ÉTALE CHERN CLASSES AT THE PRIME 2

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Abstract. Let A be a commutative ring and q a power of 2. We investigate the étale Chern classes

$$c_{ik} : K_n(A;\mathbb{Z}/q) \longrightarrow H^k_{et}(A;\mu_q^{\otimes i}),$$

which are defined whenever $i \geq 1$ and $n+k=2i$. These are group homomorphisms except when $n=2$ and q is even. The usual product formula for $c_{ik}(\{a,b\})$ remains valid, except when $q=2$ and $i\geq 3$, when there is a correction term.

Introduction.

Let A be a commutative ring. For any integer q, let $K_*(A;\mathbb{Z}/q)$ denote the mod q K-theory groups of A, and let μ_q denote the étale sheaf of q^{th} roots of unity on $Spec(A)$. In his 1979 paper [S], Soulé constructed étale Chern classes

$$c_{ik} : K_n(A;\mathbb{Z}/q) \to H^k_{et}(A;\mu_q^{\otimes i}), \qquad n+k=2i, i\geq 1$$

for any q, and proved some interesting things about the c_{ik} when q is odd. Recently, there has been an interest in these classes when $q=2^\nu$. The purpose of this paper is to describe what happens, and to provide correction terms for Soulé's results in this case.

Our most dramatic results may be illustrated with the mod 2 K-theory of the real numbers $\mathbb{R}$. It is well-known [Wp, p.396] [AT,p.79] that $K_2(\mathbb{R})\otimes\mathbb{Z}/2\cong\mathbb{Z}/2$ is a subgroup of $K_2(\mathbb{R};\mathbb{Z}/2)\cong\mathbb{Z}/4$. Let $\beta\in K_2(\mathbb{R};\mathbb{Z}/2)$ be a generator, so that 2β is the image of the nonzero element $\{-1,-1\}$ of $K_2(\mathbb{R})\otimes\mathbb{Z}/2\cong\mathbb{Z}/2$.

(0.1) The target of the Chern class c_{22} is the Brauer group $H^2_{et}(\mathbb{R};\mu_2^{\otimes i}) = Br(\mathbb{R}) \cong \mathbb{Z}/2$. One would think that the map $c_{22}:\mathbb{Z}/4\to\mathbb{Z}/2$ would send $2\beta = \{-1,-1\}$ to zero. Yet we know (say from [Sh]) that c_{22} is the Galois symbol on K_2; since $c_{22}(\{-1,-1\})$ is represented by the class of the quaternions $\mathbb{H}$ in the Brauer group of $\mathbb{R}$, it is nonzero! The resolution of this paradox is that c_{22} *is not a group homomorphism!*

(0.2) Suslin asserts in [Sus2, p.13] that Soulé's product formula remains valid for $q=2^\nu$, provided that we consider products of the type

$$K_m(A)\times K_n(A;\mathbb{Z}/q)\to K_{m+n}(A;\mathbb{Z}/q).$$

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On the other hand, we will see that

$$c_{33}(\{-1,\beta\}) = \{-1,-1,-1\} \neq 0 \qquad \text{in } H^3_{et}(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2;$$
$$c_{44}(\{-1,-1,\beta\}) = \{-1,-1,-1,-1\} \neq 0 \qquad \text{in } H^4_{et}(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

In both cases, the product formula gives zero. The resolution of this paradox is that *the product formula requires a correction term when* $i \geq 3$, $m = 2$ *and coefficients are taken in* $\mathbb{Z}/2$. Fortunately, the only applications of Suslin's assertion to date (to our knowledge) have been to use the correct formula for the Chern class c_{21} on $K_3(F;\mathbb{Z}/q)$: if $\zeta = \partial\beta$ is a q^{th} root of unity and $f \in F^*$,

$$c_{21}(\{f,\beta\}) = -\{f,\zeta\} \qquad \text{in } H^1_{et}(F;\mu_q^{\otimes 2}).$$

See [Sus2, 3.2], [MS, 7.1, 7.2], [L, 4.5, 5.1] as well as [K] and [PW, 4.5].

(0.3) Consider the noncommutative diagram

$$\begin{array}{ccc} K_3(\mathbb{R}(t);\mathbb{Z}/q) & \xrightarrow{\partial} & K_2(\mathbb{R};\mathbb{Z}/2) \\ {\scriptstyle c_{33}}\downarrow & & {\scriptstyle c_{22}}\downarrow \\ H^3(\mathbb{R}(t);\mu_q^{\otimes 3}) & \xrightarrow{\partial} & H^2(\mathbb{R};\mu_q^{\otimes 2}) \end{array}$$

Here the maps ∂ come from localization sequences for the discrete valuation ring $\mathbb{R}[t]_{(t)}$. If q is odd, Soulé proved that $\partial c_{33} = -2c_{22}\partial$. This formula does not hold for $q = 2$, because in fact:

$$\partial c_{33}(\{t,\beta\}) = \{-1,-1\} = c_{22}(\partial\{t,\beta\}) \neq 0.$$

Here the difficulty is that Soulé's proof uses the product formula for c_{33}. We will see that if $4|q$ then Soulé's formula for localization sequences continues to hold:

$$\partial c_{ik} = (1-i)c_{i-1,k-1}\partial.$$

(0.4) Let $A = \mathbb{R}[x,y]/(x^2+y^2 = 1)$ be the coordinate ring of the circle. The natural continuation of the diagram of (0.3), described in 4.3.1, would be the diagram

$$\begin{array}{ccc} K_2(\mathbb{R};\mathbb{Z}/2) & \xrightarrow{\tau} & K_2(A;\mathbb{Z}/2) \\ \downarrow{\scriptstyle c_{22}} & & \downarrow{\scriptstyle c_{34}} \\ H^2(\mathbb{R};\mathbb{Z}/2) & \longrightarrow & H^4(A;\mathbb{Z}/2). \end{array}$$

Here τ is the transfer associated to a point on the circle. By now, the reader should not be surprised to know that something is amiss. In fact $c_{34}\tau(\beta) \neq 0$ but $c_{22}(\beta) = 0$. What is surprising is not that there is a correction term for the product formula for $c_{34}(\{a,\beta\})$ with $a \in K_0(A)$ – there is – but that the product formula given in [S] is actually off by a factor of $\pm i_1$.

(0.5) Let F be a number field and let $w_i = w_i(F)$ denote the order of the finite cyclic group $H^0_{et}(F;\mathbb{Q}/\mathbb{Z}(i))$. Harris and Segal found a natural cyclic direct summand E^i of $K_{2i-1}(F)$ whose order was either w_i, $2w_i$ or $w_i/2$. We define the (real and complex) e-invariants and show that they detect all of E^i; this eliminates some of the uncertainty in the order of the summand E^i.

When F is totally imaginary and $i \neq 0 \pmod 4$, we show that the order of E^i is always w_i. If $4|i$ the order is either w_i or $\frac{1}{2}w_i$. (See 6.5 and 6.6.)

When $F = \mathbb{Q}$ and $i = 2+4k$, the uncertainty in $E^i \subseteq K_{8k+3}(\mathbb{Q})$ was clarified in [Br]: the order is $2w_i$. We clarify things even further: if $i = 4k$ the order is w_i, because $E^i = Im(J)_{8k-1}$. We expect, but cannot quite prove, that the same is true for any number field F having a real embedding (see 6.8).

(0.6) Let A be the ring of $\mathbb{Z}[\frac{1}{2}]$-integers in a totally real number field F. Lichtenbaum's conjectures for the prime $\ell = 2$ involve the 2-Sylow subgroups $H^k_{et}(A;W^{(i)})$ of the groups $H^k_{et}(A;\mathbb{Q}/\mathbb{Z}(i))$. We observe that whenever $k+i$ is odd ($k \geq 3$ and $i \geq 1$) these groups are $(\mathbb{Z}/2)^{r_1} \neq 0$. This quantifies how periodically false the conjecture (L1) of [Licht] was at $\ell = 2$. On the other hand we show in 7.3 that (L1) holds for totally imaginary number fields F, in the sense that for all $i \geq 2$ we have

$$H^2_{et}(A;W^{(i)}) = H^2_{et}(F;W^{(i)}) = 0.$$

The organization of this paper parallels the organization of Soulé's paper [S]. In §1 we study the mod q Hurewicz maps $h_n : \pi_n(X;\mathbb{Z}/q) \to H_n(X;\mathbb{Z}/q)$. We show that h_2 is not always a homomorphism when q is even. We also discuss the behavior of the Hurewicz map with respect to products. Using this, §2 introduces Soulé's chern classes on $K_n(A;\mathbb{Z}/q)$. These are homomorphisms except when $n = 2$ and q is even, when there is a correction term. This resolves paradox (0.1) above.

In §3 we discuss (0.2), providing the correction term for Soulé's product formulas (3.1) and (3.10). This allows us to extend Soulé's results on localization and transfer in §4, resolving (0.3) and (0.4). Since it fits naturally here, we give a short proof, due to Soulé, that if A is the ring of integers A in a global field F then $K_n(A) = K_n(F)$ for all odd n. This extension of [S, III.3] is well-known to the experts, and written down in [Sherm] and [S2].

In §5 we study the image of the Chern classes, using [MS], [MS2] and [L] to update the results of [S, IV]. Of course at $\ell = 2$ the image of the Chern class c_{ik} tends to have index $(i-1)!$, so it is less interesting than in the cases studied by Soulé. When $i = 0$ we can effectively divide the Chern class c_{i0} by $(i-1)!$ to obtain the e-invariant, studied in §6. The relation with the Harris-Segal summands E^i of (0.5) is analyzed.

In §7 we address point (0.6). With the exception of 7.3, which is the 2-primary analogue of Soulé's proof of (L1) for $\ell = 2$, most of this material should be unsurprising to the experts.

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1. The Hurewicz "morphism" at the prime 2.

Fix an integer q and let P^n denote the n-dimensional mod q Moore space. Recall that the topological space P^n is defined only for $n \geq 2$, and may be obtained from the sphere S^{n-1} by attaching an n-cell by a map of degree q. The mod q homotopy "groups" of a pointed space X are defined by

$$\pi_n(X;\mathbb{Z}/q) = [P^n, X], \qquad n \geq 2.$$

If $n \geq 3$ then $P^n = \Sigma P^{n-1}$ is a co-H-space and therefore $\pi_n(X;\mathbb{Z}/q)$ is a group. If X is an H-space, then $\pi_2(X;\mathbb{Z}/q)$ is also a group. In either case, the quotient map $P^n \to S^n$ yields a homomorphism $\pi_n(X) \to \pi_n(X;\mathbb{Z}/q)$.

The *Bockstein map* $\partial : \pi_n(X;\mathbb{Z}/q) \to \pi_{n-1}(X)$ is obtained by composing with the inclusion $S^{n-1} \subset P^n$. Again supposing either that $n \geq 3$ or that $n = 2$ and X is an H-space, ∂ is a group homomorphism and its image is the q-torsion subgroup of $\pi_{n-1}(X)$. This all fits together into a universal coefficient theorem [N, 1.4]:

$$0 \to \pi_n(X) \otimes \mathbb{Z}/q \to \pi_n(X;\mathbb{Z}/q) \xrightarrow{\partial} \mathrm{Tor}(\pi_{n-1}(X), \mathbb{Z}/q) \to 0.$$

The *mod q Hurewicz map* is the set map

$$h_n : \pi_n(X;\mathbb{Z}/q) \longrightarrow H_n(X;\mathbb{Z}/q)$$

which sends $\alpha \in [P^n, X]$ to $\alpha_*(\varepsilon_n)$, the image of the canonical generator $\varepsilon_n \in H_n(P^n;\mathbb{Z}/q)$. By definition, ε_n corresponds to $1 \in \mathbb{Z}/q$ under the natural isomorphism $H_n(P^n;\mathbb{Z}/q) \cong H_n(S^n;\mathbb{Z}/q) \cong \mathbb{Z}/q$ induced by the quotient map $P^n \to S^n$. If $n \geq 3$ it is easy to see that h_n is a group homomorphism [N, 3.3].

Now suppose that $n = 2$ and that X is an H-space. If q is odd, h_2 is also a homomorphism (see 1.2 below). When q is even, h_2 need not be a group homomorphism, because the image of h_2 may not be primitive when q is even. To see this, write H_* for $H_*(\ ;\mathbb{Z}/q)$, so that the diagonal map Δ_* goes from $H_*(X)$ to $H_*(X \times X) \cong H_*(X) \otimes H_*(X)$. An element $h \in H_*$ is called *primitive* if $\Delta_*(h) = 1 \otimes h + h \otimes 1$. If $n \geq 3$ the image of h_n consists of primitive elements [N, 3.4].

Let $h_1 : \pi_1(X) \to H_1(X)$ denote the integral Hurewicz map, reduced modulo q. We shall need to evaluate $h_1(\partial\alpha)$, where $\alpha \in \pi_2(X;\mathbb{Z}/q)$. If $\tilde{\varepsilon}_1$ is the generator of $H_1(S^1)$ and $\iota : S^1 \subset P^2$ is the standard inclusion, then $\varepsilon_1 = \iota_*(\tilde{\varepsilon}_1)$ is a canonical generator of the group $H_1(P^2) \cong \mathbb{Z}/q$. We have:

$$h_1(\partial\alpha) = h_1(\alpha\iota) = (\alpha\iota)_*(\tilde{\varepsilon}_1) = \alpha_*(\varepsilon_1).$$

Proposition 1.1. *Suppose that $\alpha \in \pi_2(X;\mathbb{Z}/q)$. When q is odd, the element $h_2(\alpha) \in H_2(X;\mathbb{Z}/q)$ is primitive. When q is even then*

$$\Delta_*(h_2\alpha) = (h_2\alpha) \otimes 1 + 1 \otimes (h_2\alpha) + (\frac{q}{2})h_1(\partial\alpha) \otimes h_1(\partial\alpha).$$

In particular, the element $\varepsilon_2 = h_2(id)$ of $H_2(P^2)$ is not primitive for <u>any</u> even q, because $H_2(P^2 \otimes P^2) \cong (\mathbb{Z}/q)^3$ on basis $\{\varepsilon_2 \otimes 1, 1 \otimes \varepsilon_2, \varepsilon_1 \otimes \varepsilon_1\}$, and

$$\Delta_*(\varepsilon_2) = \varepsilon_2 \otimes 1 + 1 \otimes \varepsilon_2 + (\frac{q}{2})\varepsilon_1 \otimes \varepsilon_1.$$

Proof. Since $\Delta_*\alpha_* = (\alpha_* \otimes \alpha_*)\Delta_*$, it suffices to assume that $X = P^2$ and compute $\Delta_*(\varepsilon_2)$. Set $G = \mathbb{Z}/q$ with generator t. The canonical embedding of P^2 in the classifying space BG is an injection on H_*, so it suffices to compute the diagonal map from $H_*(G) = H_*(BG)$ to $H_*(G \times G)$. Let $C_*(G) \xrightarrow{\varepsilon} \mathbb{Z}$ be the G-projective resolution of $\mathbb{Z}$:

$$\cdots \mathbb{Z}G \xrightarrow{1-t} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{1-t} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

so that $C_*(G) \otimes C_*(G)$ is a $(G \times G)$-projective resolution of $\mathbb{Z}$. To compute Δ_* we need a G-invariant chain map $\Delta : C_*(G) \longrightarrow C_*(G) \otimes C_*(G)$ lifting the identity on $\mathbb{Z}$. If e_i is a generator of $C_i(G)$, we may find a lifting with $\Delta(e_0) = e_0 \otimes e_0$, $\Delta(e_1) = e_1 \otimes e_0 + te_0 \otimes e_1$ and

$$\Delta(e_2) = e_2 \otimes e_0 + e_0 \otimes e_2 + \sum_{i<j} t^i e_1 \otimes t^j e_1.$$

(We leave it as a combinational exercise for the reader to check that $\partial\Delta(e_1) = e_0 \otimes e_0 - te_0 \otimes te_0$ and $\partial\Delta(e_2) = \Delta(Ne_1)$.) Passing to homology yields the calculation:

$$\Delta_*(\varepsilon_2) = \varepsilon_2 \otimes 1 + 1 \otimes \varepsilon_2 + \binom{q}{2}\varepsilon_1 \otimes \varepsilon_1.$$

Finally, observe that if q is odd then $\binom{q}{2} \equiv 0 \pmod q$, while if q is even then $\binom{q}{2} \equiv q/2 \pmod q$.

In order to state the next result, write $\bullet$ for the homology product on the ring $H_*(X;\mathbb{Z}/q)$. If $\alpha, \beta \in \pi_2(X;\mathbb{Z}/q)$, write $\{\partial\alpha, \partial\beta\}$ for $\mu(\partial\alpha \wedge \partial\beta)$. Note that

$$h_2(\{\partial\alpha, \partial\beta\}) = h_1(\partial\alpha) \bullet h_1(\partial\beta) = \alpha_*(\varepsilon_1) \bullet \beta_*(\varepsilon_1)$$

because the integral Hurewicz homomorphisms commute with products.

