

# CYCLIC HOMOLOGY, *cdh*-COHOMOLOGY AND NEGATIVE *K*-THEORY

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ABSTRACT. We prove a blow-up formula for cyclic homology which we use to show that infinitesimal *K*-theory satisfies *cdh*-descent. Combining that result with some computations of the *cdh*-cohomology of the sheaf of regular functions, we verify a conjecture of Weibel predicting the vanishing of algebraic *K*-theory of a scheme in degrees less than minus the dimension of the scheme, for schemes essentially of finite type over a field of characteristic zero.

## INTRODUCTION

The negative algebraic *K*-theory of a singular variety is related to its geometry. This observation goes back to the classic study by Bass and Murthy [1], which implicitly calculated the negative *K*-theory of a curve *X*. By definition, the group  $K_{-n}(X)$  describes a subgroup of the Grothendieck group  $K_0(Y)$  of vector bundles on  $Y = X \times (\mathbb{A}^1 - \{0\})^n$ .

The following conjecture was made in 1980, based upon the Bass-Murthy calculations, and appeared in [38, 2.9]. Recall that if *F* is any contravariant functor on schemes, a scheme *X* is called *F*-regular if  $F(X) \rightarrow F(X \times \mathbb{A}^r)$  is an isomorphism for all  $r \geq 0$ .

***K*-dimension Conjecture 0.1.** *Let  $X$  be a Noetherian scheme of dimension  $d$ . Then  $K_m(X) = 0$  for  $m < -d$  and  $X$  is  $K_{-d}$ -regular.*

In this paper we give a proof of this conjecture for *X* essentially of finite type over a field *F* of characteristic 0; see Theorem 6.2. We remark that this conjecture is still open in characteristic  $p > 0$ , except for curves and surfaces; see [44]. We also remark that this conjecture is sharp in the sense that for any field *k* there are *n*-dimensional schemes of finite type over *k* with an isolated singularity and non-trivial  $K_{-n}$ , see [29].

Much of this paper involves cohomology with respect to Voevodsky's *cdh*-topology. The following statement summarizes some of our results in this direction:

**Theorem 0.2.** *Let  $F$  be a field of characteristic 0,  $X$  a  $d$ -dimensional scheme, essentially of finite type over  $F$ . Then:*

$$(1) \ K_{-d}(X) \cong H_{cdh}^d(X, \mathbb{Z}) \text{ (see 6.2);}$$

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- (2)  $H_{Zar}^d(X, \mathcal{O}_X) \rightarrow H_{cdh}^d(X, \mathcal{O}_X)$  is surjective (see 6.1);  
 (3) If  $X$  is smooth then  $H_{Zar}^n(X, \mathcal{O}_X) \cong H_{cdh}^n(X, \mathcal{O}_X)$  for all  $n$  (see 6.3).

In addition to our use of the  $cdh$ -topology, our key technical innovation is the use of Cortiñas' infinitesimal  $K$ -theory [4] to interpolate between  $K$ -theory and cyclic homology. We prove (in Theorem 4.6) that infinitesimal  $K$ -theory satisfies descent for the  $cdh$ -topology. Since we are in characteristic zero, every scheme is locally smooth for the  $cdh$ -topology, and therefore locally  $K_n$ -regular for every  $n$ . In addition, periodic cyclic homology is locally de Rham cohomology in the  $cdh$ -topology. These features allow us to deduce conjecture 0.1 from Theorem 0.2.

This paper is organized as follows. The first two sections study the behavior of cyclic homology and its variants under blow-ups. We then recall some elementary facts about descent for the  $cdh$ -topology in section 3, and provide some examples of functors satisfying  $cdh$ -descent, like periodic cyclic homology (3.13) and homotopy  $K$ -theory (3.14). We introduce infinitesimal  $K$ -theory in section 4 and prove that it satisfies  $cdh$ -descent. This already suffices to prove that  $X$  is  $K_{-d-1}$ -regular and  $K_n(X) = 0$  for  $n < -d$ , as demonstrated in section 5. The remaining step, involving  $K_{-d}$ , requires an analysis of the  $cdh$ -cohomology of the structure sheaf  $\mathcal{O}_X$  and is carried out in section 6.

### Notation

The category of spectra we use in this paper will not be critical. In order to minimize technical issues, we will use the terminology that a *spectrum*  $E$  is a sequence  $E_n$  of simplicial sets together with bonding maps  $b_n : E_n \rightarrow \Omega E_{n+1}$ . We say that  $E$  is an  $\Omega$ -*spectrum* if all bonding maps are weak equivalences. A map of spectra is a strict map. We will use the model structure on the category of spectra defined in [3]. Note that in this model structure, every fibrant spectrum is an  $\Omega$ -spectrum.

If  $A$  is a ring,  $I \subset A$  a two-sided ideal and  $\mathcal{E}$  a functor from rings to spectra, we write  $\mathcal{E}(A, I)$  for the homotopy fiber of  $\mathcal{E}(A) \rightarrow \mathcal{E}(A/I)$ . If moreover  $f : A \rightarrow B$  is a ring homomorphism mapping  $I$  isomorphically to a two-sided ideal (also called  $I$ ) of  $B$ , then we write  $\mathcal{E}(A, B, I)$  for the homotopy fiber of the natural map  $\mathcal{E}(A, I) \rightarrow \mathcal{E}(B, I)$ . We say that  $\mathcal{E}$  *satisfies excision* provided that  $\mathcal{E}(A, B, I) \simeq 0$  for all  $A, I$  and  $f : A \rightarrow B$  as above. Of course, if  $\mathcal{E}$  is only defined on a smaller category of rings, such as commutative  $F$ -algebras of finite type, then these notions still make sense and we say that  $\mathcal{E}$  satisfies excision for that category.

We shall write  $\text{Sch}/F$  for the category of schemes essentially of finite type over a field  $F$ . We say a presheaf  $\mathcal{E}$  of spectra on  $\text{Sch}/F$  *satisfies the Mayer-Vietoris-property* (or MV-property, for short) for a cartesian square of schemes

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

if applying  $\mathcal{E}$  to this square results in a homotopy cartesian square of spectra. We say that  $\mathcal{E}$  satisfies the Mayer-Vietoris property for a class of squares provided it satisfies the MV-property for each square in the class. For example, the MV-property for affine squares in which  $Y \rightarrow X$  is a closed immersion is the same as the excision property for commutative algebras of finite type, combined with invariance under infinitesimal extensions.

We say that  $\mathcal{E}$  *satisfies Nisnevich descent* for  $\text{Sch}/F$  if  $\mathcal{E}$  satisfies the MV-property for all elementary Nisnevich squares in  $\text{Sch}/F$ ; an *elementary Nisnevich square* is a cartesian square of schemes as above for which  $Y \rightarrow X$  is an open embedding,  $X' \rightarrow X$  is étale and  $(X' - Y') \rightarrow (X - Y)$  is an isomorphism. By [27, 4.4], this is equivalent to the assertion that  $\mathcal{E}(X) \rightarrow \mathbb{H}_{\text{nis}}(X, \mathcal{E})$  is a weak equivalence for each scheme  $X$ , where  $\mathbb{H}_{\text{nis}}(-, \mathcal{E})$  is a fibrant replacement for the presheaf  $\mathcal{E}$  in a suitable model structure.

We say that  $\mathcal{E}$  *satisfies *cdh*-descent* for  $\text{Sch}/F$  if  $\mathcal{E}$  satisfies the MV-property for all elementary Nisnevich squares (Nisnevich descent) and for all abstract blow-up squares in  $\text{Sch}/F$ . Here an *abstract blow-up square* is a square as above such that  $Y \rightarrow X$  is a closed embedding,  $X' \rightarrow X$  is proper and the induced morphism  $(X' - Y')_{\text{red}} \rightarrow (X - Y)_{\text{red}}$  is an isomorphism. We will see in Theorem 3.4 that this is equivalent to the assertion that  $\mathcal{E}(X) \rightarrow \mathbb{H}_{\text{cdh}}(X, \mathcal{E})$  is a weak equivalence for each scheme  $X$ , where  $\mathbb{H}_{\text{cdh}}(-, \mathcal{E})$  is a fibrant replacement for the presheaf  $\mathcal{E}$  in a suitable model structure.

It is well known that there is an Eilenberg-Mac Lane functor from chain complexes of abelian groups to spectra, and from presheaves of chain complexes of abelian groups to presheaves of spectra. This functor sends quasi-isomorphisms of complexes to weak homotopy equivalences of spectra. In this spirit, we will use the above descent terminology for presheaves of complexes. Because we will eventually be interested in hypercohomology, we use cohomological indexing for all complexes in this paper; in particular, for a complex  $A$ ,  $A[p]^q = A^{p+q}$ .

## 1. PERFECT COMPLEXES AND REGULAR BLOWUPS

In this section, we compute the categories of perfect complexes for blow-ups along regularly embedded centers. Our computation slightly differs from that of Thomason ([32], see also [28]) in that we use a different filtration which is more useful for our purposes. We don't claim much originality.

In this section, "scheme" means "quasi-separated and quasi-compact scheme". For such a scheme  $X$ , we write  $\text{D}_{\text{parf}}(X)$  for the derived category of perfect complexes on  $X$  [34]. Let  $i : Y \subset X$  be a regular embedding of schemes of pure codimension  $d$ , and let  $p : X' \rightarrow X$  be the blow-up of  $X$  along  $Y$  and  $j : Y' \subset X'$  the exceptional divisor. We write  $q$  for the map  $Y' \rightarrow Y$ .

Recall that the exact sequence of  $\mathcal{O}_{X'}$ -modules  $0 \rightarrow \mathcal{O}_{X'}(1) \rightarrow \mathcal{O}_{X'} \rightarrow j_* \mathcal{O}_{Y'} \rightarrow 0$  gives rise to the fundamental exact triangle in  $\text{D}_{\text{parf}}(X')$ :

$$(1.1) \quad \mathcal{O}_{X'}(l+1) \rightarrow \mathcal{O}_{X'}(l) \rightarrow \text{R}j_*(\mathcal{O}_{Y'}(l)) \rightarrow \mathcal{O}_{X'}(l+1)[1],$$

where  $\text{R}j_*(\mathcal{O}_{Y'}(l)) = (j_* \mathcal{O}_{Y'}(l))$  by the projection formula.

We say that a triangulated subcategory  $\mathcal{S} \subset \mathcal{T}$  of a triangulated category  $\mathcal{T}$  is generated by a specified set of objects of  $\mathcal{T}$  if  $\mathcal{S}$  is the smallest *thick* (that is, closed under direct factors) triangulated subcategory of  $\mathcal{T}$  containing that set.

