, §1] that there is a surjection from $\tilde{K}_1(R) = K_1(R)/K_1(k)$ to $H^0(X, \Omega_{X/k}^1(1))$ and hence that $\tilde{K}_1(R) \neq 0$ if $d \geq 2g + 1$.

Finally, in Theorem 4.3 we give a complete calculation of the K -theory of the homogeneous coordinate ring of the plane conic, $R = k[x, y, z]/(xy - z^2)$.

This paper is organized as follows. In Section 1, we reduce the calculation of $K_n(R)$ to a cdh -cohomology computation and knowledge of $HC_{n-1}(R)$. This relies

on the basic observation that cones are \mathbb{A}^1 -contractible, so that the reduced K -theory $\tilde{K}_n(R) = K_n(R)/K_n(k)$ can be calculated in terms of $NK_n(R)$, making our previous calculations (see [1], [2], [4]) applicable. Several of the formulas we obtain are valid for general graded algebras of the form $R = k \oplus R_1 \oplus \cdots$. We also specialize these formulas to the case when $\dim R = 2$, and obtain an expression for $K_n(R)$ in terms of cdh cohomology and cyclic homology ($n \geq 1$).

In Section 2 we compute the cdh terms in the formulas of the previous sections for the case when R is the affine cone of a smooth variety. In Section 3, we return to the case when the graded coordinate ring has dimension 2, that is, we investigate cones over smooth projective curves. Finally, in Section 4 we apply the techniques of this paper to completely determine the K -theory of $R = k[x, y, z]/(xy - z^2)$.

Notations: Throughout this paper we consider (commutative, unital) algebras over a fixed ground field k , which we assume has characteristic zero. Undecorated tensor products \otimes and differential forms Ω^* are taken over \mathbb{Q} ; we write \otimes_k and $\Omega^*_{/k}$ for tensor product and forms relative to k . Similarly, cyclic homology is always taken over \mathbb{Q} . If F is a functor defined on schemes over k , we will write $F(R)$ for $F(\text{Spec}(R))$. If R is an augmented k -algebra (for example, the homogeneous coordinate ring of a variety), and F is a functor from rings to some abelian category, then we write $\tilde{F}(R)$ for the (split) quotient $F(R)/F(k)$.

1. K -THEORY OF GRADED ALGEBRAS

Throughout this section, we let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely generated graded algebra over a field k of characteristic 0 such that R_0 is a local, artinian k -algebra whose residue field is isomorphic to k as a k -algebra. These conditions ensure that the map $K_n(R) \rightarrow K_n(k)$ induced by the composition of $R \twoheadrightarrow R_0 \twoheadrightarrow k$ is a split surjection. For example, R_0 might be k itself, and indeed for most of the calculations in this paper, one may as well assume $R_0 = k$. Let \mathfrak{m}_R denote the unique graded maximal ideal of R ; that is, \mathfrak{m}_R is the kernel of the split surjection $R \twoheadrightarrow k$.

We let R_{red} denote the reduced ring associated to R . It is a graded ring whose degree 0 piece is the field k . We let \tilde{R} denote the normalization of R_{red} (i.e., the integral closure of R_{red} in its ring of total quotients). It is well known that $\tilde{R} = \tilde{R}_0 \oplus \tilde{R}_1 \oplus \cdots$ is graded, that \tilde{R}_0 is a product of fields, and that $\text{Pic}(\tilde{R}) = 0$.

We let R^+ denote the semi-normalization of R_{red} , that is, the maximal extension of R_{red} inside its total quotient ring Q such that for all $x \in Q$, $x^2, x^3 \in R^+$ implies $x \in R^+$; see [20]. Alternatively, $\text{Spec}(R^+) \rightarrow \text{Spec}(R_{red})$ is a universal homeomorphism.

We are interested in computing the kernel $\tilde{K}_n(R)$ of the split surjection $K_n(R) \rightarrow K_n(k)$, for $n = 1, 0, -1, \dots, 1 - d$. (By [1], $K_n(R) = NK_n(R) = 0$ for $n \leq -d$.) In general, for any graded ring $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$, the groups $\tilde{K}_n(R)$ are known to be R_0 -modules (see [22]), and hence (since R_0 contains \mathbb{Q}) they are uniquely divisible as abelian groups. Thus there is a decomposition $\tilde{K}_n(R) \cong \bigoplus_i \tilde{K}_n^{(i)}(R)$ according to the eigenvalues k^i of the Adams operations ψ^k .

Remark 1.1. Suppose that the punctured spectrum, $\text{Spec}(R_{red}) \setminus \{\mathfrak{m}_R\}$, is non-singular. Then the conductor \mathfrak{c} to the normalization \tilde{R} of R_{red} is \mathfrak{m}_R -primary. An easy calculation shows that the seminormalization of R_{red} is

$$R^+ = k \oplus \tilde{R}_1 \oplus \tilde{R}_2 \oplus \cdots,$$

with $\tilde{R}/R^+ = \tilde{R}_0/k$ and $R^+/R_{red} = \tilde{R}/(\tilde{R}_0 + R_{red})$. Then $K_n^{(i)}(R) \cong \tilde{K}_n^{(i)}(R)$ for $n \leq 1$, with two exceptions: $\tilde{K}_0^{(0)}(R) = 0$, and $\tilde{K}_1^{(1)}(R) \cong \text{nil}(R)/\text{nil}(R_0)$. The problem of computing \tilde{R}/R_{red} (and hence R^+/R_{red}) is hard.

The main results of this section, Theorems 1.2 and 1.13, are formulated in terms of the *cdh* cohomology groups $H_{\text{cdh}}^*(R, \Omega^i)$ introduced in [1] and [2], where the Kähler differentials, $\Omega^i = \Omega^i_{-/Q}$, are taken relative to the base field \mathbb{Q} . By [4, 2.5], we have that $H_{\text{cdh}}^0(R, \mathcal{O}) = R^+$. For simplicity, we write $H_{\text{cdh}}^m(R, \Omega^i)/dH_{\text{cdh}}^m(R, \Omega^{i-1})$ for the cokernel of the map $d: H_{\text{cdh}}^m(R, \Omega^{i-1}) \rightarrow H_{\text{cdh}}^m(R, \Omega^i)$ induced by the Kähler differential. Theorem 1.2 will follow from Proposition 1.5 and Theorem 1.12 below.

Theorem 1.2. *Let $R = R_0 \oplus R_1 \oplus \dots$ be a finitely generated graded algebra over a field k of characteristic 0. Assume R_0 is local artinian with residue field k . Then the Adams operations induce an eigenspace decomposition:*

$$K_0(R) = \mathbb{Z} \oplus R^+/R_{red} \oplus \bigoplus_{i=1}^{\dim R-1} H_{\text{cdh}}^i(R, \Omega^i)/dH_{\text{cdh}}^i(R, \Omega^{i-1}).$$

The negative K -groups are given by

$$K_{-m}(R) = H_{\text{cdh}}^m(R, \mathcal{O}) \oplus \bigoplus_{i=1}^{\dim R-m-1} H_{\text{cdh}}^{m+i}(R, \Omega^i)/dH_{\text{cdh}}^{m+i}(R, \Omega^{i-1}).$$

for $m > 0$. Here, $K_0^{(0)}(R) = \mathbb{Z}$, $K_0^{(1)}(R) \cong R^+/R_{red}$, $K_{-m}^{(1)}(R) \cong H_{\text{cdh}}^m(R, \mathcal{O})$ and the groups indexed by i are $K_0^{(i+1)}$ and $K_{-m}^{(i+1)}(R)$, respectively.

By [23, 1.2 and 2.3] we have $KH_*(R) \cong KH_*(R_0) \cong K_*(k)$, and thus by [2, 1.6], we have

$$(1.3) \quad \tilde{K}_n(R) \cong \pi_n \mathcal{F}_K(R) \cong \pi_{n-1} \mathcal{F}_{HC}(R) \quad \text{for all } n.$$

Here, $\mathcal{F}_{HC}(R) = \mathcal{F}_{HC}(R/\mathbb{Q})$ is the homotopy fiber of $HC(R) \rightarrow \mathbb{H}_{\text{cdh}}(R, HC)$, with cyclic homology taken relative to the subfield \mathbb{Q} of k , so that there is a long exact sequence

$$\dots \rightarrow HC_n(R) \rightarrow \mathbb{H}_{\text{cdh}}^{-n}(R, HC) \rightarrow \tilde{K}_n(R) \rightarrow HC_{n-1}(R) \rightarrow \dots$$

These groups all have λ -decompositions and the maps in this sequence are compatible with these decompositions (see [3]), but there is a weight shift in that $\tilde{K}_n^{(i)}(R)$ maps to $HC_{n-1}^{(i-1)}(R)$. We have $\tilde{K}_n^{(0)}(R) = 0$ for all n because $\mathcal{F}_{HC}^{(-1)}(R) \simeq 0$. Moreover, by [2, 2.2] we have $\mathbb{H}_{\text{cdh}}^m(R, HC^{(i)}) \cong \mathbb{H}_{\text{cdh}}^{2i+m}(R, \Omega^{\leq i})$, so the long exact sequence becomes

$$(1.4) \quad \dots HC_n^{(i-1)}(R) \rightarrow \mathbb{H}_{\text{cdh}}^{2i-n-2}(R, \Omega^{<i}) \rightarrow \tilde{K}_n^{(i)}(R) \rightarrow HC_{n-1}^{(i-1)}(R) \dots$$

The general picture is given by the following proposition.