Theorem 1.2. *Let X be an H-space. If q is odd, the Hurewicz map*

$$h_2 : \pi_2(X;\mathbb{Z}/q) \longrightarrow H_2(X;\mathbb{Z}/q)$$

is a group homomorphism. If q is even we have

$$h_2(\alpha + \beta) = h_2(\alpha) + h_2(\beta) + \frac{q}{2} h_2(\{\partial\alpha, \partial\beta\})$$

Proof. $h_2(\alpha + \beta)$ is the image of ε_2 under the composition

$$H_2(P^2) \xrightarrow{\Delta_*} H_2(P^2 \times P^2) \xrightarrow{(\alpha\times\beta)_*} H_2(X \times X) \xrightarrow{\mu_*} H_2(X).$$

By the following proposition, we compute, with $\lambda = \binom{q}{2}$:

$$\begin{aligned} h_2(\alpha + \beta) &= \mu_*(\alpha \times \beta)_*(\varepsilon_2 \otimes 1 + 1 \otimes \varepsilon_2 + \lambda\varepsilon_1 \otimes \varepsilon_1) \\ &= \mu_*\{\alpha_*(\varepsilon_2) \otimes 1 + 1 \otimes \beta_*(\varepsilon_2) + \lambda\alpha_*(\varepsilon_1) \otimes \beta_*(\varepsilon_1)\} \\ &= h_2(\alpha) + h_2(\beta) + \lambda\alpha_*(\varepsilon_1) \bullet \beta_*(\varepsilon_1). \end{aligned}$$

Example 1.3. Let X be either of the H-spaces BO or $\mathbb{RP}^\infty$. It is well known [AT, 1.1] that $\pi_2(X;\mathbb{Z}/2) \cong \mathbb{Z}/4$ on the standard inclusion $\iota : P^2 \subset X$ and that $h_2(\iota) = \varepsilon_2$. Since $h_2(0) = 0$, the theorem gives us

$$h_2(2\iota) = 2h_2(\iota) + \varepsilon_1 \bullet \varepsilon_1 = \varepsilon_1 \bullet \varepsilon_1 = \varepsilon_2.$$
$$h_2(-\iota) = h_2(\iota) + h_2(2\iota) + \varepsilon_1 \bullet 0 = 2\varepsilon_2 = 0.$$

Example 1.4. The mod q K-theory of an associative ring A is defined to be $K_n(A;\mathbb{Z}/q) = \pi_n(X;\mathbb{Z}/q)$ for the space $X = BGL(A)^+$. For $n \geq 2$, the mod q Hurewicz map goes from

$$h_n : K_n(A;\mathbb{Z}/q) \longrightarrow H_n(X;\mathbb{Z}/q) \cong H_n(GL(A);\mathbb{Z}/q).$$

This is a group homomorphism unless $n = 2$ and q is even. The failure of the Hurewicz map h_2 to be a homomorphism underlies the Chern class paradox mentioned in the introduction. Indeed, when $q = 2$ there is an element $\beta \in K_2(A;\mathbb{Z}/2)$ such that $\partial\beta = [-1] \in K_1(A)$; h_2 fails to be a homomorphism whenever $h_2(\beta+\beta) = h_2(\{-1,-1\})$ is not equal to $2h_2(\beta) = 0$.

When q is even and $4|q$, we can do slightly better, using the following elementary lemma.

Lemma 1.4.1. *If F is a $\mathbb{Z}[\frac{1}{2}]$-algebra and $x, y \in \mu_q(F)$, then $2\{x, y\} = 0$ in $K_2(F)$.*

Proof. It is classical that $2\{t, t\} = 2\{-1, t\} = 0$ and hence that $2\{t^i, t^j\}$ for every $t \in F^*$. If F is a domain then $\mu_q(F)$ is cyclic, and the result follows. In general, we may assume that F is finitely generated over $\mathbb{Z}[\frac{1}{2}]$ and restrict to each component of F to assume F has only one minimal prime ideal. But then $\mu_q(F) = \mu_q(F_{red})$ is again cyclic, and the result follows.

Corollary 1.4.2. $(4|q)$ *Let F be a $\mathbb{Z}[\frac{1}{2}]$-algebra such that $K_1(F) = F^*$. If $4|q$ then the Hurewicz map*

$$h_2 : K_2(F;\mathbb{Z}/q) \to H_2(GL(F);\mathbb{Z}/q)$$

is a group homomorphism.

To conclude this section, we discuss the behavior of the mod q Hurewicz map under products. Assume we are given a pairing of spaces $\mu : X \wedge Y \to Z$. For example, if $X = BGL(A)^+$ we can consider Loday's pairing $X \wedge X \to X$. There is a bilinear pairing $\pi_*(X) \otimes \pi_*(Y) \to \pi_*(Z)$ sending $a \otimes b$ to $\{a, b\} = \mu(a \wedge b)$. If q is odd or $4|q$, there is also a bilinear pairing

$$\mu : \pi_*(X;\mathbb{Z}/q) \otimes \pi_*(Y;\mathbb{Z}/q) \to \pi_*(Z;\mathbb{Z}/q)$$

(see [Br, 1.6]). In Lemma 2 of [S], Soulé proves that this pairing fits into a commutative diagram (for $m, n \geq 2$):

$$(1.5)\qquad \begin{array}{ccc} \pi_m(X;\mathbb{Z}/q) \times \pi_n(Y;\mathbb{Z}/q) & \xrightarrow{\mu} & \pi_{m+n}(Z;\mathbb{Z}/q) \\ {\scriptstyle h_m \times h_n}\downarrow & & {\scriptstyle h_{m+n}}\downarrow \\ H_m(X;\mathbb{Z}/q) \otimes H_n(Y;\mathbb{Z}/q) & \xrightarrow{\mu} & H_{m+n}(Z;\mathbb{Z}/q). \end{array}$$

There is no $\otimes$ in the upper left corner of (1.5) because h_2 need not be a homomorphism when q is even. We shall refer to this property as "bilinearity," even though this is a slight abuse of terminology.

When $q = 2$ there is no such product on $\pi_*(\ ;\mathbb{Z}/2)$, essentially because $P^m \wedge P^n$ no longer decomposes as $P^{m+n} \vee P^{m+n-1}$. However, there is still a pairing

$$\pi_m(X) \times \pi_n(Y;\mathbb{Z}/2) \to \pi_{m+n}(Z;\mathbb{Z}/2), (m \geq 0, n \geq 2)$$

arising from the isomorphism $S^m \wedge P^n \cong P^{m+n}$; again (1.5) commutes. More generally, let $\pi_m \times \pi_n$ denote the following subset of $\pi_m(X;\mathbb{Z}/2) \times \pi_n(Y;\mathbb{Z}/2)$:

$$\pi_m \times \pi_n = \{(\alpha, \beta) : \{\partial\alpha, \partial\beta\} = 0 \text{ in } \pi_{m+n-2}(Z)\}.$$

Proposition 1.6. *(Mod 2 products) Suppose given a pairing of spaces $X \wedge Y \to Z$. Then we can define a partial pairing $\mu : \pi_m \times \pi_n \to \pi_{m+n}(Z;\mathbb{Z}/2)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \pi_m \times \pi_n & \xrightarrow{\mu} & \pi_{m+n}(Z;\mathbb{Z}/2) \\ {\scriptstyle h_m \times h_n}\downarrow & & {\scriptstyle h_{m+n}}\downarrow \\ H_m(X;\mathbb{Z}/2) \times H_n(Y;\mathbb{Z}/2) & \xrightarrow{\mu_*} & H_{m+n}(Z;\mathbb{Z}/2). \end{array}$$

This partial pairing is well-defined and bilinear if $\pi_{m+n}(Z)$ has no 2-torsion, or more generally if the Hopf map $\pi_{m+n-1}(Z) \to \pi_{m+n}(Z)$ is zero.

Proof. (see [Br, 1.9]) If $\alpha \in \pi_m(X;\mathbb{Z}/2)$ and $\beta \in \pi_n(Y;\mathbb{Z}/2)$ the element $\{\partial\alpha, \partial\beta\}$ represents the composition

$$S^{m+n-2} = S^{m-1} \wedge S^{n-1} \subset P^m \wedge P^n \xrightarrow{\alpha\wedge\beta} X \wedge Y \xrightarrow{\mu} Z.$$

Set $Q = (P^m \wedge P^n)/(S^{m+n-2}) \simeq P^{m+n} \vee S^{m+n-1}$. If $\{\partial\alpha, \partial\beta\} = 0$ in $\pi_{m+n-2}(Z)$ then $\mu(\alpha \wedge \beta)$ factors through a map $\varphi : Q \to Z$. Any other choice of φ will differ by a map $Q \to S^{m+n-1} \to Z$. Since the composition $\eta : P^{m+n} \subset Q \to S^{m+n-1}$ is $P^{m+n} \to S^{m+n}$ followed by the Hopf map, the restriction $\varphi|P^{m+n}$ represents an element $\{\alpha, \beta\} \in \pi_{m+n}(Z;\mathbb{Z}/2)$ which is well-defined up to an element of the form $\gamma \circ \eta$, where $\gamma \in \pi_{m+n-1}(Z)$. We set $\mu(\alpha, \beta) = \{\alpha, \beta\}$. Note that $h_{m+n}(\{\alpha, \beta\}) \in H_{m+n}(Z;\mathbb{Z}/2)$ is independent of this choice because $h_{m+n}(\gamma \circ \eta) = 0$.

To see that the diagram commutes, we first suppose that the pairing is the natural quotient $\mu' : P^m \wedge P^n \to Q$ and α, β are the identity maps. In this case we can take $Q \to Z$ to be the identity map, and represent $\{id_m, id_n\} \in \pi_{m+n}(Q;\mathbb{Z}/2)$ by the inclusion $P^{m+N} \subset Q$. Clearly $h_{m+n}(\{id_m, id_n\})$ is the natural generator $\varepsilon_{m+n} = \mu'_*(\varepsilon_m \otimes \varepsilon_n)$ of $H_{m+n}(Q;\mathbb{Z}/2)$.

For a general pairing μ, suppose given α, β so that $\{\partial\alpha, \partial\beta\} = 0$. The commutativity of (1.6) follows from the calculation:

$$h_{m+n}(\{\alpha, \beta\}) = \varphi_*(\mu'_*(\varepsilon_m \otimes \varepsilon_n)) = \mu_*(\alpha \wedge \beta)_*(\varepsilon_m \wedge \varepsilon_n) = \mu_*(h_m(\alpha) \otimes h_n(\beta)).$$

Corollary 1.6.1. *(Browder [Br, 1.9]) Suppose that there is no 2-torsion in the even homotopy groups $\pi_{2m}(X)$ of a space X. Then any pairing $X \wedge X \to X$ induces a well-defined product on $\pi_*(X;\mathbb{Z}/2)$ such that (1.5) commutes.*

Indeed, if $m+n$ is odd then either $\partial\alpha$ or $\partial\beta$ is zero.

Corollary 1.6.2. *(Suslin [Sus1]) Let k be an algebraically closed field containing $1/q$, and let $\beta \in K_2$ be a "Bott" element, i.e., an element such that $\partial\beta$ is a primitive q^{th} root of unity. Then there are ring isomorphisms for all q, including $q=2$:*

$$K_*(k;\mathbb{Z}/q) \cong \mathbb{Z}/q[\beta].$$

Corollary 1.6.3. *(Browder [Br, 2.6]) If $\mathbb{F}$ is a finite field of characteristic $\neq 2$ then*

$$K_*(\mathbb{F};\mathbb{Z}/2) \cong \mathbb{Z}/2[\eta,\beta]/(\eta^2=0),$$

where $\eta \in K_1$ is the class of the unit -1 and $\beta \in K_2$ is the "Bott" element.

Example 1.6.4. The graded abelian group $K_*(\mathbb{R};\mathbb{Z}/2)$ is determined in [Sus1]; prominent elements are the class $\eta \in K_1$ of the unit -1, the "Bott" element $\beta \in K_2$ and the "periodicity" element $\gamma \in K_8$. By (1.6) $K_*(\mathbb{R};\mathbb{Z}/2)$ contains the commutative ring

$$\mathbb{Z}/2[\eta,\gamma]/(\eta^3=0),$$

The product $\{\beta,\gamma\}$ exists in K_{10}, but is only defined up to a multiple of $\eta^2\gamma = \{-1,-1,\gamma\}$. The product $\{\beta,\beta\}$ does not exist, and we only have a partial product $\pi_2 \times \pi_2 \to K_4(\mathbb{R};\mathbb{Z}/2)$, where $\pi_2 \times \pi_2$ is the set of all pairs (x,y) of elements in $K_2(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/4$ except (β,β). The group $K_4(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2$ is generated by $\eta^2\beta = \{-1,-1,\beta\}$, but the product $K_4(\mathbb{R};\mathbb{Z}/2) \otimes K_4(\mathbb{R};\mathbb{Z}/2) \to K_8(\mathbb{R};\mathbb{Z}/2)$ is zero.

2. Chern classes at the prime 2.

Let A be a commutative ring containing $1/\ell$ and set $q=\ell^\nu$. Chern classes in K-theory may be constructed using Grothendieck's theory of Chern classes for representations in [G]. For every representation $\rho : G \to GL_n(A)$ of a group G Grothendieck constructs elements $c_i(\rho) \in H^{2i}_{et}(A,G;\mu_q^{\otimes i})$. Via the Künneth homomorphism

$$\Phi : H^{2i}_{et}(A,G;\mu_q^{\otimes i}) \to \coprod_{k=0}^{2i} Hom\,(H_{2i-k}(G;\mathbb{Z}/q), H^k_{et}(A;\mu_q^{\otimes i}))$$

these elements yield homomorphisms $c_{ik}(\rho)$ from $H_{2i-k}(G;\mathbb{Z}/q)$ to $H^k_{et}(A;\mu_q^{\otimes i})$; see [S, II.1]. Taking $G = GL_N(A)$ and $\rho = id_N$, and passing to the limit as $N \to \infty$, we obtain homomorphisms

$$c_{ik}(id) : H_n(GL(A);\mathbb{Z}/q) \to H^k_{et}(A,\mu_q^{\otimes i}), \qquad n+k=2i.$$

Composing with the Hurewicz map h_n of (1.4) yields Soulé's Chern classes,

$$\text{(2.1)} \qquad c_{ik} : K_n(A;\mathbb{Z}/q) \to H^k_{et}(A,\mu_q^{\otimes i}), \qquad n+k=2i, \quad n \geq 2.$$

Proposition 2.1.1. *The Chern classes are natural in A. That is, if $f : A \to B$ is a ring homomorphism then the following diagram commutes:*

$$\begin{array}{ccc} K_n(A;\mathbb{Z}/q) & \xrightarrow{c_{ik}} & H^k_{et}(A;\mu_q^{\otimes i}) \\ \downarrow & & \downarrow \\ K_n(B;\mathbb{Z}/q) & \xrightarrow{c_{ik}} & H^k_{et}(B;\mu_q^{\otimes i}) \end{array}$$

Proof. Set $G_1 = GL_N(A)$ and $G_2 = GL_N(B)$. Given compatible representations

$$\begin{array}{ccc} G_1 & \xrightarrow{\rho_1} & GL_n(A) \\ \varphi\downarrow & & f\downarrow \\ G_2 & \xrightarrow{\rho_2} & GL_n(B) \end{array}$$

we have a commutative diagram:

$$\begin{array}{ccccc} H^{2i}_{et}(A,G_1;\mu_q^{\otimes i}) & \xrightarrow{f^*} & H^{2i}_{et}(B,G_1;\mu_q^{\otimes i}) & \xleftarrow{\varphi^*} & H^{2i}(B,G_2;\mu_q^{\otimes i}) \\ \Phi\downarrow & & \Phi\downarrow & & \Phi\downarrow \\ Hom(H_n(G_1),H^k(A)) & \xrightarrow{f^*} & Hom(H_n(G_1),H^k(B)) & \xleftarrow{\varphi^*} & Hom(H_n(G_2),H^k(B)). \end{array}$$

By functoriality of Grothendieck's classes c_i ([G,2.3]) we have

$$f^*c_i(\rho_1) = c_i(f^*\rho_1) = c_i(\varphi^*\rho_2) = \varphi^*c_i(\rho_2).$$

Hence $f^*\Phi c_i(\rho_1) = \varphi^*\Phi c_i(\rho_2)$ as maps from $H_n(GL_N(A);\mathbb{Z}/q)$ to $H^k_{et}(B;\mu_q^{\otimes i})$. Letting $N \to \infty$ and composing with the Hurewicz map yields the Proposition.

Gillet and Shekhtman used the same construction in [Gi] and [Sh], but with the integral Hurewicz map $K_n(A) \to H_n(GL(A);\mathbb{Z})$, to construct integral Chern classes

$$c_{ik} : K_n(A) \to H^k_{et}(A;\mu_q^{\otimes i}), \qquad n+k = 2i, \quad n \geq 0. \tag{2.2}$$

Example 2.2.1. ($n = 1$) By construction ([Sh, 1.b],[S, p. 279]), c_{11} is the homomorphism

$$c_{11} : K_1(A) \xrightarrow{det} A^* \to A^*/A^{*q} \subset H^1_{et}(A,\mu_q).$$

All the other Chern classes $c_{ik} : K_1(A) \to H^k_{et}(A,\mu_q^{\otimes i})$ vanish on $A^* \subset K_1(A)$. This is because the Hurewicz homomorphism maps A^* into $H_1(GL_1(A))$, and $c_{ik}(id)|H_*(GL_1)$ is the map $c_{ik}(id_1)$ arising via Φ from $c_i(id_1)$. But since id_1 is a 1-dimensional representation of $GL_1(A)$, $c_i(id_1) = 0$ for $i > 1$.

Lemma 2.3. *For $n \geq 2$, the integral Chern classes (2.2) are the composition of Soulé's Chern classes (2.1) with mod q reduction:*

$$K_n(A) \to K_n(A;\mathbb{Z}/q) \xrightarrow{c_{ik}} H^k(A;\mu_q^{\otimes i})$$

The integral Chern classes are group homomorphisms for all $n \geq 0$.

Proof. The first assertion follows from compatability of the two Hurewicz maps. The second results from the fact that the integral Hurewicz map is always a group homomorphism.

The mod q Chern classes (2.1) are not always homomorphisms, because the Hurewicz maps may not be. Here is the correct assertion, which follows immediately from theorem (1.2).

Proposition 2.4. *Soulé's Chern classes (2.1) are group homomorphisms if either q is odd or if q is even and $n \neq 2$.*

If q is even and $n = 2$ then Soulé's Chern classes $c_{ik} : K_2(A;\mathbb{Z}/q) \to H^k_{et}(A;\mu_q^{\otimes i})$ satisfy:

$$c_{ik}(a+b) = c_{ik}(a) + c_{ik}(b) + \frac{q}{2}c_{ik}(\{\partial a, \partial b\}).$$

Here $a, b \in K_2(A;\mathbb{Z}/q)$, $i \geq 1$ and $k = 2i-2$.

Corollary 2.5. *($4|q$). Let F be a $\mathbb{Z}[\frac{1}{2}]$-algebra such that $K_1(F) = F^*$. If $4|q$ then the Chern classes*

$$c_{i,2i-2} : K_2(F;\mathbb{Z}/q) \to H^{2i-2}_{et}(F;\mathbb{Z}/q)$$

are all group homomorphisms.

