

**$K$ -REGULARITY,  
 $cdh$ -FIBRANT HOCHSCHILD HOMOLOGY,  
AND A CONJECTURE OF VORST**

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ABSTRACT. In this paper we prove that for an affine scheme essentially of finite type over a field  $F$  and of dimension  $d$ ,  $K_{d+1}$ -regularity implies regularity, assuming that the characteristic of  $F$  is zero. This verifies a conjecture of Vorst.

INTRODUCTION

It is a well-known fact that algebraic  $K$ -theory is homotopy invariant as a functor on regular schemes; if  $X$  is a regular scheme then the natural map  $K_n(X) \rightarrow K_n(X \times \mathbb{A}^1)$  is an isomorphism for all  $n \in \mathbb{Z}$ . This is false in general for nonregular schemes and rings.

To express this failure, Bass introduced the terminology that, for any contravariant functor  $\mathcal{P}$  defined on schemes, a scheme  $X$  is called  $\mathcal{P}$ -regular if the pullback maps  $\mathcal{P}(X) \rightarrow \mathcal{P}(X \times \mathbb{A}^r)$  are isomorphisms for all  $r \geq 0$ . If  $X = \text{Spec}(R)$ , we also say that  $R$  is  $\mathcal{P}$ -regular. Thus regular schemes are  $K_n$ -regular for every  $n$ . In contrast, it was observed as long ago as [2] that a nonreduced affine scheme can never be  $K_1$ -regular. In particular, if  $A$  is an Artinian ring (that is, a 0-dimensional Noetherian ring) then  $A$  is regular (that is, reduced) if and only if  $A$  is  $K_1$ -regular. In [17], Vorst conjectured that for an affine scheme  $X$ , of finite type over a field  $F$  and of dimension  $d$ , regularity and  $K_{d+1}$ -regularity are equivalent; Vorst proved this conjecture for  $d = 1$  (by proving that  $K_2$ -regularity implies normality).

In this paper, we prove Vorst's conjecture in all dimensions provided the characteristic of the ground field  $F$  is zero. In fact we prove a stronger statement. We say that  $X$  is *regular in codimension  $< n$*  if  $\text{Sing}(X)$  has codimension  $\geq n$  in  $X$ . Note that for all  $n \in \mathbb{Z}$ , if a ring  $R$  is  $K_n$ -regular, then it is  $K_{n-1}$ -regular. This is proved in [17] for  $n \geq 1$ , and in [6, 4.4] for  $n \leq 0$ .

Let  $\mathcal{F}_K$  denote the presheaf of spectra such that  $\mathcal{F}_K(X)$  is the homotopy fiber of the natural map  $K(X) \rightarrow KH(X)$ , where  $K(X)$  is the algebraic  $K$ -theory spectrum of  $X$  and  $KH(X)$  is the homotopy  $K$ -theory of  $X$  defined in [19]. We write  $\mathcal{F}_K(R)$  for  $\mathcal{F}_K(\text{Spec}(R))$ .

**Theorem 0.1.** *Let  $R$  be a commutative ring which is essentially of finite type over a field  $F$  of characteristic 0. Then:*

(a) *If  $\mathcal{F}_K(R)$  is  $n$ -connected, then  $R$  is regular in codimension  $< n$ .*

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*Date:* December 16, 2006.

*2000 Mathematics Subject Classification.* 19D35 (primary), 14F20, 13D03, 19D55.

Cortiñas' research was partially supported by FSE and by grants ANPCyT PICT 03-12330, UBACyT-X294, JCyL VA091A05, and MEC MTM00958.

Weibel's research was partially supported by NSA grant MSPF-04G-184.

- (b) If  $R$  is  $K_n$ -regular, then  $R$  is regular in codimension  $< n$ .  
(c) (Vorst's conjecture) If  $R$  is  $K_{1+\dim(R)}$ -regular, then  $R$  is regular.

It was observed in [19, Proposition 1.5] that if  $X$  is  $K_n$ -regular then  $K_i(X) \rightarrow KH_i(X)$  is an isomorphism for  $i \leq n$ , and a surjection for  $i = n + 1$ , so that  $\mathcal{F}_K(X)$  is  $n$ -connected. Thus (a) implies (b) in this theorem, and (c) is a special case of (b).

The bounds in (a) and (b) are the best possible, because it follows from Vorst's results ([18, Thm. A], [17, Thm. 3.6]) that for an affine singular seminormal curve  $X$ ,  $\mathcal{F}_K(X)$  is 1-connected, but not 2-connected. The converse of (c) is trivial, but those of (a) and (b) are false. Indeed, affine normal surfaces are regular in codimension 1 but may not be  $K_{-1}$ -regular, much less  $K_2$ -regular; see [23, Affine Cones 5.8.1].

Finally the analogue of (c) –and thus also of (a) and (b)– for  $K$ -theory of general nonaffine schemes is false. Indeed we give the following example of a nonreduced (and in particular nonregular) projective curve which is  $K_n$ -regular for all  $n$ .

**Theorem 0.2.** *Let  $(X, Q)$  be an elliptic curve over a field of characteristic 0, and  $P$  a rational point on  $X$  such that the line bundle  $L = L(P - Q)$  does not have odd order in the Picard group  $\text{Pic}(X)$ . Write  $Y$  for the nonreduced scheme with the same underlying space as  $X$  but with structure sheaf  $\mathcal{O}_Y = \mathcal{O} \oplus L$ , where  $L$  is regarded as a square-zero ideal.*

*Then  $Y$  is  $K_n$ -regular for all  $n$ , and  $\mathcal{F}_K(Y)$  is contractible.*

The idea of the proof of Theorem 0.1 is to use results from [4] to translate  $K$ -theoretic statements into statements about Hochschild homology and ultimately, into statements about differentials. Thus the original  $K$ -theoretic problem is reinterpreted as a characterization of regularity in terms of Hochschild homology and differential forms, which we solve.

In more detail, the structure of the proof of 0.1 is as follows. Using results from [4], we translate the failure of  $K$ -theory to be homotopy invariant into a statement about the failure of Hochschild homology taken over  $\mathbb{Q}$  to satisfy  $cdh$ -descent (roughly speaking this means that the Hochschild homology of a singularity cannot be computed from that of a resolution of its singularities; see Section 1). By hypothesis, the  $K$ -theory of  $R$  is homotopy invariant up to some degree; therefore, its Hochschild homology over  $\mathbb{Q}$  satisfies  $cdh$ -descent up to some degree (Corollary 1.7). Using the transitivity spectral sequence of [11] and its  $cdh$ -sheafified version (Lemmas 4.1 and 4.2), we conclude that this implies that the Hochschild homology of  $R$  taken over  $F$  satisfies  $cdh$ -descent up to some degree (Proposition 4.8). Finally, we show that this has implications for the algebra of differentials of  $R$  over  $F$  (Lemmas 2.3 and 3.2) that imply the regularity of  $R$  (Theorems 4.11 and 4.12).

As mentioned above, the proof employs results from our paper with M. Schlichting [4] that allow us to describe  $\mathcal{F}_K$  in terms of cyclic homology; the necessary statements will be recalled in Section 1. In Section 2, we study the  $cdh$ -fibrant version of Hochschild homology and its Hodge decomposition. Section 3 contains a smoothness criterion (Theorem 3.1) using  $cdh$ -fibrant Hochschild homology, which is of independent interest and generalized in Theorem 4.11. Section 4 (see Theorem 4.12) contains the discussion of the transitivity spectral sequence and the proof of part (a) of Theorem 0.1. As explained above, parts (b) and (c) follow from this.