Proposition 1.5. *Let $R = R_0 \oplus R_1 \oplus \dots$ be as in Theorem 1.2. Then $\tilde{K}_n^{(0)}(R) = 0$ for all n . For $n \leq 0$, or for $n > 0$ and $i \geq n + 2$, we have*

$$\tilde{K}_n^{(i)}(R) \cong \mathbb{H}_{\text{cdh}}^{2i-n-2}(R, \Omega^{<i}), \quad \text{except for } (n, i) = (0, 1),$$

In the exceptional case, $\tilde{K}_0^{(1)}(R) = \text{Pic}(R) = R^+/R_{red}$.

Proof. The group $HC_n(R)$ vanishes for $n < 0$ and is R for $n = 0$. Similarly, $HC_n^{(i)}(R)$ vanishes for $i > n > 0$ (see [24, 9.8.14]). The proposition now follows from (1.4) and the fact that $H_{\text{cdh}}^0(R, \mathcal{O}) = R^+$ by [4, 2.5]. \square

To go further, it is useful to invoke the following trick, using the standard \mathbb{A}^1 -contraction of a cone to its vertex.

Standard Trick 1.6. If R is a positively graded algebra, there is an algebra map $\nu : R \rightarrow R[t]$ sending $r \in R_n$ to rt^n . If F is a functor on algebras, then the composition of ν with evaluation at $t = 0$ factors as $R \rightarrow R_0 \rightarrow R$, so $F(R) \xrightarrow{\nu} F(R[t]) \xrightarrow{t=0} F(R)$ is zero on the kernel $\tilde{F}(R)$ of $F(R) \rightarrow F(R_0)$. Similarly, the composition of ν with evaluation at $t = 1$ is the identity. That is, ν maps $\tilde{F}(R)$ isomorphically onto a summand of $NF(R)$, and $\tilde{F}(R)$ is in the image of the map $(t = 1) : NF(R) \rightarrow F(R)$.

The following technical result is crucial for our calculations; it asserts that many SBI sequences ([24, 9.6.11]) decompose into split short exact sequences. We write \mathcal{F}_{HH} and \mathcal{F}_{HC} for the homotopy fibers of $HH(R) \rightarrow \mathbb{H}_{\text{cdh}}(R, HH)$ and $HC(R) \rightarrow \mathbb{H}_{\text{cdh}}(R, HC)$, respectively. Then we have distinguished cohomological triangles

$$\begin{aligned} \mathcal{F}_{HC}[-1] \xrightarrow{S} \mathcal{F}_{HC}[1] \xrightarrow{B} \mathcal{F}_{HH} \xrightarrow{I} \mathcal{F}_{HC}, \\ \mathbb{H}_{\text{cdh}}(R, HC)[-1] \xrightarrow{S} \mathbb{H}_{\text{cdh}}(R, HC)[1] \xrightarrow{B} \mathbb{H}_{\text{cdh}}(R, HH) \xrightarrow{I} \mathbb{H}_{\text{cdh}}(R, HC). \end{aligned}$$

Lemma 1.7. *If $R = R_0 \oplus R_1 \oplus \cdots$ is a graded algebra then for each m the map $\pi_m \mathcal{F}_{HC}(R) \xrightarrow{S} \pi_{m-2} \mathcal{F}_{HC}(R)$ is zero, and there is a split short exact sequence:*

$$0 \rightarrow \pi_{m-1} \mathcal{F}_{HC}(R) \xrightarrow{B} \pi_m \mathcal{F}_{HH}(R) \xrightarrow{I} \pi_m \mathcal{F}_{HC}(R) \rightarrow 0.$$

Similarly, there are split short exact sequences:

$$0 \rightarrow \tilde{\mathbb{H}}_{\text{cdh}}^{m+1}(R, HC) \xrightarrow{B} \tilde{\mathbb{H}}_{\text{cdh}}^m(R, HH) \xrightarrow{I} \tilde{\mathbb{H}}_{\text{cdh}}^m(R, HC) \rightarrow 0.$$

and

$$0 \rightarrow \tilde{\mathbb{H}}_{\text{cdh}}^{n-1}(R, \Omega^{<i}) \xrightarrow{B} \tilde{H}_{\text{cdh}}^{n-i}(R, \Omega^i) \xrightarrow{I} \tilde{\mathbb{H}}_{\text{cdh}}^n(R, \Omega^{\leq i}) \rightarrow 0.$$

Proof. The third sequence is obtained from the second one by taking the i^{th} component in the Hodge decomposition, described in [2, 2.2], and setting $n = 2i + m$. For the first two sequences to split, it suffices to show that I is onto and split.

By [2, 2.4], $\mathcal{F}_{HH}(k) = \mathcal{F}_{HC}(k) = 0$, so $\tilde{\mathcal{F}}_{HH} = \mathcal{F}_{HH}$ and $\tilde{\mathcal{F}}_{HC} = \mathcal{F}_{HC}$. By the standard trick 1.6, it suffices to show that the maps $N\pi_m \mathcal{F}_{HH}(R) \rightarrow N\pi_m \mathcal{F}_{HC}(R)$ and $N\mathbb{H}_{\text{cdh}}^m(R, HH) \rightarrow N\mathbb{H}_{\text{cdh}}^m(R, HC)$ are onto and split. But they are split surjections, as is evident from the respective decompositions of their terms in [4, 3.2] and [4, 2.2]; $\mathbb{H}_{\text{cdh}}(R, NHH^{(i)}) \simeq \mathbb{H}_{\text{cdh}}(R, NHC^{(i)}) \oplus \mathbb{H}_{\text{cdh}}(R, NHC^{(i-1)})$ and $N\mathcal{F}_{HH}^{(i)}(R) \simeq N\mathcal{F}_{HC}^{(i)}(R) \oplus N\mathcal{F}_{HC}^{(i-1)}(R)$. (Note that $H_{\text{cdh}}(-, NF) = NH_{\text{cdh}}(-, F)$ for any presheaf F .) \square

Splicing the final sequences of Lemma 1.7 together, we see that the de Rham complexes are exact in *cdh*-cohomology:

Proposition 1.8. *The following sequences are exact:*

$$(1.8a) \quad 0 \rightarrow k \rightarrow R^+ \xrightarrow{d} \tilde{H}_{\text{cdh}}^0(R, \Omega^1) \xrightarrow{d} \tilde{H}_{\text{cdh}}^0(R, \Omega^2) \rightarrow \cdots$$

$$(1.8b) \quad 0 \rightarrow H_{\text{cdh}}^m(R, \mathcal{O}) \xrightarrow{d} H_{\text{cdh}}^m(R, \Omega^1) \xrightarrow{d} H_{\text{cdh}}^m(R, \Omega^2) \rightarrow \cdots, \quad m > 0.$$

Note that the first complex is the *cdh* reduced de Rham complex.

An analogous exact sequence

$$\cdots \rightarrow \pi_{m-1} \mathcal{F}_{HH}(R) \xrightarrow{d} \pi_m \mathcal{F}_{HH}(R) \xrightarrow{d} \pi_{m+1} \mathcal{F}_{HH}(R) \rightarrow \cdots$$

is obtained by splicing the other sequences in 1.7. Using the interpretation of their Hodge components, described in [4, 3.4], produces two more exact sequences:

Proposition 1.9. *The following sequences are exact:*

$$(1.9a) \quad 0 \rightarrow \text{nil}(R) \rightarrow \text{tors } \Omega_R^1 \rightarrow \text{tors } \Omega_R^2 \rightarrow \text{tors } \Omega_R^3 \rightarrow \cdots$$

$$(1.9b) \quad 0 \rightarrow (R^+/R) \rightarrow \Omega_{\text{cdh}}^1(R)/\Omega_R^1 \rightarrow \Omega_{\text{cdh}}^2(R)/\Omega_R^2 \rightarrow \cdots$$

Here we have used the following notation

$$(1.10) \quad \Omega_{\text{cdh}}^i(R) = H_{\text{cdh}}^0(R, \Omega^i)$$

$$(1.11) \quad \text{tors } \Omega_R^i = \ker(\Omega_R^i \rightarrow \Omega_{\text{cdh}}^i(R))$$

If R is reduced then $\text{tors } \Omega_R^i$ is the usual torsion submodule, by [4, 5.6.1].

We can now make the calculations necessary to deduce Theorem 1.2.