Proof. The correction term in (2.4) is zero by (1.4.1). Alternatively, this follows by applying $c_{ik}(id)$ to (1.4.2).

Porism 2.5.1. If $4|q$, the proof of 2.5 goes through to show that the Chern classes $c_{i,2i-2}$ are all homomorphisms when restricted to the subgroup of $K_2(A;\mathbb{Z}/q)$ consisting of elements a such that $\partial a \in \mu_q(A)$.

We shall now describe the Chern classes on $K_2(A;\mathbb{Z}/2)$.

Observation 2.6. ($n = 2$) If A is a field (or more generally a semilocal ring), then the only nonzero Chern classes on $K_2(A;\mathbb{Z}/q)$ are

$$c_{10} : K_2(A;\mathbb{Z}/q) \to H^0(A;\mu_q) = \mu_q(A),$$
$$c_{22} : K_2(A;\mathbb{Z}/q) \to H^2(A;\mu_q^{\otimes 2}).$$

This follows from the fact that $K_2(A;\mathbb{Z}/q)$ is a quotient of $\pi_2(BGL_2(A)^+;\mathbb{Z}/q)$, and c_{ik} on $H_*(GL_2)$ arises via Φ from $c_i(id_2)$. Since id_2 is a 2-dimensional representation, $c_i(id_2) = 0$ for $i > 2$. The same reasoning applies to $K_2(\mathbb{Z};\mathbb{Z}/q) = K_2(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/q)$. Similar reasoning shows that if A is a field then c_{ik} vanishes on $K_n(A;\mathbb{Z}/q)$ unless $n/2 \leq i \leq n$. We will describe the Chern classes c_{10} and c_{22} in more detail below.

Example 2.6.1. (c_{10}) The Chern class map c_{10} is always a homomorphism — even when $q = 2$. Moreover, $c_{10}(a) = 0$ for every $a \in K_2(A)$. Indeed, it follows easily from the argument on p. 279 of [S] that c_{10} is the Bockstein map, followed by the determinant map:

$$c_{10} : K_2(A;\mathbb{Z}/q) \xrightarrow{\partial} {}_qK_1(A) \xrightarrow{det} \mu_q(A) = H^0_{et}(A;\mu_q).$$

But since $BGL(A)^+$ is an H-space, the Bockstein map $\partial : K_2(A;\mathbb{Z}/q) \to K_1(A)$ is a homomorphism.

Example 2.6.2. (c_{22}) Suppose that F is a field. By (2.5), c_{22} is a homomorphism when $4|q$, so suppose that $q = 2$. By [Sh], c_{22} is the Galois symbol (up to sign) on the integral group $K_2(F)$. For the real numbers $\mathbb{R}$, this implies that $c_{22}(\{-1,-1\}) = [\mathbb{H}]$, which is the nonzero element of $H^2_{et}(\mathbb{R};\mu_2^{\otimes 2}) \cong Br(\mathbb{R}) \cong \mathbb{Z}/2$. Choose an element $\beta \in K_2(\mathbb{R};\mathbb{Z}/2)$ such that $\partial\beta = [-1] \in K_1(\mathbb{R})$. It is well-known ([Wp, p. 396] [AT, p. 79]) that β generates $K_2(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/4$ and that 2β is the Steinberg symbol $\{-1,-1\}$. By (2.4) we have $c_{22}(\beta) \neq c_{22}(-\beta)$, because

$$c_{22}(\beta) + c_{22}(-\beta) = c_{22}(0) - c_{22}(\{-1,-1\}) = [\mathbb{H}] \neq 0.$$

Note the resolution of the paradox mentioned in the introduction:

$$c_{22}(\beta+\beta) = 2c_{22}(\beta) + c_{22}(\{-1,-1\}) = c_{22}(\{-1,-1\}).$$

Example 2.6.3. Let $SK_2(A;\mathbb{Z}/q)$ denote the kernel of the map $c_{10} : K_2(A;\mathbb{Z}/q) \to \mu_q(A)$. There is an exact sequence, which is split if $q \not\equiv 2 \pmod 4$:

$$0 \to K_2(A) \otimes \mathbb{Z}/q \to SK_2(A;\mathbb{Z}/q) \xrightarrow{\partial} {}_qSK_1(A) \to 0.$$

Here ${}_qSK_1(A)$ denotes the subgroup $Tor(SK_1(A),\mathbb{Z}/q)$ of $SK_1(A)$, which is in turn the kernel of $det : K_1(A) \to A^*$. The map $c_{22} : SK_2(A;\mathbb{Z}/q) \to H^2_{et}(A;\mu_q^{\otimes 2})$ is a homomorphism for every A and every q, because the error term in 2.4 vanishes by the product formula for integral Chern classes [Sh] and 2.2.1:

$$c_{22}(\{\partial a, \partial b\}) = -c_{11}(\partial a)c_{11}(\partial b) = 0.$$

Definition 2.7 (Mod 2 Bott element). There are two generators of the cyclic group $K_2(\mathbb{Z};\mathbb{Z}/2) \cong K_2(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/4$. Let $\beta_{\mathbb{Z}}$ be that generator of $K_2(\mathbb{Z};\mathbb{Z}/2)$ such that in $H^2_{et}(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2$ we have $c_{22}(\beta_{\mathbb{R}}) = 0$. This generator exists and is unique by (2.6.2). If A is any commutative ring containing 1/2, we write $\beta = \beta_A$ for the image of $\beta_{\mathbb{Z}}$ in $K_2(A;\mathbb{Z}/2)$, and call it the *mod* 2 *Bott element*. Since $H^2_{et}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2) \cong H^2_{et}(\mathbb{R};\mathbb{Z}/2)$, the observations of 2.6 imply that

$$\begin{aligned} c_{10}(\beta) &= [-1] && \text{in } H^0_{et}(A;\mathbb{Z}/2) = \mu_2(A) \\ c_{i,2i-2}(\beta) &= 0 && \text{in } H^{2i-2}_{et}(A;\mathbb{Z}/2) \text{ for } i \neq 1 \end{aligned}$$

By construction, we have $\partial\beta = [-1]$ in $K_1(A)$ and $2\beta = \{-1,-1\}$ in $K_2(A)$.

Remark 2.7.1. We warn the reader that $c_{22}(-\beta)$ need not be zero, as illustrated in (2.6.2). Indeed, by 2.4 we have $c_{22}(-\beta) = \{-1,-1\} \in H^2_{et}(A;\mathbb{Z}/2)$. See Remark 4.2.2 below for a discussion of when this vanishes.

Definition 2.7.2 (Mod q Bott elements). We can generalize 2.7 as follows. Let ζ be a fixed primative q^{th} root of unity in a ring A. The *mod q Bott element* $\beta \in K_2(A;\mathbb{Z}/q)$ associated to ζ is the following element satisfying $\partial\beta = \zeta$. Let $\mu_q(A)$ denote the cyclic group of q^{th} units in A and $\beta_0 \in \pi_2(B\mu_q(A);\mathbb{Z}/q)$ the element corresponding to ζ under the isomorphism

$$\pi_2(B\mu_q(A);\mathbb{Z}/q) \cong \pi_1 B\mu_q(A) \cong \mu_q(A).$$

Then β is the image of β_0 under the natural map induced by $B\mu_q(A) \subseteq BGL(A)^+$. Since $h_2(\beta)$ comes from $H_2(BGL_1(A);\mathbb{Z}/q)$ and $c_i(id_1) = 0$ for $i > 0$, we have $c_{i,2i-2}(\beta) = 0$ for $i \neq 1$. If $q = 2$ this recovers the mod 2 Bott element. If $q \not\equiv 2 \pmod 4$, it will follow from the product formula (3.1) or 3.2 (iv′) that

$$\begin{array}{ll} c_{i0}(\beta^i) = (-1)^{i-1}(i-1)!\zeta^{\otimes i} & \text{in } H^0_{et}(A;\mu_q^{\otimes i}) \cong \mu_q(A)^{\otimes i} \\ c_{jk}(\beta^i) = 0 & \text{if } j \neq i. \end{array}$$

The following well–known result will be needed in §5; special cases have been used in [S,IV.2], [Sus2,3.1] and [L,pp.330,335,340].

Proposition 2.8 (Soulé). *If i, q and r are integers ≥ 1, then the cohomology sequence associated to $0 \to \mu_q^{\otimes i} \to \mu_{qr}^{\otimes i} \to \mu_r^{\otimes i} \to 0$ is compatible with the Chern classes and the K-theory Bockstein, in the sense that the following diagram commutes for $n \geq 2$.*

$$\begin{array}{ccccccc} K_n(A;\mathbb{Z}/q) & \longrightarrow & K_n(A;\mathbb{Z}/qr) & \longrightarrow & K_n(A;\mathbb{Z}/r) & \xrightarrow{\partial} & K_{n-1}(A;\mathbb{Z}/q) \\ {\scriptstyle c_{ik}}\downarrow & & {\scriptstyle c_{ik}}\downarrow & & {\scriptstyle c_{ik}}\downarrow & & {\scriptstyle c_{i,k+1}}\downarrow \\ H^k(A;\mu_q^{\otimes i}) & \longrightarrow & H^k(A;\mu_{qr}^{\otimes i}) & \longrightarrow & H^k(A;\mu_r^{\otimes i}) & \xrightarrow{\delta} & H^{k+1}(A;\mu_q^{\otimes i}) \end{array}$$

Proof. ([S,IV.3.1]). Since the Hurewicz maps are compatible with the Bockstein [N,3.2.ii], we may replace $K_n(A;\mathbb{Z}/q)$ with $H_n(G;\mathbb{Z}/q)$ in order to prove compatability with Chern classes. If $C_\bullet(G)$ is the standard resolution of BG and $\Gamma(\mathcal{L}_q^\bullet)$ is the complex of sections of a resolution of $\mu_q^{\otimes i}$, the naturality of Φ implies that the diagram

$$\begin{array}{ccccc} C_\bullet(G)\otimes\mathbb{Z}/q & \longrightarrow & C_\bullet(G)\otimes\mathbb{Z}/qr & \longrightarrow & C_\bullet(G)\otimes\mathbb{Z}/r \\ {\scriptstyle \Phi(c_i)}\downarrow & & {\scriptstyle \Phi(c_i)}\downarrow & & {\scriptstyle \Phi(c_i)}\downarrow \\ \Gamma(\mathcal{L}_q^\bullet) & \longrightarrow & \Gamma(\mathcal{L}_{qr}^\bullet) & \longrightarrow & \Gamma(\mathcal{L}_r^\bullet) \end{array}$$

not only commutes in the derived category $D(\mathbb{Z})$, but is a morphism of triangles. Passing to homology yields the Proposition.

3. The Product Formula.

In order to have a formula for $c_{ik}(\{a,b\})$, one needs the K-theory product $\{a,b\}$ to be defined in $K_*(A;\mathbb{Z}/q)$. If q is odd or if $4|q$ then this product exists (see [Br, 1.6]). When $q=2$, the mod 2 product $\{a,b\}$ only exists if the integral product $\{\partial a,\partial b\}$ vanishes; see (1.6).

Suppose that $a\in K_m(A;\mathbb{Z}/q)$ and $b\in K_n(A;\mathbb{Z}/q)$ are such that the product $\{a,b\}$ is defined. We assume that $m,n\geq 1$, using the convention that $K_1(A;\mathbb{Z}/q)=K_1(A)/qK_1(A)$.

Soulé's product formula states that

$$c_{ik}(\{a,b\})=\sum_{i_1+i_2=i}\frac{-(i-1)!}{(i_1-1)!\,(i_2-1)!}c_{i_1k_1}(a)c_{i_2k_2}(b). \tag{3.1}$$

Here the sum is over all $i_1\geq m/2$ and $i_2\geq n/2$ such that $i_1+i_2=i$, and we have set $k=2i-m-n$, $k_1=2i_1-m$, $k_2=2i_2-n$.

Soulé proves in II.3 of [S] that his product formula (3.1) holds for odd q. Shekhtman points out in [Sh] that Soulé's product formula (3.1) also holds for integral Chern classes, i.e., when both a and b come from integral K-theory. The purpose of this section is to analyze the situation when q is even.

Theorem 3.2. *Suppose that q is even. Then Soulé's product formula (3.1) holds for $c_{ik}(\{a,b\})$ in the following cases:*

(i) Neither m nor n is 2.

(i′) If $m=2$ then $a\in K_2(A)$, and if $n=2$ then $b\in K_2(A)$.

(ii) $i=2$. That is, if $k=4-m-n$ then in $H^k_{et}(A;\mu_q^{\otimes 2})$ we have

$$c_{2k}(\{a,b\})=-c_{1,2-m}(a)c_{1,2-n}(b).$$

(iii) $i=3$, $n=2$ and $m\geq 3$. That is, $c_{3k}(\{a,b\})=-2c_{2,4-m}(a)\cup c_{10}(b)$.

(iv) $(4|q)$ and $K_1(A)=A^*$.

(iv′) $(4|q)$ and: if $m=2$ then $\partial a\in A^*$; if $n=2$ then $\partial b\in A^*$.

(v) $i\geq 5$ and $K_1(A)=A^*$.

(v′) $i\geq 5$ and: if $m=2$ then $\partial a\in A^*$; if $n=2$ then $\partial b\in A^*$.

Remark 3.2.1. As mentioned in the introduction, the only version of the product formula used in [Sus2] is when $a\in K_1(A)$, $b\in K_2(A;\mathbb{Z}/q)$ and $i=2$. Indeed, by 3.2(ii) we have

$$c_{21}(\{a,b\})=-c_{11}(a)c_{10}(b)=\{det(a),det(\partial b)\}.$$

If $q=2$, $n=2$, and $i=3,4$ there is an extra term in the product formula. Note that if $b\in K_2(A;\mathbb{Z}/2)$ is such that ∂b is a unit, and $\frac{1}{2}\in A$, then $\partial b=\pm 1$ on each component of A. If $\partial b=+1$ then $b\in K_2(A)$, and Soulé's product formula holds by 3.2(i'). If $\partial b=-1$ then b differs from the mod 2 Bott element β of (2.7) by an element b' of $K_2(A)$. If $m\geq 1$ then by 2.4 we have $c_{ik}(\{a,b\})=c_{ik}(\{a,\beta\})+c_{ik}(\{a,b'\})$. The last term is described by the product formula and the next Theorem describes $c_{ik}(\{a,\beta\})$.

Theorem 3.3. *Let $\beta \in K_2(A;\mathbb{Z}/2)$ be the mod 2 Bott element.*

(i) Suppose that either $a \in K_m(A)$ for $m \geq 1$, or that $a \in K_m(A;\mathbb{Z}/2)$ for $m \geq 3$ and the product $\{a,\beta\} \in K_{m+2}(A;\mathbb{Z}/2)$ is defined. Write $[-1]$ for $c_{10}(\beta) \in H^0(A;\mu_2) = \mu_2(A)$ and $\{-1,-1\} \in H^2_{et}(A;\mathbb{Z}/2)$ for the square of the class of -1 in $A^*/A^{*2} \subseteq H^1_{et}(A;\mu_2)$. Then in $H^k_{et}(A;\mathbb{Z}/2)$ we have:

$$c_{ik}\{a,\beta\} = (1-i)c_{i-1,k}(a) \cup [-1] + \binom{i-1}{2} c_{i-2,k-2}(a) \cup \{-1,-1\}.$$

The first term represents Soulé's product formula (3.1).

(ii) ($a=\beta$) Suppose that the product $\{\beta,\beta\}$ is defined in $K_4(A;\mathbb{Z}/2)$, i.e., that $\{-1,-1\} = 0$ in $K_2(A)$. Then $c_{22}(\{\beta,\beta\}) = [-1] \otimes [-1] \neq 0$, and all other Chern classes $c_{ik}(\{\beta,\beta\})$ vanish.

Example 3.4. Theorem 3.3 corrects the assertion on p. 13 of [Sus2] – that Soulé's product formula remains valid for $q = 2^\nu$. For example, if $\beta \in K_2(\mathbb{R};\mathbb{Z}/2)$ is the mod 2 Bott element, then we get the calculations in the introduction:

$$c_{33}(\{-1,\beta\}) = \{-1,-1,-1\} \neq 0 \qquad \text{in } H^3_{et}(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2,$$
$$c_{44}(\{-1,-1,\beta\}) = \{-1,-1,-1,-1\} \neq 0 \qquad \text{in } H^4_{et}(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

In both cases Soulé's product formula (3.1) fails to hold.

(3.5.0). In order to prove Theorems 3.2 and 3.3, we shall modify the proof given by Soulé for odd q in [S, II.3]. For this, we need the maps

$$\Phi c_I : H_n(GL(A);\mathbb{Z}/q) \to H^k_{et}(A,\mu_q^{\otimes i})$$

associated to an ordered multi-index $I = (i_1,\dots,i_\alpha)$ with $i = \Sigma i_\gamma$ and $n+k = 2i$. The map $\Phi c_I = \Phi c_I(id)$ is the composition of the diagonal map $\Delta_* : H_* \to H_* \otimes \cdots \otimes H_*$ with the tensor product of the maps $\Phi(c_{i_\gamma}(id))$ constructed at the start of section 2.

We are going to apply the maps Φc_I to Hurewicz elements $h_n(a)$. If $I = (i)$ then $\Phi c_{(i)}(h_n(a)) = c_{ik}(a)$, as expected. If $h_n(a)$ is primitive and I has length ≥ 2 then $\Phi c_I(h_n a) = 0$ because all the $\Phi(c_i(id))$ vanish on $1 \in H_0$.

Lemma 3.5. *Suppose that $b \in K_2(A;\mathbb{Z}/q)$. If I has length ≥ 3 then $\Phi c_I(h_n\, b) = 0$. If $I = (i_1,i_2)$ and q is even, then*

$$\Phi c_I(h_2(b)) = (\frac{q}{2}) c_{i_1k_1}(\partial b) \cup c_{i_2k_2}(\partial b), \qquad \textit{where } k_\gamma = 2i_\gamma - 1.$$

Proof. This follows immediately from Proposition 1.1 and the fact that the $h_1(\partial b)$ are primitive elements of H_1.