Finally Section 5 is devoted to the counterexample stated in Theorem 0.2 (and restated as Theorem 5.2).

**Notation**

All rings considered in this paper are commutative and noetherian. We shall write  $\text{Sch}/F$  for the category of schemes essentially of finite type over a field  $F$ . Objects of  $\text{Sch}/F$  shall be called  $F$ -schemes.

The category of spectra we use in this paper will not be critical. In order to minimize technical issues, we will use the terminology that a *spectrum*  $\mathcal{E}$  is a sequence of simplicial sets  $\mathcal{E}_n$  together with bonding maps  $b_n : \mathcal{E}_n \rightarrow \Omega\mathcal{E}_{n+1}$ . We say that  $\mathcal{E}$  is an  $\Omega$ -spectrum if all bonding maps are weak equivalences. A map of spectra is a strict map. We will use the model structure on the category of spectra defined in [3]. Note that in this model structure, every fibrant spectrum is an  $\Omega$ -spectrum.

We say that a presheaf  $E$  of spectra on  $\text{Sch}/F$  satisfies the *Mayer-Vietoris-property* (or MV-property, for short) for a cartesian square of schemes

$$\begin{array}{ccc}
 Y' & \longrightarrow & X' \\
 \downarrow & & \downarrow \\
 Y & \longrightarrow & X
 \end{array}
 \quad (\square)$$

if applying  $E$  to this square results in a homotopy cartesian square of spectra. We say that  $E$  satisfies the Mayer-Vietoris property for a class of squares provided it satisfies the MV-property for each square in the class.

We say that  $E$  satisfies *Nisnevich descent* for  $\text{Sch}/F$  if  $E$  satisfies the MV-property for all elementary Nisnevich squares in  $\text{Sch}/F$ ; an *elementary Nisnevich square* is a cartesian square of schemes  $(\square)$  for which  $Y \rightarrow X$  is an open embedding,  $X' \rightarrow X$  is étale and  $(X' - Y') \rightarrow (X - Y)$  is an isomorphism. It is a result of Nisnevich that this is equivalent to the assertion that  $E(X) \rightarrow \mathbb{H}_{nis}(X, E)$  is a weak equivalence for each scheme  $X$ , where  $\mathbb{H}_{nis}(-, E)$  is a fibrant replacement for  $E$  in a suitable model structure; see [4, Theorem 3.4] for a proof.

We say that  $E$  satisfies *cdh-descent* for  $\text{Sch}/F$  if  $E$  satisfies the MV-property for all elementary Nisnevich squares (Nisnevich descent) and for all abstract blow-up squares in  $\text{Sch}/F$ ; an *abstract blow-up square* is a square  $(\square)$  such that  $Y \rightarrow X$  is a closed embedding,  $X' \rightarrow X$  is proper and the induced morphism  $(X' - Y')_{red} \rightarrow (X - Y)_{red}$  is an isomorphism. With M. Schlichting, we showed in Theorem 3.7 of [4] that *cdh*-descent is equivalent to the assertion that  $E(X) \rightarrow \mathbb{H}_{cdh}(X, E)$  is a weak equivalence for each scheme  $X$ , where  $\mathbb{H}_{cdh}(-, E)$  is a fibrant replacement for the presheaf  $E$  in a suitable model structure. We abbreviate  $\mathbb{H}_{cdh}(-, E)$  as  $\mathbb{H}(-, E)$  if no confusion can arise, and write  $\mathbb{H}_n(X, E) = \mathbb{H}^{-n}(X, E)$  for  $\pi_n\mathbb{H}(X, E)$ .

We use cohomological indexing for all chain complexes in this paper; for a complex  $A$ ,  $A[p]^q = A^{p+q}$ . It is well-known that there is an Eilenberg-Mac Lane functor  $A \mapsto |A|$  from chain complexes of abelian groups to spectra, and from presheaves of chain complexes of abelian groups to presheaves of spectra. See, for example, [20, 10.9.19]. This functor sends quasi-isomorphisms of complexes to weak homotopy equivalences of spectra, and satisfies  $\pi_n(|A|) = H^{-n}(A)$ . In this spirit, we will use descent terminology for presheaves of complexes.

For example, the Hochschild, cyclic, periodic and negative cyclic homology of schemes over a field  $k$  (such as  $F$ -schemes over a field  $F \supseteq k$ ) can be defined using the Zariski hypercohomology of certain presheaves of complexes; see [21] and [4,

2.7] for precise definitions. We shall write these presheaves as  $HH(/k)$ ,  $HC(/k)$ ,  $HP(/k)$  and  $HN(/k)$ , respectively and regard them as presheaves of either cochain complexes or spectra. When  $k$  is omitted, it is understood that  $k = \mathbb{Q}$  is intended. Finally, we write  $\Omega_{/k}^i$  for the presheaf  $X \mapsto \Omega_{X/k}^i$ , while  $\Omega_{/k}^\bullet$  denotes the presheaf of algebraic de Rham complexes and  $\Omega_{/k}^{\leq i}$  denotes its brutal truncation in degree  $i$ .

### 1. *cdh*-DESCENT

In this section we recall the main results from [4], and prove that the failure of  $K$ -theory to be homotopy invariant can be measured using cyclic homology. We work on the category  $Sch/F$  of  $F$ -schemes essentially of finite type over a field  $F$  of characteristic 0.

Here are two of the main results of [4]. Recall that infinitesimal  $K$ -theory  $\mathcal{K}^{inf}(X)$  is the homotopy fiber of the Jones-Goodwillie Chern character  $K(X) \rightarrow HN(X)$ . The first one is Theorem 4.6 of [4]:

**Theorem 1.1.** *The presheaf of spectra  $\mathcal{K}^{inf}$  satisfies *cdh*-descent.*

The second one is a slight modification of Corollary 3.13 of [4].

**Theorem 1.2.** *For each subfield  $k \subseteq F$ , the presheaf  $HP(/k)$  satisfies *cdh*-descent on  $Sch/F$ . In particular,  $HP$  satisfies *cdh*-descent.*

*Proof.* For  $k = F$ , this is proven in [4, Corollary 3.13]. As in *loc. cit.*, the result for  $k \subset F$  follows from [4, Theorem 3.12], once we verify that the hypotheses hold. But this follows from three observations: (1) the Cuntz-Quillen excision theorem holds over  $k$  (see [5, 5.3], noting that the condition that the base field be algebraically closed is not needed, see [5, p. 69]). Cuntz-Quillen is a theorem about bivariant periodic cyclic homology; to obtain the result we need, simply set one of the variables equal to  $k$ . (2) Goodwillie's theorem that periodic cyclic homology is invariant under infinitesimal extension does not require rings of finite type over  $k$  (see [7, II.5.1] or [12, 4.1.15]); and (3) the results of [4, Section 2] hold for (Hochschild, cyclic, periodic, negative) homology over  $k$ .  $\square$

*Example 1.3.* Consider the presheaf of spectra  $KH$  associated to Homotopy  $K$ -theory. The main theorem of [9] implies that the natural transformation  $K \rightarrow KH$  induces an equivalence  $\mathbb{H}_{cdh}(-, K) \simeq KH$ , because  $KH$  satisfies *cdh* descent. Thus the following definition of  $\mathcal{F}_K$  is compatible (up to canonical natural equivalence) with the definition of  $\mathcal{F}_K$  given in the introduction.