Theorem 1.12. *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded algebra, finitely generated over a field k of characteristic 0. Assume R_0 is local artinian with residue field k . Then we have*

$$\mathbb{H}_{\text{cdh}}^{q+i}(R, \Omega^{\leq i}) = \begin{cases} H_{dR}^{q+i}(k), & q < 0; \\ \text{coker}\{H_{\text{cdh}}^q(R, \Omega^{i-1}) \xrightarrow{d} H_{\text{cdh}}^q(R, \Omega^i)\}, & q \geq 0; \\ 0, & q \geq \dim(R). \end{cases}$$

Proof. The Cartan-Eilenberg spectral sequence for $\Omega^{\leq i}$ is

$${}^I E_1^{p,q} = H_{\text{cdh}}^q(R, \Omega^p) \implies \mathbb{H}_{\text{cdh}}^{p+q}(R, \Omega^{\leq i}) \quad (0 \leq p \leq i, q \geq 0).$$

(See [24, 5.7.9].) Since $H_{\text{cdh}}^0(R, \Omega^p) = \Omega_k^p \oplus \tilde{H}_{\text{cdh}}^0(R, \Omega^p)$, the row $q = 0$ is the brutal truncation of the direct sum of the de Rham complex of k over \mathbb{Q} and the complex (1.8a), which is acyclic by Proposition 1.8. Since $H_{\text{cdh}}^q(R, \Omega^p) = \tilde{H}_{\text{cdh}}^q(R, \Omega^p)$ for $q > 0$, the other rows on the E_1 -page are the truncations of the complex (1.8b), which is also acyclic by 1.8. Hence the spectral sequence degenerates at E_2 , yielding the calculation. Note that the last possible nonzero group is $\mathbb{H}_{\text{cdh}}^{i+\dim R-1}(R, \Omega^{\leq i}) = H_{\text{cdh}}^{\dim R-1}(R, \Omega^i)$ by the cohomological bound in [2, 2.6]. \square

Proof of Theorem 1.2. Simply plug the calculations of Theorem 1.12 into those of Proposition 1.5 to get the asserted result. \square

We conclude the section with a calculation of the higher K -theory of R in terms of Kähler differentials, the cyclic homology of R and the cdh -cohomology of $\text{Spec}(R)$. In the next section, we will reinterpret Theorems 1.13 and 1.15 in terms of the Zariski cohomology of $X = \text{Proj}(R)$.

Theorem 1.13. *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely generated graded algebra over a field k of characteristic 0. Assume R_0 is local artinian with residue field k . Then for $n \geq 1$ we have:*

$$(a) \quad K_n^{(i)}(R) \cong HC_{n-1}^{(i-1)}(R) \text{ whenever } 0 < i < n;$$

$$(b) \quad \tilde{K}_n^{(n)}(R) \cong \text{tors } \Omega_R^{n-1}/d \text{tors } \Omega_R^{n-2}. \text{ In particular, } \tilde{K}_1^{(1)}(R) \cong \text{nil}(R) \text{ and}$$

$$\tilde{K}_2^{(2)}(R) \cong \text{tors } \Omega_R^1/d \text{nil}(R).$$

$$(c) \quad K_n^{(n+1)}(R) \cong \text{coker}\{\Omega_{\text{cdh}}^{n-1}(R) \xrightarrow{d} \Omega_{\text{cdh}}^n(R)/\Omega_R^n\}.$$

$$(d) \quad K_n^{(i)}(R) \cong \text{coker}\{H_{\text{cdh}}^{i-(n+1)}(R, \Omega^{i-2}) \xrightarrow{d} H_{\text{cdh}}^{i-(n+1)}(R, \Omega^{i-1})\} \text{ when } i \geq n+2.$$

Proof. By Theorem 1.12, we have $\widetilde{\mathbb{H}}_{\text{cdh}}^m(R, \Omega^{\leq i}) = 0$ whenever $m < i$ (i.e., $q < 0$). Substituting this into (1.4) gives assertion (a), because $HC_n^{(i)}(k) \xrightarrow{\cong} H_{dR}^{2i-n}(k/\mathbb{Q})$ also holds. Taking $m = i$, it also gives exactness of the top row in the diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{K}_n^{(n)}(R) & \longrightarrow & \widetilde{HC}_{n-1}^{(n-1)}(R) & \longrightarrow & \widetilde{\Omega}_{\text{cdh}}^{n-1}(R)/d\Omega_{\text{cdh}}^{n-2}(R) \\
& & \downarrow & & B \downarrow \text{into} & & B \downarrow \text{into} \\
0 & \longrightarrow & \text{tors } \Omega_R^n & \longrightarrow & \Omega_R^n/\Omega_k^n & \longrightarrow & \Omega_{\text{cdh}}^n(R)/\Omega_k^n \\
& & d \downarrow & & d \downarrow & & d \downarrow \\
0 & \longrightarrow & \text{tors } \Omega_R^{n+1} & \longrightarrow & \Omega_R^{n+1}/\Omega_k^{n+1} & \longrightarrow & \Omega_{\text{cdh}}^{n+1}(R)/\Omega_k^{n+1}.
\end{array}$$

The other two rows are exact by definition, see (1.11). The two right columns are exact by [24, 9.9.1] and (1.8a), respectively. By a diagram chase, $\widetilde{K}_n^{(n)}(R)$ is the kernel of $\text{tors } \Omega_R^n \rightarrow \text{tors } \Omega_R^{n+1}$. Part (b) now follows from (1.9a).

Part (c) is immediate from (1.4), given the following information: $\mathbb{H}_{\text{cdh}}^n(R, \Omega^{\leq n})$ is the cokernel of $d : \Omega_{\text{cdh}}^{n-1}(R) \rightarrow \Omega_{\text{cdh}}^n(R)$ by Theorem 1.12, $HC_n^{(n)}(R) = \Omega_R^n/d\Omega_R^n$ and $HC_{n-1}^{(n)}(R) = 0$. Part (d) follows from Theorem 1.12 and the formula $\widetilde{K}_n^{(i)}(R) \cong \mathbb{H}_{\text{cdh}}^{2i-n-2}(R, \Omega^{\leq i})$ for $i \geq n+2$, which is Proposition 1.5. \square

Corollary 1.14. *If $i > n$ and $(n, i) \neq (0, 1)$, the map $K_n^{(i)}(R) \rightarrow K_n^{(i)}(R^+)$ is an isomorphism.*

Proof. For $n \geq 1$, it follows from Theorem 1.13, and for $n = 0$, it follows from Proposition 1.5. \square

If the dimension of R is 2 (for example, if R is the cone over a projective curve), then the calculations of Theorem 1.13 apply to compute the higher K -groups of R , but here the more dominant role is played by Kähler differentials. As in (1.10), we write $\Omega_{\text{cdh}}^i(R)$ for $H_{\text{cdh}}^0(R, \Omega^i)$.

Theorem 1.15. *Assume $\dim(R) = 2$ and that R is reduced. Then we have:*

- (1) $K_1(R) = k^\times \oplus K_1^{(2)}(R) \oplus K_1^{(3)}(R)$ and $K_1^{(i)}(R) = 0$ for all $i \geq 4$, with:

$$K_1^{(2)}(R) \cong \Omega_{\text{cdh}}^1(R)/(\Omega_R^1 + d(R^+)), \quad \text{and}$$

$$K_1^{(3)}(R) \cong \mathbb{H}_{\text{cdh}}^3(R, \Omega^{\leq 2}) \cong \text{coker}\{H_{\text{cdh}}^1(R, \Omega^1) \xrightarrow{d} H_{\text{cdh}}^1(R, \Omega^2)\};$$

- (2) $K_2(R) \cong K_2(k) \oplus \text{tors } \Omega_R^1 \oplus K_2^{(3)}(R) \oplus K_2^{(4)}(R)$ with

$$K_2^{(3)}(R) \cong \Omega_{\text{cdh}}^2(R)/(\Omega_R^2 + d\Omega_{\text{cdh}}^1(R)) \quad \text{and}$$

$$K_2^{(4)}(R) \cong \text{coker}\{H_{\text{cdh}}^1(R, \Omega^2) \xrightarrow{d} H_{\text{cdh}}^1(R, \Omega^3)\};$$

- (3) For all $n \geq 3$, $K_n(R) \cong K_n(k) \oplus \bigoplus_{i=2}^{n+2} \widetilde{K}_n^{(i)}(R)$, where

$$\widetilde{K}_n^{(i)}(R) = \begin{cases} \widetilde{HC}_{n-1}^{(i-1)}(R), & i < n, \\ \text{tors } \Omega_R^{n-1}/d\text{tors } \Omega_R^{n-2}, & i = n, \\ \text{coker}\{\Omega_{\text{cdh}}^{n-1}(R) \xrightarrow{d} \Omega_{\text{cdh}}^n(R)/\Omega_R^n\}, & i = n+1, \\ \text{coker}\{H_{\text{cdh}}^1(R, \Omega^n) \xrightarrow{d} H_{\text{cdh}}^1(R, \Omega^{n+1})\}, & i = n+2. \end{cases}$$

Proof. For $n = 1$ we see from Remark 1.1 that $\tilde{K}_1^{(1)}(R) = \text{nil}(R) = 0$, and from Theorem 1.13(c) that $K_1^{(2)}(R)$ is the cokernel of $d : R^+ \rightarrow \Omega_{\text{cdh}}^1(R)/\Omega_R^1$. Since $R^+ \rightarrow \Omega_{\text{cdh}}^1(R)$ factors through $\Omega_{R^+}^1$, the description of $K_1^{(2)}(R)$ follows. From (1.4), we have $K_1^{(3)}(R) \cong \mathbb{H}_{\text{cdh}}^3(R, \Omega^{\leq 2})$, which is described by 1.12, and $K_1^{(i)}(R) = \mathbb{H}_{\text{cdh}}^{2i-3}(R, \Omega^{< i})$ for $i \geq 4$, which vanishes because $\mathbb{H}_{\text{cdh}}^m(R, \Omega^{< i}) = 0$ for $m \geq 1 + i$ by Theorem 1.12.