Corollary 3.5.1. *Suppose that $b \in K_2(A;\mathbb{Z}/q)$ and that $u = \partial b$ is a unit of A. Then $\Phi c_{(i_1,i_2)}(h_2 b) = 0$ unless both $I = (1,1)$ and $q \equiv 2 \pmod 4$. In that case,*

$$\Phi c_{(1,1)}(h_2 b) = c_{11}(\partial b)^2 = \{-1,u\} \in H^2_{et}(A;\mu_2^{\otimes 2}) = H^2_{et}(A;\mathbb{Z}/2).$$

Proof. If $i_2 > 1$ then $c_{i_2k_2}(u) = 0$ by (2.2.1). If $4|q$ then we have a multiple of $2c_{1k}^2(\partial b)$, which vanishes in the graded ring H^*. Finally, we identify $c_{11}(\partial b)$ with $u \in A^*/A^{*2} \subseteq H^1(A;\mu_2)$ by (2.2.1) and cite the well-known identity $\{u,u\} = \{-1,u\}$ in H^2.

In [RRR], Grothendieck defined certain universal polynomials

$$Q_i(x_1,\dots,x_{i-1};y_1,\dots,y_{i-1}) = \sum a_{IJ}\, x^I y^J,$$

in order to describe the Chern classes of a product of vector bundles. Here the notation is such that I and J are ordered multi-indices, and x^I means the product of the x_{i_γ} as i_γ runs through I. The polynomial Q_i is homogeneous of degree i, provided that x_j and y_j have degree j. The first few polynomials are:

$$\begin{aligned} &Q_1 = 0\\ (3.6)\qquad &Q_2 = -x_1y_1\\ &Q_3 = (x_1^2y_1 + x_1y_1^2) - 2(x_1y_2 + x_2y_1)\\ &Q_4 = -3(x_1y_3 + x_3y_1) - 6x_2y_2 + 3(x_1^2y_2 + x_2y_1^2)\\ &\qquad + 3(x_1x_2y_1 + x_1y_1y_2) - (x_1^3y_1 + x_1^2y_1^2 + x_1y_1^3). \end{aligned}$$

These formulas will be derived in 3.8 and 3.9 below. In general, we know from [RRR, (1.18)] that the coefficient of x_my_n in Q_{m+n} is

$$(3.6.1)\qquad a_{mn} = \frac{-(m+n-1)!}{(m-1)!\,(n-1)!}.$$

Soulé uses the Q_i to define a map $\Phi_i = \Phi Q_i(c_*(id);c_*(id))$ by the formula $\Phi_i = \sum a_{IJ}\,\Phi c_I(id) \cup \Phi c_J(id)$ and observes (for odd q) that the diagram

$$\begin{array}{ccc} K_m(A;\mathbb{Z}/q) \times K_n(A;\mathbb{Z}/q) & \xrightarrow{\mu} & K_{m+n}(A;\mathbb{Z}/q) \\ {\scriptstyle h_m\times h_n}\downarrow & & \downarrow{\scriptstyle c_i} \\ H_m(GL(A)) \otimes H_n(GL(A)) & \xrightarrow{\Phi_i} & H^\bullet_{et}(A;\mu_q^{\otimes i}) \end{array}$$

commutes. That is, we have:

$$(3.7)\qquad c_{ik}(\{a,b\}) = \sum a_{IJ}\Phi c_I(h_m(a)) \cup \Phi c_J(h_n(b)).$$

Soulé's observation applies equally well if $4|q$ (when μ is defined), or even if $q = 2$, provided that we restrict to the subset $\pi_m \times \pi_n$ of $K_m(A;\mathbb{Z}/q) \times K_n(A;\mathbb{Z}/q)$ on which the product is defined by (1.6).

By (3.6.1), Soulé's product formula (3.1) is the sum of those terms in (3.7) in which both I and J have length 1. By 3.5 we can ignore terms in (3.7) in which I or J has length ≥ 3. Therefore the deviation from the product formula is given by terms in which I or J have length 2, and these are described by 3.5.

Proof of 3.2. As long as the elements h_ma and h_nb are primitive, the only nonzero terms in (3.7) are the ones in the product formula (3.1). This is the case in (i) and (i'). The explicit formula for Q_2 proves 3.2(ii). Part (iii) follows from 3.7.1 below, and part (iv) follows immediately from 3.5.1. Part (v) follows from Theorem 3.3, which we will prove below.

(3.7.1) Product formula for c_{3k}. The explicit formula for Q_3 and (3.5) shows that if $4|q$ then Soulé's product formula holds:

$$c_{3k}(\{a,b\}) = -2c_{1k_1}(a)c_{2k_2}(b) - 2c_{2k_2}(a)c_{1k_1}(b).$$

When $q=2$, things are more complicated. If we fix $b \in K_2(A;\mathbb{Z}/2)$, then $c_{3k}(\{a,b\})$ is only defined for $m \le 4$. If $a \in K_1(A)$, then

$$c_{33}(\{a,b\}) = c_{11}(a)c_{11}^2(\partial b) = [det(a)] \cup [det(\partial b)]^2.$$

This yields the formula $c_{33}(\{a,\beta\}) = \{det(a),-1,-1\}$ of Theorem 3.3(i) and 3.4.

Suppose that $a \in K_2(A;\mathbb{Z}/2)$, and $\{a,b\}$ is defined, i.e., that $\{\partial a, \partial b\} = 0$ in $K_2(A)$. Then

$$c_{32}(\{a,b\}) = c_{11}^2(\partial a)c_{10}(b) + c_{10}(a)c_{11}^2(\partial b).$$

This formula has two consequences of interest. On the one hand, if $a \in K_2(A)$ this yields $c_{32}(\{a,b\}) = 0$ for every b. On the other hand, whenever $\{b,b\}$ is defined we have $c_{32}(\{b,b\}) = 0$. This implies the case $i=3$ of 3.3(ii).

Finally, suppose that a is in $K_3(A;\mathbb{Z}/2)$ or $K_4(A;\mathbb{Z}/2)$ and $\{a,b\}$ is defined. The explicit formula for Q_3 shows that $c_{3k}(\{a,b\}) = 0$, and yields 3.2(iii).

(3.7.2) Product formula for c_{4k}. Suppose first that $a \in A^*$ and that $b \in K_2(A;\mathbb{Z}/q)$. By 2.2.1 the only nonzero Chern class on a is $c_{11}(a) = [a]$. Using 3.5, the explicit formula for Q_4 shows that in $H^5(A;\mu_q^{\otimes 4})$:

$$c_{45}(\{a,b\}) = -3[a] \cup c_{34}(b) + 3\frac{q}{2}[a] \cup [det(\partial b)] \cup c_{23}(\partial b).$$

Soulé's product formula (3.1) holds iff the second term vanishes. This will be the case whenever ∂b is a unit, for then $c_{23}(\partial b) = 0$ by 2.2.1.

Next, suppose that $a \in K_2(A)$ and β is the mod 2 Bott element of $K_2(A;\mathbb{Z}/2)$. The explicit formula for Q_4 shows that if $4|q$ then the product formula (3.1) holds:

$$c_{44}(\{a,\beta\}) = -3c_{34}(a) \cup [-1],$$

while if $q=2$ we have:

$$c_{44}(\{a,\beta\}) = c_{34}(a) \cup [-1] + c_{22}(a) \cup \{-1,-1\}.$$

In particular, if $x,y \in A^*$ then $c_{34}(\{x,y\}) = 0$ by 2.6, so

$$c_{44}(\{x,y,\beta\} = \{x,y,-1,-1\} \qquad \text{in } H^4(A;\mathbb{Z}/2).$$

Finally, suppose that $\{\beta,\beta\}$ is defined in $K_4(A;\mathbb{Z}/2)$. Then the explicit formula for Q_4 yields the case $i=4$ of 3.3(ii):

$$c_{44}(\{\beta,\beta\}) = -c_{11}(\partial\beta)^2 = c_{22}(\{-1,-1\}) = 0.$$

For the proof of the next result, which is used in the proof of Theorem 3.3 (and hence in the proof of 3.2), we shall need a few observations about symmetric polynomials. Formally factoring $1 + xt^m = \prod(1+\alpha_i t)$, note that the symmetric polynomials σ_i in the α_i's are: $\sigma_i = 0$ for $i \ne m$, and $\sigma_m = x$. Setting $s_k = \sum \alpha_i^k$, Newton's formula (see [CC, p. 195]) is:

$$s_n - \sigma_1 s_{n-1} + \cdots + (-1)^n n\sigma_n = 0.$$

This easily yields $s_k = 0$ for $k < m$, $s_m = (-1)^{m-1}mx$, and $s_k = 0$ if $k > m$ and $m \nmid k$. By induction, we also see that if $k = jm$ then $s_{jm} = (-1)^{j(m-1)}mx^j$. With these preliminaries out of the way, we are ready to proceed.

Proposition 3.8. *The coefficient of* $x_m y_1^n$ *in* $Q_{m+n}(x_1,\dots;y_1,\dots)$ *is*

$$a_{m,(1,\dots,1)} = (-1)^n \binom{m+n-1}{n}.$$

In particular, the coefficient of $x_m y_1^2$ *in* Q_{m+2} *is* $a_{m,(1,1)} = \binom{m+1}{2}$.

Proof. It suffices to evaluate $Q_i = Q_i(0,\dots,x,0;y,0,\dots)$ modulo x^2. According to (1.17 bis) of [RRR], Q_i is the coefficient of t^i in the formal expansion of

$$(1+xt^m)*(1+yt) = \prod_{i=1}^{m} \frac{(1+(\alpha_i+y)t)}{(1+\alpha_i t)(1+yt)}.$$

The logarithm of this power series is

$$L = \sum_{i,k} \frac{(-1)}{k}^{k-1}(\alpha_i+y)^k t^k - \sum_{i,k} \frac{(-1)}{k}^{k-1}\alpha_i^k t^k - m\sum_k \frac{(-1)}{k}^{k-1} y^k t^k.$$

The above remarks about the s_k show that the leading term of L is of degree $m+1$, namely t^{m+1} times:

$$Q_{m+1} = \frac{(-1)^m}{m+1}\left\{\sum_j (\alpha_i+y)^{m+1} - s_{m+1} - my^{m+1}\right\} = (-1)^m s_m y = -mxy.$$

This is no surprise, as it recovers Grothendieck's observation in (1.18) of [RRR] that $Q_{m+1} = -mxy$. We are interested in the term of degree $m+n$ in t:

(3.8.1)
$$\frac{(-1)^{m+n-1}}{m+n}\left\{\sum(\alpha_i+y)^{m+n} - s_{m+n} - my^{m+n}\right\} t^{m+n}$$
$$= \frac{(-1)^{m+n-1}}{m+n}\sum_{k=0}^{m+n}\binom{m+n}{k} s_k y^{m+n-k} t^{m+n}.$$

Modulo x^2, only the term $k=m$ is nonzero, and this equals

$$(-1)^n \frac{m}{m+n}\binom{m+n}{n} xy^n t^{m+n} = (-1)^n\binom{m+n-1}{n} xy^n t^{m+n}.$$

Therefore modulo x^2 we have:

$$(1+xt^m)*(1+yt) = \exp(-mxyt^{m+1} + \binom{m+1}{2} xy^2 t^{m+2} - \cdots)$$
$$= 1 - mxyt^{m+1} + \binom{m+1}{2} xy^2 t^{m+2} - \cdots.$$

It follows that $Q_{m+n}(0,\dots,x,0;y,\dots) \equiv (-1)^n \binom{m+n-1}{n} xy^n t^{m+n}$, as required.

Exercise 3.8.2. Use (3.8.1) to show that the coefficient of $x_1x_2y_1$ in Q_4 is 3.

Proof of Theorem 3.3. Let $\beta \in K_2(A;\mathbb{Z}/2)$ be the mod 2 Bott element (see 2.7). By 3.5.1, the only nonzero terms in (3.7) in which J has length 2 are those with $J=(1,1)$, and have the form $a_{IJ}\Phi c_I(h_m a)\cup\{-1,-1\}$. To prove (i), when $h_m(a)$ is primitive, we only need the coefficients a_{iJ}, which equal $\binom{i-1}{2}$ by 3.8.

To prove (ii), suppose that $\{\beta,\beta\}$ is defined in $K_4(A;\mathbb{Z}/2)$. By 3.2(ii) we have $c_{20}(\{\beta,\beta\}) = -c_{10}(\beta)^2$. This is the nonzero element of $H^0_{et}(A;\mu_2^{\otimes 2})\cong \mu_2\otimes\mu_2$. We saw in (3.7.1) and (3.7.2) that c_{32} and c_{44} also vanish on $\{\beta,\beta\}$. For $i\ge 5$ we argue as follows. By 2.7, $c_{ik}(\beta)=0$ for $i\ne 1$. By 3.5 and 3.5.1 this means that $\Phi c_I(h_2\beta)=0$ unless I is $(1),(2)$ or $(1,1)$. These belong to $H^k(A;\mathbb{Z}/2)$ for $k=0,1$ and 2, respectively. Since $c_{ik}(\{\beta,\beta\})$ belongs to $H^{2i-4}(A;\mathbb{Z}/2)$, the right-hand side of (3.7) must be zero for dimensional reasons.

Lemma 3.9. *The coefficient of $x_1^i y_1^{m-i}$ in $Q_m(x_1,...;y_1,...)$ is: zero if $i=0,m$ and $(-1)^{m-1}$ when $1\le i<m$.*

Proof. According to (1.17bis) of [RRR], $Q_m(x,0,...;y,0,...)$ is the coefficient of t^m in the formal power series

$$(1+xt)*(1+yt)=\frac{(1+(x+y)t)}{(1+xt)(1+yt)}=(1+(x+y)t)\sum(-xt)^i\sum(-yt)^j.$$

Hence $Q_m=(-1)^{m-1}(x^{m-1}y+\cdots+xy^{m-1})$.

Example 3.9.1. The explicit formulas (3.6) for the homogeneous polynomials Q_3 and Q_4 follow from (3.6.1), 3.8, 3.8.2 and the following cases of 3.9:

$$Q_3(x,0;y,0)=x^2y+xy^2$$
$$Q_4(x,0,0;y,0,0)=-(x^3y+x^2y^2+xy^3).$$

We conclude this section with a product formula for $c_{ik}(\{a,b\})$ when $\pm a\in\tilde K_0(A)$ is the reduced class $[L]-1$ of a line bundle L. By definition, $c_{12}(a)$ is the class of $L^{\pm1}$ in $Pic(A)/q\subseteq H^2_{et}(A;\mu_q)$ and $c_{i,2i}(a)=0$ for $i\ne 1$. Write λ for $c_{12}(a)\in Pic(A)/q\subseteq H^2(A;\mu_q)$, and consider the following product formula:

$$c_{ik}(\{a,b\})=\sum_{i_1+i_2=i}(-1)^{i_1}\binom{i-1}{i_1}\lambda^{i_1}c_{i_2k_2}(b). \tag{3.10}$$

The sum here is over all $i_1\ge 1$ and $i_2\ge n/2$ such that $i_1+i_2=i$, and we have set $k=2i-n$, $k_2=2i_2-n$.

Soulé gave a slightly different (incorrect) product formula in II.3 of [S], differing from (3.10) by $\pm i_1$. We have expanded his rather terse proof in order to show the key role played by Lemma 3.8 above. Soulé has remarked that Suslin has also noticed the formula (3.10) (personal communication).

Remark 3.10.1. We must have $rank(a)=0$ in (3.10). For example, 3.10 fails for $a=1$: $\{1,b\}=b$ but the right-hand side of (3.10) vanishes as $c_{i,2i}(1)=0$ for all i. This corrects a typo in [S, II.3].

Theorem 3.11. *Suppose that* $\pm a \in \tilde{K}_0(A)$ *is the reduced class* $[L]-1$ *of a line bundle, and that* $b \in K_n(A;\mathbb{Z}/q)$*. Then the product formula (3.10) holds for* $c_{ik}(\{a,b\})$ *in the following cases:*

(i) If $n \neq 2$*, or if* q *is odd.*
(ii) If $n = 2$ *and* $b \in K_2(A)$*.*
(iii) If $n = 2$*,* $4|q$ *and* $\partial b \in A^*$*.*
(iv) If $n = 2$ *and* $i = 2$*, when (3.10) becomes*

$$c_{22}(\{a,b\}) = -\lambda \cup [\partial b].$$

If $b \in K_2(A;\mathbb{Z}/2)$*,* $\partial b = -1 \in A^*$ *and* $i \geq 3$ *then there is an extra term:*

$$\text{(3.11.1)}\quad c_{i,2i-2}\{a,b\} = \sum (-1)^{i_1} \binom{i-1}{i_1} \lambda^{i_1} c_{i_2k_2}(b) + (-1)^{i-1}\lambda^{i-2} \cup \{-1,-1\}.$$

Proof. If P is a finitely generated projective A-module, let 1_P denote the trivial representation of G in $Aut(P) \subseteq GL(A)$. The corresponding homomorphisms

$$c_{ik}(1_P) : H_n(G;\mathbb{Z}/q) \to H^k_{et}(A;\mu_q^{\otimes i})$$

are zero unless $n = 0$ and $k = 2i$, when $1 \in H_0(G;\mathbb{Z}/q)$ maps to $c_{i,2i}(P)$. As in definition (3.5.0), if $J = (j_1, ...)$ is a multi-index let $\Phi c_J(1_P)$ denote the composition of the diagonal map $\Delta : H_n \to H_* \otimes \cdots \otimes H_*$ with the tensor product of the maps $\Phi(c_{j_\gamma}(1_P))$. Clearly this map is zero on $H_n(G;\mathbb{Z}/q)$ unless $n = 0$, when $\Phi c_J(1_P)$ maps 1 to $\prod c_{j_\gamma,2j_\gamma}(P)$. With this notation, Soulé proves on p. 265 of [S] that if $a = [P] - rank(P)$ then

$$\text{(3.11.2)}\quad \begin{aligned} c_{ik}(\{a,b\}) &= c_{ik}(id_m \otimes 1_P)(h_n b) \\ &= \Phi Q_i(c_*(id_m); c_*(1_P))(\Delta h_n b) \\ &= \Sigma a_{IJ} \Phi c_I(h_n b) \cup c_J(1_P)(1) \end{aligned}$$

If the $c_{i,2i}(P)$ vanish for $i \neq 1$, for example if P is a line bundle, we can restrict to the terms in which J is the multi-index $(1, ..., 1)$ of length i_1. If $I = (i_2)$ and $i_1 + i_2 = i$, Lemma 3.8 shows that the corresponding term is

$$(-1)^{i_1} \binom{i-1}{i_1} \lambda^{i_1} c_{i_2k_2}(b).$$

The sum of all these terms is the product formula (3.10). Therefore the formula (3.10) holds whenever $h_n b$ is primitive. This proves (i) and (ii); (iv) follows from the explicit formula (3.6) for Q_2.