**Lemma 1.2.** (1) *The triangulated category  $\text{D}_{\text{parf}}(X')$  is generated by  $\text{L}p^*F$ ,  $\text{R}j_*\text{L}q^*G \otimes \mathcal{O}_{X'}(-l)$ , for  $F \in \text{D}_{\text{parf}}(X)$ ,  $G \in \text{D}_{\text{parf}}(Y)$  and  $l = 1, \dots, d-1$ .*  
(2) *The triangulated category  $\text{D}_{\text{parf}}(Y')$  is generated by  $\text{L}q^*G \otimes \mathcal{O}_{Y'}(-l)$ , for  $G \in \text{D}_{\text{parf}}(Y)$  and  $l = 0, \dots, d-1$ .*

*Proof.* (Thomason [32]) For  $k = 0, \dots, d$ , let  $\mathcal{A}'_k$  denote the full triangulated subcategory of  $\text{D}_{\text{parf}}(X')$  of those complexes  $E$  for which  $\text{R}p_*(E \otimes \mathcal{O}_{X'}(l)) = 0$  for

$0 \leq l < k$ . In particular,  $D_{\text{parf}}(X') = \mathcal{A}'_0$ . By [32, Lemme 2.5(b)],  $\mathcal{A}'_d = 0$ . Using [32, Lemme 2.4(a)], and descending induction on  $k$ , we see that for  $k \geq 1$ ,  $\mathcal{A}'_k$  is generated by  $Rj_*Lq^*G \otimes \mathcal{O}_{X'}(-l)$ , for some  $G$  in  $D_{\text{parf}}(Y)$  and  $l = k, \dots, d-1$ . For  $k = 0$ , we use the fact that the unit map  $1 \rightarrow Rp_*Lp^*$  is an isomorphism [32, Lemme 2.3(a)] to see that  $\mathcal{A}'_0 = D_{\text{parf}}(X')$  is generated by the image of  $Lp^*$  and the kernel of  $Rp_*$ . But  $\mathcal{A}'_1$  is the kernel of  $Rp_*$ .

Similarly, for  $k = 0, \dots, d$ , let  $\mathcal{A}_k$  be the full triangulated subcategory of  $D_{\text{parf}}(Y')$  of those complexes  $E$  for which  $Rq_*(E \otimes \mathcal{O}_{Y'}(l)) = 0$  for  $0 \leq l < k$ . In particular,  $D_{\text{parf}}(Y') = \mathcal{A}_0$ . By [32, Lemme 2.5(a)],  $\mathcal{A}'_d = 0$ . Using [33, p.247, from “Soit  $F$  un objet dans  $\underline{\mathcal{A}}'_k$ ” to “Alors  $G$  est un objet dans  $\underline{\mathcal{A}}'_{k+1}$ ”], and descending induction on  $k$ , we have that  $\mathcal{A}_k$  is generated by  $Lq^*G \otimes \mathcal{O}_{Y'}(-l)$ ,  $l = k, \dots, d-1$ .  $\square$

*Remark 1.3.* As a consequence of the proof of 1.2, we note the following. Let  $k = 0, \dots, d-1$  and  $m$  any integer. The full triangulated subcategory of  $D_{\text{parf}}(Y')$  of those complexes  $E$  with  $Rq_*(E \otimes \mathcal{O}_{Y'}(l)) = 0$  for  $m \leq l < k+m$  is the same as the full triangulated subcategory generated by  $Lq^*G \otimes \mathcal{O}_{Y'}(n)$ , for  $G \in D_{\text{parf}}(Y)$  and  $k+m \leq n \leq d-1+m$ . In particular, the condition that a complex be in the latter category is local in  $Y$ .

**Lemma 1.4.** *The functors  $Lp^* : D_{\text{parf}}(X) \rightarrow D_{\text{parf}}(X')$ ,  $Lq^* : D_{\text{parf}}(Y) \rightarrow D_{\text{parf}}(Y')$  and  $Rj_*Lq^* : D_{\text{parf}}(Y) \rightarrow D_{\text{parf}}(X')$  are fully faithful.*

*Proof.* The functors  $Lp^*$  and  $Lq^*$  are fully faithful, since the unit maps  $1 \rightarrow Rp_*Lp^*$  and  $1 \rightarrow Rq_*Lq^*$  are isomorphisms [32, Lemme 2.3].

By the fundamental exact triangle (1.1), the cone of the counit  $Lj^*Rj_*\mathcal{O}_{Y'} \rightarrow \mathcal{O}_{Y'}$  is in the triangulated subcategory generated by  $\mathcal{O}_{Y'}(1)$ , since the counit map is a retraction of  $Lj^*\mathcal{O}_{X'} \rightarrow Lj^*Rj_*\mathcal{O}_{Y'}$ . It follows that the cone of the counit map  $Lj^*Rj_*Lq^*E \rightarrow Lq^*E$  is in the triangulated subcategory generated by  $Lq^*E \otimes \mathcal{O}_{Y'}(1)$ , since the latter condition is local in  $Y$  (see Remark 1.3), and  $D_{\text{parf}}(Y)$  is generated by  $\mathcal{O}_Y$  for affine  $Y$ . Since  $Rq_*(Lq^*G \otimes \mathcal{O}(-1)) = G \otimes Rq_*\mathcal{O}(-1) = 0$ , we have  $\text{Hom}(A, B) = 0$  for  $A$  (respectively  $B$ ) in the triangulated subcategory of  $D_{\text{parf}}(Y')$  generated by  $Lq^*G \otimes \mathcal{O}(1)$  (respectively, generated by  $Lq^*G$ ), for  $G \in D_{\text{parf}}(Y)$ . Applying this observation to the cone of  $Lj^*Rj_*Lq^*E \rightarrow Lq^*E$  justifies the second equality in the display:

$$\begin{aligned} \text{Hom}(E, F) &= \text{Hom}(Lq^*E, Lq^*F) = \text{Hom}(Lj^*Rj_*Lq^*E, Lq^*F) \\ &= \text{Hom}(Rj_*Lq^*E, Rj_*Lq^*F). \end{aligned}$$

The first equality holds because  $Lq^*$  is fully faithful, and the final equality is an adjunction. The composition is an equality, showing that  $Rj_*Lq^*$  is fully faithful.  $\square$

For  $l = 0, \dots, d-1$ , let  $D_{\text{parf}}^l(X') \subset D_{\text{parf}}(X')$  be the full triangulated subcategory generated by  $Lp^*F$  and  $Rj_*Lq^*G \otimes \mathcal{O}_{X'}(-k)$  for  $F \in D_{\text{parf}}(X)$ ,  $G \in D_{\text{parf}}(Y)$  and  $k = 1, \dots, l$ . For  $l = 0, \dots, d-1$ , let  $D_{\text{parf}}^l(Y') \subset D_{\text{parf}}(Y')$  be the full triangulated subcategory generated by  $Lq^*G \otimes \mathcal{O}_{Y'}(-k)$  for  $G \in D(Y)$  and  $k = 0, \dots, l$ .

By Lemma 1.4,  $Lp^* : D_{\text{parf}}(X) \rightarrow D_{\text{parf}}^0(X')$  and  $Lq^* : D_{\text{parf}}(Y) \rightarrow D_{\text{parf}}^0(Y')$  are equivalences. By Lemma 1.2,  $D_{\text{parf}}^{d-1}(X') = D_{\text{parf}}(X')$  and  $D_{\text{parf}}^{d-1}(Y) = D_{\text{parf}}(Y')$ .

**Proposition 1.5.** *The functor  $Lj^*$  is compatible with the filtrations on  $D_{\text{parf}}(X')$  and  $D_{\text{parf}}(Y')$ :*

$$\begin{array}{ccccccc} D_{\text{parf}}(X) & \xrightarrow[\sim]{Lp^*} & D_{\text{parf}}^0(X') & \subset & D_{\text{parf}}^1(X') & \subset & \cdots \subset D_{\text{parf}}^{d-1}(X') = D_{\text{parf}}(X') \\ \downarrow Li^* & & \downarrow Lj^* & & \downarrow Lj^* & & \downarrow Lj^* \\ D_{\text{parf}}(Y) & \xrightarrow[\sim]{Lq^*} & D_{\text{parf}}^0(Y') & \subset & D_{\text{parf}}^1(Y') & \subset & \cdots \subset D_{\text{parf}}^{d-1}(Y') = D_{\text{parf}}(Y'). \end{array}$$

For  $l = 0, \dots, d-2$ ,  $Lj^*$  induces equivalences on successive quotient triangulated categories:

$$Lj^* : D_{\text{parf}}^{l+1}(X')/D_{\text{parf}}^l(X') \xrightarrow{\sim} D_{\text{parf}}^{l+1}(Y')/D_{\text{parf}}^l(Y').$$

*Proof.* The commutativity of the left hand square follows from  $Lq^*Li^* = Lj^*Lp^*$ . The compatibility of  $Lj^*$  with the filtrations only needs to be checked on generators, that is, we need to check that  $Lj^*[Rj_*Lq^*G \otimes \mathcal{O}_{X'}(-l)]$  is in  $D_{\text{parf}}^l(Y')$ ,  $l = 1, \dots, d-1$ . The last condition is local in  $Y$  (see Remark 1.3), a fortiori it is local in  $X$ . So we can assume that  $X$  and  $Y$  are affine, and  $G = \mathcal{O}_Y$ . In this case, the claim follows from the fundamental exact triangle (1.1).

For  $l-k = 1, \dots, d-1$ ,  $E \in D_{\text{parf}}(X)$  and  $G \in D_{\text{parf}}(Y)$ , we have  $\text{Hom}(Lp^*E \otimes \mathcal{O}(-k), Rj_*Lq^*G \otimes \mathcal{O}(-l)) = \text{Hom}(Lp^*E \otimes \mathcal{O}(l-k), Rj_*Lq^*G) = \text{Hom}(Lj^*Lp^*E \otimes \mathcal{O}(l-k), Lq^*G) = \text{Hom}(Lq^*Li^*E \otimes \mathcal{O}(l-k), Lq^*G) = 0$  since  $Rq_*\mathcal{O}(k-l) = 0$ . Therefore, all maps from objects of  $D_{\text{parf}}^{l+1}(X')$  to an object of  $\mathcal{O}(-l-1) \otimes Rj_*Lq^*D_{\text{parf}}(Y) \subset D_{\text{parf}}^{l+1}(X')$  are trivial. It follows that the composition

$$\mathcal{O}(-l-1) \otimes Rj_*Lq^*D_{\text{parf}}(Y) \subset D_{\text{parf}}^{l+1}(X') \rightarrow D_{\text{parf}}^{l+1}(X')/D_{\text{parf}}^l(X')$$

is an equivalence (it is fully faithful, both categories have the same set of generators, and the source category is idempotent complete). Similarly, the composition

$$\mathcal{O}(-l-1) \otimes Lq^*D_{\text{parf}}(Y) \subset D_{\text{parf}}^{l+1}(Y') \rightarrow D_{\text{parf}}^{l+1}(Y')/D_{\text{parf}}^l(Y')$$

is an equivalence.