For $n \geq 2$, $K_n^{(i)}(R)$ was described in Proposition 1.5 and Theorem 1.13. \square

Lemma 1.16. *Assume that $R = k \oplus R_1 \oplus \cdots$ is graded and $\dim(R) = 2$. Then for all $i \geq 2$:*

$$\Omega_{R/k}^i/d(\Omega_{R/k}^{i-1}) \cong \text{tors } \Omega_{R/k}^i/d(\text{tors } \Omega_{R/k}^{i-1}).$$

Proof. For $i \geq 3$ the R -module $\Omega_{R/k}^i$ is torsion because $\Omega_{\text{cdh}}^i(R/k) = 0$. For $i = 2$ we simply chase the diagram

$$\begin{array}{ccccccc} \text{tors } \Omega_{R/k}^1 & \longrightarrow & \text{tors } \Omega_{R/k}^2 & \longrightarrow & \text{tors } \Omega_{R/k}^3 & \longrightarrow & \text{tors } \Omega_{R/k}^4 \\ \downarrow \text{into} & & \downarrow \text{into} & & \parallel & & \parallel \\ \Omega_{R/k}^1 & \xrightarrow{d} & \Omega_{R/k}^2 & \xrightarrow{d} & \Omega_{R/k}^3 & \xrightarrow{d} & \Omega_{R/k}^4, \end{array}$$

comparing the exact sequence for $\text{tors } \Omega_{R/k}^*$, analogous to (1.9a), to the de Rham sequence for $\Omega_{R/k}^*$ (which is exact by [24, 9.9.3]). \square

Proposition 1.17. *If k is algebraic over \mathbb{Q} and $R = k \oplus R_1 \oplus \cdots$ is seminormal of dimension 2, then:*

- a) $K_1(R) \cong k^\times \oplus \Omega_{\text{cdh}}^1(R)/\Omega_R^1$;
- b) $K_2(R) \cong K_2(k) \oplus \text{tors } \Omega_R^1$;
- c) $K_n(R) \cong K_n(k) \oplus \widehat{HC}_{n-1}(R)$, $n \geq 3$.

Proof. These assertions are special cases of Theorem 1.15. Using Lemma 1.16 for $n \geq 3$ we have

$$\tilde{K}_n^{(n)}(R) \cong \text{tors } \Omega_R^{n-1}/d \text{tors } \Omega_R^{n-2} \cong \Omega_R^{n-1}/d\Omega_R^{n-2} = HC_{n-1}^{(n-1)}(R).$$

By (1.8a), $K_n^{(n+1)}(R)$ is a subquotient of $\Omega_{\text{cdh}}^{n+1}(R)$ and vanishes for $n \geq 2$; by (1.8b), $K_n^{(n+2)}(R)$ is a subgroup of $H_{\text{cdh}}^1(R, \Omega^{n+2})$ and vanishes for $n \geq 1$. \square

We conclude this section with two classical examples for which $\text{Spec}(R)$ has a smooth affine *cdh* cover, so that Ω_{cdh}^* is easy to determine.

Example 1.18. The cusp $R = k[t^2, t^3]$ has $R^+ = k[t]$ and $K_1^{(2)}(R) = \Omega_{\text{cdh}}^1/d(R^+) = \Omega_k^1$ (cf. [12, 12.1]). The computation of $K_n(R)$ for $n \geq 2$ is also easily derived from Theorem 1.13, and stated explicitly in [6, 6.7].

Example 1.19. The seminormal ring $R = k[x_1, x_2, y_1, y_2]/(\{x_i y_j : 1 \leq i, j \leq 2\})$ is the homogeneous coordinate ring of a pair of skew lines in \mathbb{P}_k^3 . Its normalization is $\tilde{R} = k[x_1, x_2] \times k[y_1, y_2]$, and $\text{Spec}(\tilde{R}) \rightarrow \text{Spec}(R)$ is a *cdh* cover. It is easy to see that $H_{\text{cdh}}^1(R, \Omega^i) = 0$, and $\Omega_R^i \rightarrow \Omega_{\text{cdh}}^i(R)$ is onto for $i \neq 0$. Applying Theorem 1.15, we see that $K_0(R) = \mathbb{Z}$, $K_1(R) = k^\times$ and $K_{-1}(R) = 0$. This recovers a classic result of Murthy in [15]. If k is algebraic over \mathbb{Q} then we also have $\text{tors } \Omega_R^1 \cong k^4$

(on the $x_i dy_j$), tors $\Omega_R^2 \cong k^4$ (on the $dx_i dy_j$) and $\Omega_R^3 = 0$, so by Proposition 1.17 we have

$$K_2(R) = K_2(k) \oplus k^4, \quad \text{while} \quad \widetilde{K}_n(R) = \widetilde{HC}_{n-1}(R) \quad \text{for all } n \geq 3.$$

2. AFFINE CONES OF SMOOTH VARIETIES

Let X be a smooth projective variety in \mathbb{P}_k^N , and let $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ be the associated homogeneous coordinate ring. We will write L for the pullback to X of the ample bundle $\mathcal{O}(1)$ on \mathbb{P}_k^N , and if \mathcal{F} is a quasi-coherent sheaf on X , we write $\mathcal{F}(t)$ for $\mathcal{F} \otimes_{\mathcal{O}_X} L^t$. In this section we compute the *cdh* cohomology of $\text{Spec}(R)$ and use it to compute the K -theory of R , via Proposition 1.5. The main result is the theorem below, computing the non-positive K -groups of R . Later in this section, we give partial calculations of the positive K -groups.

Recall from Proposition 1.5 that $K_{-m}^{(0)}(R) = 0$ for all $m > 0$ and $\widetilde{K}_n^{(0)}(R) = 0$ for $n \geq 0$. Thus we are interested in $K_{-m}^{(i+1)}(R)$ for $i \geq 0$.

Theorem 2.1. *Let X be a smooth projective variety in \mathbb{P}_k^N with homogeneous coordinate ring R . Then*

$$\begin{aligned} K_0^{(1)}(R) &\cong R^+/R = \bigoplus_{t=1}^{\infty} H^0(X, \mathcal{O}_X(t))/R_t, \quad \text{and} \\ K_0^{(i+1)}(R) &\cong \bigoplus_{t=1}^{\infty} H^i(X, \Omega_X^i(t)), \quad \text{for all } i \geq 1. \end{aligned}$$

For any $m > 0$, and all $i \geq 0$, we have:

$$K_{-m}^{(i+1)}(R) \cong \bigoplus_{t=1}^{\infty} H^{m+i}(X, \Omega_X^i(t)).$$

If k has finite transcendence degree over \mathbb{Q} then each vector space $K_0(R)/\mathbb{Z}$ and $K_{-m}(R)$ is finite-dimensional.

A few parts of Theorem 2.1 are easy to prove. The formula $K_0^{(1)}(R) = R^+/R$ is given in Proposition 1.5. Since $\text{Spec}(R) \setminus \{\mathfrak{m}_R\}$ is regular, we see from Remark 1.1 that R^+ agrees with the normalization \widetilde{R} of R in degrees $t > 0$, and it is well known that $\widetilde{R} = \bigoplus_{t=0}^{\infty} H^0(X, \mathcal{O}(t))$; see [7, Theorem 7.16] and [26, Ch VII, §2, Remark at the bottom of page 159]. This yields the first display. The final assertion, when $\text{tr. deg.}(k/\mathbb{Q}) < \infty$, follows from the fact that each Ω_X^i is a coherent sheaf; for each $q > 0$ the $H^q(X, \Omega_X^i(t))$ are finite-dimensional, and only finitely many are nonzero, by Serre's Theorem B ([5, III.5.2]).

The proof of the rest of the theorem will be given in Corollary 2.5 and Proposition 2.10, building upon several intermediate results.

To compute the *cdh* cohomology of $\text{Spec}(R)$, we will use the blowup Y of $\text{Spec}(R)$ at the origin (i.e., at \mathfrak{m}_R). The following description of Y is well known.

Lemma 2.2. *The exceptional fiber of $\pi : Y \rightarrow \text{Spec}(R)$ is isomorphic to X and there is a projection $p : Y \rightarrow X$ identifying Y with the geometric line bundle $\text{Spec}_X(\text{Sym}(L))$ over X , with sheaf of sections L^* . Moreover, the inclusion of the exceptional fiber X into Y is the zero section of the bundle $p : Y \rightarrow X$.*

Proof. The exceptional fiber is Proj of the Rees algebra $\bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1}$, which is just R , and $X = \text{Proj}(R)$ by construction. For each $x \in R_1$, the affine open $D_+(x)$ of X is $\text{Spec}(A)$, where $R[1/x] = A[x, 1/x]$, and the line bundle L^n restricts to the A -submodule $x^n A$ of $R[1/x]$.