Now suppose that $b \in K_2(A;\mathbb{Z}/q)$ and ∂b is a unit of A. By 3.5.1 there are no more terms in (3.11.2) if $4|q$, proving (iii). If $q = 2$ there is an extra term, when $I = (1,1)$ and $J = (1, ..., 1)$. In this case we know by 3.9 that $a_{IJ} = (-1)^{i-1}$, so by 3.5.1 the extra term is $\pm\lambda^{i-2} \cup \{-1, \partial b\}$. This establishes formula (3.11.1).

Remark 3.11.3. As the proof shows, Theorem 3.11 also applies to any $a \in \tilde{K}_0(A)$ such that $c_{i,2i}(a) = 0$ for $i \neq 1$.

4. Localization and Transfer.

In this section, A will always denote a Dedekind domain, and F will denote its field of fractions. The residue fields of A will be written as k_v; if A is a discrete valuation ring (DVR) we will drop the subscript v and just write k.

All the preliminary material in III.1 of [S] on étale cohomology remains true if $q = 2^\nu$, except for the comments on cohomological dimension of a global field at the end of III.1.1 and III.1.3. In particular, the localization sequences for a Dedekind domain A are valid for all q with $\frac{1}{q} \in A$:

$$\text{(4.0)}\quad \begin{array}{c} \cdots K_n(A;\mathbb{Z}/q) \to K_n(F;\mathbb{Z}/q) \xrightarrow{\partial} \bigoplus_v K_{n-1}(k_v;\mathbb{Z}/q) \to K_{n-1}(A;\mathbb{Z}/q)\cdots \\ 0 \to H^1_{et}(A,\mu_q^{\otimes i}) \to H^1_{et}(F,\mu_q^{\otimes i}) \xrightarrow{\partial} \bigoplus_v H^0_{et}(k_v,\mu_q^{\otimes i-1}) \to H^2_{et}(A,\mu_q^{\otimes i})\cdots \end{array}$$

The following result summarizes Propositions 2(i), 3 and 4 of [S,III.2-3]; case (a) is due to Quillen. Soulé's proof goes through for all q, although in the proof of (c) we need to observe that condition (iii) on p. 27 of [HS] holds when $q = 2^\nu$.

Proposition 4.1. *Suppose one of the following hypotheses holds:*
(a) $A = k[t]$;
(b) A is a DVR and $char(F) = char(k)$;
(c) A is a complete DVR and k is a finite field.
Then the localization sequences (4.0) for the mod q K-theory and étale cohomology of A and F split up into short exact sequences:

$$0 \to K_n(A;\mathbb{Z}/q) \to K_n(F;\mathbb{Z}/q) \xrightarrow{\partial} \bigoplus_v K_{n-1}(k_v;\mathbb{Z}/q) \to 0$$

$$0 \to H^k_{et}(A;\mu_q^{\otimes i}) \to H^k_{et}(F;\mu_q^{\otimes i}) \xrightarrow{\partial} \bigoplus_v H^{k-1}_{et}(k_v;\mu_q^{\otimes i}) \to 0.$$

In case (a) the Weil reciprocity law $\Sigma N_v\partial_v + \partial_\infty = 0$ also holds, where ∂_∞ corresponds to the place at infinity for $F = k(t)$. In case (c) the map $K_n(A;\mathbb{Z}/q) \to K_n(k;\mathbb{Z}/q)$ is a split surjection.

The next result concerns the compatability of these sequences with Chern classes. Consider the (noncommutative) diagrams arising from (4.0):

$$\begin{array}{ccccc} K_n(A;\mathbb{Z}/q) & \longrightarrow & K_n(F;\mathbb{Z}/q) & \xrightarrow{\partial} & \oplus K_{n-1}(k_v;\mathbb{Z}/q) \\ \downarrow{\scriptstyle c_{ik}} & & \downarrow{\scriptstyle c_{ik}} & & \downarrow{\scriptstyle c_{i-1,k-1}} \\ H^k_{et}(A;\mu_q^{\otimes i}) & \longrightarrow & H^k_{et}(F;\mu_q^{\otimes i}) & \xrightarrow{\partial} & \oplus H^{k-1}_{et}(k_v;\mu_q^{\otimes i}) \end{array}$$

Proposition 4.2. *Suppose that A is a Dedekind domain containing $\frac{1}{q}$. Then with one exception, the Chern classes c_{ik} on $K_n(F;\mathbb{Z}/q)$ satisfy*

$$\partial c_{ik} = (1-i)c_{i-1,k-1}\partial.$$

This exception is when $q = 2, n = 3, c_{ik} = c_{33}$ *and* $\mathbb{Q} \subseteq A$. *In this case, the formula* $\partial c_{33} = -2c_{22}\partial = 0$ *holds iff* $\{-1,-1\} = 0$ *in every* $H^2_{et}(k_v; \mathbb{Z}/2)$.

Proof. (Cf. pp. 271 and 274 of [S].) By naturality, we may assume that A is a complete DVR with parameter π. Now $K_*(A; \mathbb{Z}/q) \cong K_*(k; \mathbb{Z}/q)$ by [Gab]. Given $b \in K_{n-1}(A; \mathbb{Z}/q)$, form $\{\pi, b\} \in K_n(F; \mathbb{Z}/q)$ and note that $\partial\{\pi, b\}$ is the image $\bar{b}$ of b in $K_{n-1}(k; \mathbb{Z}/q)$. Unless $q = 2$, $n = 3$ and $i = 3$, Soulé's product formula (3.1) applies by 3.2(i,ii,iv,v) and 3.7.2, yielding

$$c_{ik}\{\pi, b\} = (1-i)c_{11}(\pi)c_{i-1,k-1}(b),$$
$$\partial c_{ik}\{\pi, b\} = (1-i)c_{i-1,k-1}(\bar{b}).$$

By a diagram chase we get $\partial c_{ik} = (1-i)c_{i-1,k-1}\partial$.

In the exceptional case ($q = 2, n = i = 3$), we argue as follows. Given $b \in K_2(A; \mathbb{Z}/2)$, the product formula (3.7.1) yields

$$c_{33}\{\pi, b\} = c_{11}(\pi)c_{11}^2(\partial b) \qquad \text{in } H^3_{et}(F, \mathbb{Z}/2), \text{and}$$
$$\partial c_{33}\{\pi, b\} = c_{11}^2(\partial\bar{b}) \qquad \text{in } H^2_{et}(k, \mathbb{Z}/2).$$

Since $\partial\bar{b} \in \mu_2(k) = \{\pm 1\}$, we see that formula (4.2) holds ($\partial c_{33} = 0$) iff $\{-1,-1\} = 0$ in $H^2_{et}(k; \mathbb{Z}/2)$. This is impossible if $char(k) \neq 0$.

Remark 4.2.1. Levine uses this proposition for c_{21} in the proof of Theorem 4.5 and 4.12 of [L]. For $F = k(t)$ the following diagram commutes, even if $q = 2$.

$$\begin{array}{ccccc}
K_3(F; \mathbb{Z}/q) & \xrightarrow{\partial} & \bigoplus_x K_2(k(x); \mathbb{Z}/2) & \longrightarrow & 0 \\
\downarrow{\scriptstyle c_{21}} & & \downarrow{\scriptstyle -c_{10}} & & \\
H'(F; \mu_q^{\otimes 2}) & \xrightarrow{\partial} & \bigoplus_x H^0_{et}(k(x); \mu_q) & \longrightarrow & 0
\end{array}$$

Remark 4.2.2. The symbol $\{-1,-1\} \in H^2_{et}(k; \mathbb{Z}/2)$ is nonzero for $k = \mathbb{R}$ as well as for $k = \hat{\mathbb{Q}}_2$ and any extension of $\hat{\mathbb{Q}}_2$ of odd degree. It is zero for every other local field. Therefore if k is a number field, this symbol is nonzero if and only if either k has a real embedding, or there is an unramified prime $\mathfrak{p}$ over 2 with residue field $\mathbb{F}_{2^f}$, f odd. This observation is due to W. Raskind.

Proposition 4.3. *Suppose that* A *is a Dedekind domain containing* $\frac{1}{q}$ *and a primitive* q^{th} *root of unity* ζ_q, $q \not\equiv 2 (mod\ 4)$. *Then for every finite residue field* $k = A/\mathfrak{p}$ *the following diagrams commute for all* $i \geq 2$, *with* n *and* k *so that* $n + k = 2i$.

$$\begin{array}{ccc}
K_n(k; \mathbb{Z}/q) & \xrightarrow{\tau} & K_n(A; \mathbb{Z}/q) \\
\downarrow{\scriptstyle (1-i)c_{i-1,k-2}} & & \downarrow{\scriptstyle c_{ik}} \\
H^{k-2}_{et}(k; \mu_q^{\otimes i-1}) & \xrightarrow{\tau} & H^k_{et}(A; \mu_q^{\otimes i})
\end{array}$$

Proof. By [Br, 2.6] and (3.1), every element $\bar{b}$ of $K_n(k; \mathbb{Z}/q)$ lifts to an element b of $K_n(A; \mathbb{Z}/q)$ with the property that $c_{jk}(b) = 0$ unless $k \leq 1$. Now $[k] = 1 - [\mathfrak{p}]$ in $K_0(A)$, so if we set $\lambda = c_{12}([k]) \in H^2(A; \mu_q)$ then the product formula 3.11(i,iii) yields:

$$c_{i,k}\tau(\bar{b}) = c_{i,k}(\{[k], b\}) = -(i-1)\lambda \cup c_{i-1,k-2}(b) = (1-i)\tau c_{i-1,k-2}(\bar{b}).$$

Corollary 4.3.1. *Suppose that A is a Dedekind domain containing $\frac{1}{q}$ and ζ_q, $q \not\equiv 2 (mod\ 4)$. If all the residue fields k_v of A are finite, there is a map of localization sequences (4.0):*

$$\begin{array}{ccccccccc}
\cdots K_{n+1}(F;\mathbb{Z}/q) & \xrightarrow{\partial} & \oplus K_n(k_v;\mathbb{Z}/q) & \xrightarrow{\tau} & K_n(A;\mathbb{Z}/q) & \to & K_n(F;\mathbb{Z}/q) & \xrightarrow{\partial} & \cdots \\
\downarrow c_{i,k-1} & & \downarrow (1-i)c_{i-1,k-2} & & \downarrow c_{ik} & & \downarrow c_{ik} & & \\
H^{k-1}_{et}(F;\mu_q^{\otimes i}) & \xrightarrow{\partial} & \oplus H^{k-2}_{et}(k_v;\mu_q^{\otimes i-1}) & \to & H^k_{et}(A;\mu_q^{\otimes i}) & \to & H^k_{et}(F;\mu_q^{\otimes i}) & \xrightarrow{\partial} & \cdots
\end{array}$$

Proof. Combine 2.1.1, 4.2 and 4.3.

Remark 4.3.2. See (5.3.1) for a version with $q = 2$ and $i = 2$. Clearly, the result (4.3.1) holds in much greater generality. For example if A is the coordinate ring of a smooth affine curve over an algebraically closed field then 4.3 and 4.3.1 hold for all q. However, care must be taken when $q = n = 2$, as the following example shows.

Example 4.3.3. When $A = \mathbb{R}[x, y]/(x^2 + y^2 = 1)$ and $k = A/(x - \cos\theta,\ y - \sin\theta)$ is any residue field isomorphic to $\mathbb{R}$, the following diagram does <u>not</u> commute.

$$\begin{array}{ccc}
K_2(\mathbb{R};\mathbb{Z}/2) & \xrightarrow{\tau} & K_2(A;\mathbb{Z}/2) \\
\downarrow -2c_{22}=0 & & \downarrow c_{34} \\
H^2_{et}(\mathbb{R};\mu_2^{\otimes 2}) & \xrightarrow{\tau} & H^4_{et}(A;\mu_2^{\otimes 3})
\end{array}$$

Indeed, $\lambda = c_{12}([k])$ is the nonzero element of $Pic(A) \cong \mathbb{Z}/2$ in $H^2_{et}(A;\mu_2)$. By [PW, 1.11] the cup product with $\{-1,-1\}$ induces an isomorphism $H^2_{et}(A;\mu_2) \cong H^4_{et}(A;\mu_2^{\otimes 3}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Letting β denote the mod 2 Bott element, the product formula (3.11.1) yields

$$c_{34}\tau(\beta) = c_{34}(\{[k], \beta\}) = \lambda \cup \{-1,-1\} \neq 0.$$

We remark that by [RW, 2.1] we have $K_2(A;\mathbb{Z}/2) \cong K_2(\mathbb{R};\mathbb{Z}/2) \oplus \mathbb{Z}/2$, the last summand being generated by $\tau(\beta)$. Since $c_{22}\tau(\beta) = -\lambda \cup c_{10}(\beta)$ by 3.11(iv), it follows that $c_{22} : SK_2(A;\mathbb{Z}/2) \cong H^2_{et}(A;\mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. This isomorphism should be compared with 5.3 below.

Proposition 4.4. *Let $k \subseteq \ell$ be a finite separable extension of fields containing $1/q$. Let N denote the transfer map in either K-theory or étale cohomology, and fix $x \in K_n(\ell;\mathbb{Z}/q)$, i and $k = 2i - n$. Then*

$$i\big\{c_{ik}(Nx) - Nc_{ik}(x)\big\} = 0.$$

Proof. Following the proof of [S, Thm. 2], choose a residue field k_w of $A = k[t]$ isomorphic to ℓ. By 4.1(a) there is a $y \in K_{n+1}(k(t);\mathbb{Z}/q)$ mapping to $x \cdot 1_\ell$ in

$\oplus K_n(k_v;\mathbb{Z}/q)$, and Weil reciprocity shows that $\partial_\infty(y) = -Nx$. Writing ∂_v for the summands of the cohomology boundary map in (4.0), we see by 4.2 that

$$\partial_v c_{i+1,k+1}(y) = \begin{cases} -ic_{ik}(x), & \text{if } v = w \\ +ic_{ik}(Nx), & \text{if } v = \infty \\ 0 & \text{otherwise} \end{cases}$$

unless $q = 2$, $n = 2$ and $i = 2$ (in which case the conclusion is trivial). Weil reciprocity for étale cohomology (4.1.a) finishes the proof.

Theorem 4.5. *Let A be a Dedekind ring whose field of fractions F is a global field. Then for all* <u>*even*</u> *n and all q the boundary map ∂ in the localization sequence for* mod q *K-theory is a surjection.*

$$K_n(F;\mathbb{Z}/q) \xrightarrow{\partial} \underset{v}{\oplus} K_{n-1}(k_v;\mathbb{Z}/q) \to 0.$$

Proof. Soulé proves this for odd q in [S], Theorem 3; his argument remains valid when $4|q$ because in that case $K_*(A;\mathbb{Z}/q)$ still has a product ([S2,p.327]). The case $q = 4$ implies the case $q = 2$ via the commutative diagram

$$\begin{array}{ccc} K_n(F;\mathbb{Z}/4) & \longrightarrow & \oplus K_{n-1}(k_v;\mathbb{Z}/4) \\ \downarrow & & \downarrow \\ K_n(F;\mathbb{Z}/2) & \longrightarrow & \oplus K_{n-1}(k_v;\mathbb{Z}/2), \end{array}$$

because when n is even the right vertical map is onto by (1.6.2).

Remark 4.5.1. The proof of 4.5 remains valid if A is any Dedekind domain for which all the residue fields are finite, $SK_1(A) = 0$ and $SK_1(B_q) = 0$, where B_q is the integral closure of $A[\zeta_q]$ in $F(\zeta_q)$, ζ_q a qth root of unity. I learned this from Clay Sherman [Sherm].

The following theorem is due to Soulé ([S,Thm 1], [S2,Thm 3], [Sherm]); I have included the short proof for the convenience of the reader.

Theorem 4.6. *Let A be a Dedekind ring whose field of fractions F is a global field. Then the localization sequence for K-theory breaks up into exact sequences:*

$$0 \to K_n(A) \to K_n(F) \to \underset{v}{\oplus} K_{n-1}(k_v) \to 0.$$

In particular, if n is odd and $n \neq 1$ then $K_n(A) \cong K_n(F)$.

Proof. Let $SK_n(A)$ denote the kernel of $K_n(A) \to K_n(F)$; it suffices to prove that $SK_n(A) = 0$ for all n. From Quillen's computation of $K_*(\mathbb{F}_t)$, $SK_n(A)$ is zero if n is even and finite if n is odd. If ℓ is a prime and $S \neq 0$ is the ℓ-Sylow subgroup of $SK_n(A)$, choose $q = \ell^\nu$ annihilating S. From the diagram

$$\begin{array}{ccccc} K_n(A) & \xrightarrow{q} & K_n(A) & \longrightarrow & K_n(A;\mathbb{Z}/q) \\ & & \downarrow & & \downarrow \\ & & K_n(F) & \longrightarrow & K_n(F;\mathbb{Z}/q) \end{array}$$

we see that S maps trivially into $K_n(F)$ and nontrivially into $K_n(A;\mathbb{Z}/q)$. This contradicts Theorem 4.5 because n is odd.

Remark 4.6.1. As in (4.5.1), it suffices to assume that the residue fields k_v are finite and that for every q the integral closure B_q of $A[\zeta_q]$ satisfies $SK_1(B_q)=0$.