The counit map  $Lj^*Rj_*Lq^* \rightarrow Lq^*$  has cone in the triangulated subcategory generated by  $Lq^*G \otimes \mathcal{O}(1)$  (see proof of 1.4),  $G \in D_{\text{parf}}(Y)$ . It follows that the natural map of functors  $Lj^*[\mathcal{O}(-l-1) \otimes Rj_*Lq^*] \rightarrow \mathcal{O}_{Y'}(-l-1) \otimes Lq^*$ , induced by the counit of adjunction, has cone in  $D_{\text{parf}}^l(Y')$ . Thus, the composition  $Lj^* \circ [\mathcal{O}(-l-1) \otimes Rj_*Lq^*] : D_{\text{parf}}(Y) \rightarrow D_{\text{parf}}^{l+1}(X')/D_{\text{parf}}^l(X') \rightarrow D_{\text{parf}}^{l+1}(Y')/D_{\text{parf}}^l(Y')$  agrees, up to natural equivalence of functors, with  $\mathcal{O}_{Y'}(-l-1) \otimes Lq^* : D_{\text{parf}}(Y) \rightarrow D_{\text{parf}}^{l+1}(Y')/D_{\text{parf}}^l(Y')$ . Since two of the three functors are equivalences, so is the third:  $Lj^* : D_{\text{parf}}^{l+1}(X')/D_{\text{parf}}^l(X') \xrightarrow{\sim} D_{\text{parf}}^{l+1}(Y')/D_{\text{parf}}^l(Y')$ .  $\square$

*Remark 1.6.* Proposition 1.5 yields *K*-theory descent for blow-ups along regularly embedded centers. This follows from Thomason's theorem in [34] (see [11, 10]), because every square in 1.5 induces a homotopy cartesian square of *K*-theory spectra.

Several people have remarked that this descent also follows from the main theorem of [32] by a simple manipulation.

## 2. THOMASON'S THEOREM FOR (NEGATIVE) CYCLIC HOMOLOGY

In this section we prove that negative cyclic, periodic cyclic and cyclic homology satisfy the Mayer-Vietoris property for blow-ups along regularly embedded centers. We will work over a ground field  $k$ , so that all schemes are  $k$ -schemes, all linear categories are  $k$ -linear, and tensor product  $\otimes$  means tensor product over  $k$ .

**Mixed Complexes.** In order to fix our notation, we recall some standard definitions (see [25] and [41]). We remind the reader that we are using cohomological notation, with the homology of  $C$  being given by  $H_n(C) = H^{-n}(C)$ .

A *mixed complex*  $\mathbf{C} = (C, b, B)$  is a cochain complex  $(C, b)$ , together with a chain map  $B : C \rightarrow C[-1]$  satisfying  $B^2 = 0$ . There is an evident notion of a map of mixed complexes, and we write  $Mix$  for the category of mixed complexes.

The complexes for cyclic, periodic cyclic and negative cyclic homology of  $(C, b, B)$  are obtained using the total complex:

$$\begin{aligned} HC(C, b, B) &= \text{Tot}(\cdots \rightarrow C[+1] \xrightarrow{B} C \rightarrow 0 \rightarrow 0 \rightarrow \cdots) \\ HP(C, b, B) &= \text{Tot}(\cdots \rightarrow C[+1] \xrightarrow{B} C \xrightarrow{B} C[-1] \xrightarrow{B} C[-2] \rightarrow \cdots) \\ HN(C, b, B) &= \text{Tot}(\cdots \rightarrow 0 \rightarrow C \xrightarrow{B} C[-1] \xrightarrow{B} C[-2] \rightarrow \cdots) \end{aligned}$$

where  $C$  is placed in horizontal degree 0 and where for a bicomplex  $E$ ,  $\text{Tot}E$  is the subcomplex of the usual product total complex (see [41]) which in degree  $n$  is

$$\text{Tot}^n E = \{(x_{p,q}) \in \prod_{p+q=n} E^{p,q} \mid x_{p,q} = 0, q \gg 0\}.$$

In addition to the familiar exact sequence  $0 \rightarrow C \rightarrow HC(\mathbf{C}) \rightarrow HC(\mathbf{C})[+2] \rightarrow 0$  we have a natural exact sequence of complexes

$$0 \rightarrow HN(\mathbf{C}) \rightarrow HP(\mathbf{C}) \rightarrow HC(\mathbf{C})[+2] \rightarrow 0.$$

Short exact sequences and quasi-isomorphisms of mixed complexes yield short exact sequences and quasi-isomorphisms of  $HC$ ,  $HP$  and  $HN$  complexes, respectively. Of course, the cyclic, periodic cyclic and negative cyclic homology groups of  $\mathbf{C}$  are the homology groups of  $HC$ ,  $HP$  and  $HN$ , respectively.

We say that a map  $(C, b, B) \rightarrow (C', b', B')$  is a quasi-isomorphism in  $Mix$  if the underlying complexes are quasi-isomorphic via  $(C, b) \rightarrow (C', b')$ ; following [24], we write  $\mathcal{D}Mix$  for the localization of  $Mix$  with respect to quasi-isomorphisms; it is a triangulated category with shift  $\mathbf{C} \mapsto \mathbf{C}[1]$ . The reader should beware that  $\mathcal{D}Mix$  is *not* the derived category of the underlying abelian category of  $Mix$ .

It is sometimes useful to use the equivalence between the category  $Mix$  of mixed complexes and the category of left dg  $\Lambda$ -modules, where  $\Lambda$  is the dg-algebra

$$\cdots 0 \rightarrow k\varepsilon \xrightarrow{0} k \rightarrow 0 \rightarrow \cdots$$

with  $k$  placed in degree zero [22, 2.2]. A left dg  $\Lambda$  module  $(C, d)$  corresponds to the mixed complex  $(C, b, B)$  with  $b = d$  and  $Bc = \varepsilon c$ , for  $c \in C$ . Under this identification, the triangulated category of mixed complexes  $\mathcal{D}Mix$  is equivalent to the derived category of left dg  $\Lambda$ -modules. With this interpretation of mixed complexes as left dg- $\Lambda$ -modules, we have  $HC(C) = k \otimes_{\Lambda}^L C$  and  $HN(C) = R\text{Hom}_{\Lambda}(k, C)$ .

Let  $\mathcal{B}$  be a small dg-category, i.e., a small category enriched over complexes. When  $\mathcal{B}$  is concentrated in degree 0 (i.e., when  $\mathcal{B}$  is a  $k$ -linear category), McCarthy defined a cyclic module and hence a mixed complex  $C_{us}(\mathcal{B})$  associated to  $\mathcal{B}$  by

$$C_{us}(\mathcal{B})_n = \coprod \text{Hom}_{\mathcal{B}}(B_n, B_0) \otimes \text{Hom}_{\mathcal{B}}(B_{n-1}, B_n) \otimes \cdots \otimes \text{Hom}_{\mathcal{B}}(B_0, B_1),$$

where the coproduct is taken over all  $n + 1$ -tuples  $(B_0, \dots, B_n)$  of objects in  $\mathcal{B}$ , and the face maps and cyclic operators are given by the usual rules; see [26]. Keller observed in [24, 1.3] that that this formula also defines a cyclic module for general dg-categories. (Since we are working over a field, Keller's flatness hypothesis is satisfied.)

*Exact categories 2.1.* If  $\mathcal{A}$  is a  $k$ -linear exact category in the sense of Quillen, Keller defines the mixed complex  $C(\mathcal{A})$  in [24, 1.4] to be the cone of  $C_{us}(\text{Ac}^b \mathcal{A}) \rightarrow C_{us}(\mathbf{Ch}^b \mathcal{A})$ , where  $\mathbf{Ch}^b \mathcal{A}$  is the dg-category of bounded chain complexes in  $\mathcal{A}$  and  $\text{Ac}^b \mathcal{A}$  is the sub dg-category of acyclic complexes. He also proves in [24, 1.5] that, up to quasi-isomorphism,  $C(\mathcal{A})$  only depends upon the idempotent completion  $\mathcal{A}^+$  of  $\mathcal{A}$ .

*Example 2.2.* Let  $A$  be a  $k$ -algebra; viewing it as a (dg) category with one object,  $C_{us}(A)$  is the usual mixed complex of  $A$  (see [25] or [41]). Now let  $\mathbf{P}(A)$  denote the exact category of finitely generated projective  $A$ -modules. By McCarthy's theorem [26, 2.4.3], the natural map  $C_{us}(A) \rightarrow C_{us}(\mathbf{P}(A))$  is a quasi-isomorphism of mixed complexes. Keller proves in [24, 2.4] that  $C_{us}(\mathbf{P}(A)) \rightarrow C(\mathbf{P}(A))$  and hence  $C_{us}(A) \rightarrow C(\mathbf{P}(A))$  is a quasi-isomorphism of mixed complexes. In particular, it induces quasi-isomorphisms of *HC*, *HP* and *HN* complexes.

*Exact dg categories 2.3.* Let  $\mathcal{B}$  be a small dg-category, and let  $DG(\mathcal{B})$  denote the category of left dg  $\mathcal{B}$ -modules. There is a Yoneda embedding  $Y : Z^0 \mathcal{B} \rightarrow DG(\mathcal{B})$ ,  $Y(B)(A) = \mathcal{B}(A, B)$ , where  $Z^0 \mathcal{B}$  is the subcategory of  $\mathcal{B}$  whose morphisms from  $A$  to  $B$  are  $Z^0 \mathcal{B}(A, B)$ . Following Keller [24, 2.1], we say that a dg-category is *exact* if  $Z^0 \mathcal{B}$  (the full subcategory of representable modules  $Y(B)$ ) is closed under extensions and the shift functor in  $DG(\mathcal{B})$ . The triangulated category  $\mathcal{T}(\mathcal{B})$  of an exact dg-category  $\mathcal{B}$  is defined to be Keller's stable category  $Z^0 \mathcal{B}/B^0 \mathcal{B}$ .