We now consider $Y = \text{Proj}(R[\mathfrak{m}t])$. For $x \in R_1$, and $xt \in R_1t$, the affine open $D_+(xt)$ in Y is $\text{Spec}(B)$, where $R[\mathfrak{m}t][1/xt] = B[xt, 1/xt]$. The graded map $R \cong \bigoplus R_i t^i \rightarrow R[\mathfrak{m}t]$ induces a projection $Y \rightarrow X$ as well as an inclusion of $A[x]$ in B . This is onto, since B is generated by elements of the form $rt^m/(xt)^m = (r/x^n)x^{n-m}$ for $r \in R_n$, $n \geq m$. Hence $B = A[x]$. This shows that Y is the geometric line bundle over X , associated to the locally free sheaf L (see [5, Ex. II.5.18]). \square

By [1, 6.3] and [2, 2.5], the splitting of $X \rightarrow Y$ in Lemma 2.2 induces split exact sequences

$$(2.3) \quad 0 \rightarrow H_{\text{cdh}}^0(R, \mathcal{F}) \rightarrow H_{\text{zar}}^0(Y, \mathcal{F}) \oplus \mathcal{F}(k) \rightarrow H_{\text{zar}}^0(X, \mathcal{F}) \rightarrow 0, \\ 0 \rightarrow H_{\text{cdh}}^m(R, \mathcal{F}) \rightarrow H_{\text{zar}}^m(Y, \mathcal{F}) \rightarrow H_{\text{zar}}^m(X, \mathcal{F}) \rightarrow 0, \quad \text{for } m > 0,$$

when \mathcal{F} is one of the *cdh* sheaves \mathcal{O} or Ω^i , or a complex of *cdh* sheaves of the form $\Omega^{\leq i}$. Thus the calculation of $H_{\text{cdh}}^*(R, \mathcal{F})$ is reduced to the calculation of $H_{\text{zar}}^*(Y, \mathcal{F})$.

Lemma 2.4. *We have $H_{\text{cdh}}^0(R, \mathcal{O}) = R^+$. For $m > 0$ we have $H_{\text{cdh}}^m(R, \mathcal{O}) = \bigoplus_{t=1}^{\infty} H^m(X, \mathcal{O}_X(t))$.*

Proof. Since p is affine, $H_{\text{zar}}^*(Y, \mathcal{O}_Y) = H_{\text{zar}}^*(X, p_*\mathcal{O}_Y)$, and $p_*\mathcal{O}_Y = \text{Sym}(L)$ by Lemma 2.2. Hence $H^m(Y, \mathcal{O}) = \bigoplus_{t=0}^{\infty} H_{\text{zar}}^m(X, \mathcal{O}(t))$ for all m ; if $m = 0$, this equals R^+ . Now apply (2.3). \square

From Proposition 1.5 and 2.4 we deduce the case $K_*^{(1)}$ of Theorem 2.1. For comparison, recall that $K_0^{(1)}(R) = \text{Pic}(R)$, $K_1^{(1)}(R) = R^\times = k^\times$ and $K_n^{(1)}(R) = 0$ for all $n \geq 2$ by Soulé [16].

Corollary 2.5. *For $m > 0$ we have*

$$K_{-m}^{(1)}(R) = H_{\text{cdh}}^m(R, \mathcal{O}) = \bigoplus_{t=1}^{\infty} H^m(X, \mathcal{O}_X(t)).$$

Remark 2.6. This clarifies results of Srinivas in [17, Thm. 3], [18] and Weibel [25], which observed (when X is a curve) that the right side of the display in Corollary 2.5 is an obstruction to the vanishing of $\tilde{K}_0(R)$ and $K_{-1}(R)$.

There is an exact sequence $0 \rightarrow p^*\Omega_X^1 \rightarrow \Omega_Y^1 \rightarrow \Omega_{Y/X}^1 \rightarrow 0$ of sheaves on Y . The relative sheaf $\Omega_{Y/X}^1$ is the line bundle p^*L , and so we deduce exact sequences for all $i \geq 1$:

$$0 \rightarrow p^*\Omega_X^i \rightarrow \Omega_Y^i \rightarrow p^*(\Omega_X^{i-1} \otimes L) \rightarrow 0.$$

Since $p_*p^*\mathcal{F} = \mathcal{F} \otimes \text{Sym}(L)$, applying p_* yields (graded) exact sequences of sheaves on X for all $i \geq 1$:

$$(2.7) \quad 0 \rightarrow \Omega_X^i \otimes \text{Sym}(L) \rightarrow p_*\Omega_Y^i \rightarrow \Omega_X^{i-1} \otimes L \otimes \text{Sym}(L) \rightarrow 0.$$

Lemma 2.8. *The sequence (2.7) determines a graded split exact sequence*

$$0 \rightarrow \bigoplus_{t=0}^{\infty} H_{\text{zar}}^*(X, \Omega_X^i(t)) \rightarrow H_{\text{zar}}^*(Y, \Omega_Y^i) \rightarrow \bigoplus_{t=1}^{\infty} H_{\text{zar}}^*(X, \Omega_X^{i-1}(t)) \rightarrow 0$$

for each $i \geq 1$. The left-hand map is an isomorphism in degree 0 and, in degrees $t \geq 1$, its splitting is a consequence of the fact that the composition

$$(2.8.1) \quad H_{\text{zar}}^*(X, \Omega_X^i(t)) \rightarrow H_{\text{zar}}^*(Y, \Omega_Y^i) \xrightarrow{d} H_{\text{zar}}^*(Y, \Omega_Y^{i+1}) \rightarrow H_{\text{zar}}^*(X, \Omega_X^i(t))$$

is an isomorphism.

Proof. It follows from (2.7) that we have a (graded) exact sequence

$$\cdots \xrightarrow{\partial} \bigoplus_{t=0}^{\infty} H_{\text{zar}}^*(X, \Omega_X^i(t)) \xrightarrow{p^*} H_{\text{zar}}^*(Y, \Omega_Y^i) \rightarrow \bigoplus_{t=1}^{\infty} H_{\text{zar}}^*(X, \Omega_X^{i-1}(t)) \xrightarrow{\partial} \cdots$$

Therefore, the assertion that (2.8.1) is an isomorphism implies the first assertion. Referring to the maps of (2.7), it suffices to show that the composition

$$\Omega_X^i \otimes \text{Sym}(L) \rightarrow p_* \Omega_Y^i \xrightarrow{d} p_* \Omega_Y^{i+1} \rightarrow \Omega_X^i \otimes L \otimes \text{Sym}(L)$$

is the evident graded surjection, with kernel Ω_X^i . But, in the notation of the proof of Lemma 2.2, it suffices to look on the affine $D_+(x) = \text{Spec}(A)$ of X , and here this is the map $\Omega_A^* \otimes_A A[x] \rightarrow \Omega_A^* \otimes_A \Omega_{A[x]/A}^1$ sending $\omega \otimes x^n$ to $\omega \otimes nx^{n-1} dx$. \square

Example 2.8.2. In particular, $0 \rightarrow H^0(X, \Omega_X^1(t)) \rightarrow H^0(Y, \Omega_Y^1)_t \rightarrow R_t \rightarrow 0$ is exact for $t \geq 1$, and the composition $R_t \xrightarrow{d} H^0(Y, \Omega_Y^1)_t \rightarrow R_t$ is an isomorphism.

Corollary 2.9. *For $i \geq 1$ and $m \geq 1$ we have:*

$$\begin{aligned} \Omega_{\text{cdh}}^i(R) &\cong \Omega_k^i \oplus \bigoplus_{t=1}^{\infty} H_{\text{zar}}^0(X, \Omega^i(t)) \oplus H_{\text{zar}}^0(X, \Omega^{i-1}(t)); \\ H_{\text{cdh}}^m(R, \Omega^i) &\cong \bigoplus_{t=1}^{\infty} H_{\text{zar}}^m(X, \Omega^i(t)) \oplus H_{\text{zar}}^m(X, \Omega^{i-1}(t)). \end{aligned}$$

The cokernel of $\Omega_{\text{cdh}}^{i-1}(R) \xrightarrow{d} \Omega_{\text{cdh}}^i(R)$ is $\Omega_k^i/d\Omega_k^{i-1} \oplus \bigoplus_{t=1}^{\infty} H_{\text{zar}}^0(X, \Omega_X^i(t))$, and the cokernel of $H_{\text{cdh}}^m(R, \Omega^{i-1}) \xrightarrow{d} H_{\text{cdh}}^m(R, \Omega^i)$ is the summand $\bigoplus_{t=1}^{\infty} H_{\text{zar}}^m(X, \Omega_X^i(t))$.

Proof. The first assertions follow from Lemma 2.8 and (2.3). The cokernel assertions follow from this using (1.8a), (1.8b) and induction on i . \square

We may now deduce the remaining cases of Theorem 2.1, the main theorem of this section. Recall that $K_{-m}^{(1)}(R)$ is $\bigoplus_t H^m(X, \mathcal{O}(t))$ by Corollary 2.5.

Proposition 2.10. *For $i \geq 1$, we have*

$$K_{-m}^{(i+1)}(R) \cong \mathbb{H}_{\text{cdh}}^{m+2i}(R, \Omega^{\leq i}) \cong \bigoplus_{t=1}^{\infty} H^{m+i}(X, \Omega_X^i(t)), \quad m \geq 0.$$

Proof. The first isomorphism is Proposition 1.5. The second isomorphism is established in Lemma 2.8, using the isomorphism

$$\mathbb{H}_{\text{cdh}}^{m+2i}(R, \Omega^{\leq i}) \cong \text{coker} \{ H_{\text{cdh}}^{m+i}(R, \Omega^{i-1}) \xrightarrow{d} H_{\text{cdh}}^{m+i}(R, \Omega^i) \}$$

of Theorem 1.12. \square

The proof of Theorem 2.1 is now complete. We next deduce partial information about the groups $K_n(R)$ for $n \geq 1$.