5. The image of the Chern classes.

It is unrealistic to expect 2-primary Chern classes c_{ik} to be onto unless $i\le 2$. This is illustrated by any finite field $\mathbb{F}$ of order r. By Quillen, we have

$$K_{2i}(\mathbb{F};\mathbb{Z}/q)\cong K_{2i-1}(\mathbb{F};\mathbb{Z}/q)\cong K_{2i-1}(\mathbb{F})\otimes\mathbb{Z}/q\cong\mathbb{Z}/(q,r^i-1).$$

As observed in III.1.3 of [S], these groups are abstractly isomorphic to

$$H^0_{et}(\mathbb{F};\mu_q^{\otimes i})\cong H^1_{et}(\mathbb{F};\mu_q^{\otimes i})\cong\mathbb{Z}/(q,r^i-1).$$

Proposition 5.1. *Let $\mathbb{F}$ be a finite field of order r. The maps c_{i0} and c_{i1} may be identified with multiplication by $\pm(i-1)!$ on the cyclic group $\mathbb{Z}/(q,r^i-1)$.*

Proof ([S, IV.2]). First, suppose that $\mathbb{F}$ contains a primitive q^{th} root of unity ζ. If $\beta\in K_2(\mathbb{F};\mathbb{Z}/q)$ is such that $\partial\beta=\zeta$ then β^i and $\zeta\beta^{i-1}$ generate $K_{2i}(\mathbb{F};\mathbb{Z}/q)$ and $K_{2i-1}(\mathbb{F};\mathbb{Z}/q)$. On the other hand $H^0_{et}(\mathbb{F};\mu_q^{\otimes i})\cong\mu_q^{\otimes i}$ is generated by $\zeta^{\otimes i}$ and $H^1_{et}(\mathbb{F};\mu_q^{\otimes i})$ is generated by $c_{11}(\zeta)\otimes\zeta^{\otimes i-1}$. Therefore it suffices to observe that the product formula (3.1), which is valid by 3.2(iv$'$) and 3.3(ii), yields:

$$c_{i0}(\beta^i)=(-1)^{i-1}(i-1)!\zeta^{\otimes i},$$
$$c_{i1}(\zeta\beta^{i-1})=(-1)^{i-1}(i-1)!c_{11}(\zeta)\otimes\zeta^{\otimes i-1}.$$

If $\mathbb{F}$ does not contain ζ, we argue as follows. The inclusion $\mathbb{F}\subseteq\mathbb{F}(\zeta)$ induces inclusions of $K_{2i}(\mathbb{F};\mathbb{Z}/q)$ and $H^0(\mathbb{F};\mu_q^{\otimes i})$ into $K_{2i}(\mathbb{F}(\zeta);\mathbb{Z}/q)$ and $H^0(\mathbb{F}(\zeta);\mu_q^{\otimes i})$. By naturality, $c_{i0}=(-1)^{i-1}(i-1)!$. This implies the same result for c_{i1}, for if we set $s=r^i-1$ then by 2.8 there is a commutative diagram:

$$\begin{array}{ccc} K_{2i}(\mathbb{F};\mathbb{Z}/s) & \xrightarrow{\partial} & K_{2i-1}(\mathbb{F};\mathbb{Z}/q)\to 0 \\ \downarrow{\scriptstyle c_{i0}} & & \downarrow{\scriptstyle c_{i1}} \\ H^0(\mathbb{F};\mu_s^{\otimes i}) & \xrightarrow{\delta} & H^1(\mathbb{F};\mu_q^{\otimes i})\to 0. \end{array}$$

We have already described the Chern classes c_{11} and c_{10} in 2.2.1 and 2.6.1. We now turn to a description of the classes c_{2k}.

If F is a field, the Murkurjev-Suslin Theorem of [MS2] states that c_{22} induces an isomorphism $K_2(F)\otimes\mathbb{Z}/q\to H^2_{et}(A;\mu_q^{\otimes 2})$ for all q. More recently, Levine [L] and Merkurjev-Suslin [MS] proved that for any field F the map c_{21} induces an exact sequence for all q:

$$K_3^M(F)\to K_3(F;\mathbb{Z}/q)\overset{c_{21}}{\to}H^1_{et}(F;\mu_q^{\otimes 2})\to 0.$$

Corollary 5.1.1. *If A is the ring of integers in a local field F with residue field k, and $\frac{1}{q} \in A$, then $K_*(A;\mathbb{Z}/q) \cong K_*(k;\mathbb{Z}/q)$ by [Gab] and $H^*_{et}(A;\mu_q^{\otimes i}) \cong H^*_{et}(k;\mu_q^{\otimes i})$, so again c_{i0} and c_{i1} may be identified with multiplication by $\pm(i-1)!$.*

The groups $K_{2i-1}(F;\mathbb{Z}/q)$ and $H^1_{et}(F;\mu_q^{\otimes i})$ are also abstractly isomorphic by 4.1(c), and under this isomorphism we can again identify c_{i1} with multiplication by $\pm(i-1)!$.

Proposition 5.2. *(Soulé [S,IV.1.5]). Let F be a field containing $\frac{1}{q}$ and a primitive q^{th} root of unity ζ, $q \not\equiv 2 \pmod 4$. Then for $i \geq 2$ the cokernels of the maps*

$$\begin{aligned} c_{i0} &: K_{2i}(F;\mathbb{Z}/q) \to H^0_{et}(F;\mu_q^{\otimes i}) \cong \mu_q^{\otimes i} \\ c_{i1} &: K_{2i-1}(F;\mathbb{Z}/q) \to H^1_{et}(F;\mu_q^{\otimes i}) \cong F^* \otimes \mu_q^{\otimes i-1} \\ c_{i2} &: K_{2i-2}(F;\mathbb{Z}/q) \to H^2_{et}(F;\mu_q^{\otimes i}) \cong {}_qBr(F) \otimes \mu_q^{\otimes i-1} \end{aligned}$$

have exponents dividing $(i-1)!$.

Proof. Let $\beta \in K_2(F;\mathbb{Z}/q)$ be the mod q Bott element associated to ζ, as in 2.7.2. Let $x \in K_1(F)$ and $y \in K_2(F)$. The product formula (3.1), which holds by 3.2(iv′), yields

$$\begin{aligned} c_{i0}(\beta^i) &= (-1)^{i-1}(i-1)!\zeta^{\otimes i} \\ c_{i1}(\{x,\beta^{i-1}\}) &= (-1)^{i-1}(i-1)!c_{11}(x) \otimes \zeta^{\otimes i-1} \\ c_{i2}(\{y,\beta^{i-2}\}) &= (-1)^{i}(i-1)!c_{22}(y) \otimes \zeta^{\otimes i-2}. \end{aligned}$$

Since c_{11} and c_{22} are onto by 2.2.1 and [MS2], the result follows.

Remark 5.2.1. If $q = 2$ $(\zeta = -1)$ the assertions of 5.2 are vacuous unless $i = 2$. We know that c_{22} is onto by [MS2], and c_{21} is onto since $c_{21}(\{x,\beta\}) = c_{11}(x) \otimes \zeta$ ([MS], [L]). But the conclusion of 5.2 fails for c_{20} since c_{20} need not be onto. If $F = \mathbb{R}$ the map $c_{20} : K_4(\mathbb{R};\mathbb{Z}/2) \to H^0_{et}(\mathbb{R};\mu_q^{\otimes 2}) \cong \mu_2^{\otimes 2}$ is zero because $c_{20}(\{-1,-1,\beta\}) = 0$ by 3.2(ii) and (2.6.1) and $\{-1,-1,\beta\}$ is the only nonzero element of $K_4(\mathbb{R};\mathbb{Z}/2) \cong \mathbb{Z}/2$. However if $\{-1,-1\} = 0$ in $K_2(F)$ then β^2 is defined and c_{20} is onto because $c_{20}(\beta^2) = \zeta^{\otimes 2}$ is the generator of $\mu_2^{\otimes 2}$.

Remark 5.2.2. let A be any commutative ring containing $\frac{1}{q}$ and ζ. Then the assertion in 5.2 about c_{i0} remains valid for A. If $Pic(A)$ has no q-torsion, the assertion in 5.2 about c_{i1} remains valid.

For the next result, recall from 2.6.3 that the restriction of c_{22} to $SK_2(A;\mathbb{Z}/q)$ is a homomorphism.

Proposition 5.3. *Let A be a Dedekind domain containing $\frac{1}{q}$ and a primitive q^{th} root of unity ζ_q.*

(1) *Assume that for every residue field k_v the map $K_2(A) \to K_2(k_v) \otimes \mathbb{Z}/q$ is onto. Then there is an exact sequence:*

$$\oplus K_2(k_v) \otimes \mathbb{Z}/q \to SK_2(A;\mathbb{Z}/q) \xrightarrow{c_{22}} H^2_{et}(A;\mu_q^{\otimes 2}) \to 0.$$

(2) *If the residue fields of A are finite, there is an isomorphism:*

$$SK_2(A;\mathbb{Z}/q) \underset{c_{22}}{\xrightarrow{\cong}} H^2_{et}(A;\mu_q^{\otimes 2}).$$

(3) *If A lies in a global field then* $\quad c_{22} : K_2(A) \otimes \mathbb{Z}/q \cong H^2_{et}(A;\mu_q^{\otimes 2}).$

Proof. The result will follow from a diagram chase, once we establish that the diagram (5.3.1) is commutative with exact rows. Note that the maps c_{10} and c_{21} in (5.3.1) are onto, and the kernel of c_{10} is $\oplus K_2(k_v)\otimes\mathbb{Z}/q$; if the k_v are finite then c_{10} is an isomorphism.

(5.3.1)

$$\begin{array}{ccccccccc}
K_3(F;\mathbb{Z}/q) & \xrightarrow{\partial} & \oplus K_2(k_v;\mathbb{Z}/q) & \xrightarrow{\tau} & SK_2(A;\mathbb{Z}/q) & \to & SK_2(F;\mathbb{Z}/q) & \xrightarrow{\partial} & \oplus k_v^*\otimes\mathbb{Z}/q \\
\downarrow{\scriptstyle c_{21}} & & \downarrow{\scriptstyle -c_{10}} & & \downarrow{\scriptstyle c_{22}} & & \cong\downarrow{\scriptstyle c_{22}} & & \cong\downarrow{\scriptstyle -c_{11}} \\
H^1(F;\mu_q^{\otimes 2}) & \to & \oplus H^0(k_v;\mu_q) & \xrightarrow{\tau} & H^2(A;\mu_q^{\otimes 2}) & \to & H^2(F;\mu_q^{\otimes 2}) & \xrightarrow{\partial} & \oplus H^1(k_v;\mu_q)
\end{array}$$

The rows are the (exact) localization sequences (4.0). The outer squares commute by 4.2(b), and the third square by naturality (2.1.1). To see that the second square commutes, we mimic the argument of 4.3. Fix $\beta\in K_2(A;\mathbb{Z}/q)$, and write β_v for its image in $K_2(k_v;\mathbb{Z}/q)$. Now $[k_v]=1-[\mathfrak{p}]$ in $K_0(A)$, so if we set $\lambda=c_{12}([k_v])\in H^2_{et}(A;\mu_q)$ the product formula 3.11(iv) yields

$$c_{22}\tau(\beta_v)=c_{22}(\{[k_v],\beta\})=-\lambda\cup c_{10}(\beta)=-\tau c_{10}(\beta_v).$$

Porism 5.3.2. (Suslin [Sus2, 4.4]). Writing $H^0(A,\mathcal{K}_2)$ for the image of $K_2(A)$ in $K_2(F)$, we may use 2.3.6 to rewrite 5.3 as the exact sequence

$$0\to H^0(A,\mathcal{K}_2)\otimes\mathbb{Z}/q\xrightarrow{c_{22}} H^2_{et}(A;\mu_q^{\otimes 2})\xrightarrow{\partial} {}_qSK_1(A)\to 0.$$

Corollary 5.3.3. *(Soulé [S, IV.1.5]). Let A be a Dedekind domain satisfying the hypotheses of 5.3. Then the cokernel of*

$$c_{i2}: K_{2i-2}(A;\mathbb{Z}/q)\to H^2_{et}(A;\mu_q^{\otimes i})\cong H^2_{et}(A;\mu_q^{\otimes 2})\otimes\mu_q^{\otimes(i-2)}$$

has exponent dividing $(i-1)!$.

Proof. If $q=2$ there is nothing to prove, so assume $q\not\equiv 2\pmod 4$. Let $\beta\in K_2(A;\mathbb{Z}/q)$ be a mod q Bott element in the sense of 2.7.2. If $\alpha\in K_2(A;\mathbb{Z}/q)$ and $i\geq 3$ we can cite 3.2(iv$'$) and 2.7.2 to see that

$$c_{i2}(\{\alpha,\beta^{i-2}\})=(-1)^i(i-1)!c_{22}(\alpha)\otimes[\zeta]^{\otimes i-2}.$$

Since c_{22} maps onto $H^2_{et}(A;\mu_q^{\otimes i})$ by 5.3, the result follows.

Corollary 5.4. *(Soulé [S, IV.1.4]). Let A be a Dedekind domain whose field of fractions is a local or global field. Then for all q with* $\frac{1}{q}\in A$:

$$c_{22}: K_2(A)\otimes\mathbb{Z}/q\xrightarrow{\cong} H^2_{et}(A;\mu_q^{\otimes 2}).$$

Proof. Since $SK_1(A)=0$, this is 5.2 when $\mu_q\subseteq A$. In particular, this follows when q is prime (using a transfer argument for odd primes). The general case follows from induction, as in [MS2, 11.5], using the (slightly modified) diagram of 2.8

$$\begin{array}{ccccccccc}
K_3(A;\mathbb{Z}/\ell) & \to & K_2(A)\otimes\mathbb{Z}/\ell^\nu & \to & K_2(A)\otimes\mathbb{Z}/\ell^{\nu+1} & \to & K_2(A)\otimes\mathbb{Z}/\ell & \to & 0 \\
\downarrow{\scriptstyle c_{21}} & & \downarrow{\scriptstyle c_{22}} & & \downarrow{\scriptstyle c_{22}} & & \downarrow{\scriptstyle c_{22}} & & \\
H^1(A;\mu_\ell^{\otimes 2}) & \to & H^2(A;\mu_{\ell^\nu}^{\otimes 2}) & \to & H^2(A;\mu_{\ell^{\nu+1}}^{\otimes 2}) & \to & H^2(A;\mu_\ell^{\otimes 2}) & &
\end{array}$$

and 5.5 below, which says that c_{21} is onto.

Corollary 5.4.1. *(Soulé [S, IV.1.7]). Let F be a global field. Then the cokernel of $c_{i2}: K_{2i-2}(A;\mathbb{Z}/q) \to H^2_{et}(A;\mu_q^{\otimes i})$ has exponent dividing:*

$$\begin{array}{rl} i! & \textit{if } F \textit{ is totally imaginary, or} \\ 2i! & \textit{if } F \textit{ has a real embedding.} \end{array}$$

Proof. If q is odd, we cite [S, IV.1.7], so we can assume that $q = 2^\nu$. Let $N: K_*(F(\zeta);\mathbb{Z}/q) \to K_*(F;\mathbb{Z}/q)$ be the transfer map. If $x \in K_{2i-2}(F(\zeta);\mathbb{Z}/q)$ we know by 4.3 that

$$i\{c_{i2}(Nx) - Nc_{i2}(x)\} = 0.$$

If F is totally imaginary then the transfer map $N: H^2_{et}(F(\zeta);\mu_q^{\otimes i}) \to H^2_{et}(F;\mu_q^{\otimes i})$ is onto by [S, III.1.5]. By 5.3.3 the cokernel of c_{i2} has exponent dividing $i!$.

If F has a real embedding, the transfer factors as the surjection $H^2_{et}(F(\zeta);\mu_q^{\otimes i}) \to H^2_{et}(F(\sqrt{-1});\mu_q^{\otimes i})$ followed by a degree 2 transfer. Hence the image contains $2H^2_{et}(F;\mu_q^{\otimes i})$, and need not be onto. Therefore we lose another factor of 2.

Now suppose that A is a Dedekind domain with finite residue fields. Write $K_3(A)_{ind}$ and $K_3(A;\mathbb{Z}/q)_{ind}$ respectively for the quotients of $K_3(A)$ and $K_3(A;\mathbb{Z}/q)$ by the image of $K_3^M(F) \oplus \coprod_v K_3(k_v)$. These are respectively subgroups of $K_3(F)_{ind}$ and $K_3(F;\mathbb{Z}/q)_{ind}$, and there is a short exact sequence, split if $q \neq 2$:

$$0 \to K_3(A)_{ind} \otimes \mathbb{Z}/q \to K_3(A;\mathbb{Z}/q)_{ind} \to Tor(K_2(A),\mathbb{Z}/q) \to 0,$$

Theorem 5.5. *Let A be a Dedekind domain with finite residue fields. Then c_{21} induces an isomorphism for all q:*

$$K_3(A;\mathbb{Z}/q)_{ind} \xrightarrow{\cong} H^1_{et}(A;\mu_q^{\otimes 2}).$$

Proof. The localization sequences (4.0) fit into the diagram

$$\begin{array}{ccccccc} & & & & K_3^M(F) & & \\ & & & & \downarrow & & \\ & & K_3(A;\mathbb{Z}/q) & \longrightarrow & K_3(F;\mathbb{Z}/q) & \longrightarrow & \oplus K_2(k_v;\mathbb{Z}/q) \\ & & \downarrow c_{21} & & \downarrow c_{21} & & \cong \downarrow -c_{10} \\ 0 & \longrightarrow & H^1(A;\mu_q^{\otimes 2}) & \longrightarrow & H^1(F;\mu_q^{\otimes 2}) & \longrightarrow & \oplus H^0(k_v;\mu_q^{\otimes 2}) \end{array}$$

which commutes by 2.2.1 and 4.2. The result for A follows from the corresponding isomorphism $K_3(F;\mathbb{Z}/q)_{ind} \cong H^1(F,\mu_q^{\otimes 2})$ mentioned above.