*Localization pairs 2.4.* A *localization pair*  $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_0)$  is an exact dg-category  $\mathcal{B}_1$  endowed with a full dg-subcategory  $\mathcal{B}_0 \subset \mathcal{B}_1$  such that  $Z^0 \mathcal{B}_0$  is an exact subcategory of  $Z^0 \mathcal{B}_1$  closed under shifts and extensions. For a localization pair  $\mathcal{B}$ , the induced functor on associated triangulated categories  $\mathcal{T}(\mathcal{B}_0) \subset \mathcal{T}(\mathcal{B}_1)$  is fully faithful, and the *associated triangulated category*  $\mathcal{T}(\mathcal{B})$  of  $\mathcal{B}$  is defined to be the Verdier quotient  $\mathcal{T}(\mathcal{B}_1)/\mathcal{T}(\mathcal{B}_0)$ .

*Sub and quotient localization pairs 2.5.* Let  $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_0)$  be a localization pair, and let  $\mathcal{S} \subset \mathcal{T}(\mathcal{B})$  be a full triangulated subcategory. Let  $\mathcal{C} \subset \mathcal{B}_1$  be the full dg subcategory whose objects are isomorphic in  $\mathcal{T}(\mathcal{B})$  to objects of  $\mathcal{S}$ . Then  $\mathcal{B}_0 \subset \mathcal{C}$  and  $\mathcal{C} \subset \mathcal{B}_1$  are localization pairs, and the sequence  $(\mathcal{C}, \mathcal{B}_0) \rightarrow \mathcal{B} \rightarrow (\mathcal{B}_1, \mathcal{C})$  has an associated sequence of triangulated categories which is naturally equivalent to the exact sequence of triangulated categories  $\mathcal{S} \rightarrow \mathcal{T}(\mathcal{B}) \rightarrow \mathcal{T}(\mathcal{B})/\mathcal{S}$ .

A dg category  $\mathcal{B}$  over a ring  $R$  is said to be *flat* if each  $H = \mathcal{B}(A, B)$  is flat in the sense that  $H \otimes_R -$  preserves quasi-isomorphisms of graded  $R$ -modules. A localization pair  $\mathcal{B}$  is flat if  $\mathcal{B}_1$  (and hence  $\mathcal{B}_0$ ) is flat. When the ground ring is a field, as it is in this article, every localization pair is flat.

In [24, 2.4], Keller associates to a flat localization pair  $\mathcal{B}$  a mixed complex  $C(\mathcal{B})$ , the cone of  $C(\mathcal{B}_0) \rightarrow C(\mathcal{B}_1)$ , and proves the following in [24, Theorem 2.4]:

**Theorem 2.6.** *Let  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  be a sequence of localization pairs such that the associated sequence of triangulated categories is exact up to factors. Then the induced sequence  $C(\mathcal{A}) \rightarrow C(\mathcal{B}) \rightarrow C(\mathcal{C})$  of mixed complexes extends to a canonical distinguished triangle in  $\mathcal{DMix}$*

$$C(\mathcal{A}) \rightarrow C(\mathcal{B}) \rightarrow C(\mathcal{C}) \rightarrow C(\mathcal{A})[1].$$

*Example 2.7.* The category  $\mathbf{Ch}_{\text{parf}}(X)$  of perfect complexes on  $X$  is an exact dg-category if we ignore cardinality issues. We need a more precise choice for the category of perfect complexes. Let  $F$  be a field of characteristic zero containing  $k$ . For  $X \in \text{Sch}/F$ , we choose  $\mathbf{Ch}_{\text{parf}}(X)$  to be the category of perfect bounded above complexes (under cohomological indexing) of flat  $\mathcal{O}_X$ -modules whose stalks have cardinality at most the cardinality of  $F$ . (Since  $F$  is infinite, all algebras essentially of finite type over  $F$  have cardinality at most the cardinality of  $F$ ). This is an exact dg-category over  $k$ . Let  $f : X \rightarrow Y$  be a map of schemes essentially of finite type over  $F$ . Then  $Lf^*$  is  $f^*$  on  $\mathbf{Ch}_{\text{parf}}(X)$ , so  $\mathbf{Ch}_{\text{parf}}$  is functorial up to (unique) natural isomorphism of functors on  $\text{Sch}/F$ . If we want to get a real presheaf of dg categories on  $\text{Sch}/F$ , we can replace  $\mathbf{Ch}_{\text{parf}}$  by some rectification as for example done in [40, Appendix].

Let  $\text{Ac}(X) \subset \mathbf{Ch}_{\text{parf}}(X)$  be the full dg-subcategory of acyclic complexes. Then  $\mathbf{Ch}_{\text{parf}}(X) = (\mathbf{Ch}_{\text{parf}}(X), \text{Ac}(X))$  is a localization pair over  $k$  whose associated triangulated category is naturally equivalent to  $\mathbf{D}_{\text{parf}}(X)$  ([34, 3.5.3], except for the cardinality part). We define  $\mathbf{C}(X)$  to be the mixed complex (over  $k$ ) associated to  $\mathbf{Ch}_{\text{parf}}(X)$ .

We define  $\mathbf{HC}(X)$ ,  $\mathbf{HP}(X)$ ,  $\mathbf{HN}(X)$  to be the cyclic, periodic cyclic, negative cyclic homology complexes associated with the mixed complex  $\mathbf{C}(X)$ . In particular,  $\mathbf{HC}$ ,  $\mathbf{HP}$  and  $\mathbf{HN}$  are presheaves of complexes on  $\text{Sch}/F$ . Keller proves in [23, 5.2] that these definitions agree with the definitions in [42], with  $HC_n(X) = H^{-n}\mathbf{HC}(X)$ , etc. In addition, the Hochschild homology of  $X$  is the homology of the complex underlying  $\mathbf{C}(X)$ .

*Example 2.8.* If  $Z \subset X$  is closed, let  $\mathbf{Ch}_{\text{parf}}(X \text{ on } Z)$  be the localization pair formed by the category of perfect complexes on  $X$  which are acyclic on  $X - Z$ , and its full subcategory of acyclic complexes. We define  $\mathbf{C}(X \text{ on } Z)$  to be the mixed complex associated to this localization pair.

If  $U \subset X$  is the open complement of  $Z$ , then Thomason and Trobaugh proved in [34, §5] that the sequence  $\mathbf{Ch}_{\text{parf}}(X \text{ on } Z) \rightarrow \mathbf{Ch}_{\text{parf}}(X) \rightarrow \mathbf{Ch}_{\text{parf}}(U)$  is such that the associated sequence of triangulated categories is exact up to factors. As pointed out in [23, 5.5], Keller's Theorem 2.6 implies that  $\mathbf{C}(X \text{ on } Z) \rightarrow \mathbf{C}(X) \rightarrow \mathbf{C}(U)$  fits into a distinguished triangle in  $\mathcal{DMix}$ .

Suppose that we are given an étale neighborhood  $q : V \rightarrow X$  of a closed subscheme  $Z$  of  $X$ , i.e., an étale morphism which is an isomorphism over  $Z$ . Then  $\mathbf{C}(X \text{ on } Z) \rightarrow \mathbf{C}(V \text{ on } Z)$  is a quasi-isomorphism. This is a consequence of the fact, demonstrated by Thomason and Trobaugh in [34, Theorem 2.6.3], that the functors  $Lq^*$  and  $Rq_*$  induce quasi-inverse equivalences on derived categories  $\mathbf{D}_{\text{parf}}(X \text{ on } Z) \cong \mathbf{D}_{\text{parf}}(V \text{ on } Z)$ .

As a consequence of 2.7 and 2.8, and a standard argument involving étale covers, we recover the following theorem, which was originally proven by Geller and Weibel

in [37, 4.2.1 and 4.8]. (The term “étale descent” used in [37] implies Nisnevich descent; for presheaves of  $\mathbb{Q}$ -modules, they are equivalent notions.)

**Theorem 2.9.** *Hochschild, cyclic, periodic and negative cyclic homology satisfy Nisnevich descent.*

We are now ready to prove the cyclic homology analogue of Thomason’s theorem for regular embeddings.

**Theorem 2.10.** *Let  $Y \subset X$  be a regular embedding of  $F$ -schemes of pure codimension  $d$ , let  $X' \rightarrow X$  the blow-up of  $X$  along  $Y$  and  $Y'$  the exceptional divisor. Then the presheaves of cyclic, periodic cyclic and negative cyclic homology complexes satisfy the Mayer-Vietoris property for the square*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

*Proof.* By section 2.5, the filtrations in Proposition 1.5 induce filtrations on both  $\mathbf{Ch}_{\text{parf}}(X')$  and on  $\mathbf{Ch}_{\text{parf}}(Y')$ , and  $Lf^* = f^*$  is compatible with these filtrations. Moreover,  $f^*$  induces a map on associated graded localization pairs. By Theorem 2.6 and Proposition 1.5, each square in the map of filtrations induces a homotopy cartesian square of mixed complexes, hence the outer square is homotopy cartesian, too.  $\square$

*Remark 2.11.* The filtrations in Proposition 1.5 split (see proof of 1.5), and induce the usual projective space bundle and blow-up formulas

$$\mathbf{HC}(Y') = \mathbf{HC}(\mathbb{P}_Y^{d-1}) \simeq \bigoplus_{0 \leq i \leq d-1} \mathbf{HC}(Y), \text{ and } \mathbf{HC}(X') \simeq \mathbf{HC}(X) \oplus \bigoplus_{1 \leq i \leq d-1} \mathbf{HC}(Y).$$

Similarly for  $\mathbf{HP}$  and  $\mathbf{HN}$  in place of  $\mathbf{HC}$ . For more details in the  $K$ -theory case, see [32].

*Remark 2.12.* Combining the Mayer-Vietoris property for the usual covering of  $X \times \mathbb{P}^1$  with the decomposition of 2.11 yields the Fundamental Theorem for negative cyclic homology, which states that there is a short exact sequence,

$$0 \rightarrow \mathbf{HN}(X \times \mathbb{A}^1) \cup_{\mathbf{HN}(X)} \mathbf{HN}(X \times \mathbb{A}^1) \rightarrow \mathbf{HN}(X \times (\mathbb{A}^1 - \{0\})) \rightarrow \mathbf{HN}(X)[1] \rightarrow 0.$$

This sequence is split up to homotopy; the splitting  $\mathbf{HN}(X)[1] \rightarrow \mathbf{HN}(X \times (\mathbb{A}^1 - \{0\}))$  is multiplication by the class of  $dt/t \in \mathbf{HN}_1(k[t, 1/t])$ . The same argument shows that there are similar Fundamental Theorems for cyclic and periodic cyclic homology.

### 3. DESCENT FOR THE *cdh*-TOPOLOGY

We recall the definition of a *cd*-structure, given in [35] and [36].

**Definition 3.1.** Let  $\mathcal{C}$  be a small category. A *cd-structure* on  $\mathcal{C}$  is a class  $\mathcal{P}$  of commutative squares in  $\mathcal{C}$  that is closed under isomorphism.