Proposition 2.11. *Let X be a smooth projective variety in \mathbb{P}_k^N with homogeneous coordinate ring R . Then for all $n \geq 1$ we have graded isomorphisms:*

$$\begin{aligned} K_n^{(n+1)}(R) &\cong \text{coker} \left\{ \Omega_R^n/d\Omega_R^{n-1} \rightarrow \bigoplus_{t=1}^{\infty} H^0(X, \Omega_X^n(t)) \right\}; \\ K_n^{(i)}(R) &\cong \bigoplus_{t=1}^{\infty} H^{i-n-1}(X, \Omega_X^{i-1}(t)), \quad i \geq n+2. \end{aligned}$$

The graded decomposition of $K_n^{(n+1)}(R) = \bigoplus_{t=1}^{\infty} K_n^{(n+1)}(R)_t$ is:

$$K_n^{(n+1)}(R)_t \cong \text{coker} \left\{ (\Omega_R^n / d\Omega_R^{n-1})_t \rightarrow H^0(X, \Omega_X^n(t)) \right\}.$$

Proof. By Theorem 1.13(c),

$$\begin{aligned} K_n^{(n+1)}(R) &\cong \Omega_{\text{cdh}}^n(R) / (\Omega_R^n + d\Omega_{\text{cdh}}^{n-1}(R)) \\ &= \text{coker} \left(\Omega_R^n / d\Omega_R^{n-1} \rightarrow \Omega_{\text{cdh}}^n(R) / d\Omega_{\text{cdh}}^{n-1}(R) \right). \end{aligned}$$

Since $\widetilde{H}_{\text{cdh}}^0(R, \Omega^n) = \Omega_{\text{cdh}}^n(R) / \Omega_k^n$ and $\Omega_k^n \subset \Omega_R^n$, we see from Corollary 2.9 that this is the cokernel of $\Omega_R^n / d\Omega_R^{n-1} \rightarrow \bigoplus_{t=1}^{\infty} H^0(X, \Omega_X^1(t))$, as claimed. \square

Remark 2.12. When $X = \mathbb{P}_k^r$ is embedded in \mathbb{P}_k^N as a subvariety of degree $d > r$, our $L^t = \mathcal{O}_X(t)$ agrees with $\mathcal{O}_{\mathbb{P}_k^r}(d \cdot t)$, because it is the pullback of $\mathcal{O}_{\mathbb{P}_k^N}(t)$ to $X = \mathbb{P}_k^r$. Similarly, the terms written as $\Omega_X^i(t)$ in Proposition 2.11 should be read as $\Omega_{\mathbb{P}_k^r}^i \otimes \mathcal{O}_{\mathbb{P}_k^r}(d \cdot t)$.

3. CONES OVER SMOOTH CURVES

In this section, we focus on the case when X is a curve (i.e., a smooth projective variety of dimension one, embedded in \mathbb{P}_k^N), and apply the results of Sections 1 and 2 in this case. Recall from Theorem 2.1 that $K_{-m}(R) = 0$ for $m > 1$.

The simplest case is when k is algebraic over \mathbb{Q} . In this case, we know from Proposition 1.17 that $\widetilde{K}_2(R) \cong \text{tors } \Omega_R^1$ and if $n \geq 3$ then $\widetilde{K}_n(R) \cong \widetilde{HC}_{n-1}(R)$. It remains to describe the situation when $-1 \leq n \leq 1$.

Lemma 3.1. *Suppose that k is algebraic over \mathbb{Q} and that R is the homogeneous coordinate ring of a smooth curve X over k . Then $K_{-1}(R) = \bigoplus_{t=1}^{\infty} H^1(X, \mathcal{O}(t))$, $K_0(R) = \mathbb{Z} \oplus (R^+ / R)$ and $\widetilde{K}_1(R) = \bigoplus_{t=1}^{\infty} H^0(X, \Omega_{X/k}^1(t)) / \Omega_{R/k}^1$.*

Proof. By Theorem 2.1, $K_0^{(i)}(R) = 0$ for $i \geq 3$ and $K_{-1}^{(i)}(R)$ is zero for $i \geq 2$, while $K_{-1}^{(1)}(R)$ is the sum of the $H^1(X, \mathcal{O}(t))$ by 2.5. By Serre Duality, $K_0^{(2)}(R)$ is the sum of the $H^1(X, \Omega_{X/k}^1(t)) = H^0(X, \mathcal{O}_X(-t))^*$, which are zero for all $t > 0$.

The formula for $\widetilde{K}_1(R)$ is immediate from Propositions 1.17 and 2.11. \square

Proposition 3.2. *Suppose that R is the homogeneous coordinate ring of a smooth curve X over a number field F contained in k . Then for $R_k = R \otimes_F k$:*

- (a) For $i < n$ we have $\widetilde{K}_n^{(i)}(R_k) \cong \bigoplus_{p=0}^i \Omega_k^p \otimes_F \widetilde{K}_{n-p}^{(i-p)}(R)$.
- (b) For all $n \geq 2$, $\widetilde{K}_n^{(n)}(R_k) \cong \bigoplus_{p=0}^{n-2} \Omega_k^p \otimes_F \widetilde{K}_{n-p}^{(n-p)}(R)$.
- (c) For all $n \geq 1$, $K_n^{(n+1)}(R_k) \cong \Omega_k^{n-1} \otimes_F K_1^{(2)}(R)$.
- (d) For all $n \geq 0$, $K_n^{(n+2)}(R_k) \cong \Omega_k^{n+1} \otimes_F K_{-1}(R) \cong \Omega_k^{n+1} \otimes_k K_{-1}(R_k)$.

Proof. Write \otimes for \otimes_F . Part (a) is immediate from Theorem 1.13(a) and Kassel's base change formula $\widetilde{HC}_*(R_k) \cong \Omega_k^* \otimes \widetilde{HC}_*(R)$. (See [8, (3.2)].)

For (b), recall that $\widetilde{K}_n^{(n)}(R_k) \cong \text{tors } \Omega_{R_k}^{n-1} / d \text{tors } \Omega_{R_k}^{n-2}$ by Theorem 1.13(b). By the Künneth formula, $\text{tors } \Omega_{R_k}^p = \bigoplus_{p+q=n} \Omega_k^p \otimes \text{tors } \Omega_R^q$. Filtering by $p \geq 0$ yields a 2-diagonal spectral sequence computing the kernel and cokernel of $d : \text{tors } \Omega_{R_k}^{n-1} \rightarrow \text{tors } \Omega_{R_k}^n$, with $E_0^{p,-p} = \Omega_k^p \otimes \text{tors } \Omega_R^{n-p}$ and $E_0^{p,-1-p} = \Omega_k^p \otimes \text{tors } \Omega_R^{n-p-1}$. By (1.9a), we have $E_1^{p,-p} = \Omega_k^p \otimes \widetilde{K}_{n+1}^{(n+1)}(R)$ and $E_1^{p,-1-p} = \Omega_k^p \otimes d \text{tors } \Omega_R^{n-p-2}$. Given α in

Ω_k^p and $d\tau$ in d tors Ω_R^{n-p-2} , $d(\alpha \otimes d\tau) = d\alpha \otimes d\tau = d(d\alpha \otimes \tau)$ in tors $\Omega_{R,k}^n$, which shows that $d^1 = 0$ and establishes (b).

By the Künneth formula and Proposition 2.11, $K_n^{(n+1)}(R_k)$ is the direct sum over $p+q=n$ of the cokernels of the maps

$$\Omega_k^p \otimes \Omega_R^q \rightarrow \Omega_k^p \otimes H^0(Y, \Omega_Y^q) \rightarrow \Omega_k^p \otimes \oplus_t H^0(X, \Omega_X^q(t)).$$

For $q=0$, the composite is the identity map of $\Omega_k^n \otimes R$. For $q=1$, the composite is Ω_k^{n-1} tensored with the map $\Omega_R^1 \rightarrow \oplus_t H^0(X, \Omega_X^1(t))$ defining $K_1^{(2)}(R)$. For $q \geq 2$, the right side is zero. This establishes part (c).

Since $K_{-1}(R) \cong \oplus H^1(X, \mathcal{O}(t))$, part (d) is just Proposition 2.11, together with the Künneth formula that $H^1(X_k, \Omega_{X_k}^n(t))$ is the direct sum of $\Omega_k^n \otimes H^1(X, \mathcal{O}(t))$ and $\Omega_k^{n-1} \otimes H^1(X, \Omega_X^1(t))$, which is zero for $t > 0$ by Serre Duality. \square

When k/\mathbb{Q} is transcendental, we will use a variant of the arithmetic Gauss-Manin connection $H_{dR}^1(X/k) \rightarrow \Omega_k^1 \otimes H_{dR}^1(X/k)$, or rather its (k -linear) filtered piece

$$\nabla : H^0(X, \Omega_{X/k}^1) \rightarrow \Omega_k^1 \otimes H^1(X, \mathcal{O}_X)$$

as described in [9, Thm. 2] and [13, 3.2]. When $k = \mathbb{C}$, this can be interpreted in terms of the Hodge filtration as a map $H^{1,0}(X, \mathbb{C}) \rightarrow \Omega_{\mathbb{C}/k}^1 \otimes H^{0,1}(X, \mathbb{C})$.