Remark 5.5.1. If the residue fields are infinite, the appropriate statement involves K-cohomology, and is analogous to the analysis of 5.3.2. For example, the proof yields a natural surjection from $H^1(A;\mu_q^{\otimes 2})$ onto the q-torsion subgroup of $H^0(A,\mathcal{K}_2)$, and the analysis of [Sus2, §5] applies.

Corollary 5.6. *Let A be a Dedekind domain with finite residue fields. If A contains $\frac{1}{q}$ and a primitive q^{th} root of unity, then the cokernel of*

$$c_{i1} : K_{2i-1}(A;\mathbb{Z}/q) \to H^1_{et}(A;\mu_q^{\otimes i})$$

has exponent dividing $(i-1)!$.

Proof. Combine the proofs of 5.2 and 5.3.

6. The e-invariant and Harris-Segal summands.

Let F be a field of characteristic 0, and $\bar{F}$ its algebraic closure. By [Sus1], the torsion subgroup of $K_{2i-1}(\bar{F})$ is $\mathbb{Q}/\mathbb{Z}(i)$, the union over q of the Galois groups $\mu_q^{\otimes i}$. The map $K_{2i-1}(F) \to K_{2i-1}(\bar{F})$ must send the torsion subgroup ${}_{tor}K_{2i-1}(F)$ of $K_{2i-1}(F)$ to $H^0_{et}(F;\mathbb{Q}/\mathbb{Z}(i))$. We shall use the term *e-invariant* to describe either the map (which is well-defined up to automorphism of $\mathbb{Q}/\mathbb{Z}(i)$)

$${}_{tor}K_{2i-1}(F) \to H^0_{et}(F;\mathbb{Q}/\mathbb{Z}(i))$$

or the map it induces on mod q K-theory, i.e., the horizontal maps of

$$\begin{array}{ccccc} K_{2i}(F;\mathbb{Z}/q) & \xrightarrow{e} & H^0_{et}(F;\mu_q^{\otimes i}) & \subseteq \mu_q^{\otimes i}(\bar{F}) = & K_{2i}(\bar{F};\mathbb{Z}/q) \\ \downarrow & & \downarrow & & \downarrow \\ {}_{tor}K_{2i-1}(F) & \longrightarrow & H^0_{et}(F;\mathbb{Q}/\mathbb{Z}(i)) & \subseteq & \mathbb{Q}/\mathbb{Z}(i). \end{array}$$

By [Q], the map e is a generalization of Adam's (complex) e-invariant, whence the name. The e-invariant for $i=2$ is the same as the map "d" of [MS, §9].

If F is a field of characteristic $p \neq 0$, with separable closure $\bar{F}$, we can repeat the above by restricting to those q not divisible by p. In this case the e-invariant refers either to the map $e : K_{2i}(F;\mathbb{Z}/q) \to H^0_{et}(F;\mu_q^{\otimes i}) \subseteq K_{2i}(\bar{F};\mathbb{Z}/q)$ or to the map ${}_{tor}K_{2i-1}(F) \to \mathbb{Q}/\mathbb{Z}[\frac{1}{p}](i)$. Our interest in the e-invariant stems from the following result.

Proposition 6.1. *If F is a field containing $\frac{1}{q}$, then $c_{i0} : K_{2i}(F;\mathbb{Z}/q) \to H^0_{et}(F;\mu_q^{\otimes i})$ is $(-1)^{i-1}(i-1)!$ times the e-invariant. In particular, the integral Chern class $c_{i0} : K_{2i}(F) \to H^0_{et}(F;\mu_q^{\otimes i})$ is always zero.*

Proof. If $F = \bar{F}$ this follows from 1.6.2 and the product formula, as in the proof of 5.1. The result for general F follows by naturality:

$$\begin{array}{ccc} K_{2i}(F;\mathbb{Z}/q) & \to & K_{2i}(\bar{F};\mathbb{Z}/q) \\ \downarrow c_{i0} & & \downarrow c_{i0} \\ H^0_{et}(F;\mu_q^{\otimes i}) & \hookrightarrow & \mu_q^{\otimes i} \cong \mathbb{Z}/q. \end{array}$$

Remark 6.1.1. For any scheme X over $\mathbb{Z}[\frac{1}{q}]$, $\mu_q^{\otimes i}$ is isomorphic to the étale sheaf associated to the presheaf $U \mapsto K_{2i}(U;\mathbb{Z}/q)$. (This observation follows from Theorem 1 of [Gab].) Therefore we may define the *e-invariant* on X to be the resulting map

$$K_{2i}(X;\mathbb{Z}/q) \to {}_qK_{2i-1}(X) \to H^0_{et}(X;\mu_q^{\otimes i}).$$

Corollary 6.1.2. *If A is any commutative ring containing $\frac{1}{q}$ then the integral Chern class $c_{i0} : K_{2i}(A) \to H^0_{et}(A;\mu_q^{\otimes i})$ is zero, and Soulé's Chern class factors through the q-torsion subgroup ${}_qK_{2i-1}(A)$ of $K_{2i-1}(A)$. In fact, c_{i0} is $(-1)^{i-1}(i-1)!$ times the e-invariant.*

Example 6.2. Let F be a global field, and consider the induced map c_{20} from ${}_qK_3(F)$ to $H^0(F;\mu_q^{\otimes 2}) \cong \mathbb{Z}/(q,w_2)$. Replacing F by $F(\zeta)$ and using 5.2, we see that c_{20} is the e-invariant; see [MS,§9]. By Theorem 10.2 of [MS] we have:

- If F is totally imaginary or if q is odd we have an isomorphism:

$$c_{20} : {}_qK_3(F) \xrightarrow{\cong} H^0_{et}(F;\mu_q^{\otimes 2})$$

- If F has $r_1 \neq 0$ real embeddings and $q = 2^\nu$, the kernel of c_{20} is the Milnor K-group $K_3^M(F) \cong (\mathbb{Z}/2)^{r_1}$. If ν is sufficiently large, then $H^0_{et}(F;\mu_q^{\otimes 2}) \cong \mathbb{Z}/w_2$ is independent of q and $K_3(F) \cong (\mathbb{Z}/2w_2) \oplus (\mathbb{Z}/2)^{r_1-1} \oplus \mathbb{Z}^{r_2}$. Hence c_{20} is onto for $q \geq 2w_2$ but if $q \leq w_2$ there is a non-split exact sequence

$$0 \to K_3^M(F) \to {}_qK_3(F) \xrightarrow{c_{20}} H^0_{et}(F;\mu_q^{\otimes i}) \to \mathbb{Z}/2 \to 0.$$

It is useful to have a formula for the order $w_i = w_i(F)$ of $H^0_{et}(F;\mathbb{Q}/\mathbb{Z}(i))$, or rather for the order $w_i^{(\ell)}$ of its ℓ-Sylow subgroup.

First of all, suppose that $\ell \neq 2$ and let $0 \leq m \leq \infty$ be maximal such that F contains a primitive ℓ^{th}-root of unity ζ_{ℓ^m}. The following formula is due to Harris and Segal [HS]. If $\zeta_\ell \in F$ then $\omega_i^{(\ell)} = \ell^{m+\lambda}$, where $i = \ell^\lambda s$ with s prime to ℓ. If $\zeta_\ell \notin F$ then

$$w_i^{(\ell)}(F) = \begin{cases} w_i^{(\ell)}(F(\zeta_\ell)) & \text{if } i \equiv 0 \bmod (\ell-1) \\ 1 & \text{otherwise.} \end{cases}$$

Definition 6.3.0. For the order $w_i^{(2)}$ of the 2-Sylow subgroup, we need to introduce some notation due to Harris and Segal [HS]. Call a field F *exceptional* if the Galois group $Gal(F(\zeta)/F)$ is not cyclic for some primitive 2^ν root of unity ζ, and *non-exceptional* otherwise. If F has a real embedding then F is exceptional, while if $\sqrt{-1} \in F$ then F is non-exceptional. A field F is non-exceptional iff there is a 2-primary root of unity $\zeta \neq \pm 1$ such that $\zeta - \zeta^{-1} \in F$ (see [K]).

Lemma 6.3. *(cf. [HS,p.28]) let F be a field of characteristic $\neq 2$.*

(a) If F contains $\zeta_4 = \sqrt{-1}$, write $i = 2^\lambda s$ with s odd. If m is maximal such that F contains a primitive 2^m-root of unity then $w_i^{(2)}(F) = 2^{m+\lambda}$.

(b) If $\sqrt{-1} \notin F$ and i is odd then $w_i^{(2)}(F) = 2$.

(c) If $\sqrt{-1} \notin F$, F is exceptional and i is even then $w_i^{(2)}(F) = w_i^{(2)}(F(\sqrt{-1}))$

(d) If $\sqrt{-1} \notin F$, F is non-exceptional and i is even then

$$w_i^{(2)}(F) = \frac{1}{2} w_i^{(2)}(F(\sqrt{-1})).$$

Corollary 6.3.1. *When $F=\mathbb{Q}$ and $i=2^\lambda s$ with s odd, then*

$$w_i^{(2)}(\mathbb{Q})=\begin{cases} 8\cdot 2^{\lambda-1} & \text{if } i \text{ is even}\\ 2 & \text{if } i \text{ is odd}\end{cases}$$

If A is a Dedekind domain with field of fractions F, we have $H^0_{et}(A;\mu_q^{\otimes i})\cong H^0_{et}(F;\mu_q^{\otimes i})$. If F is a local or global field then by 4.6 and 6.1.2 the map c_{i0} factors through ${}_qK_{2i-1}(A)={}_qK_{2i-1}(F)$. When $i=2$ and $r_1\neq 0$, 6.2 shows that the cokernel of c_{i0} depends on q.

Theorem 6.4. *(Harris-Segal). If F is a non-exceptional global field, the e-invariant is naturally split, so that there are natural cyclic direct summands E^i of $K_{2i-1}(F)$ isomorphic to $\mathbb{Z}/w_i(F)$. In particular, this is the case if $char(F)\neq 0$ or if F is a number field containing $\sqrt{-1}$.*

Proof. Let S be a finite nonempty set of places of F and A the ring of S-integers in F. By 4.6 the group $K_{2i-1}(A)\cong K_{2i-1}(F)$ is independent of S. Harris and Segal proved in [HS] that $K_{2i-1}(A)$ has a direct summand $E^i(F)$ isomorphic to $\mathbb{Z}/w_i(F)$. If $F\subset F'$ is a finite separable extension, their proof shows that $E^i(F)$ injects into $E^i(F')$. Now take the direct limit over all F' to see that $E^i(F)$ injects into the torsion subgroup of $K_{2i-1}(\bar F)$; this map is the e-invariant.

Now suppose that F is exceptional, hence a number field. Let $E^i_{(\ell)}$ denote the ℓ-Sylow subgroup of $E^i(F(\sqrt{-1}))$, and $\tau_*: K_{2i-1}(F(\sqrt{-1}))\to K_{2i-1}(F)$ the transfer map. If $\ell\neq 2$, $E^i_{(\ell)}$ maps isomorphically onto a summand $E^i_{(\ell)}(F)$ of $K_{2i-1}(F)$; if $\ell=2$ all we can say is that $\tau^*: K_{2i-1}(F)\to K_{2i-1}(F(\sqrt{-1}))$ maps $\tau_*(E^i_{(2)})$ onto the subgroup $2E^i_{(2)}$ [HS,p.30]. From this, Harris and Segal deduce the following result.

Corollary 6.4.1. *(Harris-Segal) If F is an exceptional number field then there is a natural cyclic direct summand E^i of $K_{2i-1}(F)$ of order either w_i, $2w_i$ or $w_i/2$.*

In the rest of this section we make some improvements upon the uncertainty in the 2-Sylow subgroup of E^i.

First suppose that i is odd, so that $w_i^{(2)}(F)=2$; Harris and Segal's assertion about $E^i_{(2)}$ is vacuous in this case as $\tau^*\tau_*E^i_{(2)}(F(\sqrt{-1}))=0$.

If $i=4k+1$, so that $2i-1=8k+1$, we define $E^i_{(2)}(F)$ to be the subgroup of $K_{8k+1}(F)$ generated by the image of the Adams element η_{8k+1} of order 2 in π^s_{8k+1}. Browder proved in [Br,4.7] (cf. [Q]) that this element maps nontrivially into $K_{8k+1}(\mathbb{C})$, so $E^i_{(2)}(F)\cong\mathbb{Z}/2$. As the target of the e-invariant is the subgroup $\mathbb{Z}/2$ of $K_{8k+1}(\bar{\mathbb{Q}})$, $E^i_{(2)}(F)$ is a direct summand of $K_{8k+1}(F)$. In this case we define the Harris-Segal summand to be $E^i=E^i_{odd}\oplus E^i_{(2)}$; it is mapped by the e-invariant isomorphically onto $H^0_{et}(F;\mathbb{Q}/\mathbb{Z}(i))\cong\mathbb{Z}/w_i$.

If $i=4k+3$, so that $2i-1=8k+5$, we consider two cases. If F has a real embedding, then $w_i(F)=2$. The e-invariant $K_{8k+5}(F)\to\mathbb{Z}/2$ is zero because it factors through the group $K_{8k+5}(\mathbb{R})$, which is uniquely divisible by [Sus1]. Since the image of $K_{8k+5}(F)$ in $K_{8k+5}(F(\sqrt{-1}))$ misses $E^i(F(\sqrt{-1}))$ and $\tau^*\tau_*$ kills $E^i(F(\sqrt{-1}))$, we may as well define $E^i(F)$ to be zero.

If F is totally imaginary the situation is completely different, as the following result shows: the e-invariant is an isomorphism, provided we define the 2-Sylow subgroup of E^i properly.

Proposition 6.5. *If F is a totally imaginary number field, then for every odd i there is a natural cyclic direct summand E^i of $K_{2i-1}(F)$ of order $w_i(F)$, mapped isomorphically by the e-invariant to $H^0_{et}(F;\mathbb{Q}/\mathbb{Z}(i)) \cong \mathbb{Z}/w_i$. The 2-Sylow subgroup of E^i has order 2.*

Proof. Given 6.4.1 and 6.3(b), it suffices to find a naturally defined element of order 2 in $K_{4k+2}(F;\mathbb{Z}/8)$ which is detected by the e-invariant. Let $\beta \in K_2(\mathbb{C};\mathbb{Z}/8)$ be a mod 8 Bott element, and $\gamma \in K_2(\mathbb{Z};\mathbb{Z}/8) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ an element mapping to 4β in $K_2(\mathbb{C};\mathbb{Z}/8)$. Let $z \in K_3(\mathbb{Z})$ be an element of order 16 mapping to $\partial(\beta^2)$ in $K_3(\mathbb{C})$. By [MS] the image $\bar{z}$ of z in $K_3(F)$ has order 8, so there also exists an element $\omega \in K_4(F;\mathbb{Z}/8)$ with $\partial\omega = \bar{z}$. By construction, the image of ω in $K_4(\mathbb{C};\mathbb{Z}/8)$ is β^2. The product $\omega^k\gamma \in K_{4k+2}(F;\mathbb{Z}/8)$ maps to the nonzero element $4\beta^{2k+1}$ of $K_{4k+2}(\mathbb{C};\mathbb{Z}/8) \cong \mathbb{Z}/8$, so $\omega^k\gamma$ is detected by the e-invariant. Since $2(\omega^k\gamma) = \omega^k(2\gamma) = 0$, $\omega^k\gamma$ is our sought-for element.

We now turn to the situation when $i = 2n$ is even. We first describe the situation when F is totally imaginary.

Theorem 6.6. *Let F be a totally imaginary number field and $w_i = w_i(F)$. Then*

(a) If $i = 4k+2$, the Harris-Segal summand E^i of $K_{8k+3}(F)$ has order w_i, and the e-invariant induces an isomorphism

$$e : E^i \cong H^0_{et}(F;\mathbb{Q}/\mathbb{Z}(i)) \cong \mathbb{Z}/w_i.$$

(b) If $i = 4k$, the Harris-Segal summand E^i of $K_{8k-1}(F)$ has order either w_i or $\frac{1}{2}w_i$, and the e-invariant maps E^i injectively into $H^0_{et}(F;\mathbb{Q}/\mathbb{Z}(i)) \cong \mathbb{Z}/w_i$.

Proof. Given 6.4.1, it suffices to determine the 2-Sylow subgroup of E^i. Set $q = w_2^{(2)}(F)$, and let $\beta \in K_2(\mathbb{C};\mathbb{Z}/q)$ be a mod q Bott element. By 6.2 there is an element $\omega \in K_4(F;\mathbb{Z}/q)$ mapping to $\beta^2 \in K_4(\mathbb{C};\mathbb{Z}/q)$. The element ω^n of $K_{4n}(F;\mathbb{Z}/q)$ maps to β^{2n}, so both ω^n and $\partial(\omega^n) \in K_{4n-1}(F;\mathbb{Z}/q)$ have order q and are mapped by the e-invariant to an element of order q. This proves (b). If n is odd then $w_{2n}^{(2)}(F) = q$, and $\partial(\omega^n)$ generates a cyclic summand of order q in $K_{4n-1}(F)$. This proves (a).

Remark 6.6.1. If F is totally imaginary then by [DF, 5.2] the 2-primary étale K-theory of $Spec(F)$ fits into an extension

$$0 \to H^2_{et}(F;\mu_q^{\otimes i+1}) \to K^{et}_{2i}(F,\mathbb{Z}/q) \to H^0_{et}(F;\mu_q^{\otimes i}) \to 0.$$

There is a natural map $\rho : K_{2i}(F;\mathbb{Z}/q) \to K^{et}_{2i}(F;\mathbb{Z}/q)$, and it is not hard to see that the e-invariant is the composition of ρ with the above surjection. If $\sqrt{-1} \in F$, then ρ is onto by [DF, 8.2]. In general, the surjectivity of ρ would imply that the Harris-Segal summand E^{4k} of $K_{8k-1}(F)$ has order w_{4k}.