A *cd*-structure defines a topology on  $\mathcal{C}$ . We use the following *cd*-structures on  $\text{Sch}/F$  and on the subcategory  $\text{Sm}/F$  of essentially smooth schemes (that is, localizations of smooth schemes) over  $F$ .

- Example 3.2.* (1) The *combined cd-structure* on the category  $\text{Sch}/F$ . This consists of all elementary Nisnevich and abstract blow-up squares. It is complete ([36, Lemma 2.2]), bounded ([36, Proposition 2.12]) and regular ([36, Lemma 2.13]). By definition, the *cdh-topology* is the topology generated by the combined *cd-structure* (see [36, Proposition 2.16]).
- (2) The combined *cd-structure* on  $\text{Sm}/F$  is the sum of the “upper” and “smooth blow-up” *cd-structures* on  $\text{Sm}/F$ . It consists of all elementary Nisnevich squares and those abstract blow-up squares of smooth schemes isomorphic to a blow-up of a smooth scheme along a smooth center (this *cd-structure* is discussed in [36, Section 4]). This *cd-structure* is complete, bounded and regular (because resolution of singularities holds over  $F$ , see the discussion following [36, Lemma 4.5]). By definition, the *scdh-topology* is the topology generated by this *cd-structure*. It coincides with the restriction of the *cdh-topology* to  $\text{Sm}/F$  (see [30, Section 5] for more on *cdh-* and *scdh-topology* and their relationship).

We shall be concerned with two notions of weak equivalence for a morphism  $f : \mathcal{E} \rightarrow \mathcal{E}'$  between presheaves of spectra (or simplicial presheaves) on a category  $\mathcal{C}$ . We say that  $f$  is a *global weak equivalence* if  $\mathcal{E}(U) \rightarrow \mathcal{E}'(U)$  is a weak equivalence for each object  $U$ . If  $\mathcal{C}$  is a site, we say that  $f$  is a *local weak equivalence* if it induces an isomorphism on sheaves of stable homotopy groups (or ordinary homotopy groups, in the case of simplicial presheaves).

We are primarily interested in the following model structures on the categories of presheaves of spectra (or simplicial presheaves) on a category  $\mathcal{C}$ ; the terminology is taken from [2]. First, there is the *global projective model structure* for global weak equivalences. A morphism  $f : \mathcal{E} \rightarrow \mathcal{E}'$  is a fibration in this global projective model structure provided  $f(U) : \mathcal{E}(U) \rightarrow \mathcal{E}'(U)$  is a fibration of spectra for each object  $U$  of  $\mathcal{C}$  (we say that weak equivalences and fibrations are defined objectwise); cofibrations are defined by the left lifting property. If  $\mathcal{E} \rightarrow \mathcal{E}'$  is a cofibration then each  $\mathcal{E}(U) \rightarrow \mathcal{E}'(U)$  is a cofibration of spectra, but the converse does not hold.

Second, for a site  $\mathcal{C}$  there is the *local injective model structure* for local weak equivalences. A morphism  $\mathcal{E} \rightarrow \mathcal{E}'$  is a cofibration in this model structure if each  $\mathcal{E}(U) \rightarrow \mathcal{E}'(U)$  is a cofibration; fibrations are defined by the right lifting property. These model structures were studied by Jardine in [19] and [21].

Third, there is the *local projective* (or *Brown-Gersten*) model structure for local weak equivalences. A morphism  $\mathcal{E} \rightarrow \mathcal{E}'$  is a cofibration in this model structure if it is a global projective cofibration; fibrations are defined by the right lifting property.

We warn the reader that our local projective model structure for presheaves is slightly different from (but Quillen-equivalent to) the corresponding model structure for sheaves discussed in [35].

Note that since a cofibration in the local projective model structure is an objectwise cofibration, it is also a cofibration in the local injective model structure. In particular, trivial cofibrations in the local projective structure are also trivial cofibrations in the local injective model structure. It follows from the lifting property that a morphism of presheaves which is a fibration in the local injective model structure is also a fibration in the local projective model structure.

Any local weak equivalence  $\mathcal{E} \rightarrow \mathcal{E}'$  between local projective fibrant presheaves is a global weak equivalence. This useful remark follows from the fact that the identity functor on the category of presheaves of spectra (respectively, simplicial

sets) is a right Quillen functor from the local projective to the global projective model structure and hence preserves weak equivalences between fibrant objects, see [18, Proposition 8.5.7].

Recall that a *fibrant replacement* of  $\mathcal{E}$  in a model category is a trivial cofibration  $\mathcal{E} \rightarrow \mathcal{E}'$  with  $\mathcal{E}'$  fibrant. Even though we don't need it, we remark that for all the model structures we consider, a fibrant replacement can be chosen functorially by the “small object argument” (see [19] for the local injective model structure, and [2] for the projective model structures). We will fix a fibrant replacement functor  $\mathcal{E} \rightarrow \mathbb{H}_{\mathcal{C}}(-, \mathcal{E})$  for the local injective model structure, and we will drop the site from the notation when the topology is clear from the context. Following Thomason [31, p. 532], we write  $\mathbb{H}^n(X, \mathcal{E})$  for  $\pi_{-n}\mathbb{H}(X, \mathcal{E})$ .

**Definition 3.3.** A presheaf of spectra (or simplicial sets)  $\mathcal{E}$  on a site  $\mathcal{C}$  is called *quasifibrant* if the local injective fibrant replacement  $\mathcal{E} \rightarrow \mathbb{H}(-, \mathcal{E})$  is a global weak equivalence, i.e., the map  $\mathcal{E}(U) \rightarrow \mathbb{H}(U, \mathcal{E})$  is a weak equivalence for all  $U$  in  $\mathcal{C}$ .

An important result of [35] is that under certain conditions, presheaves satisfying the Mayer-Vietoris property are precisely quasifibrant presheaves. Note that in that paper, presheaves satisfying the MV-property are called “flasque”.

**Theorem 3.4.** *Let  $\mathcal{C}$  be a category with a complete bounded regular *cd*-structure  $\mathcal{P}$ . Then a presheaf of simplicial sets (or spectra)  $\mathcal{E}$  on  $\mathcal{C}$  is quasifibrant (with respect to the topology induced by  $\mathcal{P}$ ) if and only if  $\mathcal{E}$  satisfies the MV-property for  $\mathcal{P}$ .*

*Proof.* We first prove this for presheaves of simplicial sets. Both properties, satisfying the MV-property and being quasifibrant, are invariant under global weak equivalence. A local injective fibrant presheaf is globally equivalent to a fibrant replacement (as sheaf) of its sheafification. Hence [35, Lemma 4.3] shows that a quasifibrant presheaf satisfies the MV-property. Conversely, [35, Lemma 3.5] asserts that a local weak equivalence between presheaves satisfying the MV-property is a global weak equivalence. As fibrant presheaves satisfy the MV-property, this implies that any local injective fibrant replacement  $\mathcal{E} \rightarrow \tilde{\mathcal{E}}$  is a global weak equivalence if the presheaf satisfies the MV-property; that is, presheaves that satisfy the MV-property are quasifibrant.

The assertion for presheaves of spectra follows from this, because a fibrant spectrum is an  $\Omega$ -spectrum. Indeed, since the properties “quasifibrant” and “satisfying the MV-property” are once again invariant under global weak equivalence, we can as well assume that all our presheaves are global projective fibrant, in particular, they are presheaves of  $\Omega$ -spectra. Now a map of  $\Omega$ -spectra is a (stable) weak equivalence if and only if it is levelwise a weak equivalence of simplicial sets, and a square of  $\Omega$ -spectra is homotopy cartesian if and only if it is levelwise a homotopy cartesian square of simplicial sets. This reduces the proof to the case of presheaves of simplicial sets, as claimed.  $\square$

*Terminology 3.5.* If a presheaf  $\mathcal{E}$  satisfies the equivalent conditions in Theorem 3.4 for a topology  $t$  generated by a complete regular bounded *cd*-structure  $\mathcal{P}$  we say that  $\mathcal{E}$  satisfies *t-descent*, or *descent for the t-topology*.

For later use, we note that the analogues of Theorem 3.4 hold for complexes of (pre)sheaves of abelian groups.

**Definition 3.6.** Let  $\mathcal{C}$  be a category with a *cd*-structure  $\mathcal{P}$ . Let  $A^\bullet$  be a presheaf of cochain complexes on  $\mathcal{C}$ . We say that  $A^\bullet$  is *quasifibrant* for the topology generated

by  $\mathcal{P}$  provided the natural map  $A^\bullet(U) \rightarrow R\Gamma(U, A^\bullet)$  is a quasi-isomorphism for each object  $U$  of  $\mathcal{C}$ . (This property is usually called “pseudoflasque” because it is satisfied by any cochain complex of flasque sheaves.)

We say that  $A^\bullet$  satisfies the MV-property for  $\mathcal{P}$ , if, for any square  $Q \in \mathcal{P}$ , the square of complexes  $A^\bullet(Q)$  is homotopy cocartesian.

The notation is explained by the following “great enlightenment,” due to Thomason [31, 5.32]. If  $\mathcal{E}$  is the presheaf of Eilenberg-Mac Lane spectra associated to  $A^\bullet$ , then  $\mathbb{H}(-, \mathcal{E})$  is the presheaf of Eilenberg-Mac Lane spectra associated to  $R\Gamma(-, A^\bullet)$ , and we have  $\mathbb{H}^n(X, \mathcal{E}) \cong H^n(X, A^\bullet)$ .

With these definitions, the exact analogue of Theorem 3.4 holds for complexes of presheaves.

**Theorem 3.7.** *Suppose that  $\mathcal{C}$  is a category with a complete bounded regular  $cd$ -structure  $\mathcal{P}$ . Then a complex of presheaves  $A^\bullet$  is quasifibrant if and only if it satisfies the MV-property for  $\mathcal{P}$ .*

*Proof.* Reduce to the result for presheaves of spectra by associating Eilenberg-Mac Lane spectra to all complexes.  $\square$

*Terminology 3.8.* Once again, we say that a presheaf  $A^\bullet$  of complexes satisfies  $t$ -descent for a topology  $t$  generated by a  $cd$ -structure  $\mathcal{P}$  if it satisfies the equivalent properties of Theorem 3.7.

**Corollary 3.9.** *Let  $A$  be a presheaf of spectra (respectively, complexes) on  $\text{Sm}/F$ . Then  $A$  satisfies  $scdh$ -descent (3.2.2) if and only if  $A$  satisfies Nisnevich descent and  $A$  satisfies the Mayer-Vietoris property for smooth blow-up squares.*

*Example 3.10.* It follows from [14, exp. Vbis, Corollaire 4.1.6] that singular cohomology satisfies  $cdh$ -descent on the category  $\text{Sch}/\mathbb{C}$ . By this we mean that the presheaf of complexes  $X \mapsto S^*(X^{an})$  assigning to a complex variety  $X$  the singular cochain complex of its associated analytic space satisfies  $cdh$ -descent.