It is known (see [9]) that ∇ is the cohomology boundary map associated to the fundamental short exact sequence $0 \rightarrow \Omega_k^1 \otimes \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow \Omega_{X/k}^1 \rightarrow 0$. Twisting this short exact sequence by $\mathcal{O}(t)$ yields a twisted version $\nabla_t : H^0(X, \Omega_{X/k}^1(t)) \rightarrow \Omega_k^1 \otimes H^1(X, \mathcal{O}(t))$. We see from Lemma 2.8 that the direct sum of the ∇_t is a component of the cohomology boundary map associated to $0 \rightarrow \Omega_k^1 \otimes \mathcal{O}_Y \rightarrow \Omega_Y^1 \rightarrow \Omega_{Y/k}^1 \rightarrow 0$; it follows that $\oplus \nabla_t$ is R -linear.

Since $\Omega_{X/k}^2 = 0$, we have fundamental exact sequences for each i :

$$(3.3) \quad 0 \rightarrow \Omega_k^i \otimes \mathcal{O}_X(t) \rightarrow \Omega_X^i(t) \rightarrow \Omega_k^{i-1} \otimes \Omega_{X/k}^1(t) \rightarrow 0.$$

The cohomology boundary maps are the k -linear homomorphisms

$$\Omega_k^{i-1} \otimes H^0(X, \Omega_{X/k}^1(t)) \xrightarrow{\nabla_t} \Omega_k^i \otimes H^1(X, \mathcal{O}(t)).$$

The sum of the ∇_t is again R -linear, as the sum of the sequences (3.3) is R -linear. Alternatively, we can use the fact that the arithmetic Gauss-Manin connection can be extended via the usual formula $\nabla_t(\omega \otimes x) = d\omega \otimes x + (-1)^{i-1} \omega \wedge \nabla_t(x)$, and the first term vanishes because it is in a lower part of the Hodge filtration.

Lemma 3.4. *If X is a smooth curve and $i \geq 1$, there is a graded exact sequence of R -modules, the sum over $t > 0$ of the exact sequences*

$$0 \rightarrow \Omega_k^i \otimes R_t \rightarrow H^0(X, \Omega_X^i(t)) \rightarrow \Omega_k^{i-1} \otimes H^0(X, \Omega_{X/k}^1(t)) \xrightarrow{\nabla_t} \Omega_k^i \otimes H^1(X, \mathcal{O}_X(t)) \rightarrow H^1(X, \Omega_X^i(t)) \rightarrow 0.$$

Moreover, we have the identity

$$\nabla_t(\omega \otimes x) = \omega \wedge \nabla_t(x), \quad \text{for } \omega \in \Omega_k^{i-1} \text{ and } x \in H^0(X, \Omega_{X/k}^1(t)).$$

Proof. This is just the cohomology exact sequence for (3.3), together with Serre Duality, which says that $H^1(X, \Omega_{X/k}^1(t)) = H^0(X, \mathcal{O}_X(-t)) = 0$ for all $t > 0$.

To prove that the boundary map is ∇ , let \mathcal{U} be a cover of X by affine open subschemes and consider the exact sequence of Čech complexes associated to (3.3). We have $\check{C}(\mathcal{U}, \Omega_k^i \otimes \mathcal{O}_X(t)) = \Omega_k^i \otimes \check{C}(\mathcal{U}, \mathcal{O}_X(t))$ and

$$\check{C}(\mathcal{U}, \Omega_k^{i-1} \otimes \Omega_X^1(t)) = \Omega_k^{i-1} \otimes \check{C}(\mathcal{U}, \Omega_{X/k}^1(t)).$$

Let $\omega \in \Omega_k^{i-1}$ and $x \in H^0(X, \Omega_{X/k}^1(t)) = H^0 \check{C}(\mathcal{U}, \Omega_{X/k}^1(t))$. If $y \in \check{C}(\mathcal{U}, \Omega_X^1(t))$ maps to x , then $\delta(y)$ is in $\Omega_k^i \otimes \check{C}(\mathcal{U}, \mathcal{O}(t))$ and represents $\nabla(x)$. Since $\omega \wedge y$ lifts $\omega \otimes x$, $\nabla(\omega \otimes x)$ is the class of $\delta(\omega \wedge y)$ in $\Omega_k^i \otimes H^1(X, \Omega_{X/k}^1)$. Since ω is globally defined, we have $\delta(\omega \wedge y) = \omega \wedge \delta(y)$. \square

Proposition 3.5. *If X is a smooth curve, we have graded exact sequences*

$$\begin{aligned} 0 \rightarrow K_1^{(2)}(R) &\rightarrow \frac{\oplus_t H^0(X, \Omega_{X/k}^1(t))}{\text{image } \Omega_{R/k}^1} \xrightarrow{\nabla} \Omega_k^1 \otimes (\oplus_t H^1(X, \mathcal{O}(t))) \rightarrow K_0^{(2)}(R) \rightarrow 0; \\ 0 \rightarrow K_{n+1}^{(n+2)}(R) &\rightarrow \frac{\Omega_k^n \otimes [\oplus_t H^0(X, \Omega_{X/k}^1(t))]}{\text{image } \Omega_R^{n+1}} \xrightarrow{\nabla} \Omega_k^{n+1} \otimes (\oplus_t H^1(X, \mathcal{O}_X(t))) \\ &\rightarrow K_n^{(n+2)}(R) \rightarrow 0, \quad n \geq 1. \end{aligned}$$

The direct sums are taken from $t = 1$ to ∞ .

Proof. This follows from the exact sequence of Lemma 3.4, using the formulas $K_n^{(n+2)}(R)_t \cong H^1(X, \Omega_X^{n+1}(t))$ and $K_{n+1}^{(n+2)}(R)_t \cong H^0(X, \Omega_X^{n+1}(t))/\text{im}(\Omega_R^{n+1})_t$ of Propositions 2.10 and 2.11, once we observe that the first map of Lemma 3.4 factors through Ω_R^i . This is because it is a quotient of $\Omega_k^i \otimes R \rightarrow \pi_*(\Omega_Y^i) = H^0(Y, \Omega_Y^i)$, which factors as $\Omega_k^i \otimes R \rightarrow \Omega_R^i \rightarrow \pi_*(\Omega_Y^i)$. \square

Example 3.6. If X is a curve definable over a number field contained in k , then the Fundamental Sequence (3.3) (with $i = 1$ and $t = 0$) splits as $\Omega_X^1 \cong \Omega_{X/k}^1 \oplus \Omega_k^1 \otimes \mathcal{O}_X$, by the Künneth formula. This implies that $\Omega_X^i \cong (\Omega_k^{i-1} \otimes \Omega_{X/k}^1) \oplus (\Omega_k^i \otimes \mathcal{O}_X)$, so the Gauss-Manin connection ∇ of Lemma 3.4 vanishes and therefore:

$$K_n^{(n+1)}(R) = \frac{\Omega_k^{n-1} \otimes [\oplus_t H^0(X, \Omega_{X/k}^1(t))]}{\text{image } \Omega_R^n}, \quad n \geq 1;$$

$$K_n^{(n+2)}(R) = \Omega_k^{n+1} \otimes [\oplus_t H^1(X, \mathcal{O}_X(t))] \cong \Omega_k^{n+1} \otimes K_{-1}(R), \quad n \geq 0.$$

Of course, the formula for $K_n^{(n+1)}(R)$ reduces to that of Proposition 3.2(c).

The formula for $K_0^{(2)}(R)$ clarifies the examples given by Srinivas in [17]. There it was shown that if X is definable over a number field, then $K_0(R)$ maps onto $\Omega_k^1 \otimes H^1(X, \mathcal{O}_X(1))$ (see page 264). From this Srinivas deduced that if $k = \mathbb{C}$ and $H^1(X, \mathcal{O}_X(1)) \neq 0$ then $\tilde{K}_0(R) \neq 0$.

The description of $K_0(R) = \mathbb{Z} \oplus K_0^{(2)}(R)$ in this special case was independently discovered by Krishna and Srinivas [11].

Lemma 3.7. *For any graded algebra $R = k \oplus R_1 \oplus \dots$, the degree 1 part of Ω_R^i decomposes as*

$$(\Omega_R^i)_1 \cong (R_1 \otimes \Omega_k^i) \oplus (\Omega_k^{i-1} \otimes R_1).$$

The inclusions of $R_1 \otimes \Omega_k^i$ and $\Omega_k^{i-1} \otimes R_1$ are given by $r \otimes \omega \mapsto r\omega$ and $\omega \otimes r \mapsto \omega \wedge dr$, respectively.

Proof. We may suppose for simplicity that $N = \dim(R_1)$ is finite, so that the polynomial ring $S = k[x_1, \dots, x_N]$ maps to R , and $S \rightarrow R$ is an isomorphism in degree 1. For every subfield ℓ of k , $\Omega_{R/\ell}^1$ is the cokernel of the Hochschild boundary $R^{\otimes 3} \rightarrow R \otimes_{\ell} R$; thus the map $\Omega_{S/\ell}^1 \rightarrow \Omega_{R/\ell}^1$ is an isomorphism in degree 1, and therefore so is $\Omega_{S/\ell}^i \rightarrow \Omega_{R/\ell}^i$. Since $\Omega_S^1 \cong (\Omega_k^1 \otimes S) \oplus \Omega_{S/k}^1$, it is easy to check that the degree 1 part of Ω_S^i is $(\Omega_k^i \otimes S_1) \oplus \Omega_k^{i-1} \otimes S_1$, via the given formulas. \square

Theorem 3.8. *Let X be a curve of genus g , embedded in \mathbb{P}_k^N by a complete linear system of degree $d > 1$. Assume that the twisted Gauss-Manin connection $\nabla : H^0(X, \Omega_{X/k}^1(1)) \rightarrow \Omega_k^1 \otimes H^1(X, \mathcal{O}_X(1))$ is zero. Then $K_1^{(2)}(R)_1 \cong k^{d+g-1} \neq 0$, and*

$$K_n^{(n+1)}(R)_1 \cong \Omega_k^{n-1} \otimes_{\mathbb{Q}} k^{d+g-1} \quad (n \geq 1).$$

In particular, $K_n^{(n+1)}(R) \neq 0$ for all n with $1 \leq n < \text{tr. deg.}(k/\mathbb{Q})$.