**Definition 1.4.** For any presheaf of spectra  $E$ , we write  $\mathcal{F}_E$  for the homotopy fiber of  $E \rightarrow \mathbb{H}_{cdh}(-, E)$ . If  $\mathcal{F}_E(X)$  is  $n$ -connected for some scheme  $X$  (for all  $X$  in some subcategory of  $Sch/F$ ), we say that  $E$  satisfies *n-cdh*-descent on  $X$  (resp., on the subcategory). Note that if  $E$  satisfies *n-cdh*-descent for all  $n$  on a subcategory, then  $E$  satisfies *cdh*-descent on that subcategory.

Since the fibrant replacement functor  $\mathbb{H}_{cdh}$  preserves (objectwise) fibration sequences, it follows that  $\mathcal{F}$  does too. (See the first paragraph of [4, Section 5].) We record this as the following observation.

**Lemma 1.5.** *Let  $E_1 \rightarrow E_2 \rightarrow E_3$  be a fibration sequence of presheaves of spectra. Then there is a natural induced fibration sequence*

$$\mathcal{F}_{E_1} \rightarrow \mathcal{F}_{E_2} \rightarrow \mathcal{F}_{E_3}.$$

**Theorem 1.6.** *For any scheme  $X$ , essentially of finite type over a field of characteristic 0, the Chern character  $K \rightarrow HN = HN(/\mathbb{Q})$  induces natural weak equivalences*

$$\mathcal{F}_K(X) \xrightarrow{\cong} \mathcal{F}_{HN}(X) \xleftarrow{\cong} \Omega^{-1}\mathcal{F}_{HC}(X).$$

*Proof.* The first weak equivalence follows from Lemma 1.5 and Theorem 1.1. The second weak equivalence follows from Lemma 1.5, Theorem 1.2 and the *SBI* fibration sequence  $\Omega HP \rightarrow \Omega^{-1}HC \rightarrow HN \rightarrow HP$ .  $\square$

**Corollary 1.7.** *Let  $X \in \text{Sch}/F$ . Then  $K$  satisfies  $(n+1)$ -*cdh*-descent on  $X$  if and only if  $HH$  satisfies  $n$ -*cdh*-descent on  $X$ .*

*Proof.* Since  $HH(X)$  and  $\mathbb{H}(X, HH)$  are  $n$ -connected for  $n < -\dim(X)$  by [21], this follows from Theorem 1.6 and the *SBI* sequence  $\Omega^{-1}HC \rightarrow HH \rightarrow HC$ . Indeed, by Lemma 1.5 and the theorem, the *SBI* sequence induces a fibration sequence  $\mathcal{F}_K(X) \rightarrow \mathcal{F}_{HH}(X) \rightarrow \Omega\mathcal{F}_K(X)$  in which all terms have bounded below homotopy. Clearly, if  $K$  satisfies  $(n+1)$ -descent, then  $HH$  satisfies  $n$ -descent. Conversely, if  $HH$  satisfies  $n$ -descent, then  $\pi_k\mathcal{F}_K(X) \cong \pi_{k-2}\mathcal{F}_K(X)$  for  $k \leq n+1$ ; since  $\pi_k\mathcal{F}_K(X) = 0$  for  $k \ll 0$ , this completes the proof.  $\square$

## 2. *cdh*-FIBRANT HOCHSCHILD AND CYCLIC HOMOLOGY.

In this section we study the *cdh*-fibrant version of Hochschild homology and its Hodge decomposition, and establish some of their basic properties. We will use the terminology of [4, Section 3].

For legibility, we will write  $a$  for the natural morphism of sites from the *cdh*-site to the Zariski site on  $\text{Sch}/F$ . If  $A$  is a Zariski sheaf, its associated *cdh* sheaf will be written as  $A_{cdh}$  or  $a^*A$ . We will simplify notation and write  $H_{cdh}^*(X, A)$  for the cohomology of  $A_{cdh}$ . In particular this applies to the sheaf  $X \mapsto \Omega_{X/k}^i$  of Kähler  $i$ -differential forms ( $i \geq 0$ );  $H_{cdh}^*(X, \Omega_{X/k}^i)$  is the cohomology of  $a^*\Omega_{X/k}^i$ . If  $A$  is a complex of presheaves of abelian groups on  $\text{Sch}/F$ , then we write  $\mathbb{H}_{cdh}(A)$  for a *cdh*-fibrant replacement of  $A$ , and  $\mathbb{H}_{cdh}(X, A)$  for its complex of sections over  $X$ ; the usual hypercohomology  $\mathbb{H}_{cdh}^n(X, A)$  is just  $H^n\mathbb{H}_{cdh}(X, A)$ . For example, if  $A$  is a presheaf, considered as a complex concentrated in degree zero, then  $\mathbb{H}_{cdh}(A)$  is just an injective resolution of the *cdh*-sheafification  $A_{cdh}$ , and  $\mathbb{H}_{cdh}^n(X, A)$  is the usual cohomology  $H_{cdh}^n(X, A_{cdh})$  of  $A_{cdh}$ .

When  $A$  is an unbounded complex, such as a complex representing Hochschild homology, then  $\mathbb{H}_{cdh}(A)$  may be constructed using product total complexes of flasque Cartan-Eilenberg resolutions. This works because the columns of the Cartan-Eilenberg double complex are locally cohomologically bounded by [15].

The *cdh* site is Noetherian (every covering has a finite subcovering), so *cdh* cohomology commutes with filtered direct limits of sheaves. A typical application of this fact is that if  $M$  is a sheaf of  $F$ -modules and  $V$  is a vector space then  $H_{cdh}^n(X, V \otimes_F M) \cong V \otimes_F H_{cdh}^n(X, M)$ . (See [1], Exp. VI, 2.11 and 5.2.)

The Hochschild and cyclic homology of schemes over a field  $k$  (such as  $F$ -schemes over a field  $F \supseteq k$ ) can be defined using the Zariski hypercohomology of certain presheaves of mixed complexes; see [21] and [4, 2.7] for precise definitions. It was observed in [22, 3.0] that, because the mixed complexes already admit a Hodge decomposition, so do the complexes  $HH(/k)$ ,  $HC(/k)$ ,  $HP(/k)$  and  $HN(/k)$ . Taking

fibrant replacements in the  $cdh$ -topology (or, in fact, in any topology) respects such product decompositions up to canonical global equivalence, as recorded in the following proposition.

**Proposition 2.1.** *Let  $X$  be a scheme over a field  $k$  of characteristic 0 (such as an  $F$ -scheme for  $F \supseteq k$ ). Then the  $cdh$ -fibrant Hochschild, cyclic, negative cyclic and periodic cyclic homology of  $X$  over  $k$  admit natural Hodge decompositions. That is, if  $H$  denotes any of  $HH(/k)$ ,  $HC(/k)$ ,  $HP(/k)$  or  $HN(/k)$  then:*

$$\mathbb{H}_{cdh}(X, H) \cong \prod \mathbb{H}_{cdh}(X, H^{(i)}).$$

*Proof.* Let  $H = \prod H^{(i)}$  be any of the product decompositions in question. Since the condition of being fibrant is defined by a right lifting property, the product  $\prod \mathbb{H}_{cdh}(-, H^{(i)})$  is fibrant. The natural map  $\mathbb{H}_{cdh}(-, H) \rightarrow \prod \mathbb{H}_{cdh}(-, H^{(i)})$  is a local weak equivalence; since both source and target are fibrant, it is a global weak equivalence, as asserted.  $\square$

Moreover, using the computations of these decompositions provided in [22, Theorem 3.3] and the fact that all  $F$ -schemes are smooth locally in the  $cdh$ -topology by resolution of singularities, it is possible to compute the Hodge decomposition explicitly in terms of the  $cdh$ -hypercohomology of the de Rham complex.