For any presheaf  $A$  on  $\text{Sch}/F$  write  $rA$  for the presheaf on the subcategory  $\text{Sm}/F$  of smooth schemes obtained by restriction. The following lemma is immediate from the observation that  $r(a_{cdh}\pi_*A) = a_{scdh}\pi_*(rA)$ .

**Lemma 3.11.** *Let  $f : A \rightarrow B$  be a morphism of presheaves of spectra on  $\text{Sch}/F$ . If  $f$  is a local weak equivalence in the  $cdh$ -topology then  $rf : rA \rightarrow rB$  is a local weak equivalence in the  $scdh$ -topology.*

We can now prove the main technical result of this section; it will be used in 4.6 to show that infinitesimal  $K$ -theory satisfies  $cdh$ -descent. Recall that the combined  $cd$ -structures on schemes and smooth schemes are complete, bounded and regular, so that Theorem 3.4 applies.

**Theorem 3.12.** *Let  $\mathcal{E}$  be a presheaf of spectra on  $\text{Sch}/F$  such that  $\mathcal{E}$  satisfies excision, is invariant under infinitesimal extension, satisfies Nisnevich descent and satisfies the Mayer-Vietoris property for every blow-up along a regular sequence. Then  $\mathcal{E}$  satisfies  $cdh$ -descent.*

*Proof.* We will prove that  $\mathcal{E}$  is  $cdh$ -quasifibrant. As  $\mathcal{E}$  satisfies Nisnevich descent and the MV-property for blow-ups along a regular sequence (in particular, for a blow-up of a smooth scheme along a smooth subscheme),  $r\mathcal{E}$  satisfies the MV-property for the combined  $cd$ -structure on  $\text{Sm}/F$ . Let  $\mathcal{E} \rightarrow \mathbb{H}_{cdh}(-, \mathcal{E})$  be a local

injective fibrant replacement of  $\mathcal{E}$ . By Theorem 3.4,  $\mathbb{H}_{cdh}(-, \mathcal{E})$  satisfies the MV-property for the combined  $cd$ -structure on  $\text{Sch}/F$ . A fortiori,  $r\mathbb{H}_{cdh}(-, \mathcal{E})$  satisfies the MV-property for the combined  $cd$ -structure on  $\text{Sm}/F$ . By Lemma 3.11, the restriction  $r\mathcal{E} \rightarrow r\mathbb{H}_{cdh}(-, \mathcal{E})$  is a local weak equivalence in the  $scdh$ -topology. As source and target satisfy the MV-property, it is a global weak equivalence on  $\text{Sm}/F$ . In other words, for any smooth scheme  $X$ , the map  $\mathcal{E}(X) \rightarrow \mathbb{H}_{cdh}(X, \mathcal{E})$  is a weak equivalence.

Now we proceed as in [16, Sections 5–6], replacing  $\mathcal{KH}$  by the presheaf  $\mathcal{E}$  everywhere. Specifically, we make the following conclusions. First of all, because  $\mathcal{E}$  satisfies excision, Nisnevich descent and is invariant under infinitesimal extensions,  $\mathcal{E}$  satisfies the MV-property for all closed covers, as well as for finite abstract blow-ups, such as normalizations. (By a *finite abstract blow-up* we mean an abstract blow-up  $p : X' \rightarrow X$  that is a finite morphism.) If  $X$  is a hypersurface inside some smooth  $F$ -scheme  $U$ , we can factor its resolution of singularities locally into a sequence of blow-ups along regular sequences and finite abstract blow-ups; using induction on the dimension of  $X$  and the length of the resolution, we conclude that  $\mathcal{E}(X) \cong \mathbb{H}_{cdh}(X, \mathcal{E})$ . (See [16, Theorem 6.1] for details of the proof in the case where  $\mathcal{E} = \mathcal{KH}$ .) Next, if  $X$  is a local complete intersection, we use induction on the embedding codimension and Mayer-Vietoris for closed covers to prove that once again  $\mathcal{E}(X) \cong \mathbb{H}_{cdh}(X, \mathcal{E})$  in this case (see [16, Corollary 6.2] for details). Finally, the general case follows from this because every integral  $F$ -scheme is locally a component of a complete intersection (see [16, Theorem 6.4] for details).  $\square$

As a typical application of this result, we prove that periodic cyclic homology satisfies  $cdh$ -descent when  $\mathbb{Q} \subseteq F$ . This can also be deduced from Feigin-Tsygan's theorem [8, Theorem 5] (see also [43, 3.4] [6, 6.8]), which identifies HP with crystalline cohomology and from known properties of the latter established in [17]. Note that **HP** here means the presheaf of complexes computing periodic cyclic homology over  $\mathbb{Q}$ .

**Corollary 3.13.** *The presheaf of complexes **HP** on  $\text{Sch}/F$  satisfies  $cdh$ -descent. Hence its associated presheaf of Eilenberg-Mac Lane spectra also satisfies  $cdh$ -descent.*

*Proof.* We have to check the hypotheses of Theorem 3.12 are satisfied by **HP**. The fact that **HP** satisfies excision is in [7, 5.3]; invariance under infinitesimal extensions is proved in [12, Theorem II.5.1]; Nisnevich descent is Theorem 2.9; and Mayer-Vietoris for blow-ups along a regular sequence is Theorem 2.10.  $\square$

*Example 3.14.* Theorem 3.12 applies in particular to prove that homotopy  $K$ -theory  $KH$  satisfies  $cdh$ -descent, as explained in [16]. As a consequence we have the following computation (see [16, Theorem 7.1]).

Suppose that  $X$  is a scheme, essentially of finite type over a field  $F$  of characteristic 0 and such that  $\dim(X) = d$ . Then  $KH_n(X) = 0$  for  $n < -d$  and  $KH_{-d}(X) = H_{cdh}^d(X, \mathbb{Z})$ .

#### 4. DESCENT FOR INFINITESIMAL $K$ -THEORY

In this section, we combine the previous sections to prove (in Theorem 4.6 below) that Cortiñas' infinitesimal  $K$ -theory satisfies  $cdh$ -descent on  $\text{Sch}/F$ , for any field  $F$  of characteristic 0. All variants of cyclic homology are taken over  $\mathbb{Q}$ .

Recall from 2.7 that  $\mathbf{HN}(X)$  is the presheaf of complexes defining negative cyclic homology; we obtain a presheaf of spectra from this by taking the associated Eilenberg-Mac Lane spectrum. There is a Chern character  $\mathcal{K}(X) \rightarrow \mathbf{HN}(X)$  (for a definition, start for example with [39, Section 4], use the Fundamental Theorem 2.12 to extend to non-connective  $K$ -theory and globalize using Zariski descent). Here  $\mathcal{K}(X)$  denotes the non-connective  $K$ -theory spectrum of perfect complexes on  $X$  as in [34, 6.4].

**Definition 4.1.** Let  $X$  be a  $\mathbb{Q}$ -scheme. We define the infinitesimal  $K$ -theory of  $X$ ,  $\mathcal{K}^{inf}(X)$ , to be the homotopy fiber of the Chern character  $\mathcal{K}(X) \rightarrow \mathbf{HN}(X)$ .

The following theorem was proven by Cortiñas in [5]. It verified the “KABI-conjecture” of Geller-Reid-Weibel ([9, 0.1]).

**Theorem 4.2** (Cortiñas).  *$\mathcal{K}^{inf}$  satisfies excision on the category of  $\mathbb{Q}$ -algebras.*

**Theorem 4.3.**  *$\mathcal{K}^{inf}$  satisfies Nisnevich descent.*

*Proof.* Both  $\mathcal{K}$  and  $\mathbf{HN}$  do, by [34, 10.8] and Theorem 2.9.  $\square$

From Theorem 2.10 and Remark 1.6, both  $\mathbf{HN}$  and  $K$ -theory have the Mayer-Vietoris property for any square associated to a blow-up along a regular embedding. This proves the following result.

**Theorem 4.4.**  *$\mathcal{K}^{inf}$  satisfies the Mayer-Vietoris property for every blow-up along a regular embedding.*

Finally, we have the following result, due to Goodwillie, see [13].

**Theorem 4.5** (Goodwillie). *Let  $A$  be a  $\mathbb{Q}$ -algebra and  $I \subset A$  a nilpotent ideal. Then  $\mathcal{K}^{inf}(A, I)$  is contractible. That is,  $\mathcal{K}^{inf}$  is invariant under infinitesimal extension.*

*Proof.* Goodwillie proves (in [13, Theorem II.3.4]) that the Chern character induces an equivalence  $\mathcal{K}(A, I) \rightarrow \mathbf{HN}(A, I)$ . This immediately implies the assertion.  $\square$

**Theorem 4.6.** *The presheaf of spectra  $\mathcal{K}^{inf}$  satisfies cdh-descent.*

*Proof.* This follows from Theorem 3.12, once we observe that the presheaf  $\mathcal{K}^{inf}$  satisfies the conditions given in the theorem. These conditions hold by Theorem 4.2, Theorem 4.4, Theorem 4.3 and Theorem 4.5.  $\square$

## 5. THE OBSTRUCTION TO HOMOTOPY INVARIANCE

We will say that a sequence of presheaves of spectra  $\mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3$  is an (objectwise) *homotopy fibration sequence* provided that for each scheme  $X$ , the sequence of spectra  $\mathcal{E}_1(X) \rightarrow \mathcal{E}_2(X) \rightarrow \mathcal{E}_3(X)$  is weakly equivalent to a fibration sequence (that is, it defines a distinguished triangle in the homotopy category of spectra). We have the following useful observation (cf. [31, 1.35], [19, p. 73], [20, p. 194]): if  $\mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3$  is a homotopy fibration sequence, then  $\mathbb{H}_{cdh}(-, \mathcal{E}_1) \rightarrow \mathbb{H}_{cdh}(-, \mathcal{E}_2) \rightarrow \mathbb{H}_{cdh}(-, \mathcal{E}_3)$  is also a homotopy fibration sequence.

For a presheaf of spectra  $\mathcal{E}$  on  $\text{Sch}/F$ , we will write  $\tilde{C}_j \mathcal{E}$  for the cofiber of the map  $\mathcal{E} \rightarrow \mathcal{E}(- \times \mathbb{A}^j)$ . Since  $\tilde{C}_j \mathcal{E}$  is a direct factor of  $\mathcal{E}(- \times \mathbb{A}^j)$ , the functor  $\mathcal{E} \mapsto \tilde{C}_j \mathcal{E}$  preserves homotopy fibration sequences. If we also use the  $\tilde{C}_j$  notation for presheaves of abelian groups, then we have  $\tilde{C}_j(\pi_r \mathcal{E}) \cong \pi_r(\tilde{C}_j \mathcal{E})$ .