Remark 6.6.2. It seems reasonable that for a totally imaginary number field F the Harris-Segal summand E^i of $K_{2i-1}(F)$ is mapped isomorphically by the e-invariant onto $H^0_{et}(F;\mathbb{Q}/\mathbb{Z}(i)) \cong \mathbb{Z}/w_i$. By 6.4, 6.5 and 6.6 this is true unless $4|i$. The next result implicitly establishes this result for any F such that $F(\sqrt{-1})$ contains no primitive 8^{th} root of unity.

Theorem 6.7 ($F = \mathbb{Q}$). *For every $n \geq 1$, let $w = w_{2n}(\mathbb{Q})$ denote the order of the cyclic group $H^0_{et}(\mathbb{Q};\mathbb{Q}/\mathbb{Z}(2n))$. Then:*

(a) $Im(J)_{4n-1} \cong \mathbb{Z}/w$.

(b) The denominator of $\zeta(1-2n)$ is $\frac{1}{2}w$.

(c) The Harris-Segal direct summand E^{2n} of $K_{4n-1}(\mathbb{Z}) = K_{4n-1}(\mathbb{Q})$ contains the image of J.

(d) If $4n-1 \equiv 7 (mod\ 8)$ then $E^{2n} = Im(J)_{4n-1} \cong \mathbb{Z}/w$ and the e-invariant induces an isomorphism

$$e : E^{2n} \cong H^0_{et}(\mathbb{Q};\mathbb{Q}/\mathbb{Z}(2n)).$$

(e) If $4n-1 \equiv 3 (mod\ 8)$ then E^{2n} is cyclic of order $2w$ and the e-invariant induces a short exact sequence:

$$0 \to \mathbb{Z}/2 \to E^{2n} \xrightarrow{e} H^0_{et}(\mathbb{Q};\mathbb{Q}/\mathbb{Z}(2n)) \to 0.$$

Proof. Let B_n denote the n^{th} Bernoulli number. It is well-known that $\zeta(1-2n) = -B_n/2n$, and that $Im(J)_{4n-1}$ is a cyclic group of order equal to the denominator of $B_n/4n$; see [CC,p.284]. Since the numerator of B_n is odd, this shows that (a) implies (b). The calculation of the denominator of B_n/n on p. 284 of [CC], compared with the calculation of w_{2n} in 6.3.1, proves (a). By [Q], $Im(J)_{4n-1}$ injects into $K_{4n-1}(\mathbb{Z})$, and is a direct summand if $4n-1 \equiv 7(\text{mod } 8)$. The construction used by Harris and Segal shows that $Im(J)_{4n-1}$ is contained in E^{2n} (see 3.2 and p. 30 of [HS], recalling that $16|w_{2n}$). Hence $Im(J)_{4n-1} = E^{2n}$, except possibly for the 2-Sylow subgroup when $4n-1 \equiv 3(\text{mod } 8)$. The 2-Sylow subgroup of $Im(J)_{8k+3}$ is $\mathbb{Z}/8$, and is contained in a $\mathbb{Z}/16$ summand of $K_{8k+3}(\mathbb{Z})$ by [Br,4.8]. Hence the 2-Sylow subgroup of E^{2n} must be that $\mathbb{Z}/16$ summand.

Application 6.7.1. If $q = 2^\nu$ the 2-primary Chern class

$$c_{2n,0} : K_{4n}(\mathbb{Q};\mathbb{Z}/q) \to {}_qE^{2n} \to H^0_{et}(\mathbb{Q};\mu_q^{\otimes 2n}) \cong \mathbb{Z}/(w_{2n}, q)$$

is zero unles n is 1, 2 or 4.

- ($n = 1$) $E^2 = K_3(\mathbb{Q}) \cong \mathbb{Z}/48$, and the kernel of

  $$c_{20} : \mathbb{Z}/(48, q) = {}_q(E^2) \to \mathbb{Z}/(24, q)$$

  has order 2. This map is onto iff $16|q$.

- ($n = 2$) $E^4 \subseteq K_7(\mathbb{Z})$ is isomorphic to $\mathbb{Z}/w_4 = \mathbb{Z}/240 \cong \mathbb{Z}/16 \oplus \mathbb{Z}/15$ and c_{40} may be identified with multiplication by 6.
- ($n = 4$) $E^8 \subseteq K_{15}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}/w_8 = \mathbb{Z}/480 \cong \mathbb{Z}/32 \oplus \mathbb{Z}/15$ and c_{80} may be identified with multiplication by $16 \cdot 15$. This map is nonzero iff $32|q$.

Remark 6.7.2. Browder asserted in [Br,4.8] that the $\mathbb{Z}/48$ in $K_3(\mathbb{Z})$ reproduces itself periodically as a direct summand of every $K_{8k+3}(\mathbb{Z})$. However, the $\mathbb{Z}/3$ is only a summand if $k \not\equiv 1$ mod 3. For example, the summand of $K_{11}(\mathbb{Z})$ is

$$E^6 \cong \mathbb{Z}/16 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/7.$$

Corollary 6.8. *Let F be a number field with a real embedding, and $w_i = w_i(F)$.*

(a) If $i = 4k + 2$, the Harris-Segal summand E^i of $K_{8k+3}(F)$ is cyclic of order w_i or $2w_i$, and injects into $K_{8k+3}(\mathbb{R})$. Hence the kernel of $e : E^i \to H^0_{et}(F; \mathbb{Q}/\mathbb{Z}(i))$ has order 2.

(b) If $i = 4k$, the Harris-Segal summand E^i of $K_{8k-1}(F)$ is cyclic of order w_i or $\frac{1}{2}w_i$, and the e-invariant maps E^i injectively into $H^0_{et}(F; \mathbb{Q}/\mathbb{Z}(i)) \cong \mathbb{Z}/w_i$.

Proof. The composition $E^i(\mathbb{Q}) \to E^i(F) \to E^i(\mathbb{R}) = \mathbb{Q}/\mathbb{Z}$ is an injection by 6.7, and $E^i(\mathbb{Q})$ is cyclic of even order. Therefore, the 2-Sylow subgroup of the cyclic group $E^i(F)$ must inject into $E^i(\mathbb{R})$. As the odd subgroup of $E^i(F)$ injects by Harris and Segal's result 6.4, $E^i(F)$ must inject into $E^i(\mathbb{R})$. Now $E^i(\mathbb{R} \to E^i(\mathbb{C})$ is an isomorphism if $i = 4k$, but a surjection with kernel $\mathbb{Z}/2$ if $i = 4k + 2$. Since the image of $E^i(F)$ in $E^i(\mathbb{C})$ has order w_i or $\frac{1}{2}w_i$, the result follows.

Remark 6.8.1. If F has a real embedding, we have shown that the "real e-invariant" $e_{\mathbb{R}} : E^i(F) \to E^i(\mathbb{R}) \cong \mathbb{Q}/\mathbb{Z}(i)$ is an injection for all i. It is not hard to see that this map is independent of the choice of real embedding, even though this choice affects the extension of $e_{\mathbb{R}}$ to the entire torsion subgroup

$${}_{tor}K_{2i-1}(F) \to E^i(\mathbb{R}) \cong \mathbb{Q}/\mathbb{Z}(i).$$

It seems reasonable to expect that the order of $E^i(F)$ is: $2w_i$ if $i = 4k+2$, and w_i if $i = 4k$. That is, we should expect the real e-invariant to be an isomorphism:

$$e_{\mathbb{R}} : E^i(F) \cong H^0(F; E^i(\mathbb{R})).$$

7. Towards the cohomology conjectures.

Let F be a number field of characteristic $\neq 2$ and write A for $\mathcal{O}_F[\frac{1}{2}]$. Lichtenbaum's conjectures at the prime $\ell = 2$ involve the 2-Sylow subgroup $W^{(i)}$ of $\mathbb{Q}/\mathbb{Z}(i)$. Note that $W^{(i)}$ is also the étale sheaf $\lim\limits_{\nu\to\infty} \mu_{2^\nu}^{\otimes i}$ of all 2-primary roots of unity, Tate twisted i times. As noted in III.1.3 of [S], the inclusion $j : Spec(F) \to Spec(A)$ induces isomorphisms $j_*\mu_q^{\otimes i} \cong \mu_q^{\otimes i}$ and hence $j_*W^{(i)} \cong W^{(i)}$.

Lichtenbaum conjectured in [Licht] that :
($L1$) if F is totally real then $H^k_{et}(A; W^{(i)}) = 0$ for $k \geq 2$ and $i \geq 2$ even, and
(L2) the 2-part of $\zeta_F(-1)$ equals the Euler characteristic

$$\chi = {}^{\#}H^1_{et}(A; W^{(i)})/{}^{\#}H^0_{et}(A; W^{(i)}).$$

Lichtenbaum's analogous conjectures for odd primes ℓ have been verified. For odd ℓ, $cd_\ell(A) = 2$ implies (L1) except for the vanishing of H^2 which was proven by Soulé in [S,IV.3.2]. Recently, A. Wiles proved the Main Conjecture of Iwasawa Theory in [Wiles], which implies that the ℓ-part of $\zeta_F(-1)$ equals the corresponding ℓ-primary Euler characteristic.

The case $i = 2$ of (L2) is equivalent to (the 2-primary part of) the Birch-Tate conjecture that ${}^\# K_2(A) = w_2(F)|\zeta_F(-1)|$. Indeed, $H^0(A; W^{(2)}) \cong (\mathbb{Z}/w_2(F))_{(2)}$ and $K_2(A)_{(2)} \cong H^1(A; W^{(2)})$ by [Sus2,3.11]. The Birch-Tate conjecture has been verified for many families of totally real F by Hurrelbrink and Kolster.

It is an easy calculation using [MS2] that $H^2(A, W^{(2)}) = 0$, but the following result shows that $H^3(A; W^{(2)}) \cong (\mathbb{Z}/2)^{r_1}$. Hence (L1) fails.

Proposition 7.1. *Let F be a number field with r_1 real embeddings. Then for $k \geq 3$ and $i \geq 1$:*

$$H^k_{et}(A; W^{(i)}) \cong H^k_{et}(F; W^{(i)}) \cong \begin{cases} (\mathbb{Z}/2)^{r_1} & \text{if } k+i \text{ is odd} \\ 0 & \text{if } k+i \text{ is even} \end{cases}$$

Proof. If k is a finite field then $H^k_{et}(k; W^{(i)}) = 0$ for $k \geq 2$. If $k = 1$ and $i \geq 1$ then $H^1(k; W^{(i)}) = 0$ by 5.1. The group $H^1(k; W^{(0)})$ is $W^{(0)} \cong \mathbb{Q}_2/\mathbb{Z}_2$. Since $\frac{1}{2} \in A$, the map from $Br(F) \cong H^2(F; W^{(1)})$ to $\oplus H^1(k_v; W^{(0)})$ is onto by local class field theory. Therefore the localization sequence (4.0) yields $H^k(A; W^{(i)}) \cong H^k(F; W^{(i)})$ for $k \geq 3$. On the other hand, Tate proved (see [AV] [Tate]) that the natural map

$$H^k_{et}(F; W^{(i)}) \to \bigoplus_{r_1} H^k_{et}(\mathbb{R}; W^{(i)})$$

is an isomorphism for $k \geq 3$ and a surjection for $k = 2$. Therefore it suffices to compute $H^*_{et}(\mathbb{R}; W^{(i)})$, which we do in the following lemma.

Lemma 7.1.1.

$$H^k_{et}(\mathbb{R}; W^{(i)}) \cong \begin{cases} W^{(i)} & \text{if } k=0 \text{ and } i \text{ is even} \\ \mathbb{Z}/2 & \text{if } k \geq 0 \text{ and } k+i \text{ is odd} \\ 0 & \text{if } k>0 \text{ and } k+i \text{ is even} \end{cases}$$

Proof. Let σ denote complex conjugation. If i is even then $\sigma = 1$ on $W^{(i)}$; if i is odd then $\sigma = -1$ on $W^{(i)}$. The Lemma follows from the observation that $H^*(\mathbb{R}, W^{(i)})$ is the cohomology of the periodic chain complex:

$$0 \to W^{(i)} \xrightarrow{1-\sigma} W^{(i)} \xrightarrow{1+\sigma} W^{(i)} \xrightarrow{1-\sigma} \cdots .$$

Remark 7.1.2. In case $i = 1$ we have an isomorphism

$$\begin{aligned} H^k(A; W^{(1)}) &\cong \text{2-torsion in } H^k(A; \mathbb{G}_m) \oplus H^{k-1}(A; \mathbb{G}_m) \otimes W^{(0)} \\ &\cong H^k(A; \mathbb{G}_m)_{(2)} \text{ for } k \neq 1. \end{aligned}$$

Thus if A has n primes over 2 we have

$$H^2(A; W^{(1)}) \cong Br(A)_{(2)} \cong (\mathbb{Z}/2)^{r_1} \oplus (\mathbb{Q}/\mathbb{Z})^{n-1},$$

and $H^3(A; \mathbb{G}_m) = 0, H^4(A; \mathbb{G}_m) \cong (\mathbb{Z}/2)^{r_1}$, etc. The reader should compare this result with the discussion on pp.108-9 of [Milne].

Remark 7.1.3. Recently, Levine [L] and Merkurjev-Suslin [MS] proved that for a totally real number field F we have a (non-split) extension

$$0 \to (\mathbb{Z}/2)^{r_1} \to K_3(F) \xrightarrow{c_{21}} H^0(F; \mathbb{Q}/\mathbb{Z}(2)) \to 0.$$

Since $K_3(A) = K_3(F)$ by 4.6 and $c_{21} : K_2(A) \cong H^1_{et}(A; \mathbb{Q}/\mathbb{Z}(2))$ by 5.5 we see that

$$\frac{{}^{\#}K_2(A)}{{}^{\#}K_3(A)} \cong \frac{{}^{\#}H^1(A; \mathbb{Q}/\mathbb{Z}(2))}{{}^{\#}H^0(A; \mathbb{Q}/\mathbb{Z}(2)) \cdot {}^{\#}H^4(A; \mathbb{Q}/\mathbb{Z}(1))}$$

If the Birch-Tate conjecture holds for F then this equals $\pm 2^{-r_1}\zeta_F(-1)$. As observed in [L], this is related to the γ-filtration on K-theory. It also suggests that perhaps the 2-part of ${}^{\#}K_{2i-2}(A)/{}^{\#}K_{2i-1}(A)$ is related to the étale Euler characteristic more closely than has been thought.

In order to describe $H^2(A; W^{(i)})$, we need to be more careful. By 7.1.2, we need to assume that $i \geq 2$ in order to get manageable results. The proof of 7.1 shows that if i is odd then there are natural surjections

$$H^2(A; W^{(i)}) \to H^2(F; W^{(i)}) \to (\mathbb{Z}/2)^{r_1}.$$

In order to proceed further, we need to distinguish between the exceptional and non-exceptional number fields (see 6.3.0).

Lemma 7.2. *(Lichtenbaum) If F is non-exceptional and $i \geq 2$ then $H^2_{et}(F; W^{(i)}) = 0$.*

Proof. ([Licht,9.5]). Let $F_\infty = \lim_{\nu \to \infty} F(\mu_{2^\nu})$ be the maximal abelian 2-primary extension of F, and let Γ denote the Galois group of F_∞/F. If F is non-exceptional, then Γ is cyclic, so that $cd_2(\Gamma) = 1$. Now F^*_∞ is a discrete Γ-module, and $i \geq 2$, so by [T342] we have

$$H^1(\Gamma, H^1_{et}(F_\infty; W^{(i)})) = H^1(\Gamma, F^*_\infty \otimes W^{(i-1)}) = 0.$$

The Hochschild-Serre spectral sequence therefore degenerates. Since $cd_2(F_\infty) = 1$ (see [S-CG,Prop.II-9]), this yields

$$H^2(F; W^{(i)}) = H^2(F_\infty; W^{(i)})^\Gamma = 0.$$

Remark 7.2.1. If F is exceptional and $i \geq 2$, the usual transfer argument shows that $H^2(F; W^{(i)})$ has exponent 2, and 2-rank $\geq r_1$ if i is odd.

Theorem 7.3. *(Soulé). If F is totally imaginary (or a global field of characteristic $\neq 0, 2$) and $i \geq 2$ then $H^2_{et}(A; W^{(i)}) = H^2_{et}(F; W^{(i)}) = 0$.*

Proof. ([S,IV.3.2]). By 4.2, we have a commutative diagram:

$$\begin{array}{ccccc} \lim\limits_{\nu\to\infty} K_{2i-1}(F;\mathbb{Z}/2^\nu) & \xrightarrow{\partial} & \lim\limits_{\nu\to\infty} \bigoplus\limits_v K_{2i-2}(k_v;\mathbb{Z}/2^\nu) & & \\ \Big\downarrow c_{i1} & & \Big\downarrow (1-i)c_{i-1,0} & & \\ H^1(F;W^{(i)}) & \xrightarrow{\partial} & \oplus H^0(k_v;W^{(i-1)}) & \to H^2(A;W^{(i)}) \to H^2(F;W^{(i)}) \to 0. \end{array}$$

Since $K_{2i-2}(k_v; \mathbb{Z}/2^\nu) \cong K_{2i-1}(k_v)$ for $\nu \gg 0$ and $K_{2i-2}(F)$ is a torsion group, the top map ∂ is onto by 4.6. By 5.1, the image of $(1-i)c_{i-1,0}$ is $(i-1)!$ times $\oplus H^0(k_v; W^{(i-1)})$. By 7.2 and 7.2.1 this implies that $H^2(A; W^{(i)})$ has exponent dividing $2(i-1)!$. But $H^2(A; W^{(i)})$ is divisible by the following lemma, so it must be zero.

Lemma 7.3.1. *Let F be a totally imaginary number field (or a function field in one variable over a finite field of characteristic $\neq 2$), and A a ring of S-integers in F containing $\frac{1}{2}$. Then for all i the group $H^2_{et}(A, W^{(i)})$ is divisible.*

Proof. By [Tate,p.292], $cd_2(A) = 2$. Thus $H^3(A; \mu_2) = 0$. Now consider the Kummer sequence associated to $W^{(i)}$:

$$H^2(A; W^{(i)}) \xrightarrow{2} H^2(A; W^{(i)}) \to H^3(A; \mu_2).$$

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