**Theorem 2.2.** *Let  $k \subseteq F$  be a subfield. There are natural isomorphisms for every  $F$ -scheme  $X$ :*

$$\begin{aligned} \pi_n \mathbb{H}_{cdh}(X, HH^{(i)}(/k)) &\cong H_{cdh}^{i-n}(X, \Omega_{/k}^i); \\ \pi_n \mathbb{H}_{cdh}(X, HC^{(i)}(/k)) &\cong \mathbb{H}_{cdh}^{2i-n}(X, \Omega_{/k}^{\leq i}); \\ \pi_n \mathbb{H}_{cdh}(X, HN^{(i)}(/k)) &\cong \mathbb{H}_{cdh}^{2i-n}(X, \Omega_{/k}^{\geq i}); \\ \pi_n \mathbb{H}_{cdh}(X, HP^{(i)}(/k)) &\cong \mathbb{H}_{cdh}^{2i-n}(X, \Omega_{/k}^\bullet). \end{aligned}$$

*Proof.* Let  $\mathbf{C}(X)$  denote the mixed complex computing the Hochschild and cyclic homology of  $X$  over  $k$ . The functor  $\mathbf{C} : X \mapsto \mathbf{C}(X)$  is a presheaf of mixed complexes. By [20, 9.8.12], there is a Hodge decomposition  $\mathbf{C} \cong \prod_i \mathbf{C}^{(i)}$  and a natural map of mixed complexes  $e : \mathbf{C} \rightarrow (\Omega_{X/k}^*, 0, d)$  that sends the Hochschild chain complex  $HH^{(i)}(X) = (\mathbf{C}^{(i)}(X), b)$  to  $\Omega_{X/k}^i[i]$ . As observed in [22], the induced map on Connes' double complexes sends  $\mathcal{B}_{**}^{(i)}$  to  $\Omega_{X/k}^{\leq i}[2i]$ . It suffices to prove that these are locally quasi-isomorphisms for the  $cdh$  topology. This boils down to showing that  $e$  induces a quasi-isomorphism  $HH^{(i)}(R/k) \rightarrow \Omega_{R/k}^i[i]$  for every regular local  $F$ -algebra  $R$ . For  $k = F$ , this is the Hochschild-Kostant-Rosenberg theorem ([20, 9.4.7]). The general case follows from this and the fact that  $R$  is the union of smooth  $k$ -algebras. (It also follows from the Kassel-Sledsjo spectral sequence of [11, 4.3a], which we recall in 4.1 below.)  $\square$

**Lemma 2.3.** *Let  $R$  be an  $F$ -algebra essentially of finite type,  $k \subseteq F$  a subfield. Then for each  $n$ ,  $HH(/k)$  satisfies  $n$ - $cdh$  descent on  $X = \text{Spec}(R)$  if and only if the following three conditions hold simultaneously:*

$$(2.3a) \quad HH_m^{(q)}(R/k) = 0 \quad \text{if } 0 \leq q < m \leq n;$$

$$(2.3b) \quad \Omega_{R/k}^q \rightarrow H_{cdh}^0(X, \Omega_{/k}^q) \quad \text{is bijective if } q \leq n \text{ and onto if } q = n + 1;$$

$$(2.3c) \quad H_{cdh}^p(X, \Omega_{/k}^q) = 0 \quad \text{if } p > 0 \text{ and } 0 \leq q \leq p + n + 1.$$

Note that (2.3a) is vacuous if  $n \leq 0$ , and (2.3b) is vacuous if  $n \leq -2$ . In particular,  $HH(/k)$  satisfies  $(-2)$ -*cdh*-descent just in case  $H_{cdh}^p(X, \Omega_{/k}^q) = 0$  for all  $p > q \geq 0$ .

*Proof.* This follows easily from the Hodge decomposition and the isomorphisms

$$HH_q^{(q)}(R/k) \cong \Omega_{R/k}^q, \quad \text{and} \quad HH_m^{(q)}(R/k) = 0 \text{ for } q > m.$$

In more detail, we see from 2.1 and 2.2 that the maps  $HH_m^{(q)}(R/k) \rightarrow H_{cdh}^{q-m}(X, \Omega_{/k}^q)$  must be isomorphisms for  $m \leq n$  and onto for  $m = n + 1$ .  $\square$

On smooth schemes, all our functors are well-behaved. Recall from [15] that the *scdh* topology on  $\text{Sm}/F$  is the restriction of the *cdh* topology on  $\text{Sch}/F$ . Since every scheme is smooth locally in the *cdh*-topology by resolution of singularities, every *cdh*-cover has a smooth refinement and it follows that  $\mathbb{H}_{scdh}(X, A)$  is just  $\mathbb{H}_{cdh}(X, A)$  for every presheaf  $A$ . (See the argument of the first part of the proof of [4, 3.12].) If  $A$  satisfies *scdh*-descent then  $A(X) \cong \mathbb{H}_{cdh}(X, A)$  for all smooth  $X$ .

Recall that  $k$  is a subfield of  $F$ .

**Theorem 2.4.** *Let  $H$  denote any of:  $HH(/k)$ ,  $HC(/k)$ ,  $HN(/k)$  or  $HP(/k)$ , and let  $H^{(i)}$  denote the  $i^{\text{th}}$  component in the Hodge decomposition of  $H$ . Then  $H$  and  $H^{(i)}$  satisfy *scdh*-descent on  $\text{Sm}/F$ . In particular, if  $X$  is smooth over  $F$ , then  $H^{(i)}(X) \cong \mathbb{H}_{cdh}(X, H^{(i)})$ .*

*Proof.* By [4, Theorem 3.9 and Theorems 2.9, and 2.10, which hold for homology taken relative to  $k$ ], Hochschild, cyclic, negative and periodic cyclic homology (relative to  $k$ ) all satisfy *scdh*-descent on  $\text{Sm}/F$ .

By Proposition 2.1, the quasi-isomorphisms  $H(X) \cong \mathbb{H}_{cdh}(X, H) = \mathbb{H}_{scdh}(X, H)$  induce quasi-isomorphisms  $H^{(i)}(X) \cong \mathbb{H}_{cdh}^{(i)}(X, H) = \mathbb{H}_{scdh}^{(i)}(X, H)$  for all  $i$ .  $\square$

The special case  $H_{Zar}^*(X, \mathcal{O}) \cong H_{cdh}^*(X, \mathcal{O})$  (for smooth  $X$ ) of the following corollary was proven in [4, 6.3].