**Lemma 5.1.** *Suppose the presheaf of spectra  $\mathcal{E}$  satisfies descent for the  $cdh$ -topology (or Zariski, or Nisnevich topology). Then so do the presheaves  $\tilde{C}_j\mathcal{E}$ .*

*Proof.* All three topologies are generated by a complete bounded regular  $cd$  structure. By Theorem 3.4, the presheaf  $\mathcal{E}$  satisfies the MV-property, hence so do the presheaves  $\mathcal{E}(- \times \mathbb{A}^j)$ , for all  $j$ ; consequently, the presheaves  $\tilde{C}_j\mathcal{E}$  also satisfy Mayer-Vietoris, that is, they satisfy descent.  $\square$

In particular, the presheaves  $\tilde{C}_j\mathcal{K}^{inf}$  satisfy descent for the  $cdh$ -topology, and the presheaves  $\tilde{C}_j\mathbf{HN}$  satisfy descent for the Zariski topology.

We will say that a presheaf  $\mathcal{E}$  is *contractible* if  $\mathcal{E}(X) \simeq *$  for all  $X$ .

**Lemma 5.2.** *The presheaves  $\mathbb{H}_{cdh}(-, \tilde{C}_j\mathcal{K})$  are contractible for all  $j \geq 1$ .*

*Proof.* As smooth schemes are  $K_m$ -regular for any  $m$ ,  $a_{cdh}\pi_m\tilde{C}_j\mathcal{K} = a_{cdh}\tilde{C}_jK_m = 0$  for all  $j \geq 1$  and all  $m$ . The assertion follows from the general local-to-global spectral sequence in [31, 1.36], applied to the  $cdh$  site.  $\square$

In characteristic zero, we also have the following result, see [12].

**Proposition 5.3.** *For all  $j \geq 1$ , the presheaves  $\tilde{C}_j\mathbf{HP}$  are contractible, and hence so are the presheaves  $\mathbb{H}_{cdh}(-, \tilde{C}_j\mathbf{HP})$ .*

*Proof.* Since  $\mathbb{Q} \subseteq F$ ,  $\mathbf{HP}$  is  $\mathbb{A}^1$ -homotopy invariant on algebras, by [12, III.5.1] (see also [25, E.5.1.4]). As  $\mathbf{HP}$  satisfies Zariski descent, this implies the first assertion. The second assertion is an immediate consequence.  $\square$

**Corollary 5.4.** *For all  $j \geq 1$ , there is a (global) weak equivalence  $\tilde{C}_j\mathbf{HC} \cong \Omega\tilde{C}_j\mathbf{HN}$ , and hence an (objectwise) homotopy fibration sequence:*

$$\tilde{C}_j\mathbf{HC} \rightarrow \tilde{C}_j\mathcal{K}^{inf} \rightarrow \tilde{C}_j\mathcal{K}.$$

*Proof.* Immediate from Proposition 5.3 and the fundamental homotopy fibration sequences  $\mathbf{HN}(X) \rightarrow \mathbf{HP}(X) \rightarrow \Omega^{-2}\mathbf{HC}(X)$ .  $\square$

Applying  $\mathbb{H}_{cdh}(-, -)$  to this homotopy fibration sequence, and using Lemma 5.2, we see that Theorem 4.6 implies the next result.

**Theorem 5.5.** *Let  $j \geq 0$ . There is an (objectwise) homotopy fibration sequence*

$$\tilde{C}_j\mathbf{HC} \rightarrow \mathbb{H}_{cdh}(-, \tilde{C}_j\mathbf{HC}) \rightarrow \tilde{C}_j\mathcal{K}$$

**Lemma 5.6.** *Let  $j \geq 0$ . The Zariski sheaves  $a_{Zar}\pi_n\tilde{C}_j\mathbf{HC}$  (and a fortiori, the  $cdh$ -sheaves  $a_{cdh}\pi_n\tilde{C}_j\mathbf{HC}$ ) vanish for all  $n < 0$ .*

*Proof.* For any ring  $A$ ,  $\mathbf{HC}(A)$  is  $-1$ -connected. Therefore  $\tilde{C}_j\mathbf{HC}_n(A) = 0$  for  $n < 0$ . This implies the assertion.  $\square$

*Remark 5.7.* The vanishing range in 5.6 is best possible, because  $a_{Zar}\pi_0\tilde{C}_j\mathbf{HC}$  is  $\tilde{C}_j\mathcal{O}$ .

**Corollary 5.8.** *For all  $m > d = \dim(X)$  and all  $j$ ,  $\mathbf{HC}_{-m}(X \times \mathbb{A}^j) = \mathbb{H}_{cdh}^m(X \times \mathbb{A}^j, \mathbf{HC}) = 0$ . Moreover,  $\mathbb{H}_{Zar}^d(X, \tilde{C}_j\mathbf{HC}) = H_{Zar}^d(X, \tilde{C}_j\mathcal{O})$  and  $\mathbb{H}_{cdh}^d(X, \tilde{C}_j\mathbf{HC}) = H_{cdh}^d(X, a_{cdh}\tilde{C}_j\mathcal{O})$ .*

*Proof.* Since both the Zariski and  $cdh$  cohomological dimension of  $X$  are at most  $d$ , these follow from 5.6 via the Leray spectral sequences  $H^p(X, a_t\pi_{-q}\tilde{C}_j\mathbf{HC}) \Rightarrow H^{p+q}(X, \tilde{C}_j\mathbf{HC})$ .  $\square$

From this we can already conclude the following weak form of Conjecture 0.1.

**Corollary 5.9.** *Let  $F$  be a field of characteristic 0, and  $X$  a  $d$ -dimensional scheme, essentially of finite type over  $F$ . Then  $X$  is  $K_n$ -regular and  $K_n(X) = 0$  for all  $n < -d$ .*

*Proof.* The first part is an immediate consequence of Theorem 5.5 and Corollary 5.8. The second part follows from the first using the spectral sequence  $K_q(X \times \mathbb{A}^p) \Rightarrow KH_{p+q}(X)$ .  $\square$

**Corollary 5.10.** *If  $\dim(X) = d$ , there is an exact sequence for every  $j \geq 1$ :*

$$H_{Zar}^d(X, a_{Zar}\tilde{C}_jHC_0) \rightarrow H_{cdh}^d(X, a_{cdh}\tilde{C}_jHC_0) \rightarrow \tilde{C}_jK_{-d}(X) \rightarrow 0.$$

*Proof.* Combine 5.5 and 5.8.  $\square$

## 6. $cdh$ -COHOMOLOGY OF COHERENT SHEAVES AND THE $K$ -DIMENSION CONJECTURE

Most of this section will be taken up by the proof of the next result.

**Theorem 6.1.** *Let  $X$  a  $F$ -scheme, essentially of finite type, and of dimension  $d$ . Then the natural homomorphism, induced by the change of topology,*

$$H_{Zar}^d(X, \mathcal{O}_X) \longrightarrow H_{cdh}^d(X, a_{cdh}\mathcal{O}_X)$$

*is surjective.*

Before proving this theorem, we show how Theorem 6.1 and Corollary 5.10 imply the  $K$ -dimension Conjecture 0.1.

**Theorem 6.2.** *Let  $F$  be a field of characteristic 0 and  $X$  be an  $F$ -scheme, essentially of finite type and of dimension  $d$ . Then  $X$  is  $K_{-d}$ -regular and  $K_n(X) = 0$  for  $n < -d$ . Moreover,  $K_{-d}(X) \cong H_{cdh}^d(X, \mathbb{Z})$ .*

*Proof.* Fix  $j \geq 1$  and let  $V_j$  denote the  $F$ -vector space  $F[t_1, \dots, t_j]/F$ . Since  $HC_0(A) = A$  for any commutative algebra  $A$ ,  $\tilde{C}_jHC_0(A) = A \otimes_F V_j$ . Hence  $a_{Zar}\tilde{C}_jHC_0 \cong \mathcal{O}_X \otimes_F V_j$  and  $a_{cdh}\tilde{C}_jHC_0 \cong a_{cdh}\mathcal{O}_X \otimes_F V_j$ . Therefore Theorem 6.1 and Corollary 5.10 imply that  $X$  is  $K_{-d}$ -regular.

The remaining assertions follow from the calculation of  $KH_*(X)$  in Example 3.14, and the spectral sequence  $K_q(X \times \mathbb{A}^p) \Rightarrow KH_{p+q}(X)$ .  $\square$

The proof of Theorem 6.1 will be in two parts: First, we will prove a stronger result for smooth  $X$ . The second part is the proof for a general  $X$ . To simplify notation, for any topology  $t$  and presheaf  $A$ , we write  $H_t^*(X, A)$  for  $H_t^*(X, a_t A)$ . We write  $\mathcal{O}$  for the presheaf  $X \mapsto \mathcal{O}_X(X)$ , and  $R\Gamma_t(X, \mathcal{O})$  for a functorial model for the total right derived functor of the global sections functor  $X \mapsto a_t\mathcal{O}(X)$ .