Proof. By Proposition 2.11, the degree 1 part of $K_n^{(n+1)}(R)$ is

$$K_n^{(n+1)}(R)_1 = \text{coker}((\Omega_R^n/d\Omega_R^{n-1})_1 \rightarrow H^0(X, \Omega_X^n(1))).$$

By Lemmas 3.7 and 3.4, and our hypothesis, we have morphisms of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_k^n \otimes R_1 & \longrightarrow & (\Omega_R^n)_1 & \xrightarrow{\omega \wedge dr \rightarrow \omega \otimes r} & \Omega_k^{n-1} \otimes R_1 & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow 1 \otimes d & & \\ 0 & \longrightarrow & \Omega_k^n \otimes R_1 & \longrightarrow & H^0(Y, \Omega_Y^n)_1 & \longrightarrow & \Omega_k^{n-1} \otimes H^0(Y, \Omega_{Y/k}^1)_1 & \xrightarrow{\partial} & \\ & & \downarrow \text{id} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_k^n \otimes R_1 & \longrightarrow & H^0(X, \Omega_X^n(1)) & \longrightarrow & \Omega_k^{n-1} \otimes H^0(X, \Omega_{X/k}^1(1)) & \xrightarrow{\nabla} & 0 \end{array}$$

where the bottom vertical maps are given in Lemma 2.8 as the quotients by $dH^0(X, \Omega^{n-1}(1))$ and $\Omega_k^{n-1} \otimes dR_1$. It follows that the right vertical composite is zero. Hence $K_n^{(n+1)}(R)_1$, which is the cokernel of the middle vertical composite, is isomorphic to $\Omega_k^{n-1} \otimes H^0(X, \Omega_{X/k}^1(1))$. Finally, $\dim H^0(X, \Omega_{X/k}^1(1)) = d + g - 1$ by Riemann-Roch. \square

Example 3.9. Here are two cases in which the hypotheses of Theorem 3.8 above are satisfied:

- (a) X is embedded in \mathbb{P}_k^N by a complete linear system of degree $d \geq 2g - 1$. In this case $\deg(\Omega_{X/k}^1(-1)) < 0$, so $H^1(X, \mathcal{O}(t)) = 0$ for all $t \geq 1$ by Serre duality. Theorem 3.8 improves the result of Srinivas in [19] that if $d \geq 2g + 1$ then $\tilde{K}_1(R) \neq 0$.
- (b) X is definable over a number field contained in k .

4. K-THEORY OF THE PLANE CONIC

We conclude with a classical example: X is the plane conic with homogeneous coordinate ring $R = k[x, y, z]/(z^2 - xy)$. This curve is a degree 2 embedding of \mathbb{P}_k^1 in \mathbb{P}_k^2 ; as pointed out in Remark 2.12, our line bundle $\mathcal{O}_X(t)$ is the usual $\mathcal{O}_{\mathbb{P}_k^1}(2t)$.

Murthy observed long ago, in [15, 5.3], that $K_0(R) = \mathbb{Z}$ and $K_{-1}(R) = 0$; this also follows from our Theorem 2.1. Srinivas proved in [19] that $\tilde{K}_1(R)$ surjects onto

k . Theorem 4.3 below gives a complete calculation of $K_*(R)$, or rather, $\widetilde{K}_*(R) = K_*(R)/K_*(k)$.

Lemma 4.1. *For $R = k[x, y, z]/(z^2 - xy)$, $\Omega_{R/k}^1$ is a torsionfree R -module, and the map $\Omega_{R/k}^1 \rightarrow H^0(Y, \Omega_{Y/k}^1)$ is a graded injection with cokernel k in degree $t = 1$.*

Proof. As R is a normal complete intersection, a theorem of Vasconcelos ([21, 2.4]) says that Ω_R^1 is a torsionfree R -module. As such, it is a graded submodule of $\Omega_{R[1/x]}^1$. From the factorization $\text{Spec}(R[1/x]) \rightarrow Y \rightarrow \text{Spec}(R)$, we see that the graded map $\Omega_{R/k}^1 \rightarrow H^0(Y, \Omega_{Y/k}^1)$ is an injection. Since $R/k \xrightarrow{d} \Omega_{R/k}^1 \rightarrow H^0(Y, \Omega_{Y/k}^1)$ is an injection with cokernel $\oplus_t H^0(X, \Omega_{X/k}^1(t))$ by Lemma 2.8, we are reduced to comparing the Hilbert functions of both sides.

It is easy to show that $\dim(R_t) = 2t + 1$ for all $t \geq 0$. From the resolution $0 \rightarrow R(-2) \xrightarrow{dF} R(-1)^3 \rightarrow \Omega_{R/k}^1 \rightarrow 0$, we compute that $\dim(\Omega_{R/k}^1)_t$ is 3 for $t = 1$ and $4t$ for $t \geq 2$. By Riemann-Roch, we have $\dim H^0(X, \Omega_{X/k}^1(t)) = 2t - 1$ for $t > 0$. By Lemma 2.8, this yields:

$$\dim H^0(Y, \Omega_{Y/k}^1)_t = \dim H^0(X, \Omega_{X/k}^1(t)) + \dim R_t = (2t - 1) + (2t + 1) = 4t.$$

This shows that $(\Omega_{R/k}^1)_t \cong R_t \oplus H^0(X, \Omega_{X/k}^1(t))$ when $t \geq 2$, as desired. \square

Remark 4.1.1. Since Ω_R^1 is torsionfree, the exact sequence (1.9a) shows that $d : \text{tors } \Omega_{R/k}^2 \cong \Omega_{R/k}^3 \cong k$. In fact, the 2-form $\tau = z dx \wedge dy + 2y dx \wedge dz$ has $x\tau = y\tau = z\tau = 0$ and $d\tau = dx \wedge dy \wedge dz$.

Lemma 4.2. *For $R = \mathbb{Q}[x, y, z]/(z^2 - xy)$ and $n \geq 2$, $\widetilde{HC}_n^{(i)}(R)$ is \mathbb{Q} if $n = 2i - 2$ and zero otherwise. For $R_k = R \otimes k$, $\widetilde{HC}_n^{(i)}(R_k)$ is Ω_k^p , where $p = 2i - n - 2$.*

Proof. The calculation of $HC_n^{(i)}(R)$ is taken from [14, Thms. 2–3], using the elementary calculation that $\Omega_R^3 \cong \mathbb{Q}$ for $n > 3$ and exactness of the augmented Poincaré complex $\mathbb{Q} \rightarrow \Omega_R^*$ for $n = 2, 3$. The second sentence follows using the base change formula of [8, (3.2)]. \square

Theorem 4.3. *For $R_k = k[x, y, z]/(z^2 - xy)$ and all n , we have*

$$\widetilde{K}_n(R_k) \cong \Omega_k^{n-1} \oplus \Omega_k^{n-3} \oplus \Omega_k^{n-5} \oplus \dots$$

In particular, $K_1(R_k) \cong K_1(k) \oplus k$ and $K_2(R_k) \cong K_2(k) \oplus \Omega_k^1$.

Proof. By Proposition 3.2(a) and Lemma 4.2, we see that $\widetilde{K}_n^{(n-j)}(R)$ is Ω_k^{n-2j-3} for all $j > 0$. By Theorem 1.15 and Remark 4.1.1, we have $\widetilde{K}_3^{(3)}(R_{\mathbb{Q}}) \cong k$ and $\widetilde{K}_n^{(n)}(R_{\mathbb{Q}}) = 0$ for $n \neq 3$. By Proposition 3.2(b) this implies that $\widetilde{K}_n^{(n)}(R_k) \cong \Omega_k^{n-3}$ for all $n \neq 3$. By Proposition 3.5 and Lemma 4.1, we have $K_1^{(2)}(R_k) = k$. By Proposition 3.2, this implies that $K_n^{(n+1)}(R) \cong \Omega_k^{n-1}$ for all $n \geq 1$. Finally, by Proposition 2.11 we have $K_n^{(n+2)}(R_k)_t = H^1(X_k, \Omega_{X_k}^{n+1}(t))$, which vanishes for all $n, t \geq 1$ as it is the sum of $\Omega_k^n \otimes H^1(X, \Omega_X^1(t))$, which vanishes by Serre Duality, and $\Omega_k^{n+1} \otimes H^1(X, \mathcal{O}_X(t))$, which vanishes as $X = \mathbb{P}_k^1$. \square

Remark 4.3.1. When k is algebraic over \mathbb{Q} , the formulas in Theorem 4.3 reduce to: $\widetilde{K}_n(R_k) = \mathbb{Q}$ for $n \geq 1$ odd, and $\widetilde{K}_n(R_k) = 0$ otherwise.

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