**Corollary 2.5.** *If  $X$  is smooth over  $F$ , then  $H_{Zar}^p(X, \Omega_{/k}^i) \cong H_{cdh}^p(X, \Omega_{/k}^i)$  for all  $p$  and  $i$ . In particular,  $\Omega_{X/k}^i \cong H_{cdh}^0(X, \Omega_{/k}^i)$ .*

*Proof.* Consider the map  $e^{(i)} : HH^{(i)} = (\mathbf{C}^{(i)}, b) \rightarrow \Omega_{/F}^i[i]$  of complexes of Zariski sheaves. By [22, 3.3], it is a quasi-isomorphism over every smooth scheme  $X$  over  $F$ , inducing  $HH_{i-p}^{(i)}(X) \cong H_{Zar}^{p-i}(X, \mathbf{C}^{(i)}) \cong H_{Zar}^p(X, \Omega_{/k}^i)$ . The map  $e^{(i)}$  remains a quasi-isomorphism after sheafifying for the *cdh* topology, so that  $\mathbb{H}_{cdh}^{p-i}(X, HH^{(i)}) \cong H_{cdh}^{p-i}(X, \mathbf{C}^{(i)}) \cong H_{cdh}^p(X, \Omega_{/k}^i)$ . By Theorem 2.4,  $HH_n^{(i)}(X) \cong \mathbb{H}_{cdh}^{-n}(X, HH^{(i)})$ , whence the result.  $\square$

The next result is proven by copying the proof of [4, 6.1], replacing  $\mathcal{O}$  with  $\Omega_{/k}^i$ .

**Proposition 2.6.** *If  $X$  is a  $d$ -dimensional scheme, essentially of finite type over  $F$ , and  $k \subseteq F$  is a subfield, then*

$$H_{Zar}^d(X, \Omega_{/k}^i) \rightarrow H_{cdh}^d(X, \Omega_{/k}^i)$$

*is surjective. In particular, if  $X$  is affine and  $d > 0$  then  $H_{cdh}^d(X, \Omega_{/k}^i) = 0$ .*

The following useful theorem is proven in [15, 12.1].

**Theorem 2.7.** *For every abstract blow-up square  $(\square)$ , and for every complex of sheaves of abelian groups  $A$ , there is a long exact Mayer-Vietoris sequence:*

$$\cdots H_{cdh}^n(X, A) \rightarrow H_{cdh}^n(X', A) \oplus H_{cdh}^n(Y, A) \rightarrow H_{cdh}^n(Y', A) \rightarrow H_{cdh}^{n+1}(X, A) \cdots$$

Consider the change-of-topology morphism  $a : (\text{Sch}/F)_{cdh} \rightarrow (\text{Sch}/F)_{Zar}$ .

**Lemma 2.8.** *If a Zariski sheaf  $M$  on  $\text{Sch}/F$  is a quasi-coherent sheaf (resp., coherent sheaf) on each  $X_{Zar}$ , and  $M$  satisfies  $scdh$ -descent on  $\text{Sm}/F$ , then the cohomology sheaves  $R^q a_*(a^* M)$  are also quasi-coherent (resp., coherent) on each  $X_{Zar}$ .*

*If  $X = \text{Spec}(R)$  is affine, then  $R^q a_*(a^* M)$  is the quasi-coherent sheaf associated to the  $R$ -module  $H_{cdh}^q(X, M)$ , and the natural map  $M(X) \rightarrow H_{cdh}^0(X, M)$  is  $R$ -linear.*

*Proof.* We proceed by induction on  $\dim X$ , the case  $\dim(X) = 0$  being clear. Using resolution of singularities, pick a smooth abstract blow-up  $X'$  of  $X$ , and form the abstract blow-up square  $(\square)$ . By Theorem 2.7, we get a triangle on  $X_{Zar}$ :  $Ra_*(a^* M)|_X \rightarrow Ra_*(a^* M)|_{X' \amalg Y} \rightarrow Ra_*(a^* M)|_{Y'}$ . As the latter two terms have quasi-coherent (resp., coherent) cohomology sheaves, by induction and  $scdh$ -descent on  $X'$ , so does the first.

If  $X$  is affine, then  $H_{Zar}^p(X, R^q a_* M) = 0$  for  $p > 0$ . Hence the Leray spectral sequence collapses to yield  $H_{cdh}^q(X, M) = H_{Zar}^0(X, R^q a_*(a^* M))$ .  $\square$

**Corollary 2.9.** *Suppose that  $X = \text{Spec}(R)$  is affine. Then  $U \mapsto \pi_n \mathcal{F}_{HH(/k)}(U)$  and  $U \mapsto \pi_n \mathbb{H}_{cdh}(U, HH(/k))$  are quasi-coherent Zariski sheaves on  $X$  for all  $n$ .*

### 3. A CRITERION FOR SMOOTHNESS.

In this section we present a local criterion for smoothness of schemes over a field  $F$ , in terms of the Hochschild homology and  $cdh$ -fibrant Hochschild homology of their local rings over  $F$  (see 3.1). As an application we prove Vorst's conjecture for algebras of finite type over  $\mathbb{Q}$  and their localizations at maximal ideals (see 3.3).

A stronger global version of the following result shall be proved in Section 4 below (see Theorem 4.11).

Recall that  $F$  is a field of characteristic 0.

**Theorem 3.1.** *Let  $R$  be the local ring of a  $d$ -dimensional  $F$ -algebra of finite type at a maximal ideal. If  $HH(/F)$  satisfies  $d$ - $cdh$ -descent on  $R$ , then  $R$  is smooth over  $F$ .*

*Proof.* Recall that  $\Omega_{/F}^\bullet$  denotes the de Rham complex, whose terms are the Zariski sheaves  $\Omega_{/F}^i$ , while  $\Omega_{/F}^{\leq i}$  denotes its brutal truncation in degrees at most  $i$ . By 2.2 and 1.2, we have isomorphisms

$$HP_n^{(j)}(X/F) \xrightarrow{\cong} \mathbb{H}_{cdh}^{2j-n}(X, \Omega_{/F}^\bullet)$$

for any  $X \in \text{Sch}/F$ , and all  $n$  and  $j$ . Moreover, by the proof of 2.2, this isomorphism factors through a natural map  $e : HP_n^{(j)}(X/F) \rightarrow \mathbb{H}_{Zar}^{2j-n}(X, \Omega_{/F}^\bullet)$ . Now specialize to the case  $X = \text{Spec } R$ , where  $R$  is as in the theorem. By [15, Proposition 5.9 and Lemma 12.4] and resolution of singularities, every  $cdh$  cover of  $X$  has a refinement consisting of (essentially) smooth affine  $F$ -schemes of transcendence degree at most  $d$  over  $F$ . Hence we have

$$\mathbb{H}_{cdh}^*(X, \Omega_{/F}^\bullet) = \mathbb{H}_{cdh}^*(X, \Omega_{/F}^{\leq d}).$$

Moreover, Lemma 2.3 implies that the hypercohomology spectral sequence for  $\mathbb{H}_{cdh}^*$  degenerates to yield an isomorphism

$$H^*(\Omega_{R/F}^{\leq d}, d) \rightarrow \mathbb{H}_{cdh}^*(X, \Omega_{/F}^{\leq d}).$$