**Proposition 6.3.** *Let  $X$  be a smooth  $F$ -scheme. Then  $\mathcal{O}(X) \cong a_{cdh}\mathcal{O}(X)$  and the natural homomorphism*

$$H_{Zar}^*(X, \mathcal{O}) \longrightarrow H_{cdh}^*(X, \mathcal{O})$$

*is an isomorphism.*

*Proof.* We need to show that the natural map  $\mathrm{R}\Gamma_{\mathrm{Zar}}(X, \mathcal{O}) \rightarrow \mathrm{R}\Gamma_{\mathrm{cdh}}(X, \mathcal{O})$  is a quasi-isomorphism. Since the target satisfies *cdh*-descent 3.8, this amounts to showing that the presheaf of complexes  $X \mapsto \mathrm{R}\Gamma_{\mathrm{Zar}}(X, \mathcal{O})$  on  $\mathrm{Sm}/F$  satisfies the conditions of Corollary 3.9. First of all, it is classical that  $\mathrm{R}\Gamma_{\mathrm{Zar}}(X, \mathcal{O}) \cong \mathrm{R}\Gamma_{\mathrm{Nis}}(X, \mathcal{O})$ , which implies that this presheaf satisfies Nisnevich descent (it sends elementary Nisnevich squares to homotopy cocartesian squares). Hence, it suffices to show that  $\mathrm{R}\Gamma_{\mathrm{Zar}}(-, \mathcal{O})$  transforms smooth blow-up squares into homotopy cocartesian squares. To this end, let  $X$  be a smooth scheme,  $Y \subset X$  a smooth closed subscheme,  $p : X' \rightarrow X$  the blow-up along  $Y$  and  $j : Y' \subset X'$  the exceptional divisor; write  $q : Y' \rightarrow Y$  for the restriction. Then we need to show that the natural map

$$\mathrm{Cone}\left(\mathrm{R}\Gamma_{\mathrm{Zar}}(X, \mathcal{O}) \rightarrow \mathrm{R}\Gamma_{\mathrm{Zar}}(X', \mathcal{O})\right) \longrightarrow \mathrm{Cone}\left(\mathrm{R}\Gamma_{\mathrm{Zar}}(Y, \mathcal{O}) \rightarrow \mathrm{R}\Gamma_{\mathrm{Zar}}(Y', \mathcal{O})\right)$$

is a quasi-isomorphism. In fact, both of those cones are 0. Indeed,  $\mathcal{O}_X \rightarrow \mathrm{R}p_*p^*\mathcal{O}_X$  is a quasi-isomorphism by [32, Lemme 2.3(a)], and the usual computation of cohomology of projective space shows that  $\mathcal{O}_Y \rightarrow \mathrm{R}q_*\mathrm{L}q^*\mathcal{O}_Y = \mathrm{R}q_*\mathcal{O}_{Y'}$  is also a quasi-isomorphism. Hence we have a homotopy cartesian square, or alternatively,

$$\mathrm{R}\Gamma_{\mathrm{Zar}}(X, \mathcal{O}) \rightarrow \mathrm{R}\Gamma_{\mathrm{Zar}}(X', \mathcal{O}) \times \mathrm{R}\Gamma_{\mathrm{Zar}}(Y, \mathcal{O}) \rightarrow \mathrm{R}\Gamma_{\mathrm{Zar}}(Y', \mathcal{O})$$

is a homotopy fibration sequence. It follows that  $\mathrm{R}\Gamma_{\mathrm{Zar}}(-, \mathcal{O})$  satisfies *scdh* descent, and in particular that  $\mathcal{O} \cong a_{\mathrm{cdh}}\mathcal{O}$  on  $\mathrm{Sm}/F$ .  $\square$

Neither the global sections of  $\mathcal{O}$  nor the higher Zariski cohomology of  $\mathcal{O}$  satisfies *cdh*-descent for non-smooth schemes; this fails for example when  $X$  is a cusp. Nevertheless, we have the following partial result, which suffices for Theorem 6.1.

**Lemma 6.4.** *Let  $X$  be a reduced affine Noetherian scheme and  $p : X' \rightarrow X$  a proper morphism such that all the fibers of  $p$  have dimension at most  $d-1$ . Let  $j : Y' \subset X'$  be a closed subscheme. Then the restriction map  $H_{\mathrm{Zar}}^{d-1}(X', \mathcal{O}_{X'}) \rightarrow H_{\mathrm{Zar}}^{d-1}(Y', \mathcal{O}_{Y'})$  is surjective.*

*Proof.* The theorem on formal functions implies that  $R^d p_* \mathcal{F} = 0$  for any quasicoherent sheaf  $\mathcal{F}$  on  $X'$ . Hence the functor  $R^{d-1} p_*$  is right exact on quasicoherent sheaves. In particular,  $R^{d-1} p_* \mathcal{O}_{X'} \rightarrow R^{d-1} p_* j_* \mathcal{O}_{Y'}$  is onto. Because  $X$  is affine, this proves the assertion.  $\square$

For legibility, we will write  $a$  for the natural morphism of sites from the *cdh*-site to the Zariski site on  $\mathrm{Sch}/F$ . If  $\mathcal{F}$  is a Zariski sheaf, then  $a^*\mathcal{F}$  is the same as  $a_{\mathrm{cdh}}\mathcal{F}$ .

**Lemma 6.5.** *For any scheme  $X$  of finite type over  $F$ , the complex of Zariski sheaves  $\mathrm{Ra}_* a^* \mathcal{O}|_{X_{\mathrm{Zar}}}$  has coherent cohomology sheaves.*

*Proof.* If  $X$  is smooth then the assertion is an immediate consequence of Proposition 6.3. We prove the general case by induction on the dimension of  $X$ . If  $\dim(X) = 0$ , then  $X = \mathrm{Spec}(A)$  for some Artinian ring  $A$  and  $\mathrm{Ra}_* a^* \mathcal{O} \cong \mathcal{O}^{\mathrm{red}}$ , which is a coherent sheaf. Now suppose  $d = \dim(X) > 0$ . Let  $p : X' \rightarrow X$  be a resolution of singularities,  $i : Y \subset X$  the singular set and  $j : Y' \subset X'$  the exceptional divisor. Because  $\mathrm{Ra}_*$  commutes with  $\mathrm{R}f_*$  for every morphism  $f$  in  $\mathrm{Sch}/F$ , we have a distinguished triangle of complexes of sheaves of  $\mathcal{O}_X$ -modules on  $X_{\mathrm{Zar}}$ :

$$\mathrm{Ra}_* a^* \mathcal{O} \rightarrow \mathrm{R}p_* \mathrm{Ra}_* a^* \mathcal{O} \times \mathrm{R}i_* \mathrm{Ra}_* a^* \mathcal{O} \rightarrow \mathrm{R}(pj)_* \mathrm{Ra}_* a^* \mathcal{O}.$$

The second and third terms in this triangle have coherent cohomology sheaves; this follows from induction on the dimension, the assertion for the smooth  $X'$ , and the

fact that proper morphisms have coherent direct images. Hence, the first term has coherent cohomology sheaves, too (see [15, Exposé I, Cor. 3.4]).  $\square$

*Proof of Theorem 6.1.* We proceed by induction on the dimension. If  $d = 0$ , then  $X = \text{Spec}(A)$  for some Artinian ring  $A$ , and  $H^0(X, \mathcal{O}) = A \rightarrow A^{red} = H_{cdh}^0(X, \mathcal{O})$  is surjective. Now assume that we have shown the assertion for all schemes of dimension less than  $d > 0$ .

We claim that it suffices to prove the assertion when  $X$  is affine, or indeed for local  $X$  of dimension  $d$ . To see this, suppose that  $X$  is any  $d$ -dimensional scheme, essentially of finite type over  $F$ . We have a Leray spectral sequence

$$H_{Zar}^p(X, R^q a_* a^* \mathcal{O}) \implies H_{cdh}^{p+q}(X, \mathcal{O}).$$

Fix  $q > 0$  and consider the stalk of  $R^q a_* a^* \mathcal{O}$  at a point  $x \in X$  of codimension  $c$  (that is, where the local ring  $\mathcal{O}_{X,x}$  has Krull dimension  $c$ ). By assumption, the stalk is zero if  $q \geq c$ . Since the sheaf  $R^q a_* a^* \mathcal{O}$  is coherent by Lemma 6.5, this implies that  $R^q a_* a^* \mathcal{O}$  is supported on a closed subscheme of codimension  $> q$ , i.e., of dimension  $< d - q$ . This implies that  $H_{Zar}^p(X, R^q a_* a^* \mathcal{O}) = 0$  provided  $p + q \geq d$  and  $q > 0$ . Hence the Leray spectral sequence degenerates enough to show that  $H_{Zar}^d(X, a_* a^* \mathcal{O}) \rightarrow H_{cdh}^d(X, \mathcal{O})$  is surjective.

Consider the cokernel  $\mathcal{F}$  of the adjunction map  $\mathcal{O} \rightarrow a_* a^* \mathcal{O}$ , which is coherent by 6.5. It vanishes on an open dense subset of  $X$  (namely, on the complement of the singular set of  $X^{red}$ , by 6.3), so  $\mathcal{F}$  is supported in dimension  $< d$  and hence  $H_{Zar}^d(X, \mathcal{F}) = 0$ . Since  $H_{Zar}^d(X, -)$  is right exact,  $H_{Zar}^d(X, \mathcal{O}) \rightarrow H_{Zar}^d(X, a_* a^* \mathcal{O})$  must be a surjection. This establishes our claim. To summarize, it suffices to assume the result true in dimension  $< d$  and prove it for affine schemes of dimension  $d$ . To simplify matters, we can also assume that  $X$  is reduced. Indeed, since  $H_{Zar}^d(X, -)$  is right exact, the map  $H_{Zar}^d(X, \mathcal{O}) \rightarrow H_{Zar}^d(X^{red}, \mathcal{O})$  is surjective, and  $H_{cdh}^d(X, \mathcal{O}) = H_{cdh}^d(X^{red}, \mathcal{O})$ .

Let  $X$  be an affine  $d$ -dimensional scheme, and choose a resolution of singularities  $p : X' \rightarrow X$ . Let  $Y \subset X$  be the singular subscheme and  $Y' \subset X'$  the exceptional divisor. Since  $Y$  and  $Y'$  have smaller dimension,  $H_{cdh}^d(Y, \mathcal{O}) = H_{cdh}^d(Y', \mathcal{O}) = 0$  for cohomological dimension reasons [30]. Furthermore,  $p$  is proper and has fibers of dimension  $< d$ ; because  $X$  is affine, this implies that  $H_{Zar}^d(X', \mathcal{O}) = 0$ , by the theorem on formal functions. Since  $X'$  is smooth, we conclude that  $H_{cdh}^d(X', \mathcal{O}) = 0$  by Proposition 6.3. Now the long exact sequence in  $cdh$ -cohomology for the abstract blow-up  $p$  gives a diagram with exact top row.

$$\begin{array}{ccccccc} H_{cdh}^{d-1}(Y, \mathcal{O}) \times H_{cdh}^{d-1}(X', \mathcal{O}) & \longrightarrow & H_{cdh}^{d-1}(Y', \mathcal{O}) & \longrightarrow & H_{cdh}^d(X, \mathcal{O}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow_{\text{onto}} & & \\ & & H_{Zar}^{d-1}(X', \mathcal{O}) & \xrightarrow{\text{onto}} & H_{Zar}^{d-1}(Y', \mathcal{O}) & & \end{array}$$

The right vertical map in this diagram is surjective by induction. As the fibers of  $p$  have dimension less than  $d$ , Lemma 6.4 implies that the bottom horizontal map is surjective. Therefore  $H_{cdh}^{d-1}(X', \mathcal{O}) \rightarrow H_{cdh}^{d-1}(Y', \mathcal{O})$  is also surjective and hence  $H_{cdh}^d(X, \mathcal{O}) = 0$ . This finishes the induction step and the proof of Theorem 6.1.  $\square$

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