The canonical map  $S : HP_n^{(j)}(R/F) \rightarrow HC_{n-2}^{(j-1)}(R/F)$  fits into the commutative diagram

$$\begin{array}{ccccc} HP_{d+2}^{(d+1)}(R/F) & \xrightarrow{e} & H_{dR}^d(R/F) & \longrightarrow & \mathbb{H}_{cdh}^d(X, \Omega_{/F}^\bullet) \\ \downarrow S & & \downarrow & & \downarrow \wr \\ HC_d^{(d)}(R/F) & \xrightarrow[\cong]{e} & \Omega_{R/F}^d/d\Omega_{R/F}^{d-1} & \xrightarrow[\cong]{} & \mathbb{H}_{cdh}^d(X, \Omega_{/F}^{\leq d}). \end{array}$$

We have seen that the top composite, the right vertical and both bottom arrows are isomorphisms. It follows that the middle vertical inclusion is the identity map, i.e., that  $d\Omega_{R/F}^d = 0$ . On the other hand,  $d\Omega_{R/F}^d$  generates  $\Omega_{R/F}^{d+1}$  as an  $R$ -module; therefore we can infer that  $\Omega_{R/F}^{d+1} = 0$ . By Lemma 3.2 below,  $R$  is regular, and hence smooth over  $F$ .  $\square$

**Lemma 3.2.** *Let  $F$  be any perfect field. Suppose  $R$  is the local ring of a  $d$ -dimensional  $F$ -algebra of finite type at a maximal ideal. If  $\Omega_{R/F}^{d+1} = 0$ , then  $R$  is regular.*

*Proof.* Let  $\mathfrak{m}$  be the maximal ideal of  $R$ . Since  $L := R/\mathfrak{m}$  is smooth over  $F$ , the Second Fundamental Theorem [20, 9.3.5] shows that there is an isomorphism  $\mathfrak{m}/\mathfrak{m}^2 \rightarrow L \otimes_R \Omega_{R/F}^1$  sending  $x$  to  $dx$ . Consequently, there is a surjection from  $\Omega_{R/F}^{d+1}$  onto  $\Lambda_L^{d+1}(\mathfrak{m}/\mathfrak{m}^2)$ , which is a nonzero vector space unless  $R$  is regular.  $\square$

As an application, we can now verify Vorst's Conjecture for algebras of finite type over  $\mathbb{Q}$  and their localizations at maximal ideals.

**Theorem 3.3.** *Let  $R$  be a  $d$ -dimensional  $\mathbb{Q}$ -algebra which is either of finite type over  $\mathbb{Q}$ , or a localization of a  $\mathbb{Q}$ -algebra of finite type at a maximal ideal.*

*If  $R$  is  $K_{d+1}$ -regular, then  $R$  is regular.*

*Proof.* First assume  $R$  is of finite type over  $\mathbb{Q}$ , and  $K_{d+1}$ -regular. To prove  $R$  is regular, we may replace  $R$  by its localization at a maximal ideal; these local rings are also  $K_{d+1}$ -regular, by Vorst's localization theorem (see [17, 1.9]; also see [16, 9.12] as to why the condition that  $R$  be reduced is not needed in the localization theorem). Thus we are reduced to proving the theorem in the local case.

As remarked in the introduction, if  $R$  is  $K_{d+1}$ -regular, then  $\mathcal{F}_K(R)$  is  $(d+1)$ -connected (see [19]). By Corollary 1.7,  $\mathcal{F}_{HH(/F)}(R)$  is  $d$ -connected. Now Theorem 3.1 applies to prove that  $R$  is smooth over  $\mathbb{Q}$  and hence regular.  $\square$

#### 4. VORST'S CONJECTURE.

In this section we will prove Theorem 0.1. Throughout,  $F$  will be a fixed field of characteristic zero,  $k \subseteq F$  a subfield,  $R$  an  $F$ -algebra essentially of finite type, and  $X = \text{Spec}(R)$ . Note that we write  $HH$  for  $HH(/\mathbb{Q})$ .

**Lemma 4.1.** (*Kassel-Sledsjae*, [11, 4.3a]) *Let  $k \subseteq F$  and  $p \geq 1$  be fixed. Then there is a bounded second quadrant homological spectral sequence ( $0 \leq i < p$ ,  $j \geq 0$ ):*

$${}^p E_{-i, i+j}^1 = \Omega_{F/k}^i \otimes_F HH_{p-i+j}^{(p-i)}(R/F) \Rightarrow HH_{p+j}^{(p)}(R/k)$$

**Lemma 4.2.** *Let  $k \subseteq F$  and  $p \geq 1$  be fixed. Then there is a spectral sequence:*

$${}^p E_1^{i,j} = \Omega_{F/k}^i \otimes_F H_{cdh}^{i+j}(X, \Omega_{F/k}^{p-i}) \Rightarrow H_{cdh}^{i+j}(X, \Omega_{F/k}^p).$$

*Proof.* Consider the sheaf of ideals  $I := \ker(\Omega_{F/k}^* \rightarrow \Omega_{F/k}^*)$ . The  $I$ -adic filtration of  $\Omega_{F/k}^*$  induces a filtration  $\mathcal{G} = \mathcal{G}(p)$  on  $\Omega_{F/k}^p$ . If  $R$  is any  $F$ -algebra essentially of finite type, we have a natural surjection

$$(4.3) \quad \Omega_{F/k}^i \otimes_F \Omega_{R/F}^{p-i} \rightarrow \mathcal{G}^i(R)/\mathcal{G}^{i+1}(R),$$

which is an isomorphism if  $R$  is smooth. To see this, consider the fundamental sequence  $\Omega_{F/k}^1 \otimes R \rightarrow \Omega_{R/k}^1 \rightarrow \Omega_{R/F}^1 \rightarrow 0$ . Note that  $\Omega_{R/k}^n$  receives a map from  $\Omega_{F/k}^i \otimes \Omega_{R/k}^{p-i}$  and that the image of this map is  $\mathcal{G}^i \Omega_{R/k}^n$ . The surjection  $\Omega_{F/k}^i \otimes \Omega_{R/F}^{n-i} \rightarrow \mathcal{G}^i \Omega_{R/k}^n$  is clear from this description. Note also that this implies that  $\mathcal{G}^{p+1} = 0$ . If  $R/F$  is smooth, then the fundamental sequence splits, and a choice of splitting induces a direct sum decomposition which in turn induces  $\Omega_{R/k}^n = \bigoplus_i \Omega_{F/k}^i \otimes \Omega_{R/F}^{n-i}$ . By construction, this gives a filtered isomorphism between  $\Omega_{R/k}^n$  and the graded module associated to the filtration  $\mathcal{G}$ . This proves that the maps 4.3 are isomorphisms in this case.

Thus the  $cdh$ -sheafification of (4.3) is an isomorphism. Since  $\Omega_{F/k}^i$  is a vector space, the spectral sequence of the lemma is the one associated to the corresponding filtration of the  $cdh$  sheaf  $a^* \Omega_{F/k}^p$ .  $\square$

**Lemma 4.4.** *Let  $X = \text{Spec}(R)$  be affine, and fix  $n \geq 0$ . Assume that*

$$(4.4a) \quad \Omega_{R/k}^q \rightarrow H_{cdh}^0(X, \Omega_{F/k}^q) \text{ is bijective if } q \leq n \text{ and onto if } q = n+1,$$

$$(4.4b) \quad H_{cdh}^1(X, \Omega_{F/k}^q) = 0 \text{ if } q \leq n+1.$$

*Then  $\Omega_{R/F}^q \rightarrow H_{cdh}^0(X, \Omega_{F/k}^q)$  is bijective if  $q \leq n$ , and onto if  $q = n+1$ .*

*Proof.* By induction on  $q$ . If  $q = 0$ , there is nothing to prove. Fix  $q > 0$ , and consider the filtration  $\mathcal{G}^i = \mathcal{G}^i(q)$ ,  $0 \leq i \leq q$ , considered in the proof of Lemma 4.2. We have a commutative diagram

$$(4.5) \quad \begin{array}{ccc} \Omega_{F/k}^i \otimes_F \Omega_{R/F}^{q-i} & \longrightarrow & \mathcal{G}^i(R)/\mathcal{G}^{i+1}(R) \\ \downarrow & & \downarrow \\ \Omega_{F/k}^i \otimes_F H_{cdh}^0(X, \Omega_{F/k}^{q-i}) & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^i/\mathcal{G}^{i+1}). \end{array}$$

The top arrow is surjective for all  $i$ , and an isomorphism for  $i = 0$ . The bottom arrow is an isomorphism by the proof of 4.2. By the inductive hypothesis, the left vertical arrow is an isomorphism for  $0 < i$ . It follows that the top arrow is an isomorphism for all  $0 \leq i \leq q$ , and that the arrow on the right is an isomorphism

for  $i > 0$ . By (4.4b) we have an exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^{i+1}) & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^i) & \longrightarrow & \Omega_{F/k}^i \otimes_F H_{cdh}^0(X, \Omega_{F/F}^{q-i}) \\ & & & & & & \downarrow \\ & & & & 0 & \longleftarrow & H_{cdh}^1(X, \mathcal{G}^i) \longleftarrow H_{cdh}^1(X, \mathcal{G}^{i+1}) \end{array}$$

Since  $\mathcal{G}^{q+1} = 0$ , we deduce, by descending induction on  $i$ , that for all  $i > 0$ ,

$$(4.6) \quad H_{cdh}^1(X, \mathcal{G}^i) = 0$$

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}^{i+1}(R) & \longrightarrow & \mathcal{G}^i(R) & \longrightarrow & \Omega_{F/k}^i \otimes \Omega_{R/F}^{q-i} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^{i+1}) & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^i) & \longrightarrow & \Omega_{F/k}^i \otimes H_{cdh}^0(X, \Omega_{F/F}^{q-i}) & \longrightarrow & 0 \end{array}$$

Using descending induction on  $i$  again, we obtain from this diagram that

$$(4.7) \quad \mathcal{G}^i(R) \cong H_{cdh}^0(X, \mathcal{G}^i) \quad (i > 0).$$

We have a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}^1(R) & \longrightarrow & \Omega_{R/k}^q & \longrightarrow & \Omega_{R/F}^q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^1) & \longrightarrow & H_{cdh}^0(X, \Omega_{R/k}^q) & \longrightarrow & H_{cdh}^0(X, \Omega_{R/F}^q) & \longrightarrow & 0 \end{array}$$

The third map in the bottom row is onto by (4.6). The first vertical map is an isomorphism by (4.7). By (4.4a), the second is an isomorphism if  $q \leq n$  and onto if  $q = n + 1$ . It follows that the same is true of the third vertical map.  $\square$

**Proposition 4.8.** *Assume  $n \geq 0$ . If  $HH(/k)$  satisfies  $n$ -*cdh*-descent on  $R$ , then so does  $HH(/F)$ .*

*Proof.* By Lemma 2.3, the hypothesis is equivalent to saying that the following conditions hold simultaneously.

$$(4.8a) \quad HH_m^{(q)}(R/k) = 0 \quad \text{if } 0 \leq q < m \leq n$$

$$(4.8b) \quad \Omega_{R/k}^q \rightarrow H_{cdh}^0(X, \Omega_{R/k}^q) \quad \text{is bijective if } q \leq n \text{ and onto if } q = n + 1$$

$$(4.8c) \quad H_{cdh}^p(X, \Omega_{R/k}^q) = 0 \quad \text{if } p > 0 \text{ and } q \leq p + n + 1.$$

We have to prove that the following conditions hold

$$(4.9a) \quad HH_m^{(q)}(R/F) = 0 \quad \text{if } 0 \leq q < m \leq n$$

$$(4.9b) \quad \Omega_{R/F}^q \rightarrow H_{cdh}^0(X, \Omega_{R/F}^q) \quad \text{is bijective if } q \leq n \text{ and onto if } q = n + 1$$

$$(4.9c) \quad H_{cdh}^p(X, \Omega_{R/F}^q) = 0 \quad \text{if } p > 0 \text{ and } q \leq p + n + 1.$$

Using (4.8c), the spectral sequence of 4.2, and induction, we obtain (4.9c). Hence (4.8b) implies (4.9b), by Lemma 4.4. To prove (4.9a) we proceed by induction on  $q$ . The case  $q = 0$  is just the fact that  $HH_m^{(0)}(A/k) = 0$  for any  $m > 0$ , any field  $k$  and any  $k$ -algebra  $A$ . Assume  $n \geq q \geq 1$ , and that we have  $HH_m^{(q')}(R/F) = 0$

for all  $m \leq n$  and all  $q' < \min\{m, q\}$ . By (4.9b) and (4.5), the spectral sequence of Lemma 4.1 collapses for  $j = 0$  to yield

$${}_q E_{i,-i}^\infty = {}_q E_{i,-i}^1.$$

Given this, (4.9a) follows from (4.8a) by induction.  $\square$

**Lemma 4.10.** *Let  $F$  be a field, and  $R$  a local  $F$ -algebra essentially of finite type. Then there exists a field  $F \subset E \subset R$  such that  $R$  is isomorphic to the localization of a finite type algebra over  $E$  at a maximal ideal.*

*Proof.* The hypothesis on  $R$  means that there exist an  $F$ -algebra  $A$  of finite type and a prime ideal  $P \subset A$  such that  $R = A_P$ . Suppose that  $\dim(A) = k + h$  and  $ht(P) = h$ . By Noether normalization, there is a polynomial subring  $S = F[t_1, \dots, t_k]$  of  $A$  meeting  $P$  in 0, and the field  $R/P = A_P/PA_P$  is a finite extension of  $E = F(t_1, \dots, t_k)$ . There is an evident inclusion of  $E = S_{P \cap S}$  into  $R = A_P$ , and  $R$  is the localization of the finite type  $E$ -algebra  $A \otimes_S E$  at a maximal ideal.  $\square$

The results of this section, together with Theorem 3.1, allow us to prove the following global regularity criterion.

**Theorem 4.11.** *Let  $k \subseteq F$  be fields of characteristic 0, and let  $R$  be an  $F$ -algebra essentially of finite type.*

*If  $HH(/k)$  satisfies  $h$ -cdh-descent on  $R$ , then  $R$  is smooth in codimension  $h$ . That is, for every prime ideal  $P$  of height  $h$ , the local ring  $R_P$  is regular.*

*Proof.* Let  $P$  be a prime ideal of  $R$  of height  $h$ . Since  $HH(/F)$  satisfies  $h$ -cdh descent on  $R$  by Proposition 4.8, it also satisfies  $h$ -cdh descent on the localization  $R_P$ , by Corollary 2.9. By Lemma 4.10, there is a field  $F \subset E \subset R_P$  such that  $R_P$  is the localization at a maximal ideal of an algebra of finite type over  $E$ . By 4.8,  $HH(/E)$  satisfies  $h$ -cdh descent on  $R_P$ . Because  $\dim(R_P) = h$ , Theorem 3.1 implies that  $R_P$  is smooth over  $E$ , and hence is regular.  $\square$

Once again, let  $F$  be a field of characteristic 0.

**Theorem 4.12.** *Suppose  $R$  is an  $F$ -algebra essentially of finite type. If  $R$  is  $K_{h+1}$ -regular for some  $h \geq 0$ , then  $R$  is regular in codimension  $h$ . In particular, if  $R$  is  $K_{\dim(R)+1}$ -regular, then  $R$  is regular, and hence smooth over  $F$ .*

*Proof.* If  $R$  is  $K_{h+1}$ -regular, then  $HH(/\mathbb{Q})$  satisfies  $h$ -descent on  $R$ , by Corollary 1.7. The assertion now follows from Theorem 4.11.  $\square$

## 5. A NONREDUCED SCHEME WHICH IS $K$ -REGULAR

This section is devoted to the counterexample stated in Theorem 0.2, which reappears here as Theorem 5.2.

**Lemma 5.1.** *Let  $X$  be a smooth projective elliptic curve over a field  $F$  with base-point  $Q$ , and let  $L$  be a degree zero line bundle  $L$  on  $X$ . Assume that  $L$  is not an element of odd order in the Picard group. Then  $H^*(X, L^{\otimes 2n+1}) = 0$  for all  $n \in \mathbb{Z}$ .*

*Proof.* Because  $\mathcal{O}_X$  is a dualizing sheaf, we are reduced by Serre duality to proving that  $H^0(X, L^{2n+1}) = 0$ . Because  $X$  is elliptic, there exists a rational point  $P \in X$  such that  $L := L(P - Q)$ . Now if  $H^0(X, L^{\otimes 2n+1})$  were nonzero, there would exist an element  $f$  in the function field of  $X$  with  $\operatorname{div}(f) = (2n+1)(P - Q)$ . But because  $L$  is not an odd torsion element, there is no such  $f$ .  $\square$

**Theorem 5.2.** *Let  $X$  be a smooth projective elliptic curve over a field  $F$  of characteristic 0, and let  $L$  be as in Lemma 5.1. Write  $Y$  for the nonreduced scheme with the same underlying space as  $X$  but with structure sheaf  $\mathcal{O}_Y = \mathcal{O}_X \oplus L$ , where  $L$  is regarded as a square-zero ideal; that is, the product  $(f, s) \cdot (g, t) = (fg, ft + gs)$  for sections  $f$  and  $g$  of  $\mathcal{O}_X$  and  $s$  and  $t$  of  $L$ .*

*Then for all  $n$ ,  $K_n(Y) = K_n(X)$  and  $Y$  is  $K_n$ -regular.*

*Proof.* As  $X$  is regular, and hence  $K_n$ -regular, it suffices to show that  $K(Y \times \mathbb{A}^m) \rightarrow K(X \times \mathbb{A}^m) \cong K(X)$  is an equivalence for all  $m \geq 0$ . We shall prove the equivalent assertion that the relative homotopy groups  $K(Y \times \mathbb{A}^m, X \times \mathbb{A}^m)$  are zero. By Goodwillie's theorem [8] and Zariski descent, these relative  $K$ -groups are isomorphic to the corresponding relative cyclic homology groups over  $\mathbb{Q}$ . By base-change (see [10, Example (3.3)]) it suffices to show that the relative groups  $HH_n^{rel} = HH_n(Y, X)$  and  $HC_n(Y, X)$  vanish for all  $n$ . Using the SBI sequence for the relative groups and the fact that  $HC_n(Y, X)$  vanishes for  $n < 0$ , we see that it suffices to show that  $HH_n^{rel} = 0$  for all  $n$ . There is a Zariski descent spectral sequence whose  $E_2$ -term consists of the groups  $H^0(X, HH_n^{rel})$  and  $H^1(X, HH_n^{rel})$  (note that the cohomological dimension of  $X$  is 1); therefore, it suffices to prove that these vanish for all  $n$ . From Lemma 5.3 below and the fact that  $\Omega_{X/F}^1 \cong \mathcal{O}_X$ , we see that the Zariski sheaves  $HH_n^{rel}$  are sums of odd tensor powers of  $L$  when  $F$  is a number field, and odd tensor powers of  $L$  tensored over  $F$  with vector spaces  $\Omega_{F/\mathbb{Q}}^i$  in general. But the cohomology of such powers vanishes by Lemma 5.1.  $\square$

The following lemma is well-known, at least in the case when  $L$  is free. We include a proof for the sake of completeness. For simplicity, we write  $HH_*(R)$  for  $HH_*(R/k)$ .

**Lemma 5.3.** *Let  $k$  be a field with  $\text{char}(k) \neq 2$ ,  $R$  a commutative  $k$ -algebra, and  $L$  a projective  $R$ -module of rank 1. Put  $A = R \oplus L$ , with  $L$  considered as square-zero ideal. Let  $M_*$  denote the graded  $R$ -module*

$$M_p = \begin{cases} L^{\otimes_R(p+1)} & p \geq 0 \text{ even,} \\ L^{\otimes_R p} & p > 0 \text{ odd.} \end{cases}$$

*Then for relative Hochschild homology over  $k$ ,*

$$HH_n(A, L) = \bigoplus_{p+q=n} M_p \otimes_R HH_q(R).$$

*Proof.* Let  $C_*(A, L)$  be the relative Hochschild complex; the subspace  $L^{\otimes_k 2m+1}$  of  $C_{2m}(A, L)$  consists of cycles, and induces a map  $M_{2m} = L^{\otimes_R 2m+1} \rightarrow HH_{2m}(A, L)$ , because for  $x_i \in L$  and  $r \in R$  we have

$$(-1)^i b(x_0 \otimes \cdots \otimes x_i \otimes r \otimes x_{i+1} \cdots) = (x_0 \otimes \cdots \otimes x_i r \otimes x_{i+1} \cdots) - (x_0 \otimes \cdots \otimes x_i \otimes r x_{i+1} \cdots).$$

Because  $Bb + bB = 0$ , where  $B : C_*(A, L) \rightarrow C_{*+1}(A, L)$  is the Connes operator, the subspace  $B(L^{\otimes_k 2m+1})$  of  $C_{2m+1}(A, L)$  also consists of cycles and induces a map  $M_{2m+1} = L^{\otimes_R 2m+1} \rightarrow HH_{2m}(A, L)$ . Thus we have a graded map  $M_* \rightarrow HH_{2m}(A, L)$ . Because  $HH_*(A, L)$  is a graded module over  $HH_*(R)$ , we get a canonical  $R$ -module map from  $M_* \otimes_R HH_*(R)$  to  $HH_*(A, L)$ . To see that it is an isomorphism, we may assume  $R$  is local, whence  $A = R[x]/\langle x^2 \rangle$ . By the Künneth formula, we are reduced to the case  $R = k$ , which is straightforward (see [12, 1.1.6] for the necessary computation.)  $\square$

**Acknowledgements.** The authors would like to thank M. Schlichting, whose contributions go beyond the collaboration [4]. C. Weibel would like to thank his children for riding the roller coaster “Dueling Dragons” enough times for him to work out the example in Theorem 0.2.

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